Three-Loop
Hadronic Vacuum Polarization

Nordic Lattice 2026, Edinburgh **Based on [2510.12885] [2603.15252]**

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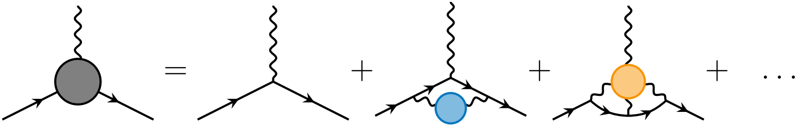
Initiative d'excellence Aix-Marseille

19 May 2026

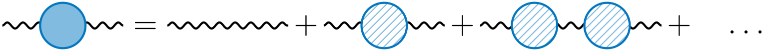
Introduction

HVP recap

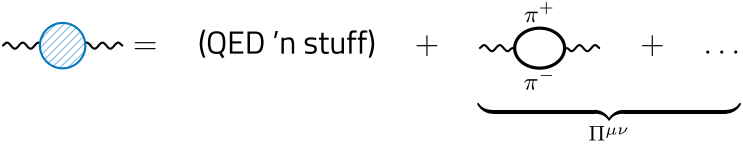
- Fermion-photon coupling (g-factor, etc.)



- ...from which the *vacuum polarization* 



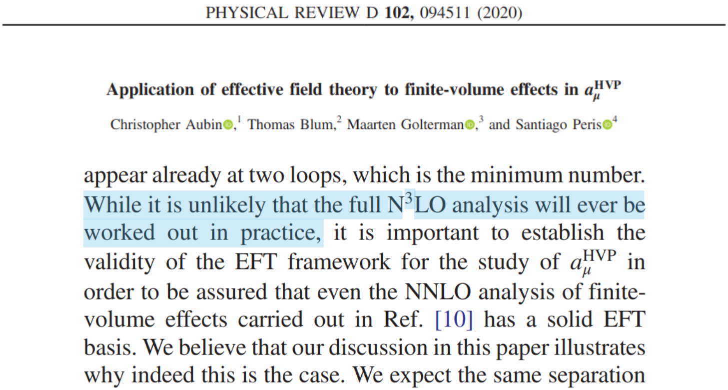
- ...from which the *hadronic* vacuum polarization $\Pi^{\mu\nu}$



Lattice QCD vs. ChPT

	Lattice QCD	**ChPT**
Physics	QCD + add-ons	QCD + add-ons
Pheno	All of it	Mainly pions
Concept	First principles	Effective field theory
Perturbative?	No	Yes
Spacetime	Euclidean	Either
Volume	Finite	Either (finite is harder)
High energy?	Discretization effects	Not valid
Low energy?	Finite-volume effects	No problems
Requirements	Collaborations & supercomputers	A laptop, a blackboard & some grit

The challenge



PHYSICAL REVIEW D **102,** 094511 (2020)

Application of effective field theory to finite-volume effects in a_μ^{HVP}

Christopher Aubin,[1] Thomas Blum,[2] Maarten Golterman,[3] and Santiago Peris[4]

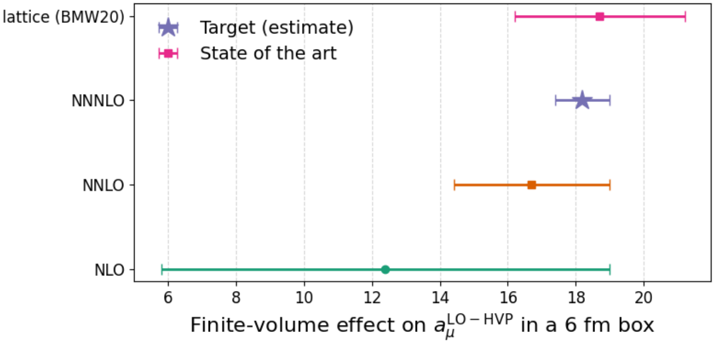
appear already at two loops, which is the minimum number. While it is unlikely that the full N^{3}LO analysis will ever be worked out in practice, it is important to establish the validity of the EFT framework for the study of a_μ^{HVP} in order to be assured that even the NNLO analysis of finite-volume effects carried out in Ref. [10] has a solid EFT basis. We believe that our discussion in this paper illustrates why indeed this is the case. We expect the same separation

Arthur C. Clarke’s 1st law

"When a distinguished [...] scientist [...] states that something is impossible, he is very probably wrong."

The ultimate goal

Assuming a geometric progression of FVE errors...



(analysis & figure by Alessandro Lupo)

The collaboration

initiated by

Kálmán Szabo

implemented by

Mattias Sjö

loop integrals by

Pierre Vanhove

phenomenology by

Alessandro Lupo

under the guidance of

Laurent Lellouch

The ChPT setup

The ChPT Lagrangian

- 2 flavors (pions only), non-dynamic photon field
- Vertices with $2n$ pions and up to 2 photons, drawn from

$$\mathcal{L}_{\text{ChPT}} = \mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} + \mathcal{L}_{\text{N}^3\text{LO}} + \cdots$$

475^* $\mathcal{O}(p^8)$ counterterms

Bijnens, (Herman-Trued)sson & Wang [1810.06384]

57^* $\mathcal{O}(p^6)$ counterterms

Bijnens, Colangelo & Ecker [hep-ph/9902437], [hep-ph/9907333]

7^* $\mathcal{O}(p^4)$ counterterms

Gasser & Leutwyler (1984)

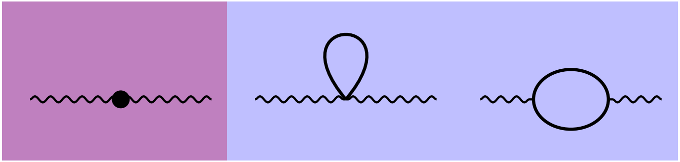
Basic $\mathcal{O}(p^2)$ vertex

Power counting (the simple way)

$$\text{N}^k\text{LO diagram:} \qquad k = (\#\text{ loops}) + \sum_{\text{vertices}} (\#\text{ N's})$$

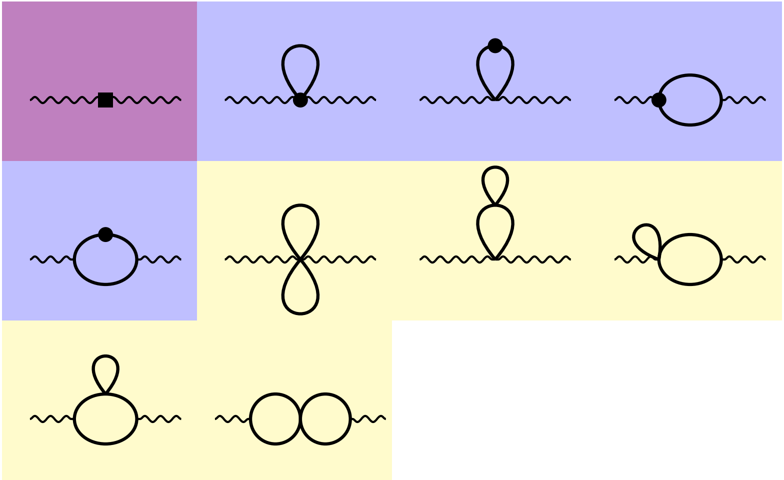
*These are 2-flavor numbers! It's 11,94,1254 for 3 flavors and 12,115,1862 for $\geq$ 4 flavors

NLO diagrams



● $=$ NLO (unmarked) $=$ LO

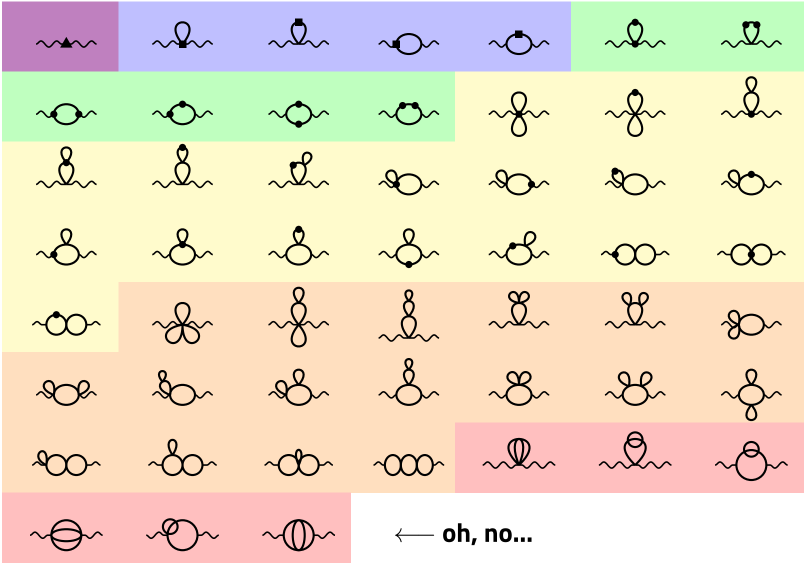
NNLO diagrams



■ = NNLO ● = NLO (unmarked) = LO

Note: no "genuine" 2-loop diagrams

N^3LO diagrams



⟵ **oh, no...**

ChPTlib

- New suite for ChPT calculations
  - "Automation of Hans Bijnens's spirit"
  - Written in FORM

    Vermaseren [math-ph/0010025]
  - Flexible implementation of $\mathcal{L}_{\text{ChPT}}$
  - Efficient extraction of Feynman rules
  - Simple input of Feynman diagrams

- Git repository: `github.com/mssjo/ChPTlib`
- HVP application: `github.com/mssjo/HVP-3loop`

Loop-wrangling

State-of-the art loop tools

IBP reduction

All integrals → small number of *master integrals*
Implemented with Roman Lee's `LiteRed2` [1212.2685]

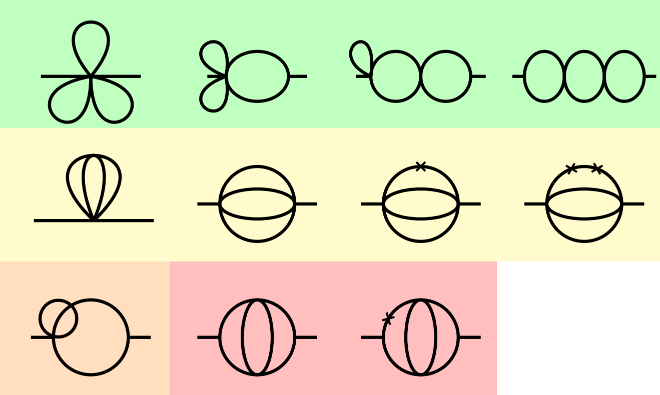
Tarasov's dimensional shift

$(4-2\epsilon)$-dimensional integrals (divergent)
$\rightarrow (2-2\epsilon)$-dimensional integrals (mostly finite)

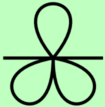
Schouten relations

Fix remaining divergences (novel!)

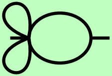
A tour of our masters



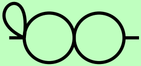
Masters: the easy ones



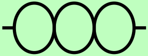
$= \mathbb{I}^3_{\underline{\circ}}$



$= \mathbb{I}^2_{\underline{\circ}}\mathbb{I}_{-\circ-}(t)$



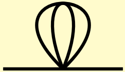
$= \mathbb{I}_{\underline{\circ}}\mathbb{I}^2_{-\circ-}(t)$

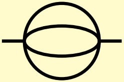


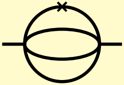
$= \mathbb{I}^3_{-\circ-}(t)$

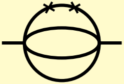
- Divergent (even in 2D) but simple
- Bubble integral required up to $\mathcal{O}(\epsilon^2)$
- Just a pile of (poly)logarithms of the photon virtuality t

Masters: the beachball


$= \mathbb{I}_{\ominus}(0) = 7\zeta(3)$


$\equiv \mathbb{I}_{\ominus}(t)$


$\sim \mathbb{I}'_{\ominus}(t)$


$\sim \mathbb{I}''_{\ominus}(t)$

- Example of the well-studied *banana graphs*
- $\mathbb{I}_{\ominus}(t)$ known (!) in terms of elliptic functions

  Bloch, Kerr & Vanhove [1406.2664]

The elliptic wonderland

Take q, $|q| < 1$ such that

$$t = -\left[\frac{\eta(q)\eta(q^3)}{\eta(q^2)\eta(q^6)}\right]^6,$$

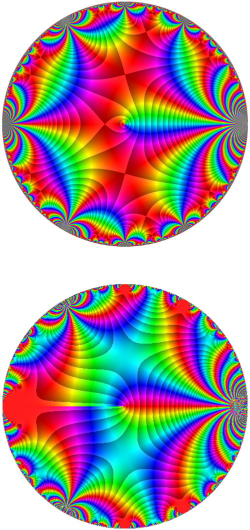
using the Dirichlet η function

Then

$$\mathbb{I}_{\ominus}(t) = \frac{\left[\eta(q^2)\eta(q^6)\right]^4}{\left[\eta(q)\eta(q^3)\right]^2}$$

$$\times\left(3(\log q)^3 - 16\zeta(3) + \sum_{n=1}^{\infty}\frac{\psi_{\ominus}(n)}{n^3}\frac{q^n}{1-q^n}\right)$$

$\psi_{\ominus} = \{1, -15, -8, -15, 1, 120\}$ repeated



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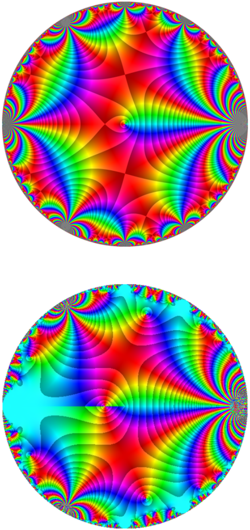

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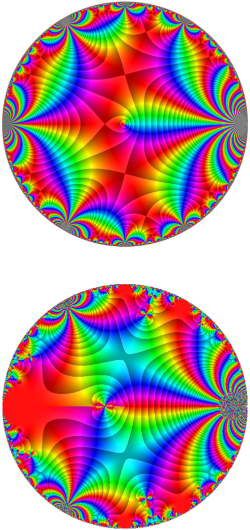

Then

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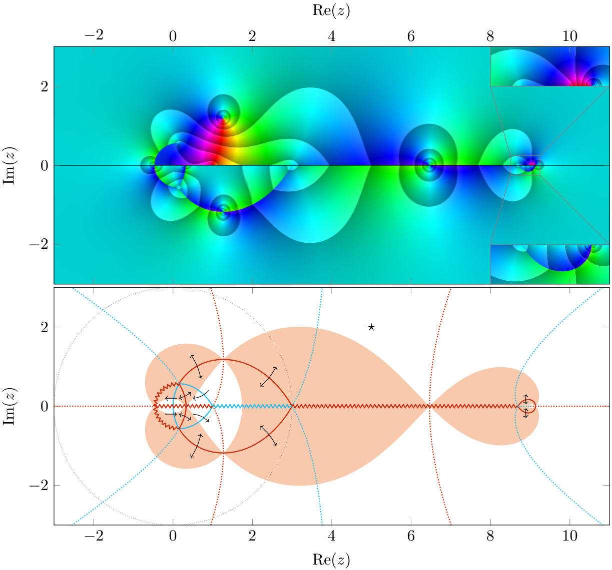
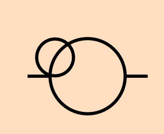
$\psi_{\ominus} = \{1, -15, -8, -15, 1, 120\}$ repeated

The road to wonderland

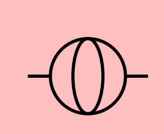


Masters: the irrelevant one

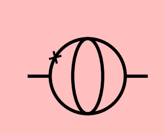


- Less well-studied, no explicit solution known before
- **Vanishes** under dimensional shift: leaves $[\mathbb{I}_{\ominus}, \mathbb{I}_{\circ}]$

Masters: the cat's eye



$\equiv \mathbb{I}_{\text{⦶}}(t)$



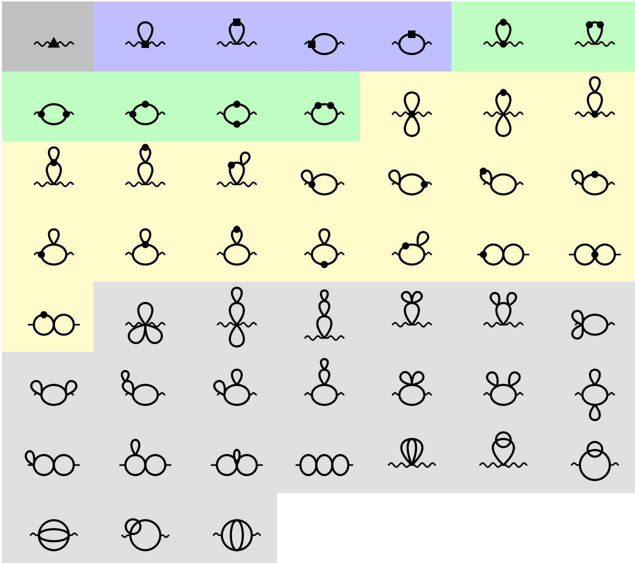
$\sim \mathbb{I}'_{\text{⦶}}(t)$

- Very difficult
- Seemingly breaks renormalization:

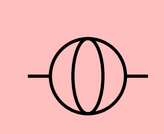
$$\mathbb{I}^{(d=4-2\epsilon)}_{\ominus} \sim \overbrace{\frac{1}{\epsilon^3} + \frac{[\mathbb{I}_{\bigcirc}]}{\epsilon^2} + \frac{[\mathbb{I}_{\bigcirc}]}{\epsilon}}^{\text{managed by counterterms}} + \overbrace{\bar{\mathbb{I}}_{\ominus}}^{\text{finite}}$$

$$\mathbb{I}^{(d=4-2\epsilon)}_{\text{⦶}} \sim \frac{1}{\epsilon^3} + \overbrace{\frac{[\mathbb{I}_{\bigcirc}, \mathbb{I}_{\ominus}, \mathbb{I}_{\text{⦶}}]}{\epsilon^2} + \frac{[\mathbb{I}_{\bigcirc}, \mathbb{I}_{\ominus}, \mathbb{I}_{\text{⦶}}]}{\epsilon}}^{\textbf{no counterterm!}} + \overbrace{\bar{\mathbb{I}}_{\text{⦶}}}^{\text{finite}}$$

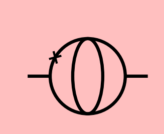
Masters: the cat's eye



Masters: the cat's eye



$\equiv \mathbb{I}_{⦶}(t)$



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$$\mathbb{I}^{(d=4-2\epsilon)}_{⦶} \sim \frac{1}{\epsilon^3} + \overbrace{\frac{[\mathbb{I}_{\circ}, \mathbb{I}_{\ominus}, \mathbb{I}_{⦶}]}{\epsilon^2} + \frac{[\mathbb{I}_{\circ}, \mathbb{I}_{\ominus}, \mathbb{I}_{⦶}]}{\epsilon}}^{\textbf{no counterterm!}} + \overbrace{\bar{\mathbb{I}}_{⦶}}^{\text{finite}}$$

Schouten relations

Gram determinant — Schouten identity

$$\mathbb{G}(u, v, w) = \begin{pmatrix} u^2 & u \cdot v & u \cdot w \\ v \cdot u & v^2 & v \cdot w \\ w \cdot u & w \cdot v & w^2 \end{pmatrix} \quad \Rightarrow \quad |\mathbb{G}(u, v, w)| = 0 \text{ in } d = 2$$

- Put $\mathbb{G}(\ell_1, \ell_2, \ell_3)$ as numerator in suitable integral
  $\Rightarrow$ linear combination $[\mathbb{I}_{\bigcirc}, \mathbb{I}_{\ominus}, \mathbb{I}_{\circledcirc}]$ vanishing at $d = 2$
  $\Rightarrow$ renormalization restored
- Effectively eliminates master integral $\mathbb{I}'_{\circledcirc}(t)$ in 2D
  $\Rightarrow$ non-IBP relation!
- Novel here, but known in 2-loop sunset integrals with 3 independent masses Remiddi & Tancredi [1311.3342]

Results

HVP overview

Parameters

M_π, F_π	Pion mass & decay constant
$t = q^2/M_\pi^2$	Normalized photon virtuality
$l_i, c_i, \tilde{c}_i$	LECs (l_i well known, c_i poorly known, $\tilde{c}_i$ unknown)

- **Transverse** due to Ward identity
- **Finite** after $\overline{MS}$ renormalization (ChPT variant)
- **Invariant** under NG manifold reparametrization
- **Subtracted** at $t = 0$

Power counting layout

$$16\pi^2\left[\Pi_T(t) - \Pi_T(0)\right] = \bar{\Pi}^{\text{NLO}}(t) + \xi\bar{\Pi}^{\text{NNLO}}(t) + \xi^2\bar{\Pi}^{\text{N}^3\text{LO}}(t) + \mathcal{O}(\xi^3)$$

(convergent since $\xi = M_\pi^2/(16\pi^2 F_\pi^2) \approx 0.03$)

Lower orders

Bubble function

$$B(t) = -\frac{1}{9} + \frac{4-t}{3t}\int_0^1 \log\left[1 - x(1-x)t\right]\, dx$$

- Leading order Gasser & Leutwyler (1985)

  $\bar{\Pi}^{\mathrm{NLO}}(t) = +2B(t) + \frac{2}{3}$

- Next-to-leading order Golowich & Kambor (1995)

  $\bar{\Pi}^{\mathrm{NNLO}}(t) = +\, tB(t)^2 - 4tB(t)l_6 - 8c_{56}t$

🟩 Function of t only

🟨 LECs known

🟧 LECs TBD

🟥 LECs unknown

⬜ Dispersive+FVE

▨ FVE only (?)

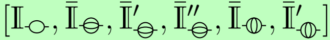
▨ No Dispersive/FVE

Three-loop order

Pion vector form factor NNLO LECs (known)

$$r_{V1} = -4(2c_{53} + 2c_{35} + 4c_6)\,, \qquad r_{V2} = 4(c_{53} - c_{51})$$

- Next-to-next-to-next-to-leading order This work (2025)

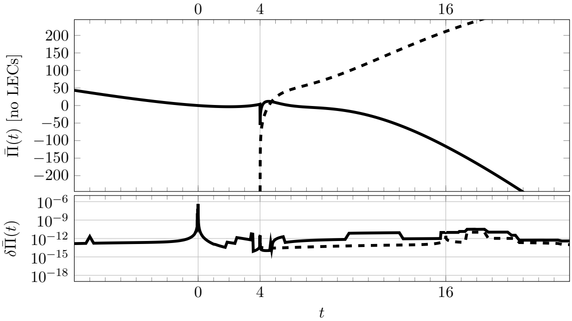


$$\begin{aligned} \bar{\Pi}^{\mathrm{N^3LO}}(t) = \;& + [\mathbb{I}, \bar{\mathbb{I}}, \bar{\mathbb{I}}', \bar{\mathbb{I}}'', \bar{\mathbb{I}}, \bar{\mathbb{I}}'] \\ & + tB(t)^2\left[t(l_2 - 2l_1 - 2l_6) + 2l_4\right] \\ & + 2tB(t)l_6(4l_4 - l_6 t) \\ & + 4tB(t)\left(r_{V1} + r_{V2}t\right) \\ & - 16 l_4 c_{56} t \\ & + 16t(\tilde{c}_{332} - \tilde{c}_{333}) - 8t^2\tilde{c}_{459} \end{aligned}$$

Function of t only | LECs known | LECs TBD | LECs unknown

Dispersive+FVE | FVE only (?) | No Dispersive/FVE

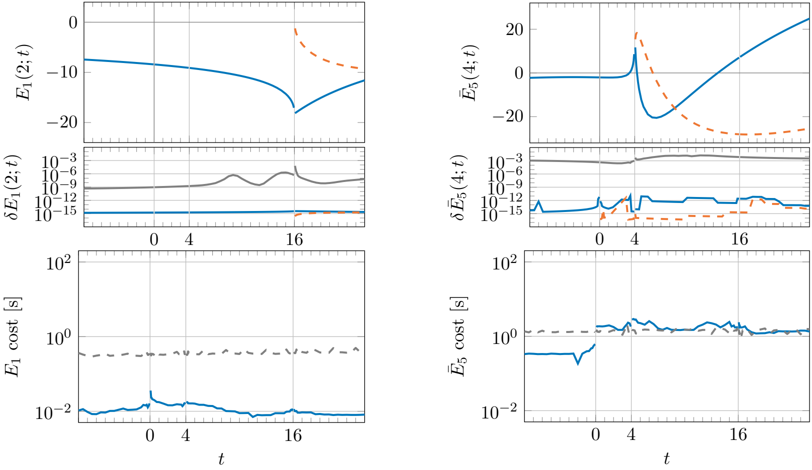
The LEC-independent part



Evaluation of master integrals

- Stable over entire complex plane
- 10^3–10^{12} times more precise than `pySecDec`
  Borowka et al. [1703.09692]
- 1–1000 times faster than `pySecDec`

Benchmarking against `pySecDec`



Loop tools in finite volume

Loop integrals (Minkowski)

$$\int_\ell \frac{1}{\prod_j (k_j^2 - m_j^2 + i\varepsilon)^{\nu_j}}$$

Integration-by-parts reduction

All integrals → small number of *master integrals*
Implemented with Roman Lee's `LiteRed2` [1212.2685]

Tarasov's dimensional shift + Schouten relations

$(4-2\epsilon)$-dimensional integrals (divergent)
$\to (2-2\epsilon)$-dimensional integrals (finite)

Loop tools in finite volume

Loop integroids (Euclid)

$$\hat{\mathcal{O}}_\ell \frac{1}{\prod_j (k_j^2 + m_j^2)^{\nu_j}}$$

Integration-by-parts reduction

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Operator-agnostic reduction

All integroids $\rightarrow$ small-ish number of *master integroids*
Novel method based on momentum rerouting

Tarasov's dimensional shift + Schouten relations

$(4 - 2\epsilon)$-dimensional integrals (divergent)
$\rightarrow (2 - 2\epsilon)$-dimensional integrals (finite)

Loop tools in finite volume

Loop integroids (Euclid)

$$\hat{\mathcal{O}}_\ell \frac{1}{\prod_j (k_j^2 + m_j^2)^{\nu_j}}$$

Operator-agnostic reduction

All integroids → small-ish number of *master integroids*
Novel method based on momentum rerouting

Divergence rearrangement

Integroids (divergent, possibly non-factorizable)
→ Integroids (finite, non-factorizable) + (divergent, factorizable)

Summary and Outlook

- Amplitude calculation published
- Loop integrals on arXiv and under review
- Low-hanging phenomenological fruit being picked
- A big step toward FVEs — to be continued...

Thank you!

The LECs

Pion form factor and charge radius

$$F_V^\pi(s) = 1 + \tfrac{1}{6}\langle r^2\rangle_V^\pi s + c_V^\pi s^2 + \mathcal{O}(s^3)$$

$$\langle r^2\rangle_V^\pi = \xi[l_6] + \xi^2\left([l_1, l_2, l_4, l_6] + 6\boldsymbol{r_{V1}}\right) + \mathcal{O}(\xi^3)$$

$$c_V^\pi = \frac{\xi}{60} + \xi^2\left([l_1, l_2, l_4, l_6] + \boldsymbol{r_{V2}}\right) + \mathcal{O}(\xi^3)$$

Vector meson dominance

$$r_{V1} \approx -6.2 \qquad r_{V2} \approx +6.5$$

Data (old)

$$r_{V2} = 4.0 \pm 1.2$$

Bijnens, Colangelo & Talavera [hep-ph/9805389]

The LECs

Pion form factor and charge radius

$$F_V^\pi(s) = 1 + \tfrac{1}{6}\langle r^2\rangle_V^\pi s + c_V^\pi s^2 + \mathcal{O}(s^3)$$

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Vector meson dominance

Data

Bijnens, Colangelo & Talavera [hep-ph/9805389]

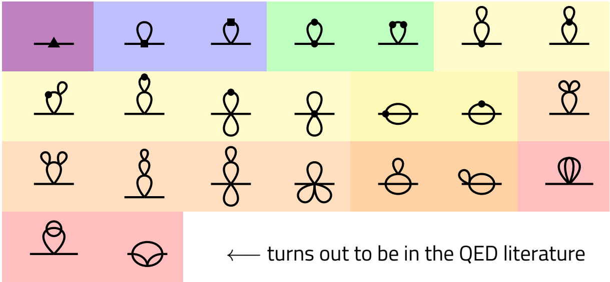
Lattice QCD

Potentially better precision (efforts underway)
Access to pion mass dependence

Colangelo, Hofericher, Kubis, Niehus, de Elvira [2110.05493]

Pion self-energy for comparison

Bijnens, (Herman-Trued)sson & Wang [1810.06384]



⟵ turns out to be in the QED literature

More on dimensional shift

IBP reduction

Integration by parts

$$\mathbb{I}_k = \left[\prod_i \int \frac{\mathrm{d}^d \ell_i}{\pi^{d/2}}\right] \mathbb{J}_k \quad \Rightarrow \quad 0 = \left[\prod_i \int \frac{\mathrm{d}^d \ell_i}{\pi^{d/2}}\right] \frac{\partial}{\partial \ell_j^\mu} \left(q^\mu \mathbb{J}_k\right)$$

$\ell_j =$ any loop momentum, $q =$ any momentum

- Gives solvable relations $0 = \sum_k \alpha_k \mathbb{I}_k$ Laporta [hep-ph/0102033]
- Yields small set of irreducible integrals

  — **master integrals** —

  (all further efforts can be limited to these)
- We used Roman Lee's `LiteRed2` [1212.2685] for reduction (requires Mathematica, but other reducers struggled)

Symanzik graph polynomials

Spanning trees

Connect all vertices without forming loops:

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{j \notin T} \alpha_j \,,$$

Spanning 2-forests

Make cut, form spanning tree for each half:

$$\mathcal{F} = \sum_{\text{2-forests } F} (-p_{\text{cut}}^2) \prod_{j \notin F} \alpha_j - \mathcal{U} \sum_j \alpha_j m_j^2 \,.$$

Schwinger form

$$\mathbb{I}_{\vec{\nu}} = \int_\alpha \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{\boldsymbol{d}/2}}$$

- **Schwinger form** of a loop integral:
- Derivative w.r.t. **mass** just drops down α:

$$\frac{\partial}{\partial m_j^2}\frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}} = \frac{\partial}{\partial m_k^2}\frac{e^{(\dots)+\sum_i \alpha_i m_i^2}}{\mathcal{U}^{d/2}} = \alpha_j \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}}$$

- Meanwhile, in the propagator picture:

$$\frac{\partial}{\partial m_i^2}\frac{1}{D_j^{\nu_j}} = \frac{\nu_j \delta_{ij}}{D_j^{\nu_j+1}} \qquad \Rightarrow \qquad \frac{\partial}{\partial m_j^2}\mathbb{I}_{\vec{\nu}} = \nu_j \mathbb{I}_{\vec{\nu}+\hat{j}}$$

Dimensional shift

Tarasov [hep-th/9606018]

- Assemble facsimile of $\mathcal{U}$ from derivatives:

$$\mathcal{U} \equiv \sum_{\text{trees } T} \prod_{j \notin T} \alpha_j \qquad \longleftrightarrow \qquad \mathcal{D} \equiv \sum_{\text{trees } T} \prod_{j \notin T} \frac{\partial}{\partial m_j^2}$$

- Propagator form: just change $\vec{\nu}$:

$$\mathcal{D}\mathbb{I}_{\vec{\nu}}(d) = \sum_{\text{trees } T} \mathbb{I}_{\vec{\nu}+\vec{\nu}_T}(d) \prod_{j \notin T} \nu_j \,, \qquad \vec{\nu}_T = \sum_{j \notin T} \hat{j}$$

- Schwinger form: formally change the dimension:

$$\mathcal{D}\mathbb{I}_{\vec{\nu}}(d) = \int_{\alpha} \frac{\mathcal{U} e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2-1}} = \mathbb{I}_{\vec{\nu}}(d-2)$$

- Allows $d = 2 - 2\epsilon$ (generally nicer than $d = 4 - 2\epsilon$)

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