Order-Order Interface in Pure $\mathrm{SU}(N)$ Gauge Theory

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Order-Order Interface in Pure $\mathrm{SU}(N)$ Gauge Theory

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- Quick recap on pure gauge theory on the lattice
- The twist and the formation of order-order interface
- Measuring interface tension of order - order interface

Introduction

Motivation

- Pure $\mathrm{SU}(N)$ gauge theory is relevant to BSM physics for example dark matter candidates
- Understanding the nature of pure $\mathrm{SU}(N)$ gauge theory such as first order phase transitions and interfaces tensions

Introduction

Goal of this research

Measure Casimir scaling hypothesis for gluon string tension (Tobias' talk)[1]

$$\frac{k(N-k)}{s(N-s)}.$$

Evaluate perfect wetting hypothyses $\sigma_{oo}(T_c) = 2\sigma_{od}(T_c)$.

1 Bursa and Teper 2005

Pure gauge theory on the lattice

Phase transition[2]

Low temperature color confining $Z(N)$-symmetric disordered phase:

$$\langle P(\vec{x}) \rangle = 0.$$

High temperature color non-confining $Z(N)$-breaking ordered phase:

$$\langle P(\vec{x}) \rangle \sim z_n.$$

[2] Polyakov 1978; Susskind 1979

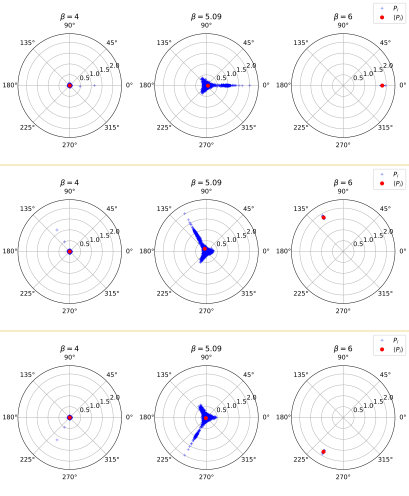
Pure gauge theory on the lattice

Phase transition

Initial: $\langle P_i \rangle = z_0$

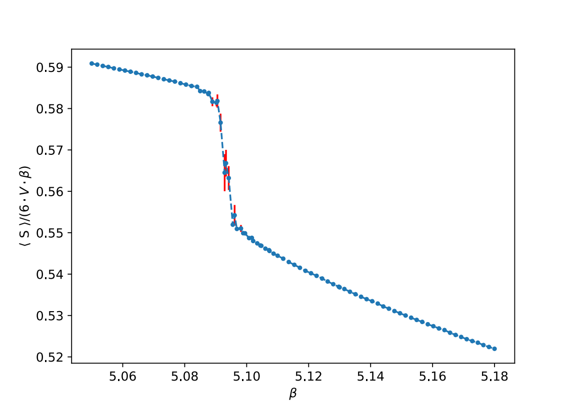
Initial: $\langle P_i \rangle = z_1$

Initial: $\langle P_i \rangle = z_2$



Pure gauge theory on the lattice

Phase transition

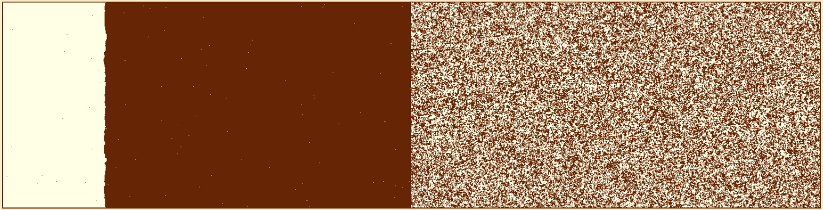


First order phase transition [3]

[3] Brown et al. 1988; Fukugita, Okawa, and Ukawa 1990

The twist and the formation of an o - o interface

The twist: ising



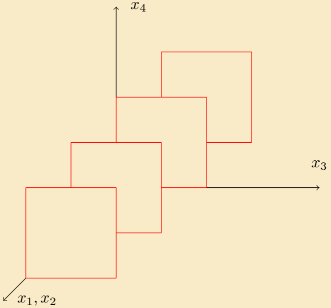
Low T: ordered phase   High T: disordered phase

order-order interface:

1. two ordered phases
2. moves freely
3. location is independent of twist

The twist and the formation of an o - o interface

The twist: $\mathrm{SU}(N)$



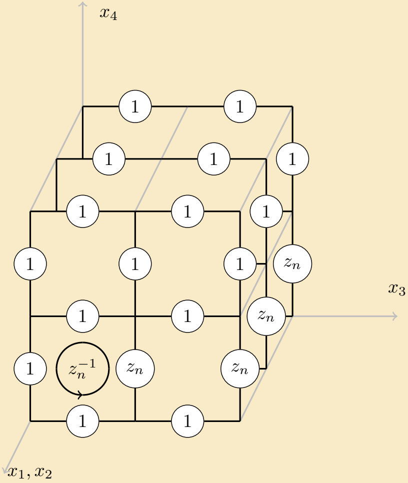
$$T = \{U_{34}(x_1, x_2, 0, 0) \mid \forall x_1 \in \{1, \cdots, N_1\},\ x_2 \in \{1, \cdots, N_2\}\}$$

$$S_T[U] = \frac{\beta}{N} \sum_x \sum_{\mu<\nu} \Re \operatorname{Tr}[1 - z_n^{-1} U_{\mu\nu}(x)].^4$$

4 Kajantie, Karkkainen, and K. Rummukainen 1991

The twist and the formation of an o - o interface

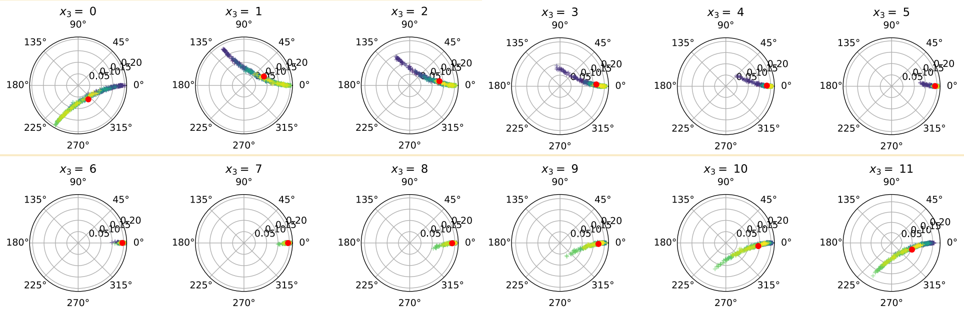
The twist: $\mathrm{SU}(N)$



The twist and the formation of an o - o interface

The twist: $\mathrm{SU}(N)$

Average polyakov loop over x_3-index at $\beta = 10 >> \beta_c$ [5]

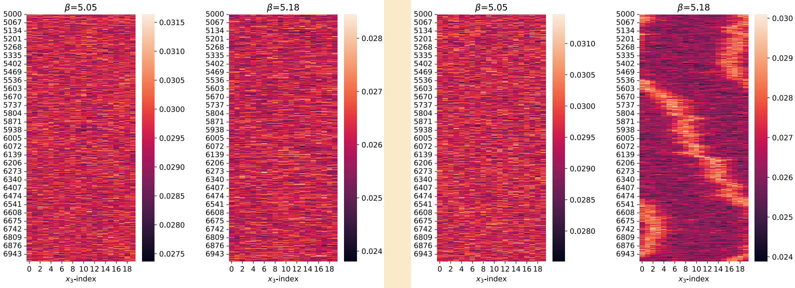


[5] $\mathrm{SU}(4)$ animation

The twist and the formation of an o - o interface

The twist: $\mathrm{SU}(N)$

Average action over x_3-index for no-twist (left) and twist (right)



Measuring interface tension of o - o interface

Integration method[6]

$$\frac{\partial}{\partial\beta}\frac{F}{T} = \frac{1}{\beta}\langle S_G[U]\rangle.$$

$$\alpha_{\text{o-o}} A = F_{\text{twist}} - F_{\text{no-twist}} = T\int d\beta \frac{1}{\beta}(\langle S_{\text{twist}}[U] - S_G[U]\rangle).$$

$$A = N_1 N_2 a^2 \text{ and } a = \tfrac{1}{N_t T}:$$

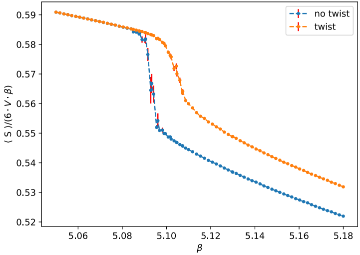
$$\Rightarrow \frac{\sigma_{o-o}}{T^3} = \frac{N_t^2}{N_1 N_2}\int_{\beta_0}^{\beta_1} d\beta \frac{1}{\beta}\langle S_{\text{twist}}[U] - S_G[U]\rangle.$$

6 Kajantie, Karkkainen, and K. Rummukainen 1991

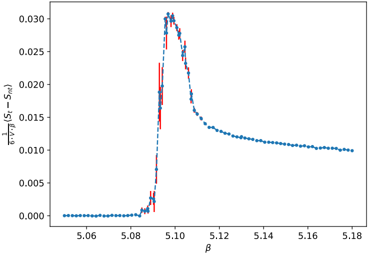
Measuring interface tension of o - o interface

Integration method

Average action



Difference in average action



Measuring interface tension of o - o interface

Integration method

FS-reweighting [7]

$$\langle O \rangle_{\beta'} = \frac{\langle O e^{-(\beta' - \beta) f_\phi} \rangle_\beta}{\langle e^{-(\beta' - \beta) f_\phi} \rangle_\beta}. \tag{1}$$

specifically multihistogram reweighting to construct the full smooth function.

Jackknife with blocking [8]

$$\sigma_{\hat{\theta}}^2 = \frac{M-1}{M} \sum_{m=1}^{M} (\theta_m - \hat{\theta})^2.$$

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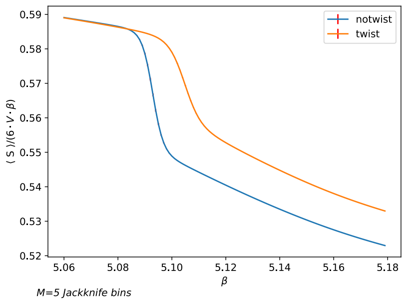
7 Ferrenberg and Swendsen 1988

8 Gattringer and Lang 2009, ch. 4.5.3

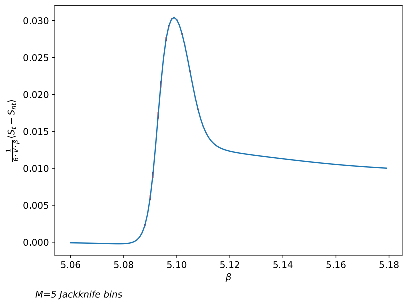
Measuring interface tension of o - o interface

Integration method

Average action



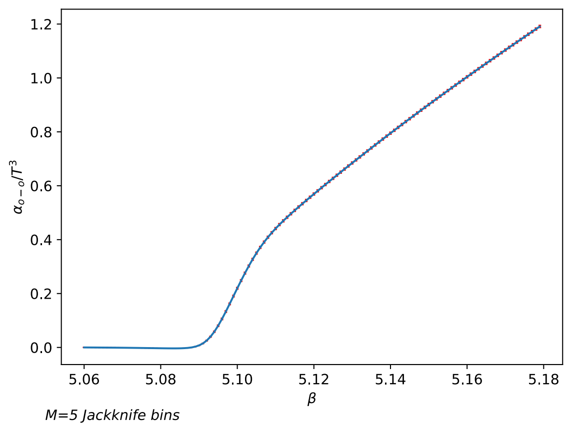
Difference in average action



Measuring interface tension of o - o interface

Integration method

Integrated surface tension



Measuring interface tension of o - o interface

Integration method

Transition barrier is dependent on confined-deconfined surface tension which scales as $\sigma_{c-d} \propto N^2$

$$\frac{P_{min}}{P_{max}} \approx \exp(-A\sigma_{c-d}/T_c + \text{finite-}V\text{corr.}).$$

In $SU(N > 3)$ standard Monte Carlo methods do not suffice.[9]

[9] Rindlisbacher, Kari Rummukainen, and Salami 2025

Measuring interface tension of o - o interface

Integration method

We use Multicanonical Monte Carlo [10] to sample the full first order phase transition [11]. Given a sufficient order parameters $\mathcal{O}(s)$ and a weight function $W(\mathcal{O}(s))$ we can measure expectation values of an observable A using the multicanonical distribution [12]

$$\langle A \rangle = \frac{\langle A(s) e^{W(\mathcal{O}(s))} \rangle}{\langle e^{W(\mathcal{O}(s))} \rangle}. \tag{2}$$

[10] Berg and Neuhaus 1992

[11] Kari Rummukainen, Seppä, and Weir 2026

[12] Hällfors and Kari Rummukainen 2025

Measuring interface tension of o - o interface

Integration method

The choice of the order parameter is a Polyakov line $|\langle P^s(x)\rangle|$ smeared perpendicular to the surface.
For each (x,y) set of z elements we use the smearing kernel

$$P_z^s(i) = \frac{P_z^{s-1}(i-1) + P_z^{s-1}(i) + P_z^{s-1}(i+1)}{3}.$$

Measuring interface tension of o - o interface

Integration method

Multi histogram reweighting with multicanonical weights

$$f_\beta = -\log\left(\sum_{i=1}^{R}\sum_{a=1}^{N_i}\frac{1}{\sum_{j=1}^{R} N_j e^{-(\beta_j-\beta)E_i^a+f_j}}\right) \Rightarrow f_\beta = -\log\left(\sum_{i=1}^{R}\sum_{a=1}^{N_i}\frac{e^{w_i^a}}{\sum_{j=1}^{R} \tilde{N}_j e^{-(\beta_j-\beta)E_i^a+f_j}}\right)$$

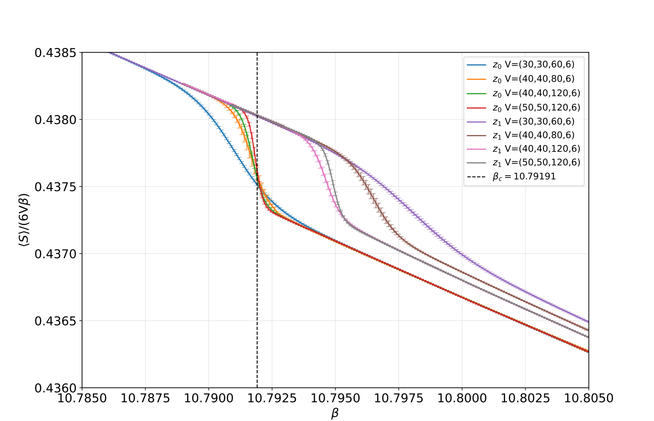
$$\langle\mathcal{O}\rangle_\beta = \sum_{i=1}^{R}\sum_{a=1}^{N_i}\frac{\mathcal{O}_i^a}{\sum_{j=1}^{R} N_j e^{-(\beta_j-\beta)E_i^a+f_j-f_\beta}} \Rightarrow \langle\mathcal{O}\rangle_\beta = \sum_{i=1}^{R}\sum_{a=1}^{N_i}\frac{\mathcal{O}_i^a e^{w_i^a}}{\sum_{j=1}^{R} \tilde{N}_j e^{-(\beta_j-\beta)E_i^a+f_j-f_\beta}}$$

$$\tilde{N}_i = \sum_{n=-\infty}^{\infty}\tilde{H}_i(n\,\Delta E) = \sum_{a=1}^{N_i} e^{w_i^a}$$

Measuring interface tension of o - o interface

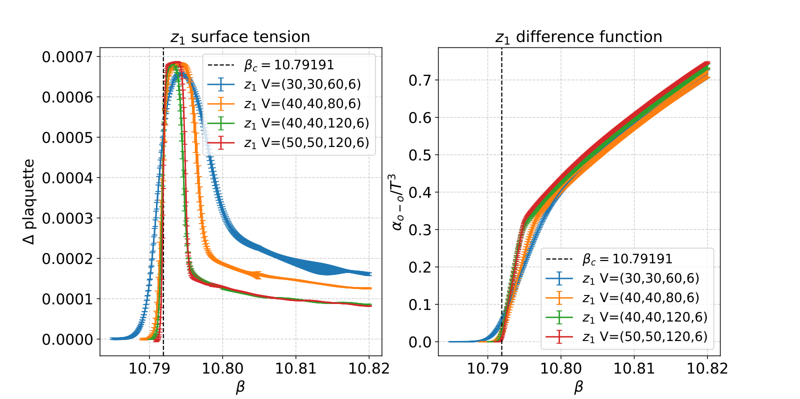
Finite volume

Multicanonical Monte Carlo and multihistogram reweighting produces very smooth functions



Measuring interface tension of o - o interface

Finite volume



- We notice $\lim_{N_z \to \infty} \Delta\beta_c = \beta_{c,t} - \beta_c = 0$
- We need an ansatz for extrapolating to β_c

Measuring interface tension of o - o interface

Finite volume effects

Consider the partition function ratios of the Z_i and Z_0 systems:

$$Z_i/Z_0 = e^{-\hat{\sigma}_V \frac{L^2}{N_t^2}}$$

in the $\lim_{V\to\infty} \hat{\sigma}_V = \sigma/T^3$ we require:

$$e^{-\hat{\sigma}_V \frac{L^2}{N_t^2}} = Ae^{-\hat{\sigma} \frac{L^2}{N_t^2}}$$

Measuring interface tension of o - o interface

Finite volume effects

$A = Z_{bulk} Z_{zm} Z_{cw}$ consists of the finite volume effects[13]

- Z_{bulk} - Gaussian fluctuations of the bulk represented by Z_0
- Z_{zm} - zeroth mode effects consisting of a IR divergence due to z-translation invariance $\propto N_z/\sqrt{L^2}$ and IR divergence of the zeroth mode CW fluctuation $\propto \sqrt{L^2}$
- Z_{cw} - Gaussian capillary wave fluctuations of the interface $Z_{cw}(L, L) := K_{cw} = \frac{4\pi^{3/2}}{\Gamma^2(1/2)}$

13 Iwasaki et al. 1994

Measuring interface tension of o - o interface

Finite volume effects

Thus we can express:

$$e^{-\hat{\sigma}_V \frac{L^2}{N_t^2}} = \tilde{c} N_z K_{cw} e^{-\hat{\sigma} \frac{L^2}{N_t^2}},$$

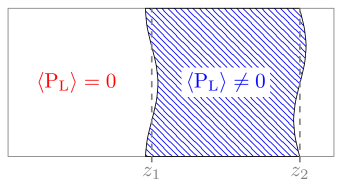
where $\tilde{c}$ contains the proportionality factors of expressing $\propto$.
Finally we can solve for the finite volume corrections:

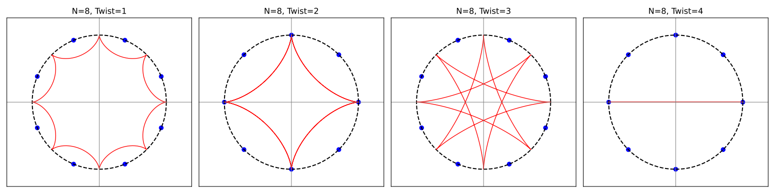
$$\sigma_V(\beta, N_t, L, N_z) = \sigma(\beta, N_t) - \frac{N_t^2}{L^2}(log(N_z) + \log(K_{cw}\tilde{c}(\beta, N_t)).$$

Measuring interface tension of o - o interface

Perfect wetting

Perfect wetting deconfined-confined surfaces at β_c join "perfectly" into a deconfined-deconfined surface with $\sigma_{oo} = 2\sigma_{od}$





Measuring interface tension of o - o interface

Perfect wetting ansatz

Using the infinite volume extrapolated function $\sigma(\beta, N_t)$ we can extrapolate to T_c based on the following ansatz that assumes perfect wetting [14]:

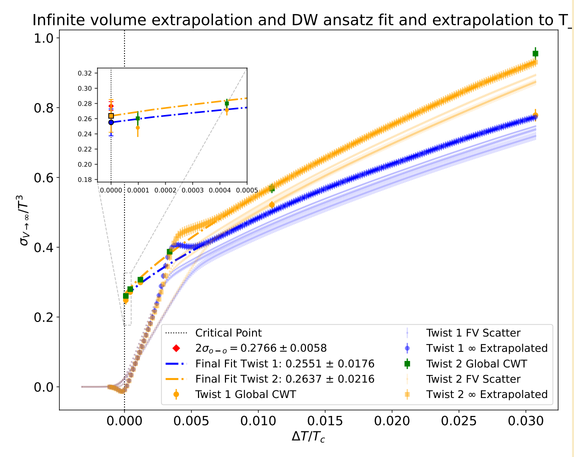
$$\frac{\sigma_{dd}}{T^3} = \frac{2\sigma_{cd}(1-3t)}{T_c^3} + \frac{\delta r_0 t}{T_c^2}(1 - log(\delta r_0 T_c t/\gamma))$$

$$\Rightarrow y = a(1-3x) + bx(1 - log(cx))$$

[14] Holland and Wiese 2000

Measuring interface tension of o - o interface

Perfect wetting result in SU(4)



Measuring interface tension of o - o interface

Conclusions

- Very good agreement with CWT method
- z_2 interface is in error bars of perfect wetting z_1 could go either way
- Next step is measuring $\mathrm{SU}(8)$ as there are 2x more twists with more perfect wetting possibilities

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