Progress understanding weakly-coupled phase transitions with lattice simulations

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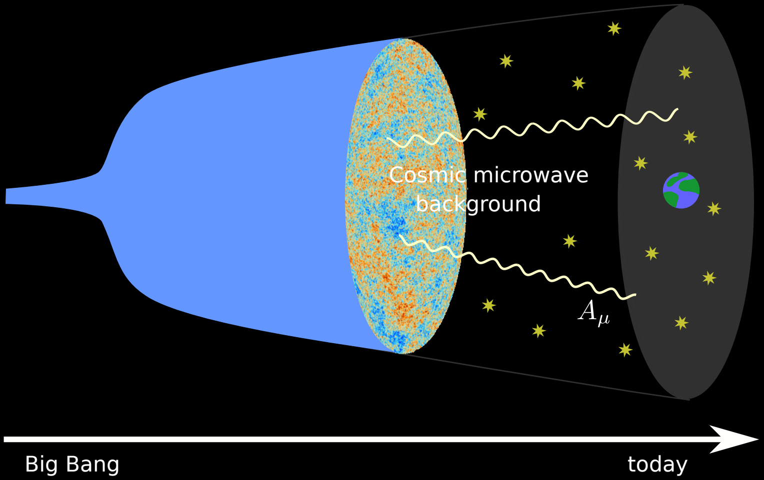
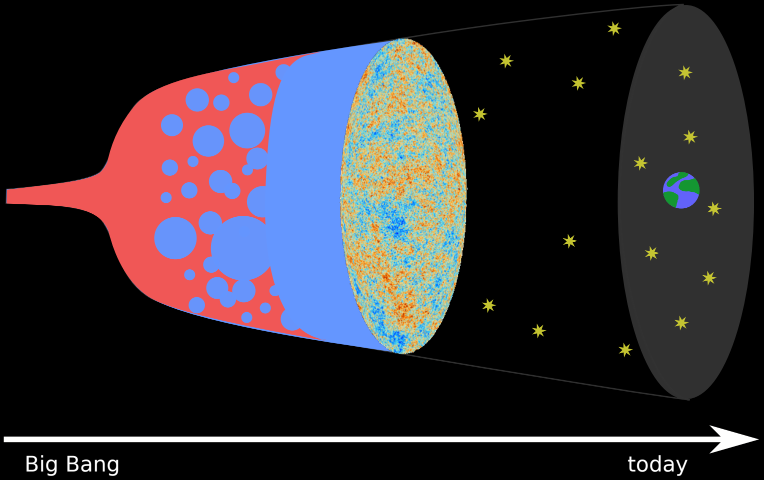
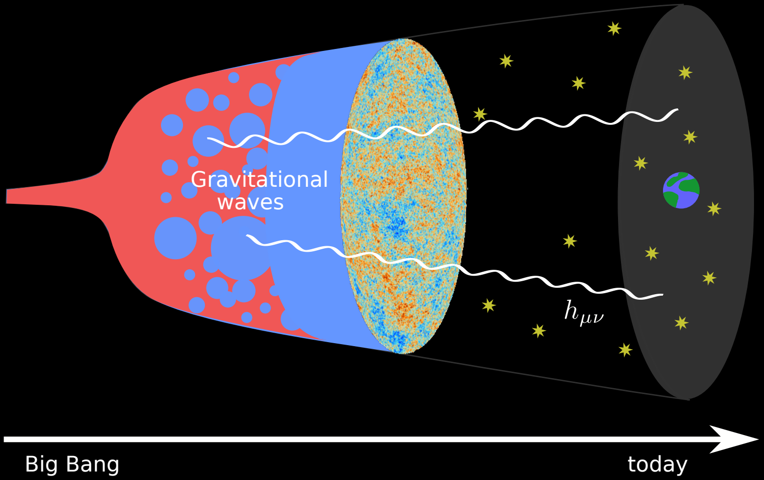
18 May 2026

University of Nottingham UK | CHINA | MALAYSIA

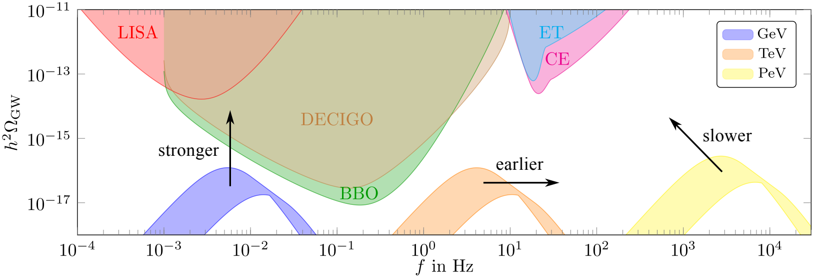
THE ROYAL SOCIETY


Cosmic microwave
background
A_μ
Big Bang
today


Big Bang
today


Gravitational waves
$h_{\mu\nu}$
Big Bang
today

Gravitational waves from phase transitions

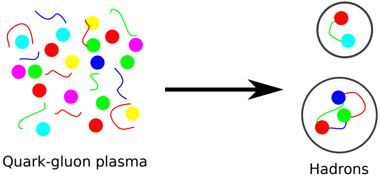


Confinement transitions in dark $\mathrm{SU}(6)$ Yang-Mills.

Huang et al. '21

Standard Model phase transitions

- Quark confinement occurs at $T \sim 155$ MeV



Aoki et al. '06

Standard Model phase transitions

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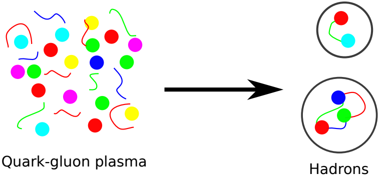
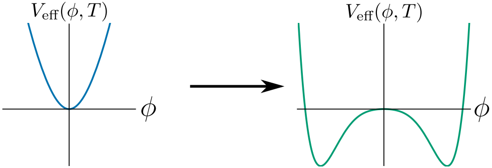


Aoki et al. '06

- Electroweak symmetry breaking occurs at $T \sim 160$ GeV

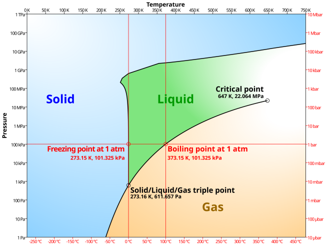


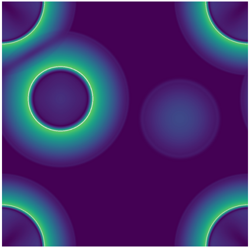
D'Onofrio & Rummukainen '16

Model space

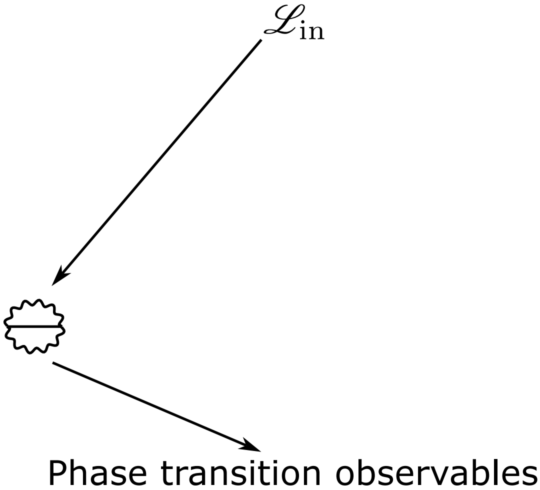

Models with cosmic phase transitions
Dark sectors
SIMPS
Stealth dark matter
WIMPS
In equilibrium with SM
Out of equilibrium with SM
FIMPS
GUT theories
SUSY
nMSSM
SUSY breaking
SU(5)× U(1)
SU(4)$_{PS}$× SU(2)× U(1)
SO(10)
Strongly-coupled models
SU($N \geq 3$) $n_f = 0$
SU($N \geq 3$) $n_f \geq 3$ $n_f \lesssim 4N$
QCD
Chemical potential
Dynamical m_q
Dynamical Λ_{QCD}
Composite Higgs
Electroweak extensions
SMEFT and HEFT
Vector like fermions
Scalar extensions
Singlet
2HDM
3HDM
SU(2) triplet
U(1)$_{B-L}$
Other unsolved problems
Quasi scale invariance
Randall-Sundrum
N-natural
Dilaton
Neutrino mass generation
Majoron
Ending inflation
Peccei-Quinn breaking

Understanding phase transitions

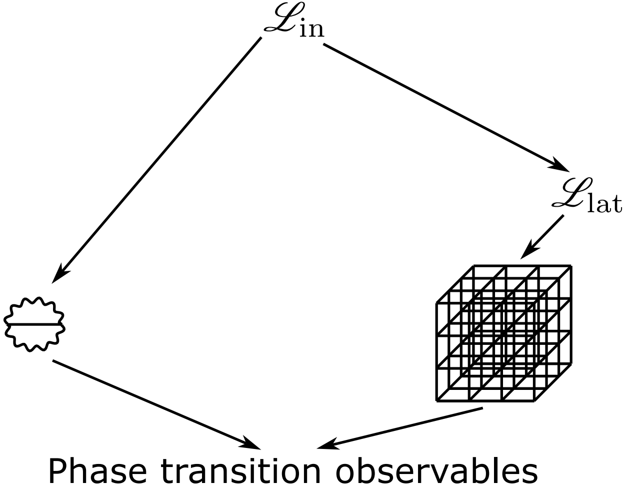
$\mathscr{L}_{\text{microphysics}} \rightarrow$ 

$\rightarrow$ 

Forks in the road



Forks in the road



Forks in the road

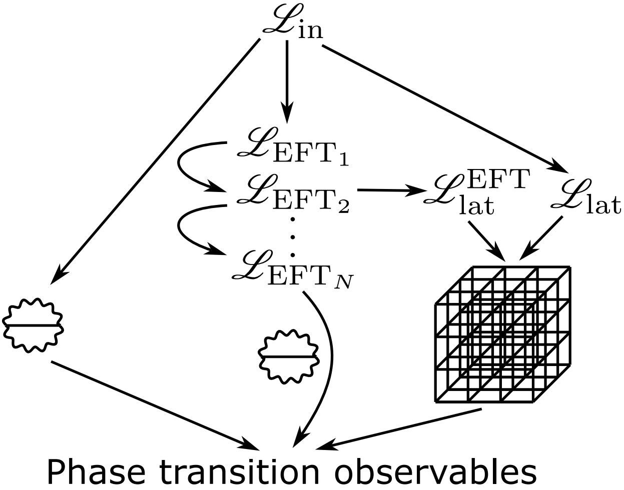


Figure idea: T.V.I. Tenkanen

Phase transitions at weak coupling

Principles of vanilla perturbation theory

Perturbation theory is organised into asymptotic expansions

$$\frac{f_{\text{obs}} - f_{\text{pert}}^{(n)}}{f_{\text{obs}}} = O(\alpha^{n+1}) \quad \text{as } \alpha \to 0_+$$

Principles of vanilla perturbation theory

Perturbation theory is organised into asymptotic expansions

$$\frac{f_{\text{obs}} - f_{\text{pert}}^{(n)}}{f_{\text{obs}}} = O(\alpha^{n+1}) \quad \text{as } \alpha \to 0_+$$

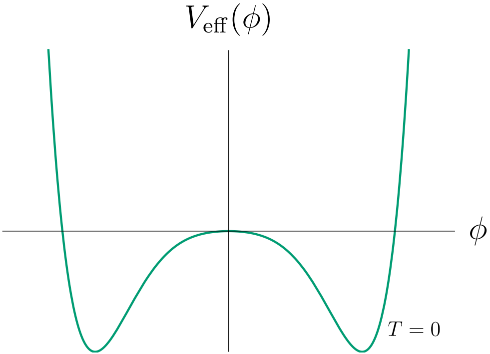
e.g. for the electron magnetic moment

$$\mu_e^{\text{th}} = \frac{-e}{m_e}\Big[\underbrace{1}_{\text{0-loop}} + \underbrace{\frac{1}{2\pi}\alpha}_{\text{1-loop}} + \underbrace{\frac{-0.328}{\pi^2}\alpha^2}_{\text{2-loop}} + \underbrace{\frac{1.181}{\pi^3}\alpha^3}_{\text{3-loop}} + \underbrace{\frac{-1.912}{\pi^4}\alpha^4}_{\text{4-loop}} + \ldots\Big]$$

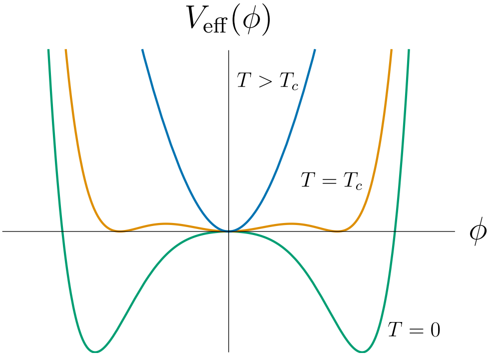
Laporta '18

The coefficients of an asymptotic expansion are unique, crucially.

Weakly coupled phase transitions

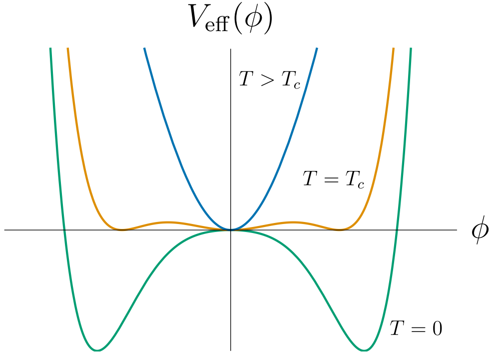


Weakly coupled phase transitions



$$V_{\text{eff}} = V_{\text{0-loop}} + \alpha V_{\text{1-loop}} + \ldots$$

Weakly coupled phase transitions



$$V_{\text{eff}} = V_{\text{0-loop}} + \alpha V_{\text{1-loop}} + \dots$$

At a weakly coupled phase transition, we must have that

$$\Rightarrow \frac{\alpha V_{\text{1-loop}} + \dots}{V_{\text{0-loop}}} = O(1) \quad \text{as } \alpha \to 0_+$$

Trouble for the vanilla loop expansion

- Vanilla loops not an asymptotic expansion at transition

⇒ Results not unique

⇒ Uncontrolled gauge fixing and RG scale dependence

- Also find spurious imaginary parts for observables

Trouble for the vanilla loop expansion

- Vanilla loops not an asymptotic expansion at transition

$\Rightarrow$ Results not unique

$\Rightarrow$ Uncontrolled gauge fixing and RG scale dependence

- Also find spurious imaginary parts for observables

This calls for new organising principles!

Hierarchies of scale

The UV dominates at fixed loop order, as $[V_{\text{eff}}] > 0$, and

$$\frac{V_{\text{1-loop}}}{V_{\text{0-loop}}} = \alpha\ F\left(\text{ratios of scales}\right) \overset{!}{=} O(1)\,,$$

$$\Rightarrow F\left(\text{ratios of scales}\right) = O\left(\frac{1}{\alpha}\right)$$

so, as $\alpha \to 0_+$, the (renormalised) loop integral must diverge $F\left(\text{ratios of scales}\right) \to \infty$.

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Weakly coupled phase transitions require scale hierarchies!

Wonders and weirdness at weak coupling

$$
\begin{aligned}
f(\alpha) = {} & a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4 + \dots \\
& + b_1\alpha\log\alpha + b_2\alpha^2\log^2\alpha + \dots \\
& + c_{\frac{3}{2}}\alpha^{\frac{3}{2}} + c_{\frac{5}{2}}\alpha^{\frac{5}{2}} + \dots + d_{\frac{3}{2}}\alpha^{\frac{3}{2}}\log\alpha + d_{\frac{5}{2}}\alpha^{\frac{5}{2}}\log^2\alpha + \dots \\
& + e_{\frac{9}{4}}\alpha^{\frac{9}{4}} + e_{\frac{11}{4}}\alpha^{\frac{11}{4}} + \dots + f_{\frac{9}{4}}\alpha^{\frac{9}{4}}\log\alpha + f_{\frac{11}{4}}\alpha^{\frac{11}{4}}\log\alpha + \dots \\
& + (e_0 + e_1\alpha + \dots)\exp(g_{-\frac{3}{2}}\alpha^{-\frac{3}{2}} + g_{-\frac{1}{2}}\alpha^{-\frac{1}{2}}) + \dots
\end{aligned}
$$

Weird powers $\alpha^{\frac{1}{2}}$

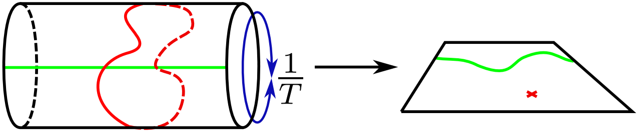
High temperature QFT

- Thermodynamics $Z = \mathrm{Tr} e^{-\hat{H}/T}$ formulated in $\mathbb{R}^3 \times S^1$

$$\Rightarrow \Psi(\tau, \mathbf{x}) = \sum_n \psi_n(\mathbf{x}) e^{i(n\pi T)\tau}$$

- These modes have masses $m_n^2 = m^2 + (n\pi T)^2$, so on energy scales $\ll \pi T$, the nonzero modes can be integrated out

$$\underbrace{\int_{\mathbb{R}^3 \times S^1} \mathscr{L}}_{\text{bosons and fermions}} \quad \to \quad \underbrace{\int_{\mathbb{R}^3} \mathscr{L}_{\text{eff}}}_{\text{just bosons}}$$

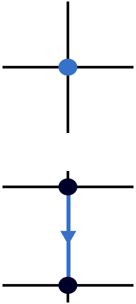


Kajantie et al. '95, Braaten & Nieto '95; Figure idea: K. Rummukainen

Simulating hot scalar fields

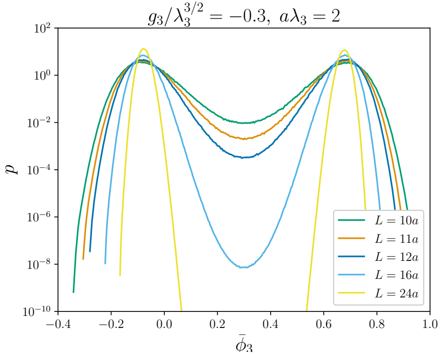
Scalar lattice action is 3d at high temperatures,

$$S = -\sum_x \sum_i \tfrac{a}{2}\phi(x)\left[\phi(x+i) + \phi(x-i) - 2\phi(x)\right] + \sum_x a^3 \left[\sigma_L \phi(x) + \tfrac{m_L^2}{2}\phi^2(x) + \tfrac{g_L}{3!}\phi^3(x) + \tfrac{\lambda_L}{4!}\phi^4(x)\right].$$



All fields living on scales $(\pi T)^{-1}$ are integrated out.

Simulating thermal first-order phase transitions



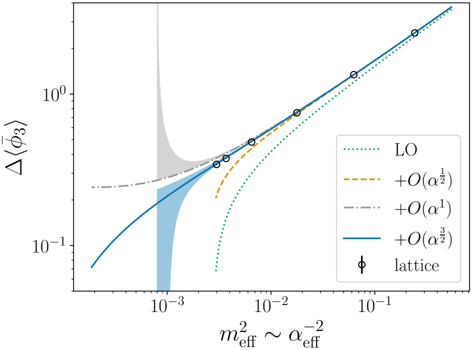
- Efficient update algorithms to sample two-peak distribution.

  Kajantie et al. '95

- Superrenormalisability $\implies$ *exact* lattice-continuum relations.

  Laine '95

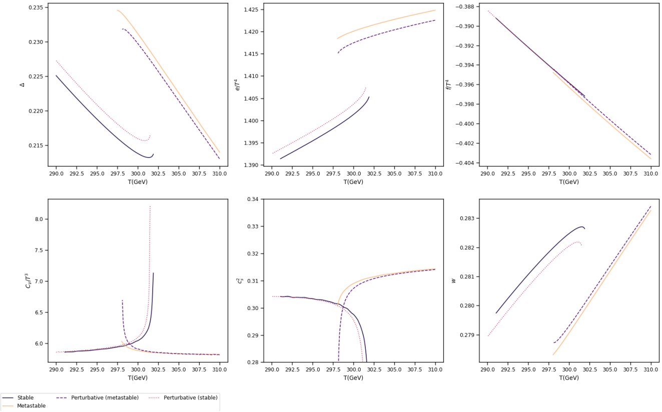
Jump in order parameter



$$\frac{1}{v_0}\Delta\langle\bar{\phi}_3\rangle = 2 + \sqrt{3}\,\alpha^{\frac{1}{2}} + \frac{1}{2}\left(1 + 2\log\tilde{\mu}_3\right)\alpha^{1} + \sqrt{3}\left[-\frac{3}{8\sqrt{2}}\xi + \frac{21}{32}\mathrm{Li}_2\frac{1}{4} - \frac{7\pi^2}{128} - \frac{1}{2} + \frac{21}{64}\log^2\frac{4}{3} + \frac{5}{8}\log\frac{4}{3}\right]\alpha^{\frac{3}{2}}$$

OG '21

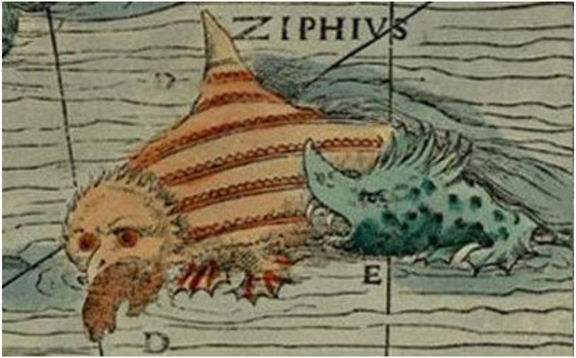
Thermodynamics of a first-order transition



Lattice versus (leading order) perturbation theory.

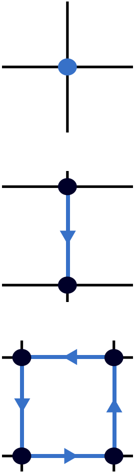
OG, Kormu & Weir (forthcoming)

Weirder powers $\alpha^{\frac{9}{4}}$



Simulating the hot electroweak model

Electroweak lattice action is 3d at high temperatures,

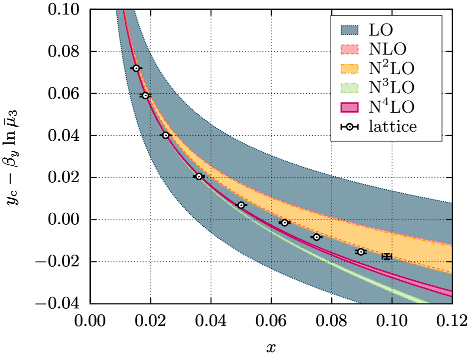


$$
\begin{aligned}
S =& \beta_G \sum_x \sum_{i<j} \left(1 - \tfrac{1}{2}\mathrm{Tr} P_{ij}\right) + \\
&- \beta_H \sum_x \sum_i \tfrac{1}{2}\mathrm{Tr}\Phi^\dagger(x) U_i(x) \Phi(x+i) + \\
&+ \sum_x \tfrac{1}{2}\mathrm{Tr}\Phi^\dagger(x)\Phi(x) + \beta_R \sum_x \left[\tfrac{1}{2}\mathrm{Tr}\Phi^\dagger(x)\Phi(x) - 1\right]^2
\end{aligned}
$$

Fermions live on scales $(\pi T)^{-1}$, so are integrated out.

Kajantie et al. '96

Electroweak phase diagram



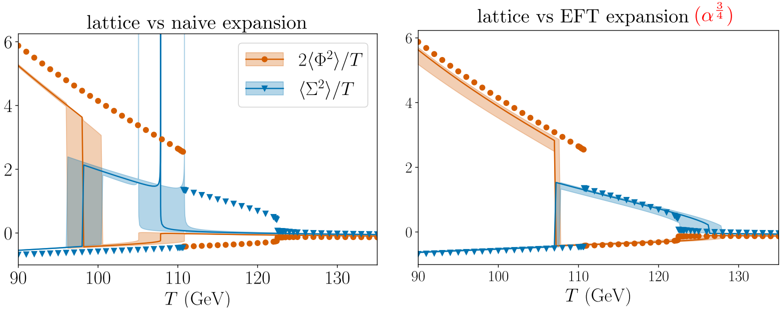
$$\frac{y_c}{y_c^{(\mathrm{LO})}} = 1 - \frac{51}{2}\alpha^{\frac{1}{2}} \log \tilde{\mu}_3 - 2^{\frac{3}{2}}\alpha^{\frac{3}{4}} + \alpha\left(42 \log \alpha + 72 \log \tilde{\mu}_3 + C\right)$$

$$+ \alpha^{\frac{5}{4}}\left(\frac{51}{2}\log \alpha + \frac{1025}{4} + 6\pi^2 + 197 \log 2 - 126 \log 3\right)$$

Ekstedt, OG, Löfgren '22; Ekstedt, Schicho & Tenkanen '24

Kajantie et al. '96; Moore & Rummukainen '00; OG, Güyer & Rummukainen '22

Beyond the Standard Model phase transitions



$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} - \frac{a_2}{2}\Phi^\dagger\Phi\Sigma^a\Sigma^a - \frac{1}{2}D_\mu\Sigma^a D^\mu\Sigma^a - \frac{m_\Sigma^2}{2}\Sigma^a\Sigma^a - \frac{b_4}{4}(\Sigma^a\Sigma^a)^2$$

Including $\alpha^{\frac{3}{4}}$ resummations is crucial for agreement with lattice.

Niemi et al. '20', OG & Tenkanen '23'

Wonderful exponentials $\exp(\alpha^{-1})$

Perturbation theory for bubble nucleation

The bubble nucleation rate takes a semiclassical form,

$$\Gamma \sim (d_0 + d_1\alpha + \dots)\, \exp(c_{-1}\alpha^{-1})\,.$$

Critical bubbles are exponentially unlikely.

Lattice simulations of bubble nucleation — a history

- Stochasic lattice simulations proposed for studying nucleation

  Grigoriev & Rubakov; Bochkarev & de Forcrand '88

- Lattice and semiclassical predictions strongly disagree on nucleation rate

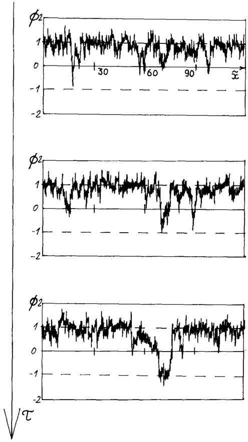
  Valls & Mazenko '90; Alford et al. '91

- Lattice counterterms and fitting improves qualitative agreement

  Alford et al. '93; Borsanyi et al. '00

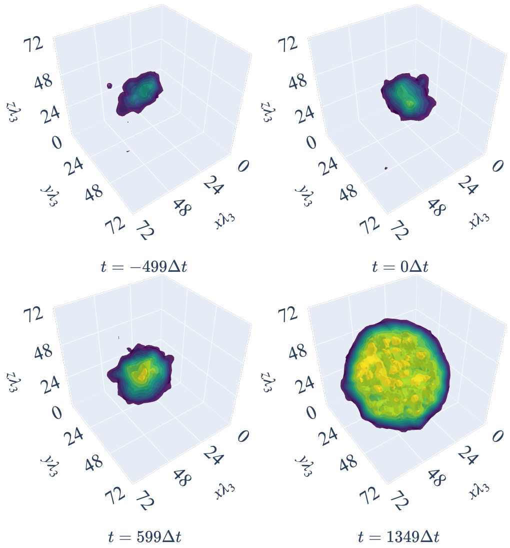
- New method for suppressed transitions, but still $e^{O(10)}$ to $e^{O(100)}$ disagreement

  Moore, Rummukainen, Tranberg '00, '01



Grigoriev & Rubakov '88

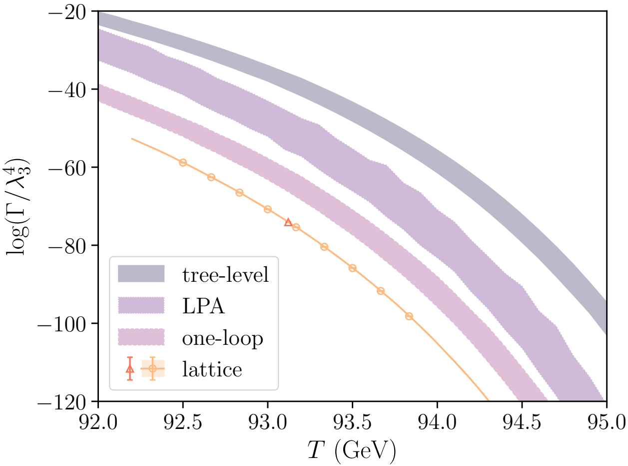
Nucleating field configurations



N.B. these are smoothed configurations.

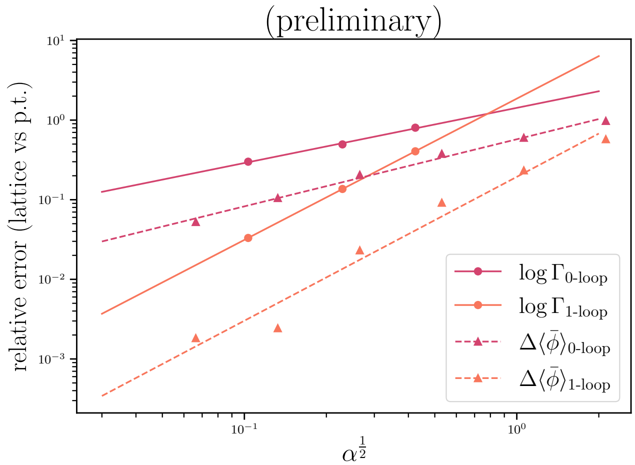
OG, Kormu & Weir '24

Comparing nucleation rates



OG, Kormu & Weir '24

Comparing convergence by observable



Errors on $\log \Gamma$ much larger, but convergence rate similar to $\Delta\langle\bar{\phi}\rangle$.

OG, Kormu & Weir (forthcoming)

Uncharted seas

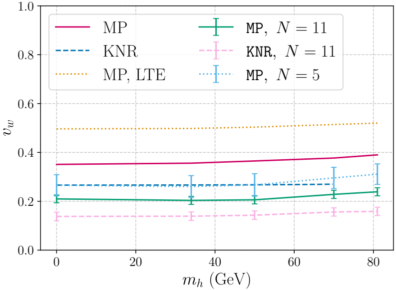
Bubble wall speed

Collision integrals can have wild structures at weak coupling,

$$\mathcal{C} = \alpha^2 T \left(c_{-1}\alpha^{-1} + c_0 + d_{-1}(\log \alpha)^{-1} + \dots \right) ,$$

so groups differ wildly on bubble speed results. Errors are $O(1)$.

Moore & Prokopec '95; Konstandin, Nardini & Rues '14; WallGo '24, '25



Lattice simulations to settle this are nascent. Mou, Saffin & Tranberg '20

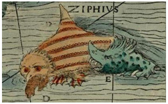
Summing up

For weakly-coupled phase transitions:

- vanilla loop expansion *always* fails
- lattice can sure-up foundations

Some things we learnt from lattice simulations:

- $\alpha^{\frac{1}{2}}$: works wonderfully when first order
- $\alpha^{\frac{9}{4}}$: tricker and tends to need tiny α
- $\exp(\alpha^{-1})$: leaves large errors, but can converge



What we don't yet know:

- is there something we're missing for $\log \Gamma$?
- can the lattice clear up the bubble speed mess?

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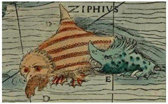

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- can the lattice clear up the bubble speed mess?

Thanks for listening!

Backup slides

3d effective theories

- In a 3d EFT, canonical mass dimensions are different

$$\int_{\mathbb{R}^3} \mathscr{L}_{\text{eff}} \supset \int_{\mathbb{R}^3} \Bigg[\underbrace{\frac{1}{2}\partial_i\phi_3\partial_i\phi_3}_{[\phi_3^a]=1/2} + \frac{m_3^2}{2}\phi_3^2 + \underbrace{\frac{\lambda_3}{4}}_{[\lambda_3]=1}\phi_3^4 \Bigg] .$$

  Marginal interactions become relevant $\rightarrow$ superrenormalisable.

- The loop expansion parameter changes

$$\alpha \equiv \frac{\lambda}{(4\pi)^2} \rightarrow \alpha_3 \approx \frac{\lambda T}{(4\pi)m_3} \sim \alpha^{\frac{1}{2}} .$$

Resummations in the thermal potential

Consider a real scalar field with potential,

$$V_0 = \sigma\phi + \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \,.$$

At high temperatures, thermal fluctuations modify the potential

$$V_T^{(\mathrm{LO})} = V_0 + $$  ,

$$= \sigma_{\mathrm{eff}}(T)\phi + \frac{m^2_{\mathrm{eff}}(T)}{2}\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \,.$$

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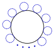
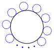
Subleading loops of ϕ must resum daisy diagrams,

$$V_T^{(\mathrm{NLO})} = V_0 +$$  $+$  ,

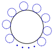
 $$= \sum_{n=0}^{\infty} c_n \alpha^n = O(\alpha^{\frac{3}{2}} T^4)\,.$$

The weird powers of $\alpha^{\frac{1}{2}}$ originate here.

Dolan & Jackiw '73; Arnold & Espinosa '92

Origins of weirder powers

In symmetry-breaking transitions, the loop-induced cubic term is of leading order,

$$V = V_0 +$$  $$+$$  $$= \frac{1}{2} m_{\text{eff}}^2 \phi^2 + \frac{1}{4} \lambda \phi^4 - \alpha^{\frac{3}{2}} T |\phi|^3 \,,$$

In the broken phase all three terms balance, so that

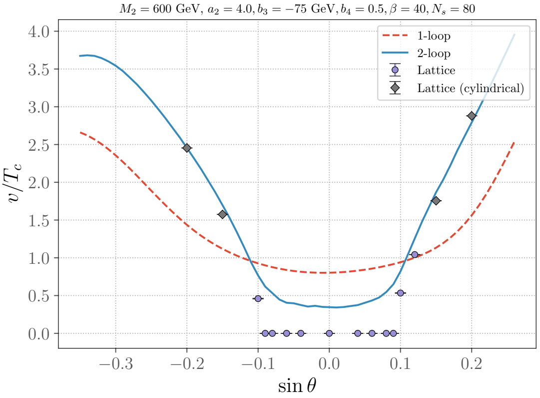
$$\implies |\phi| = O\left(\frac{\alpha^{\frac{3}{2}}}{\lambda} T \right) .$$

If the transition is reasonably strong, $\phi = O(T)$, then

$$\implies m_{\text{eff}} = O\left(\alpha^{\frac{3}{4}} T \right) .$$

The weirder powers originate from scalar loops with this mass.

Lattice simulations beyond the Standard Model



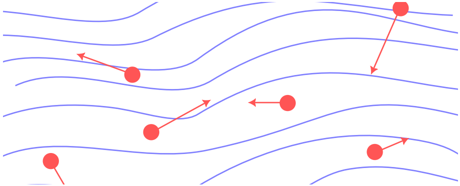
$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}} - \frac{a_1}{2}(\Phi^\dagger\Phi)S - \frac{a_2}{2}(\Phi^\dagger\Phi)S^2 - \frac{1}{2}(\partial S)^2 - \frac{m_S^2}{2}S^2 - \frac{b_3}{3}S^3 - \frac{b_4}{4}S^4$$

Niemi et al. '24

Hard thermal loops

- Generalises dimensional reduction for time-dependent quantities. Braaten, Pisarski, Frenkel, Taylor, Wong '90; Blaizot & Iancu '94

- Equivalent to fluctuating 3d fields, coupled to fluctuating Boltzmann particles

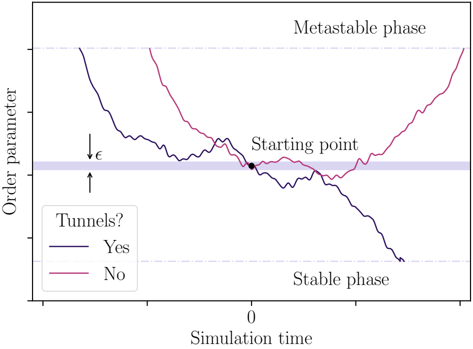


  $\rightarrow$ possible (but tricky) to put on the lattice.

  e.g. Bödeker, Moore, Rummukainen '00

- For scalars, or gauge fields further in the IR, this becomes Langevin evolution Aarts & Smit '97; Bödeker '98; Greiner & Müller '00
  $\rightarrow$ easy to put on the lattice.

Bubble nucleation à la Moore & Rummukainen



$$\text{rate} = \underbrace{\left\langle \frac{\delta_{\text{tunnel}}}{2N_{\text{crossings}}} \left| \frac{\Delta(\phi^\dagger\phi)}{\Delta t} \right|_{\varphi^2_{\text{Sep}}} \right\rangle}_{\text{dynamical}} \underbrace{\frac{P\left(|\phi^\dagger\phi - \varphi^2_{\text{Sep}}| < \epsilon/2\right)}{\epsilon P\left(\phi^\dagger\phi < \varphi^2_{\text{Sep}}\right)}}_{\text{statistical}}.$$

Moore & Rummukainen '00; Moore, Rummukainen & Tranberg '01