Nuclear Theory: Lecture 2

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UNIVERSITY OF GUELPH

23rd STFC Nuclear Physics Summer School
University of Aberdeen
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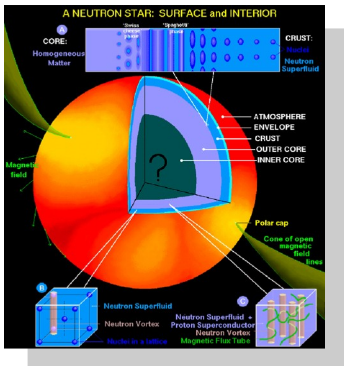
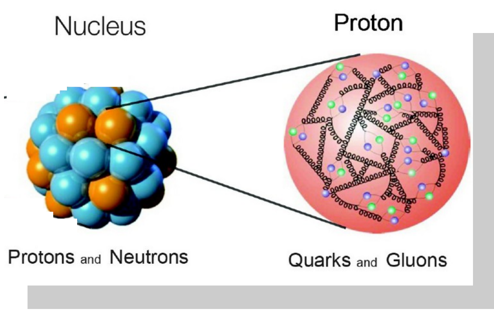
Outline

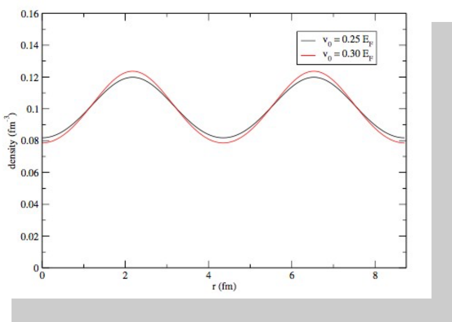


Credit: Dany Page

Motivation



Nuclear forces



Many-body physics

Textbook many-body physics:
liquid-drop formula

Liquid-drop formula

Binding-energy systematics

$$\frac{E_B}{N+Z} = a_V - \frac{a_S}{(N+Z)^{1/3}} - a_C \frac{Z(Z-1)}{(N+Z)^{4/3}} - a_{sym} \frac{(N-Z)^2}{(N+Z)^2} - \lambda \frac{a_p}{(N+Z)^{3/2}}$$

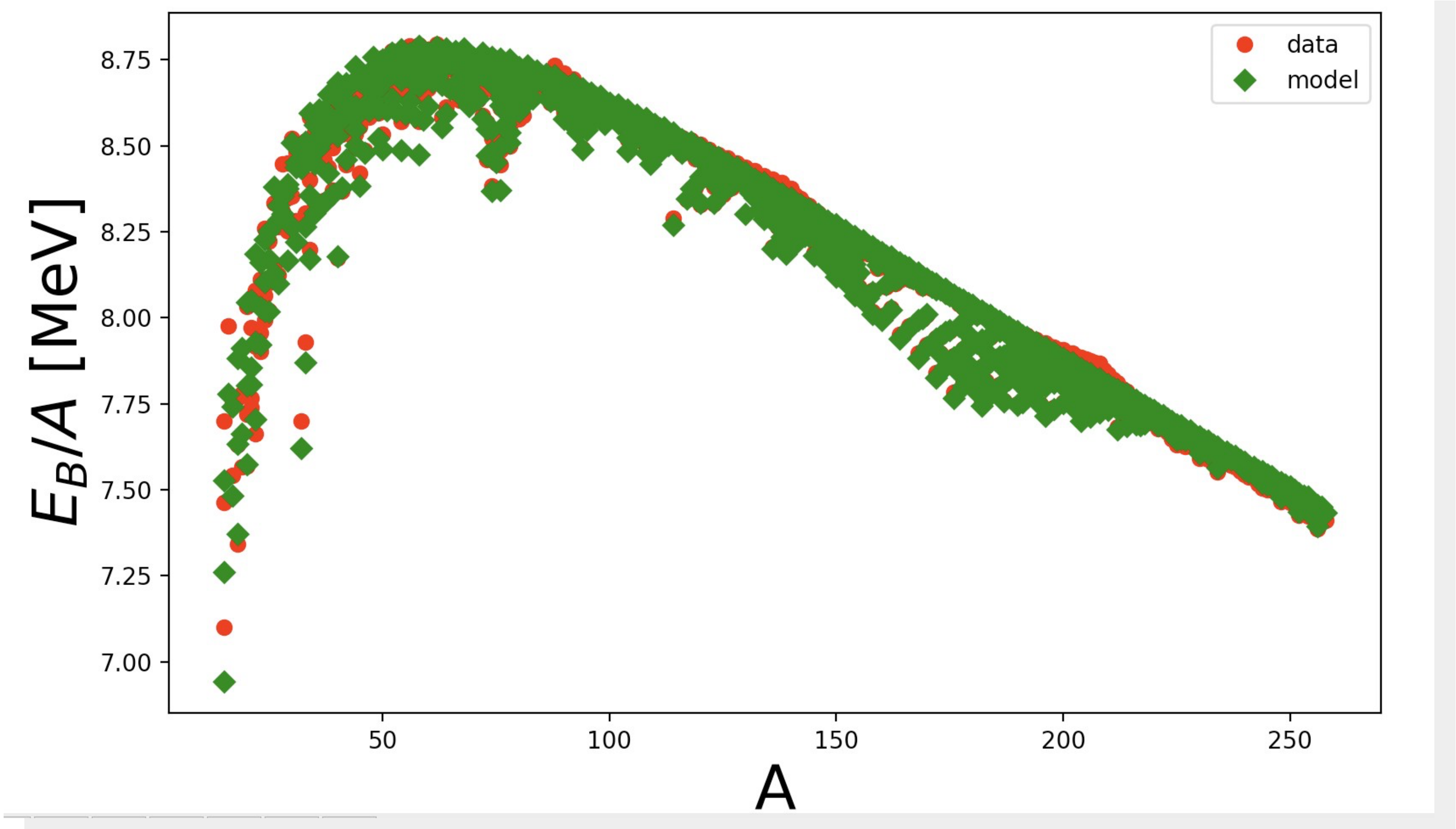
volume surface Coulomb symmetry

pairing $\lambda = \begin{cases} +1, & \text{odd } N\text{-odd } Z \\ 0, & \text{odd } N\text{-even } Z, \text{ or even } N\text{-odd } Z \\ -1, & \text{even } N\text{-even } Z \end{cases}$

Liquid-drop formula

Model checking

You'll do this during the skills session



What are you supposed do to with the interactions introduced in the previous lecture?

Nuclear many-body problem

$$H\Psi = E\Psi$$

where $H = \sum_i K_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + \cdots$

so

$$H\Psi(\mathbf{r}_1, \cdots, \mathbf{r}_A; s_1, \cdots, s_A; t_1, \cdots, t_A) = E\Psi(\mathbf{r}_1, \cdots, \mathbf{r}_A; s_1, \cdots, s_A; t_1, \cdots, t_A)$$

i.e. $2^A \binom{A}{Z}$ complex coupled second-order differential equations

Let’s take infinite nucleonic matter as an example

Degeneracy pressure

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One-dimensional box

Periodic boundary conditions:

$$\psi(x) = \psi(x + L)$$

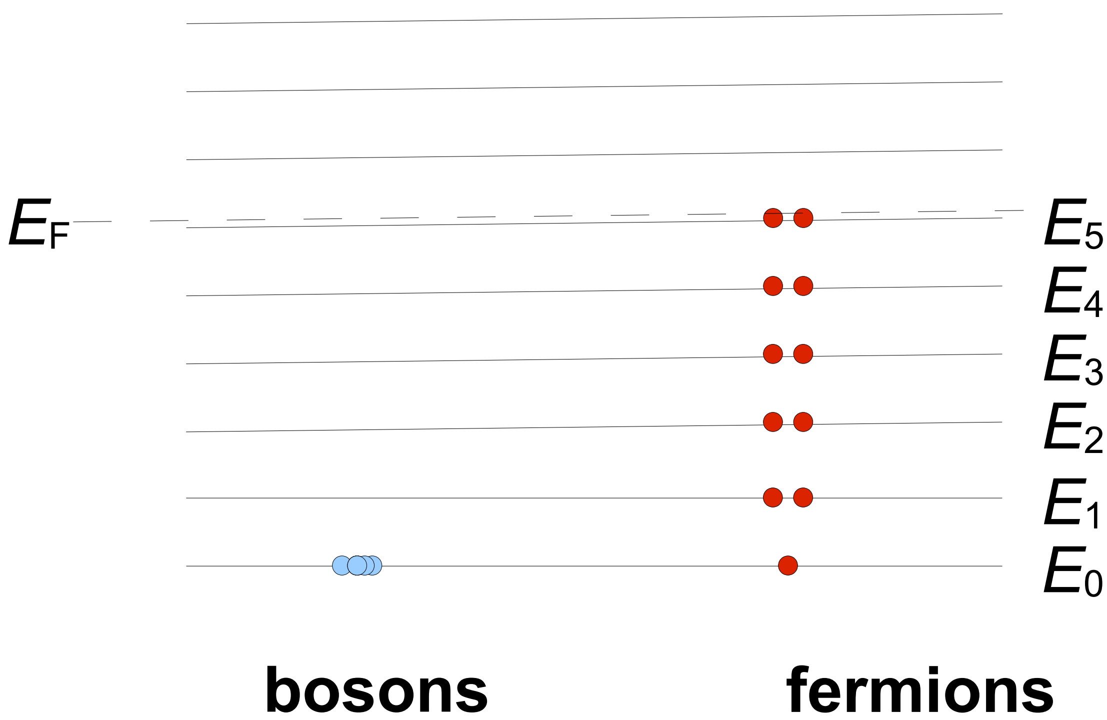
Wave function is:

$$\psi(x) = \frac{1}{\sqrt{L}} e^{i k_x x}$$

Characterized by:

$$k_x = \frac{2\pi}{L} n_x$$

$$E = \sum_{k_x}^{k_x \le k_\mathrm{F}} \frac{\hbar^2 k_x^2}{2m}$$



you *can* place as many as you want in the same state

you *cannot* place as many as you want in the same state

Degeneracy pressure

Three-dimensional box

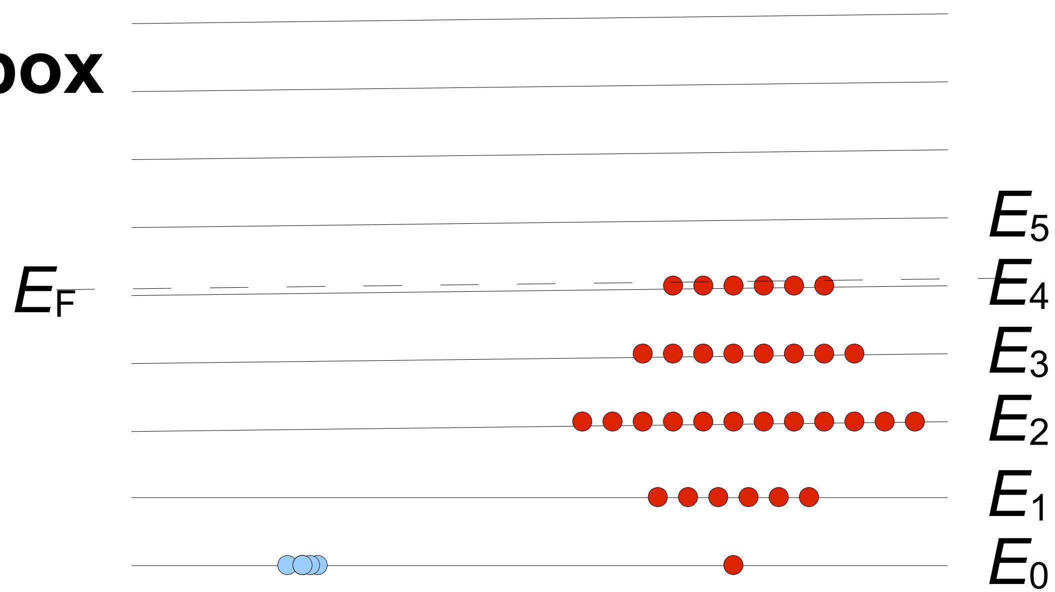
In periodic boundary conditions, the wave function is:

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{L}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

characterized by:

$$\mathbf{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$E = \sum_{k_x, k_y, k_z}^{|\mathbf{k}| \le k_F} \frac{\hbar^2 \mathbf{k}^2}{2m}$$



bosons

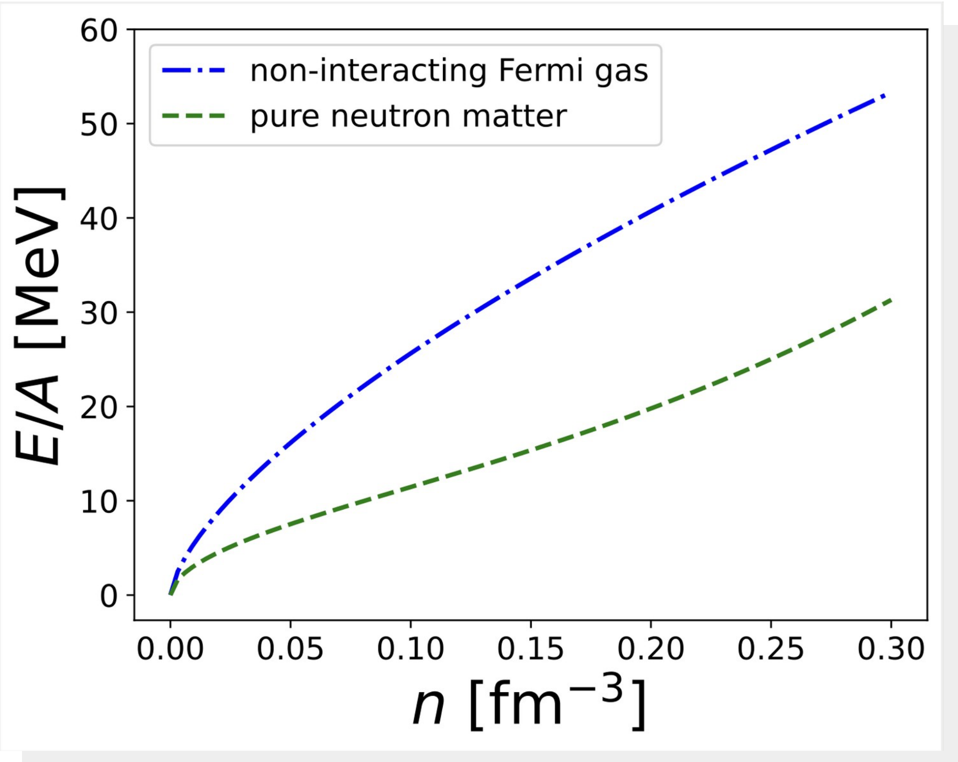
you *can* place as many as you want in the same state

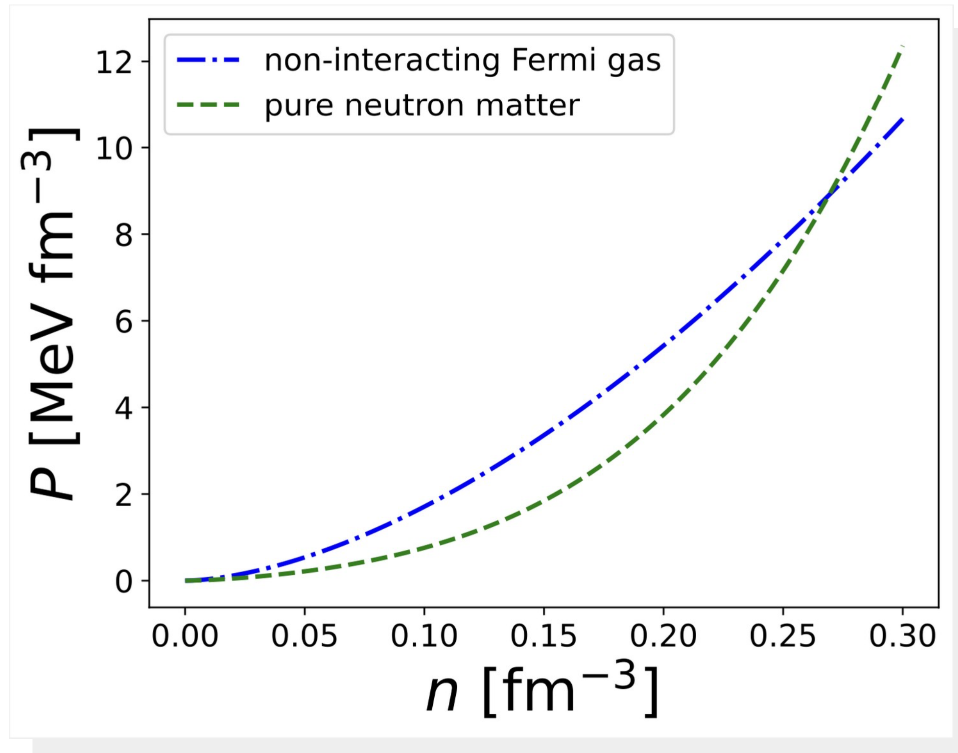
fermions

you *cannot* place as many as you want in the same state

From free fermions to neutron matter

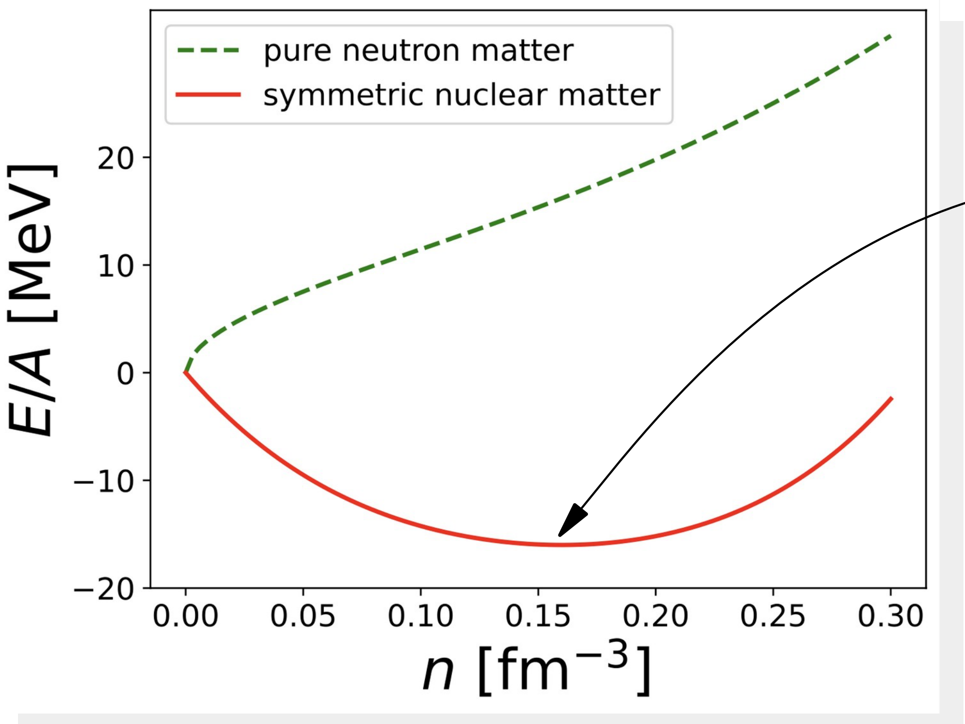
$$P = n^2 \frac{\partial(E/A)}{\partial n}$$





Pressure always positive

From the liquid drop to nuclear matter

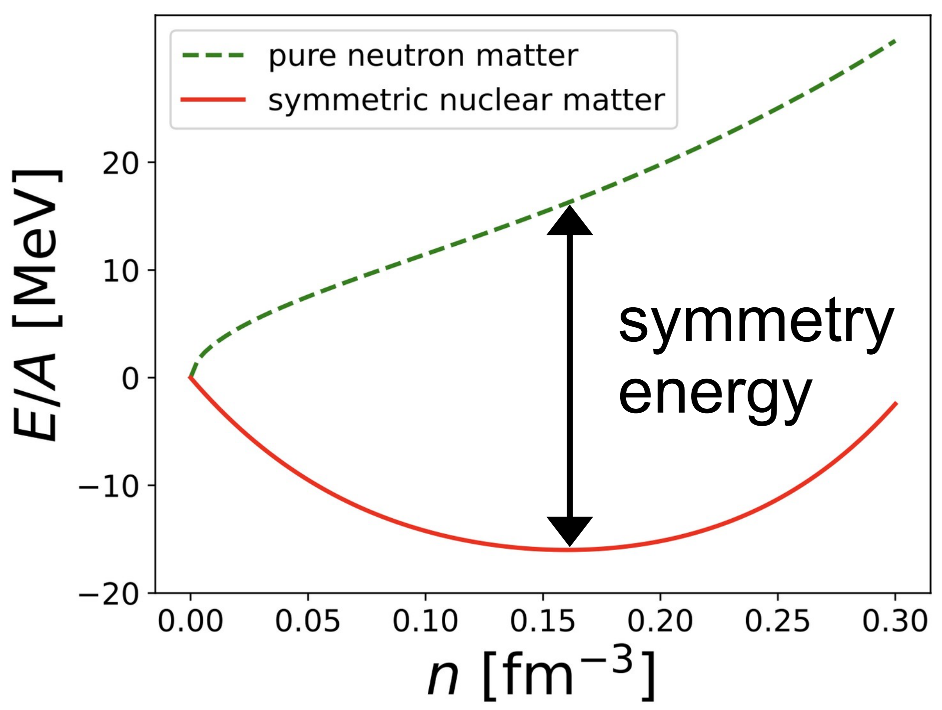


Energy always negative, exhibits a minimum

Saturation point (at 0.16 fm^{-3}) has an energy per nucleon of -16 MeV, consistent with taking *N* and *Z* to infinity in the liquid-drop formula:

$$\frac{E_B}{N+Z} = a_V - \frac{a_S}{(N+Z)^{1/3}}$$

From the liquid drop to nuclear matter



Energy always negative,
exhibits a minimum

From the liquid drop to nuclear matter

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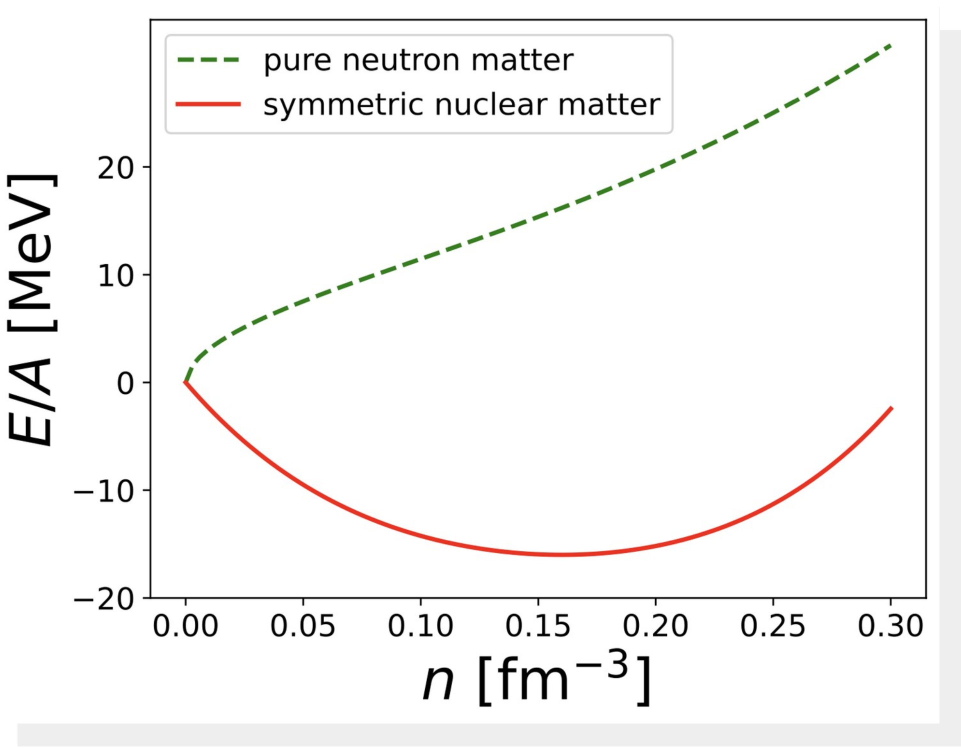
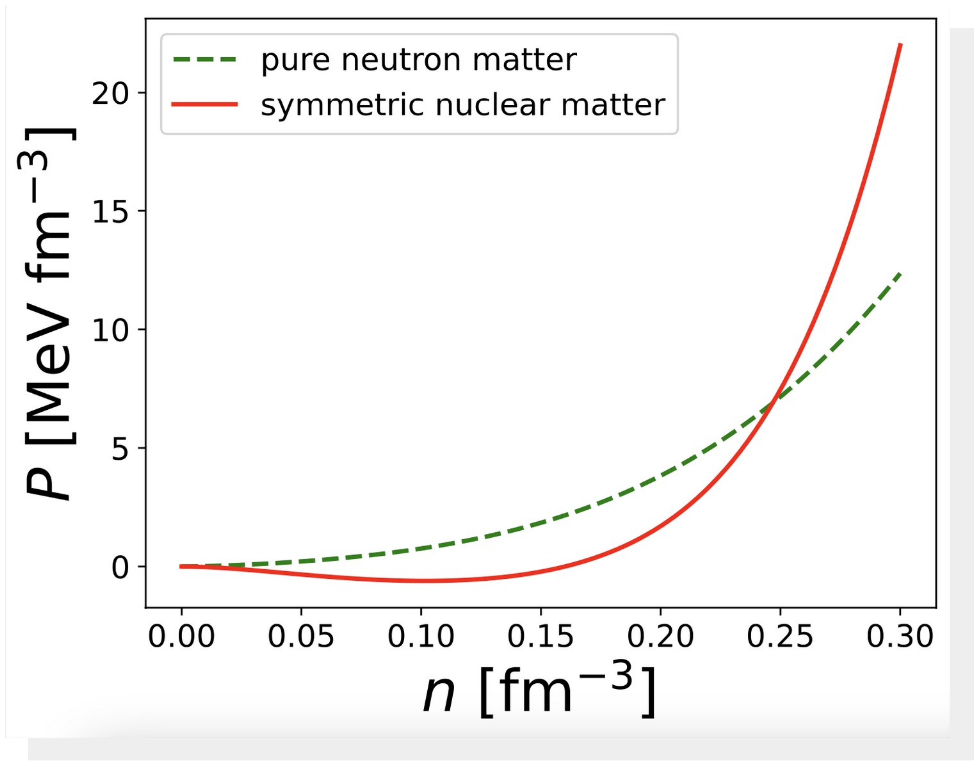
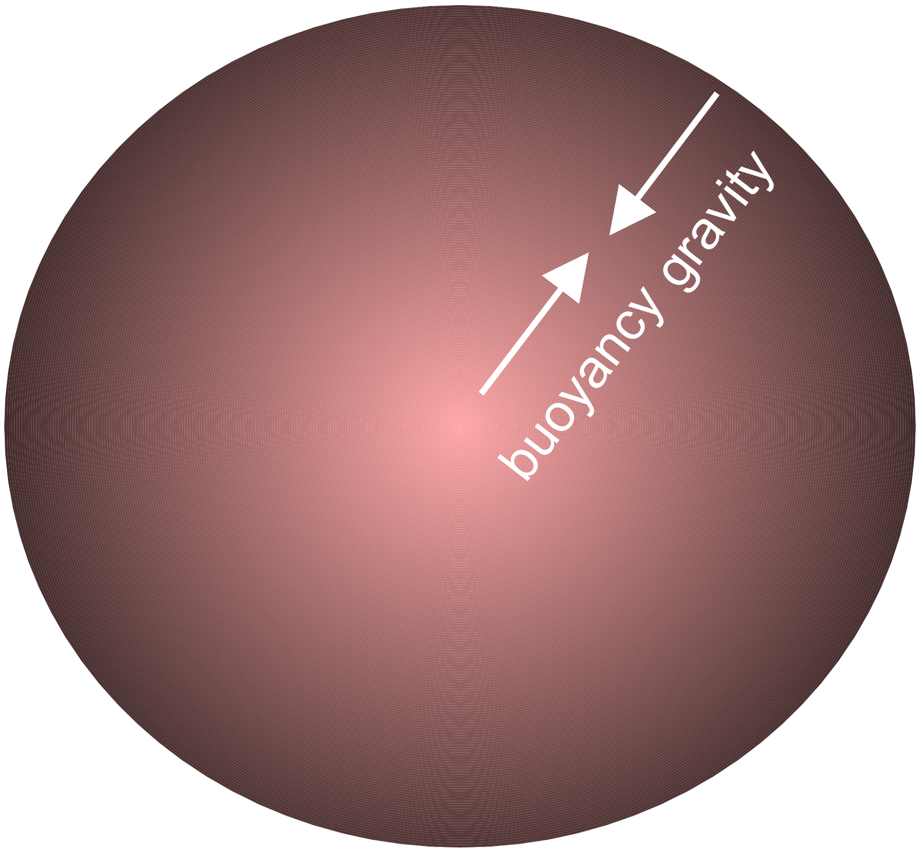
$$P = n^2 \frac{\partial(E/A)}{\partial n}$$



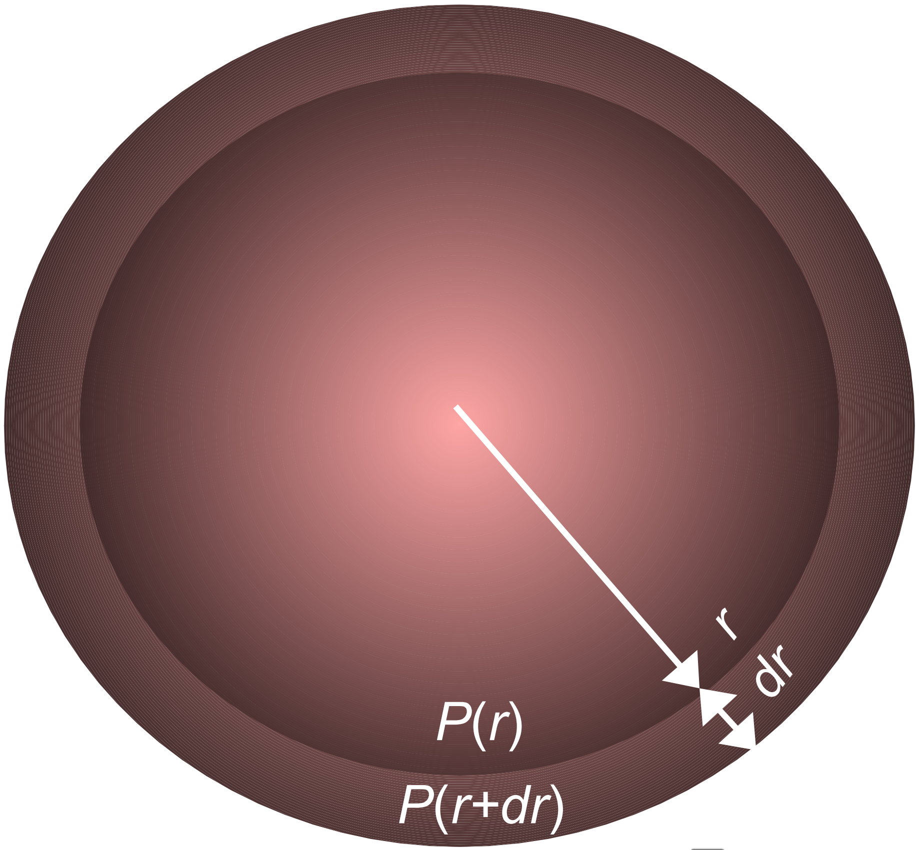


Pressure zero at saturation point

Stars



Hydrostatic equilibrium



Distinguish between the mass within *r* :

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'$$

and that of the thin shell: $4\pi r^2 \rho(r) dr$

Shell experiences gravitation:

$$F_{\text{gravitational}} = -G\frac{4\pi r^2 \rho(r) dr m(r)}{r^2} = -4\pi G \rho(r) m(r) dr$$

and buoyancy:

$$F_{\text{buoyant}} = 4\pi r^2 \left[P(r) - P(r+dr)\right] = -4\pi r^2 P'(r) dr$$

Hydrostatic equilibrium

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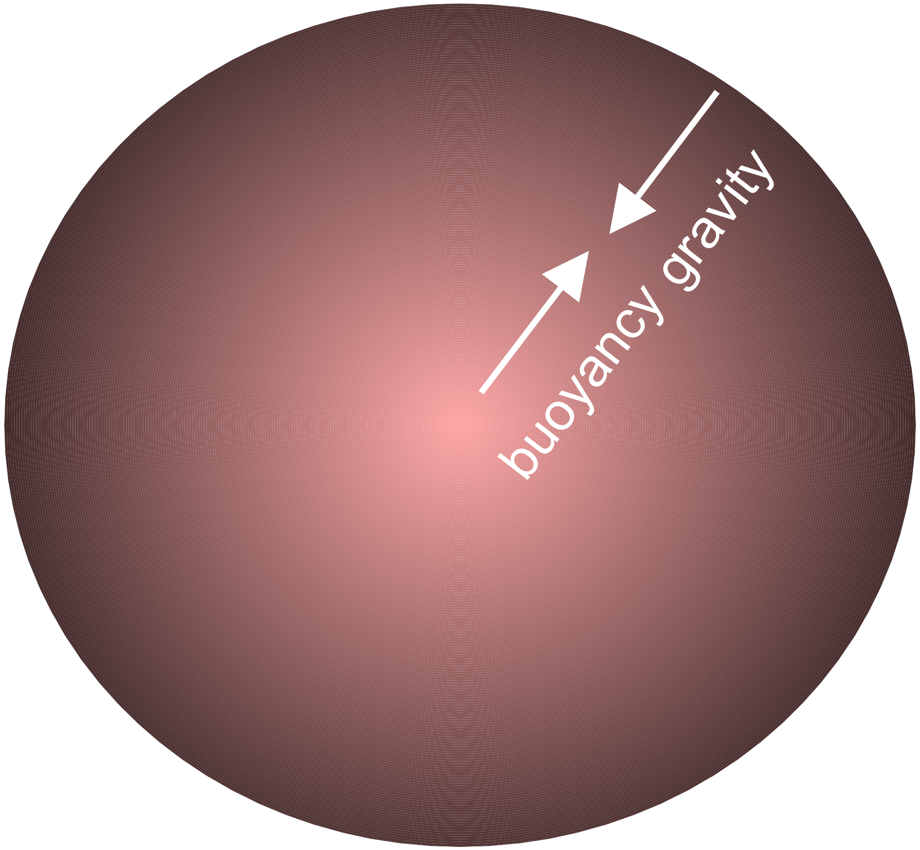


At equilibrium the sum of the two forces vanishes, so:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

To be solved together with:

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'$$

or, equivalently:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

Hydrostatic equilibrium

Our two equations:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

seem to involve three quantities $P(r)$, $m(r)$, and $\rho(r)$.

To make further progress, use *equation-of-state* (EOS):

$$P = P(\rho, T)$$

Neutron stars: gravity

Static spherically symmetric metric

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -B(r) \end{pmatrix}$$

Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ricci tensor

Energy-momentum tensor

Inside the source (for isotropic fluid with no shear forces):
Tolman-Oppenheimer-Volkoff (TOV) equation(s)

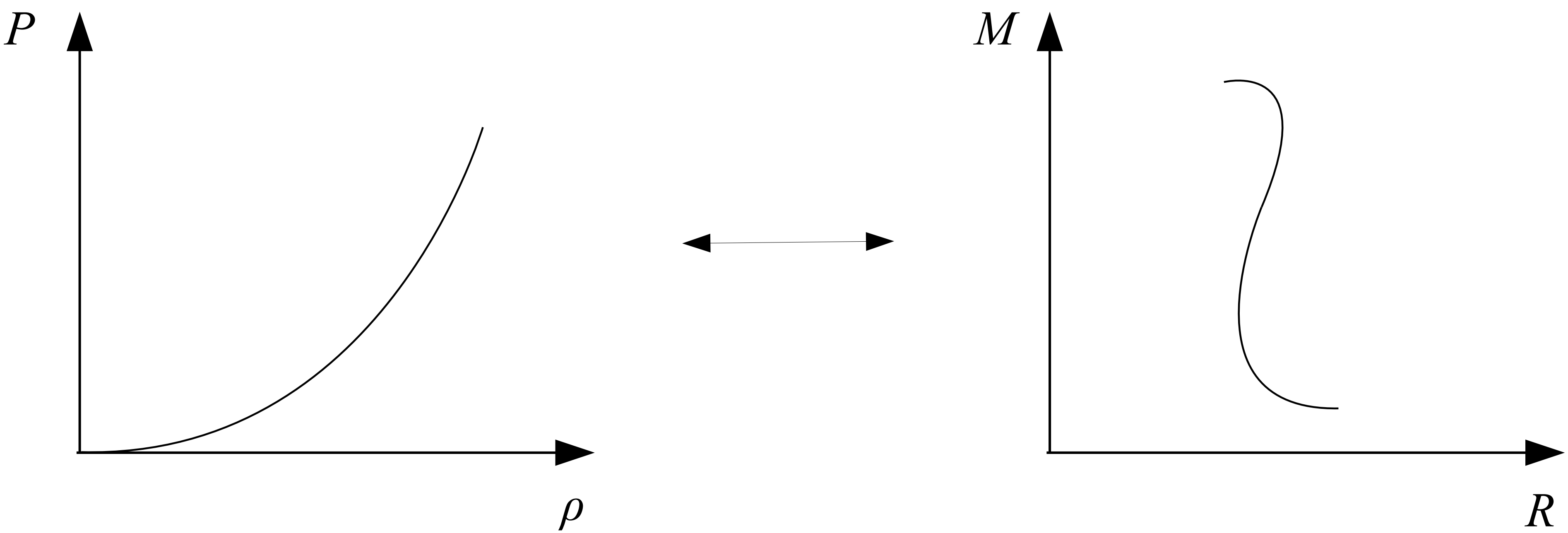
$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}\left[1 + \frac{P(r)}{\rho(r)c^2}\right]\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}$$

Neutron stars: micro-macro

TOV equations
(or Hartle-Thorne, etc)

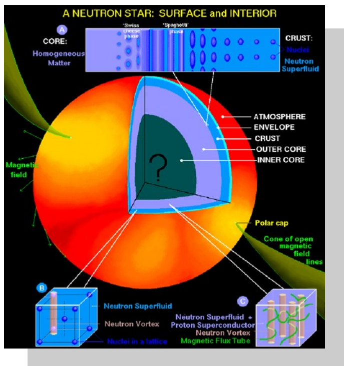
Pressure vs density

Mass vs radius



Modern goal: systematic theoretical error bars

Outline



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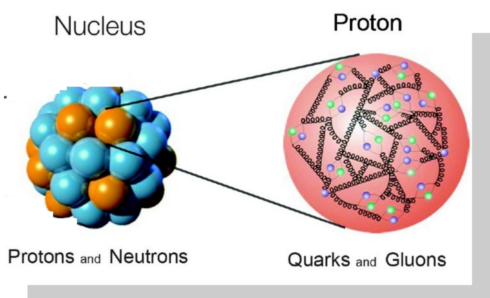
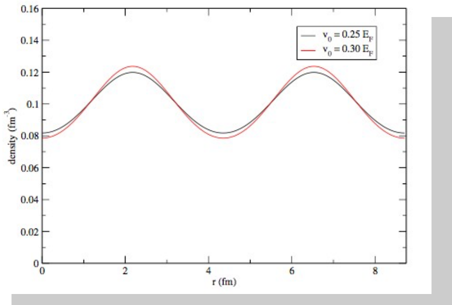


Nuclear background



Modern techniques

Nuclear many-body methods

Phenomenological (fit to A-body experiment)

Ab initio (fit to few-body experiment)

Nuclear many-body methods

Phenomenological (fit to A-body experiment)

- **Shell model**
  mainstay of nuclear physics, still very important
- **Hartree-Fock/Hartree-Fock-Bogoliubov (HF/HFB)**
  mean-field theory, a priori inapplicable, unreasonably effective
- **Energy-density functionals (EDF)**
  like mean-field but with wider applicability

Nuclear many-body methods

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Ab initio (fit to few-body experiment)

- **Quantum Monte Carlo (QMC)**
  stochastically solve the many-body problem "exactly"
- **Perturbative Theories (PT)**
  first few orders only
- **Resummation schemes (e.g. SCGF)**
  selected class of diagrams up to infinite order
- **Coupled cluster (CC)**
  generate np-nh excitations of a reference state
- **No-core shell model (NCSM)**
  fully ab initio, in contradistinction to traditional SM

Modern phenomenology:
Skyrme energy-density functionals

Skyrme EDF

Energy-density functional

Slater determinant $\phi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det|\phi_i(x_j)|$

Total energy $E = \langle\phi|\, \hat{H} \,|\phi\rangle = \int \mathcal{H}(\mathbf{r}) d^3 r$

Energy density $\mathcal{H} = \mathcal{K} + \mathcal{H}_0 + \mathcal{H}_3 + \mathcal{H}_{\text{eff}} + \mathcal{H}_{\text{fin}}$

Skyrme EDF

Energy-density functional

Energy density $$\mathcal{H} = \mathcal{K} + \mathcal{H}_0 + \mathcal{H}_3 + \mathcal{H}_{\text{eff}} + \mathcal{H}_{\text{fin}}$$

$$\mathcal{K} = \frac{\hbar^2}{2m}\tau(\mathbf{r}),$$

$$\mathcal{H}_0 = \left(C_0^{\rho,0} + C_1^{\rho,0}\right)\rho^2(\mathbf{r}) = \frac{1}{4}t_0(1-x_0)\rho^2(\mathbf{r}),$$

$$\mathcal{H}_3 = \left(C_0^{\rho,\alpha} + C_1^{\rho,\alpha}\right)\rho^{2+\alpha}(\mathbf{r}) = \frac{1}{24}t_3(1-x_3)\rho^{2+\alpha}(\mathbf{r})$$

$$\mathcal{H}_{\text{eff}} = \left(C_0^{\tau} + C_1^{\tau}\right)\rho(\mathbf{r})\tau(\mathbf{r})$$

$$= \frac{1}{8}\left[t_1(1-x_1) + 3t_2(1+x_2)\right]\rho(\mathbf{r})\tau(\mathbf{r}), \text{ and}$$

$$\mathcal{H}_{\text{fin}} = -\left(C_0^{\Delta\rho} + C_1^{\Delta\rho}\right)\left(\nabla\rho(\mathbf{r})\right)^2$$

$$= \frac{3}{32}\left[t_1(1-x_1) - t_2(1+x_2)\right]\left(\nabla\rho(\mathbf{r})\right)^2$$

Modern *ab initio*: quantum Monte Carlo

Quantum Monte Carlo

Big picture

stochastically solve the many-body Schrödinger equation in a fully non-perturbative manner

Rudiments of Diffusion Monte Carlo

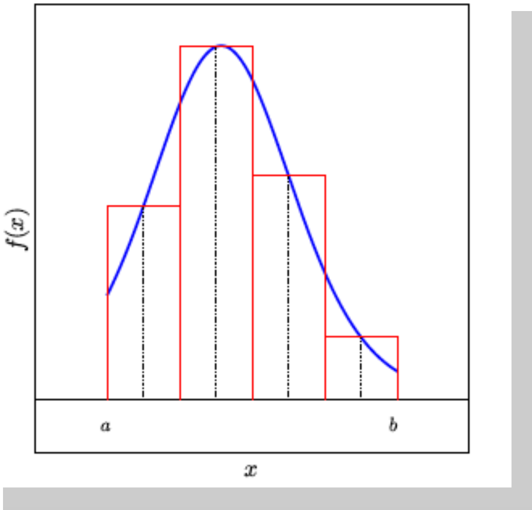
$$\Psi(\tau \to \infty) = \lim_{\tau \to \infty} e^{-(\mathcal{H}-E_T)\tau} \Psi_V$$

$$= \lim_{\tau \to \infty} \sum_i \alpha_i e^{-(E_i - E_T)\tau} \Psi_i$$

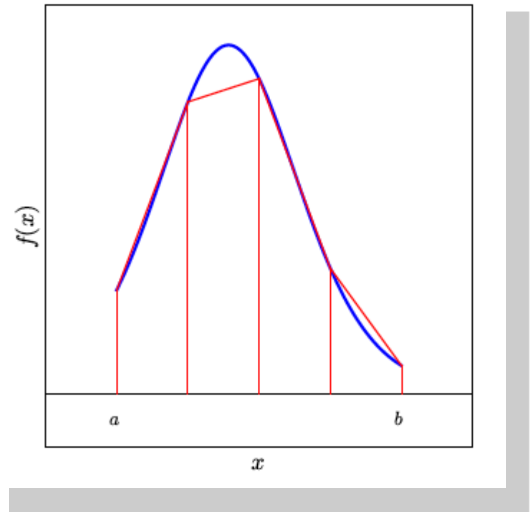
$$\to \alpha_0 e^{-(E_0 - E_T)\tau} \Psi_0$$

Probabilistic foundations

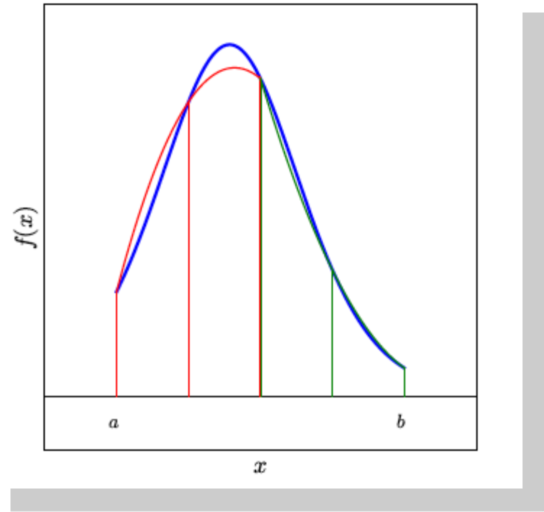
Midpoint rule



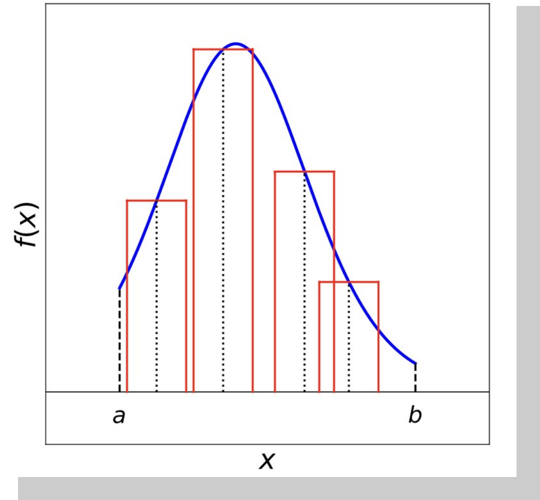
Trapezoid rule



Simpson's rule



Monte Carlo quadrature



Probabilistic foundations

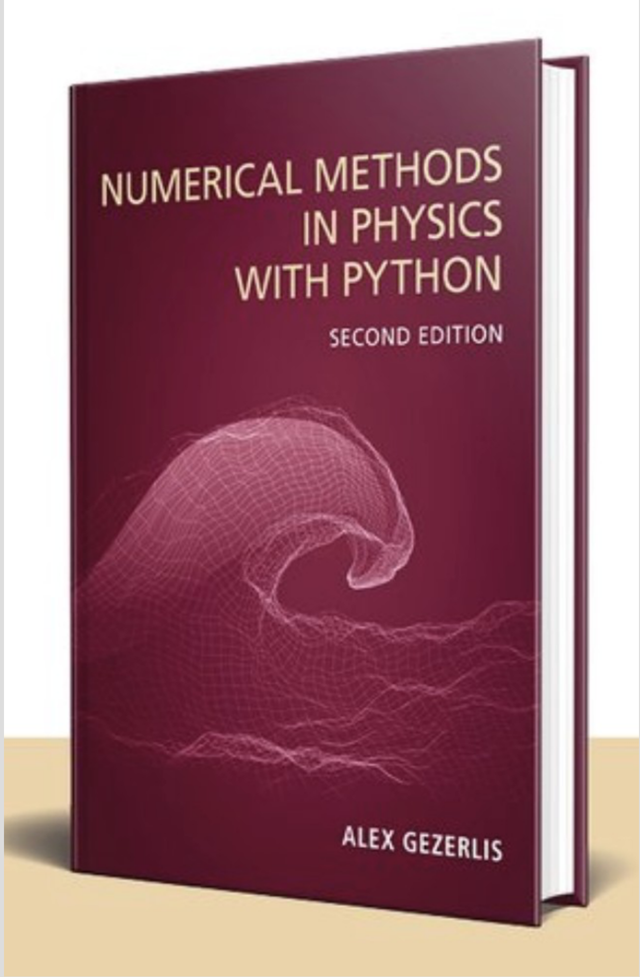
Midpoint rule **Trapezoid rule** **Simpson's rule**

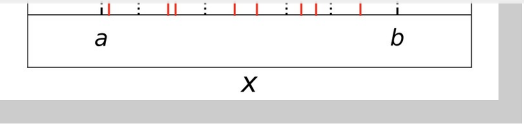
For more on foundational techniques, see:

Alex Gezerlis

Numerical Methods in Physics with Python

(2^{nd} ed, Cambridge University Press, 2023)





Quantum Monte Carlo

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Big picture

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$$= \lim_{\tau\to\infty} \sum_i \alpha_i e^{-(E_i-E_T)\tau}\Psi_i$$

$$\to \alpha_0 e^{-(E_0-E_T)\tau}\Psi_0$$

Conclusions

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- Rich connections between physics of nuclei and that of compact stars

- Exciting time in terms of interplay between nuclear interactions, QCD, and many-body approaches

- Ab initio and phenomenology are mutually beneficial

Acknowledgments

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