Factorization and Effective Field Theory

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CONTENTS

1. INTRODUCTION

Historically effective field theory was founded upon the Wilsonian idea of integrating out shells in momentum space. This strategy is encapsulated in block spin renormalization, where we average over nearest neighbor spins to produce a new Hamiltonian that effectively describes the physics at longer distance. These ideas are inherently Euclidean in the sense that large (four) momentum implies short distances/times. However if we limit ourselves to integrating out large (four) momenta, then the scope of EFT would be greatly limited. We need to generalize EFT such that we integrate out large invariant mass not large momenta. When we considered heavy quarks we integrated kept only soft modes ($k^\mu \ll m$) in the theory. However, if we are interested in high energy scattering, then we will need to generalize these ideas since, as we shall see, high energy modes, where all components are large, carry information about the IR when they have small invariant mass. While Euclidean geometry the IR constrains all momentum components to be small, if we consider two null four-vectors (p, k), then in Minkowski space the condition

$$(k+p)^2 = k^2 + p^2 + 2p_0k_0(1 - cos\theta) \sim 0 \tag{1}$$

allows for collinear radiation where all the components are large. Thus we must generalize EFT to allow for high energy (small invariant mass) modes to survive in the EFT.

In addition, so far in our discussions, we utilized the local nature of the underlying theory to separate the long distance from the short distance. The ladder being captured by the Wilson coefficients, while the former is reproduced in matrix element of local operators. That is, the low energy theory was local When we consider high energy scattering scale separation will play a new role and, much as in the case of NRQCD, the low energy theory will only be local at sufficiently small invariant masses.

If we are to allow large energies (as opposed to large virtualities) to survive into the IR, i.e. be part of the dynamics of the EFT, we will have to generalize what we did for heavy quarks where we kept the heavy quark field in the theory despite its large mass. Moreover, so far we have limited ourselves to cases where the relevant scales are explicitly present in the action [1]. In the case of high energy scattering we consider an action which has no disparate scales (say massless QCD), but we have some external momentum which is much larger than all the other external momenta, or masses, in the problem. That is, we can imagine that we are interested in a Green's function with multiple external legs and we are taking some external momenta to be large (Q) compared to all the others (p_i). We should expect that when $Q \gg p_i$ we might be able to simplify the problem using the same basic ideas utilized in studying effective field theories. After all, in the introduction when we discussed separation of scales nowhere did we assume that the scales are explicit in the action. Indeed, we would expect that, as in more canonical cases, the large invariant mass physics should decouple from the small invariant mass physics. This decoupling of external scales is sometimes called "factorization"[2]. Without factorization we would have no theoretical

[1] Strictly speaking this is not exactly true. The scales mv and mv^2 in NRQCD were generated by the bound state, but the scale m obviously arose from the action.

[2] The terms "decoupling" and "factorization" are loosely equivalent. However, the term decoupling is usually restricted to cases where one integrates out a heavy particle.

handle on collisions between protons at high energies. When we scatter at high energies and there is a collision with the exchange of highly virtual (large invariant mass) states (i.e. not forward scattering), we expect to be probing the proton at short distances, i.e., we are really scattering quarks and gluons. But to calculate quark scattering we would need to know the quarks initial state inside the proton, including its correlations with other quark and gluons. We can't just assume we have incoming beam of quarks. It seems like trying to understand proton scattering, is on the face of it, intractable. However, factorization greatly simplifies the problem. The time scales relevant for inter-nucleon reactions are on the scale of femtoseconds, which is an eternity compared to the time scale for the hard collision, as long as the momentum exchange in the hard collision is much larger then Λ_{QCD}. So we would expect that, as far as the proton interactions are concerned, the proton is being probed instantaneously. Of course, this does not mean that the proton dynamics are irrelevant, it just means that we should expect there to be no interference between long range internal proton dynamics and the short distance physics responsible for the hard scattering process. If this were true we could write the cross section as the product of the probability [3]to find quarks/gluons in a given state and the probability for such quarks/gluons to scatter.

Perhaps this should not be terribly surprising since we can see hints of such factorization in basic wave mechanics. Consider an amplitude which is a superposition of long and short wavelength modes. We would expect that if we average over a region of size R of the order of the shorter wavelength, the interference will be suppressed. Mathematically this follows from the fact that if we consider interference between two waves

$$\int_R \sin(kx)\sin(k'x)dx \sim \frac{1}{2}\left[\frac{\sin(R(k-k'))}{k-k'} - \frac{\sin(R(k+k'))}{k+k'}\right], \tag{2}$$

vanishes when $\lambda \equiv k'/k \ll 1$, when R is held fixed, as opposed to $\sin^2(kx)$.

This lack of interference will allow us to write down scattering amplitudes in terms of factorized probabilities. In general this factorization takes of form of a convolutions of functions. One function (F) which depends only upon long-distance QCD dynamics which is incalculable analytically[4], and a calculable[5] function(H), which depends on the short

[3] This implies that we are tracing over all the degrees of freedom leaving a classical mixed state for the parton.

[4] Much progress has been made in using lattice gauge theory techniques for F in certain cases.

[5] This function is calculable in perturbation theory when the theory is weakly coupled in the UV.

distance physics. Schematically, we may write the amplitude as

$$M \sim \prod \int dx_i F(x_i) H(Q, x_i) \tag{3}$$

where Q is the scale of the hard scattering. If F is incalculable, then why is (3) this useful? Because F is *universal*, i.e. it can be extracted in one process and used to make predictions in another as it depends only upon the internal dynamics of the proton. In the simple case where there is only one convolution, F will correspond to the parton distribution function (PDF). In addition, note that given a large disparity in scales, perturbation theory is bound to be poorly behaved due to the existence of large logs, [6] just as in the more canonical cases where the large scales reside in the action itself. So, as usual, we will need some sort of renormalization group techniques to resum perturbation theory in order to calculate in a systematic fashion. We have seen that EFT is tailored to perform such resummations. It is hopefully clear that, in the context of high energy scattering processes, EFT will be a very sharp tool. However, before we explore the proper EFT for this class of problems, it behooves us to discuss factorization using the classic diagrammatic arguments of Libby and Sterman[1, 2] without recourse to effective field theory. For a pedagogical review see [3]. This will let us derive equations of the form given in (3).

2. FACTORIZATION

We can get some feeling for factorization by starting to draw some Feynman diagrams. Consider a Green's function into which we inject a large momentum Q, with $|Q^2|$ much larger than all possible Lorentz invariants as well as masses. In this case we might expect that all of the lines through which Q runs could be shrunk to a point since they will be short lived. In momentum space this would amount to

$$\frac{1}{Q^2 + G(p_i \cdot p_j, Q \cdot p_i)} \approx \frac{1}{Q^2}\left(1 + \frac{G(p_i \cdot p_j, Q \cdot p_i)}{Q^2} + ...\right), \tag{4}$$

where G is some function that is determined by the diagram of interest. In kinematic situations where this approximation is valid the amplitude would lead to a simple factorized

[6] In this section we will only be considering four space-time dimensions.

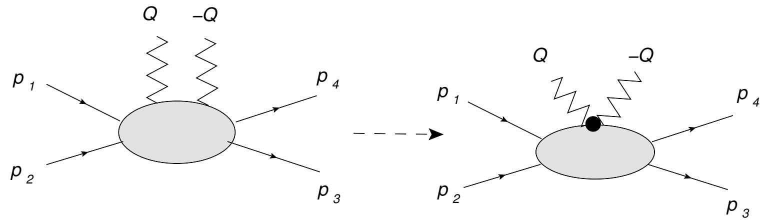


FIG. 1. A large momentum Q is injected into a Green's function in which $|\,Q^2\,|$ is larger then all other invariants.

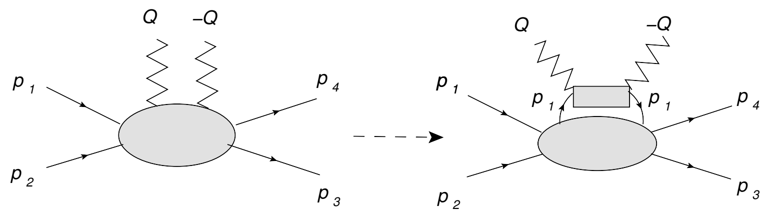


FIG. 2. A large momentum Q is injected into a Green's function in which $|\,Q^2\,|$ is larger then all other invariants except $Q \cdot p_1$. The square box is called a "bridge" and leads to a convolution in the factorization formula.

product form for the amplitude [7]

$$M(Q, p_i) = H(Q^2)S(p_i \cdot p_j) + O(p/Q) \tag{5}$$

as shown in (1). More complicated factorized forms arise when $|Q^2|$ does not dominate all invariants but is of the same order as some other invariant say, $Q \cdot p_1$. In this case it will no longer be true that all propagators through which Q flows can be contracted to a point. When this happens the factorized form will involve a convolution, as the momentum p_i will flow through a "bridge" which links the short and long distance contributions as shown in (2). The protoype process for this type of process is "deep inelastic scattering", where an electron scatters off of a nucleon via the exchange of a highly virtual spacelike photon with invariant mass $-|\,Q^2\,|$. The function F in eq. (3) in this case becomes the parton distribution function (PDF) which can be interpreted in terms of a probability to find a quark (or gluon) carrying some fraction of the momentum of the struck nucleon[3]. Note that even though a large momentum flows through the PDF, it is still independent of the large inavariant Q^2. We will be more quantitative about the distinction between figures (1) and (2) in the next section.

[7] Note that power of the first correction can vary depending upon the process.

In some sense we might expect ALL observables with a hierarchy of scales to be factorizable given our broad definition of the term. As we have argued multiple times in various contexts, scale separation is a necessary condition to have predictive powers at low energies. We might expect that the power corrections in (3) should factorize as well. This factorization is not easily seen using canonical methods, but will become manifest when we apply the tools of EFT. It is important to realize, however, that just because we can express an observable in factorized form does mean that that result is useful. One could imagine a factorized form involving multiple convolutions of various functions some of which may be incalculable in perturbation theory. It may then simply be impractical use this formalism to make any predictions.

Finally let us note that the utility of a factorization formula is limited by magnitude of the scale separations. That is, it is always true that a given factorized result is only approximately true, and that it will only be accurate up to corrections $(\mu_L/\mu_H)^n$ where μ_L and μ_H are generic high and low energy scales in the problem such that $\mu_H > \mu_L$, and n is process dependent. While it is true that we might also expect that these "power corrections" to factorize, in general these corrections will also involve a new set of long distnace matrix elements which may not be calculable in perturbation theory. So we can see that if we wish to increase our accuracy we will pay a strong price, as the amount of information needed to make a prediction will grow as we include more power corrections.

The first tool developed for factorization is the operator product expansion (OPE). As we will see in the remainder of this chapter, the OPE in conjunction with EFT, can be utilized to generate predictions for hadronic interactions in a well defined approximation scheme. We will begin by considering some classic examples where the OPE will suffice to make predictions. After which it will be demonstrated that sometimes the OPE by itself is insufficient, but that EFT can carry the rest of the burden when necessary.

3. THE OPERATOR PRODUCT EXPANSION

Historically the first factorization formulae were derived in the context of the operator product expansion (OPE). The original idea of the OPE, due to Wilson, can be schematically summarized in position space as

$$\lim_{x\to 0} O_1(x)O_2(0) \sim \sum_i C^{1,2}_{O_i}(x)O_i(0). \tag{6}$$

In Minkowksi space this expansion is asymptotic when inserted inside a matrix element, whereas in Euclidean space it is believed to be convergent [4]. The operators O_i are restricted by the transformation properties of the operator product and will in general include derivatives acting upon the fields. Whether or not the C_i's, called "Wilson Coefficients", are non-zero for a given operator will depend upon the symmetries of the underlying action. The power of the OPE stems from the fact that it is an operator statement, i.e. it does not depend upon what one chooses for external states, nor does it matter if there are other fields inside the matrix elements (as long as they are not sitting as the same points as the operators of interest). The Euclidean OPE can be proven to all orders perturbation theory by going to momentum space [5].

The OPE is at the heart of EFT. While EFT is more general, in the sense that it is formulated at the Lagrangian level, the basic notion of EFT's was inherent in the work of Kadanoff, Fisher and Wilson. The OPE states that at long distances the short distance physics, decouples (or factorizes) into Wilson coefficients. These coefficients are nothing but a matching coefficient of the type that arise in EFT. We calculate the C's by doing a matching calculation. One chooses an appropriate set of external states and takes the matrix elements of both sides of the OPE. As in EFT one can choose to match on or off shell, or even at zero momentum [8]. Also as in the EFT case, one is free to use the equations of motion on the RHS of the OPE by making field redefinitions, at the cost of generating new operators of higher dimension [6]. We can see that from a practical point of view that the OPE is most useful for theories which are weakly coupled in the UV such as QCD. In this case we can calculate the Wilson coefficients in perturbation theory and the long distance matrix elements can be extracted from the data in one process and used to make a predictions in another.

The OPE, as one can imagine, is a subject onto itself and is discussed in many texts. It is only touched upon here in order to give the reader a better understanding and appreciation

[8] This choice is dictated by what operator you wish to match. For instance if the operator entailed derivatives one can not match at zero momentum.

of factorization in high energy processes and, more importantly, explain why EFT is so useful to prove factorization in cases where the OPE is insufficient.

The utility of the OPE is predicated on our ability to make constructive use of the RHS of (6). Typically there are two cases of relevance, which were briefly mentioned in the last section: a) The simplest case arises when we can truncate the series with a minimal number of terms with the remainder terms being power suppressed[9] . These cases correspond to scenarios such as those in figure (1). For the more complicated and ubiquitous case b), we will have to keep an infinite subset of terms, resulting in a net convolution as represented in figure (2). For this to be justified we need to have a power counting which explains why all other terms are sub-leading. We can see that case (a) has more predictive power then case (b) since for (a) the long distance information necessary to reconstruct an amplitude will be minimal. (a) is called the "short distance OPE" ($x^\mu \to 0$), while (b) is called the "light-cone OPE" ($x^2 \to 0$).

To understand how these cases arise we must note that we have been rather glib in describing the limit taken in eq.(6). Consider the Fourier transform of the OPE, and suppose the operators O_1 and O_2 as well as the underlying action of the theory of interest are Lorentz invariant. Then the OPE takes the form

$$\int d^4x e^{-iq\cdot x} O_1(x) O_2(0) \sim \sum_i \tilde{C}_i^{1,2}(q^2) q_{\mu_1} ... q_{\mu_n} \hat{O}_i^{\mu_1 ... \mu_n}. \tag{7}$$

Roughly speaking, the $x \to 0$ limit now follows from taking the limit where $q \to \infty$. As q grows the x integral will be dominated by small x, since the integral over x will wash out due to large oscillations when x is away from zero. Whether or not a) or b) is the relevant case depends upon how the $q^\mu \to \infty$ limit is taken. Suppose we take the limit (using light-cone coordinates $x_\pm = (x_0 \pm x_3)$)

$$q_+ \to \infty,\ q_- \to const, \tag{8}$$

in which case only x_- is forced to zero. $x_\perp$ is unconstrained. Thus the light-cone OPE will be relevant if we consider cases where $x_\perp$ is small, e.g. if we consider an inclusive cross

[9] We must caution the reader that this is strictly true only in Euclidean space. Given the fact that physical processes are not Euclidean it is natural to ask how we justify using the OPE. In many cases, such as deep-inelastic scattering and e^+e^- to hadrons, one can use a dispersion relation which relates the unphysical Euclidean regime, where we can do the OPE, to the physical regime. See [7] for a discussion of the dispersion relation. It is interesting to note that the dispersion relation implements the averaging procedure discussed in the hand waving argument in eq. (2). When there is no averaging performed, one is making "local duality assumption". The duality being between quarks and hadrons. In such cases it is difficult to quantify the errors (for a discussion see [8]). This will be touched upon again in section 3 D.

section where the optical theorem dictates that

$$\sigma \sim Im(i\int d^4xe^{-iq\cdot x}\langle\psi \mid T(O(x)O(0)) \mid \psi\rangle) \tag{9}$$

Then assuming $q_0 > 0$

$$\sigma \propto \int d^4xe^{-iq\cdot x}\langle\psi \mid [O(x), O(0)] \mid \psi\rangle \tag{10}$$

since the unwanted term in the commutator is kinematically forbidden. (9) implies that the correlator vanishes when $x^2 < 0$ by the causal nature of the fields [10] and thus $x_\perp \to 0$ and $x^2 \sim 0$.

We said that for the short distance OPE we only need a finite number of local operator matrix elements, but for the light-cone case we need an infinite number. Let us see why this is true. Consider (7), for the moment, assuming we are working in a scale invariant theory. This approximation, allows us to use dimensional analysis to determine the relative size of terms in the sum [11]. Assign the operators $O_{1,2}$ and $\hat{O}_i$ mass dimensions d_1, d_2 and d_i respectively. Then the $\tilde{C}_i$ have mass units $d_{C_i} = d_1 + d_2 - 4 - n - d_i$, and the $\tilde{C}_i \sim (q^2)^{d_{C_i}/2}$. Now suppose we take the matrix element of $\hat{O}_i^{\mu_1\ldots\mu_n}$ between states of generic momenta. Then Lorentz covariance implies the matrix element takes the form

$$\langle p_k \mid \hat{O}_i^{\mu_1\ldots\mu_n} \mid p_j\rangle \sim A_0 p_m^{\mu_1}...p_l^{\mu_n} + A_1 g^{\mu_1\mu_2}p_m^{\mu_3}...p_l^{\mu_n} + A_2 g^{\mu_1\mu_2}g^{\mu_3\mu_4}p_m^{\mu_5}...p_l^{\mu_n} + ... \tag{11}$$

where the p_m are some set of (collinear) external momenta, such that $p_i \cdot p_j \ll q^2, q\cdot p$. On the RHS all possible permutations of the external momenta can in principle show up. We have also assumed that if the operators are composed of fermions, we have performed a spin average, otherwise would we would have to allow for Dirac spin structures on the RHS of the equation. The terms left off on the RHS have more factors of the metric chewing up indices. The A_i are functions of all the possible invariants $p_i \cdot p_j$. If all of the components of q were to grow large (holding p_i fixed), such that $q \cdot p \ll q^2$ then each term on the RHS becomes more suppressed (by powers of $1/q$ in the Wilson coefficients) as the dimensionality of $\hat{O}$ grows, i.e. this corresponds to a short distance expansion. On the other hand, if $q^2 \sim q \cdot p$ then the dimensionality of $\hat{O}_i^{\mu_1\ldots\mu_n}$ is not what fixes its relative importance in the sum. In this

[10] This means that the commutator vanishes outside light-cone since two measurements which are space-like separated can have no effect on each other.

[11] If we had a strongly coupled theory then operators can gain large anomalous dimensions dynamically, as in the case of systems with large scattering lengths discussed in the cold-atoms chapter.

case, it's the "$t \equiv$ twist" of the operator which matters. t is defined as the dimensionality minus the "spin" s, which counts the number of open indices [12]. The relevance of the operator decreases with t since as t increases the dimensionality of the A_i decreases (recall it has negative mass dimensions). Given that the A_i scales with the large scale q, when the absolute value of the dimensions of A_i grows the matrix element gets more suppressed.

So far we have ignored renormalization in our arguments, which is clearly unacceptable since the limit taken in (6) is singular. Operators composed of products of fields defined at the same point necessitate further renormalization beyond the Lagrangian counter-terms. However, for asymptotically free theories such as QCD, or any theory in which in which the UV is weakly coupled, i.e. perturbative[13], renormalization will not change our conclusions. At first we might think that the existence of a renormalization scale μ, would ruin our arguments since regulating introduces a new dimensionful parameter (μ in dimensional regularization). However, given our theory restrictions, and the fact that we work in dimensional regularization, μ can only show up in logs[14], such that our counting arguments are still valid. In fact, we expect the matching coefficients will have large logs just as in an EFT and the summation of the logs follows from running the coefficients to the low scale (in the case of QCD this scale would correspond to Λ_{QCD}).

One might worry that in an asymptotically free theory, strong coupling in the infrared can lead to large anomalous dimensions as we run down to the low scale leading to violations in our naive power counting. Typically when one runs the operators on the RHS of the OPE down to the low scale one must stop before the theory gets strongly coupled. Whether or not operator receive large enhancements before reaching the QCD scale is not known. Such enhancements would show up as numerical enhancements of their expectation values which are extracted from experiments or measured on the lattice. An operator with mass dimensions d should naively give a value of order Λ^d_{QCD} when renormalized at the scale Λ_{QCD}. Anything larger then that would be considered an enhancement. Such enhancements do occur but they are all less than an order of magnitude.

[12] On the right hand side of the OPE one should in practice write things in terms of irreducible representations of the Lorentz group. For what we are interested in here, this is not necessary.

[13] We are restricting ourselves here to four dimensions.

[14] We would not expect this to be true for theories with strongly interacting fixed point such as those discussed in the cold-atoms chapter.

A. A Short Distance Example: $e^+e^- \to H$

Let us now consider in more detail the aforementioned examples of the short distance and light-cone expansions. One way to immediately relegate oneself to a short distance expansion is to consider a case where the external state is the vacuum. In this case the OPE is an expansion in operator dimensions. This is the case for the annihilation cross section of an e^+e^- pair into hadrons.

The cross section for this process can be written as

$$d\sigma = \frac{1}{2q^2}\sum_{X_h}\int d^4y e^{iq\cdot y} L_{\mu\nu}\langle 0 \mid J_h^\mu(y) \mid X_h\rangle\langle X_h \mid J_h^\nu(0) \mid 0\rangle \tag{12}$$

where $q^2 = (k_1+k_2)^2 \equiv s$ is the square of the center of mass energy, and we have defined the leptonic tensor

$$L_{\mu\nu} = \frac{1}{4}\sum_{spins}\frac{e^2}{q^4}\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma_\nu u(k_1). \tag{13}$$

J_h^μ is the hadronic current composed of quark fields and we have ignored the weak (intermediate Z boson state) contribution to the current for simplicity. Notice that X_h are the physical states comprised of hadrons. At this point its not clear why the OPE is relevant. However, because of the inclusive nature of the process we may remove the sum over states using the completeness relation. Doing so leaves us with an operator product. We could perform the OPE on this product, but since all of our Feynman diagram technology is based upon time-ordered products, we can manipulate the cross section into a more practical form using the optical theorem such that

$$\sigma = \frac{1}{s}L_{\mu\nu}\int d^4x e^{iq\cdot x} Im\ i\langle 0 \mid T[J_h^\mu(x)J_h^\nu(0)] \mid 0\rangle. \tag{14}$$

Current conservation and Lorentz covariance imply that

$$i\int d^4x e^{iq\cdot x}\langle 0 \mid T[J_h^\mu(x)J_h^\nu(0)] \mid 0\rangle = (q^2 g_{\mu\nu} - q_\mu q_\nu)\langle 0 \mid \Pi(q^2) \mid 0\rangle. \tag{15}$$

Exercise 3.1 In the text it is mentioned that the OPE need not be performed on a time-ordered product (TOP). We turned the product of fields into a TOP using the optical theorem for convenience. Calculate the coefficient C_1 using the product of currents instead of the TOP. Show that the result is the same as the imaginary part of C_1 determined using the TOP.

Exercise 3.2 Consider the OPE when the states break Poincaré invariance, for example in a finite-temperature ensemble. In Euclidean space the time-like direction is compactified to S^1 with period β, and loop integrals become a three-dimensional integral plus a sum over discrete Matsubara modes. Determine how this compactification changes the RHS of the OPE. In particular, determine how many operators must be considered up to dimension four. Introduce a time-like vector $u^\mu = (1, 0, 0, 0)$ and use it to restore Lorentz covariance. First show that, with this additional vector, one can construct two (rather than one, cf. (15)) structures that obey current conservation. Then show that u^μ allows four operators at dimension 4. For a challenge, calculate the leading-order Wilson coefficients. If you get stuck, consult [9].

We now apply the short distance OPE to the correlator, by considering the set of lowest dimensional operators consistent with Poincare and gauge invariance,

$$\Pi(q^2) = C_1(q^2) + C_2(q^2)\bar{q}q + C_3(q^2)F^{a\mu\nu}F^a_{\mu\nu} + ... \tag{16}$$

Notice that on the right hand side we have limited ourselves to operators which can in principle have non-zero expectation values in the interacting QCD vacuum. If we were working in a background which broke Poincare invariance we would have to allow for more general structures. The operator $\bar{q}q$ breaks chiral symmetry, but since this symmetry is broken spontaneous by the QCD vacuum, leading to pseudo-Goldstone bosons (pions), we must include it. The coefficients can now be calculated by matching. Just as in an EFT we are free to choose any external states that we wish. This allows us to pick out individual operators by a judicious choice of external states. For instance, if we wish to calculate C_1

we take the matrix element in vacuum. The matrix element of the (LHS) is given by

$$\Pi(q^2) = i\frac{(eQ_f)^2 N_c}{(d-1)q^2}\int [d^dp]Tr\left(\gamma^\mu \frac{\not{p}}{p^2+i\epsilon}\gamma_\mu \frac{\not{p}+\not{q}}{(p+q)^2+i\epsilon}\right) \tag{17}$$

whereas the matrix element of the RHS of the OPE simply gives C_1. Here we have assumed only one quark flavor, the N_c results from summing over colors, and a quark, whose flavor is labelled by f, has charge Q_f in units of the electron charge. Performing the loop integration leaves

$$\Pi(q^2) = \frac{(eQ_f)^2}{4\pi^2}\left(-\frac{1}{\bar{\epsilon}} + \frac{5}{3} - \log(-(q^2+i\epsilon)/\mu^2)\right). \tag{18}$$

There are several noteworthy feature of this result. First off, we see an odd UV divergence. This divergence is an artifact of working in momentum space. Had we stayed in position space the result would simple be a product of two Green's functions. In any case this divergence is irrelevant since it is purely real. More importantly, there is no IR divergence. This must be the case since on the right hand side of the OPE there is nothing to calculate and the matrix element is simply one. Recall the the Wilson coefficients contain only hard contributions, and the matrix element must reproduce the IR, just as in a matching calculation in an EFT. The imaginary part of the matching coefficient is then given by

$$ImC_1(q^2) = \frac{(eQ_f)^2}{4\pi}, \tag{19}$$

since q^2 is timelike. Plugging this result into (14) gives $\sigma = \frac{4\pi\alpha^2\sum_f Q_f^2}{s}$, where we have summed over all possible quark flavors. This result, is accurate up to corrections in Λ/Q and α_s. We can see that in this simple case the factorization is trivial, there is nothing to factorize, the process is purely hard, at least at leading order in the Λ/Q expansion. There is a simple physical explanation for this. The inclusive nature of the observable does not probe any hadronic physics. Once we produce a quark-anti-quark pair at short distances, we don't care what happens. We're only interested in the probability to produce the pair, after that the probability to hadonize is one. At sub-leading order in the power expansion the pair creation probability becomes sensitive to vacuum flucutations. For instance, when we take the vacuum expectation value of the RHS of the OPE we encounter the "condensate" $\langle 0 \mid F^{a\mu\nu}F^a_{\mu\nu} \mid 0\rangle$. We can think of the effect of this condensate as a soft gluon arising from the vacuum, attaching to the quark line and then being reabsorbed by the vacuum.

Note these are not real gluons, nor are they virtual gluons in the standard sense of being exchanged between the quarks. These fluctuations are the result of the non-trivial nature of the QCD vacuum. In fact, it is misleading to think of the soft effects as "gluons". i.e. we should not think of vacuum effects as plane wave states created and destroyed by free field. The vacuum condensates entail all sorts of non-trivial field configurations, such as instantons (see for instance) [8], whose contributions to the action are non-analytic in the coupling constant and, as such, will not be seen at any order in perturbation theory.

B. A Light-Cone Example: Deep Inelastic Scattering

The classic example of the light-cone OPE involves the process $\gamma^\star + p \to X$, where X means fully inclusive. $\gamma^\star$ is an offshell space-like photon, with $\mid q^2 \mid \gg \Lambda^2_{QCD}$, which is emitted by an incoming electron. This process is called "deep inelastic scattering" (DIS). The off-shellness of the photon is determined by measuring the momentum of the final state electron. Treatments of DIS can be found in many standard field theory textbooks so we will not go into in great detail on this topic. We will utilize this canonical example simply to help the reader appreciate the nature of factorization, and to take a look at this subject in the lens of EFT's.

We begin by noting that since the incoming photon has a very short wavelength compared to the QCD scale (by assumption), we can think of it as of as scattering off of a quark. The time scale for the bound state dynamics (femtoseconds) is simply too long to be relevant. Moreover, thanks to asymptotic freedom, the quark is approximately free on these short time scale. However, this does not mean that the the process is not sensitive to IR physics. The hadronic physics is what determines the amount of momentum carried by the struck quark. So even at leading order in the power expansion, the scattering probability will depend upon the nature of the proton. Another way of recognizing this IR sensitivity is to note that when we calculate the partonic cross section, beyond tree level, we find that it is infra-red divergent. Whether or not a process is IR divergent depends upon the nature of the observable. The KLN theorem [10] (see [3],[11] for a discussion) states that for an observables to be IR finite, one must sum over all degenerate states, initial as well as final. That it is degeneracies that lead to IR divergences is recognized from basic quantum mechanics since degenerate states lead to vanishing energy denominators in perturbation theory. Thus we can understand why $e^+e^-\to$ hadrons is not sensitive to the IR at leading order in the power expansion. The reason this will not be the case for DIS is that now our initial states carry color [15] and since the light quarks are, for all intents and purposes massless, they can spit off an on-shell energetic collinear gluon or a very soft gluon in any direction while remaining on their mass shell and long-lived.

Thus it is hopefully clear that we can not hope to calculate, at any order in a power expansion, the rate for DIS without input from IR physics. However, thanks to factorization, we can separate the IR physics from the calculable UV physics. This factorization is achieved by using the light-cone OPE as we shall now discuss.

The DIS scattering cross section can be written as

$$\sigma = \frac{1}{2s}\int \frac{[d^3k']}{2k'_0}\frac{L^{\mu\nu}}{q^4} Im(W_{\mu\nu}), \tag{20}$$

where $L_{\mu\nu}$ is the leptonic tensor given by

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2}e^2\sum_{spins}\bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k) \\ &= 2e^2(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k'), \end{aligned} \tag{21}$$

and $q = k_\mu - k'_\mu$. The hadronic tensor is given by

$$W_{\mu\nu}(q^2, q\cdot p) = i\int d^4x e^{iq\cdot x}\langle P(p)\mid T(J^h_\mu(x)J^h_\nu(0)\mid P(p)\rangle). \tag{22}$$

Using Lorentz covariance and current conservation we can reduce the tensor to the following form

$$W_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)W_1(x,Q^2) + \left((p_\mu + \frac{q_\mu}{2x})(p_\nu + \frac{q_\nu}{2x})\right)W_2(x,Q^2). \tag{23}$$

where $x = \frac{Q^2}{2p\cdot q}$ and $Q^2 = -q^2$.

We're going to perform an OPE on $W_{\mu\nu}$, so let us consider the matching calculation. Recall that we're free to match with any state we wish, so we choose to use quarks as external states with momentum p (not the proton momentum). So that *partonically* at tree

[15] That is, the states with which we calculate, quarks, carry color. Of course the physical initial states are colorless protons.

level we have

$$
\begin{aligned}
ImW_{\mu\nu} &= -e^2Q_f^2\sum_{spins} Im[\bar{u}(p)\gamma_\mu\frac{(\not{p}+\not{q})}{(p+q)^2+i\epsilon}\gamma_\nu u(p)] + (\mu\leftrightarrow\nu, q\to -q) + O(g^2) \\
&= e^2Q_f^2\sum_{spins}\pi\delta((p+q)^2)[\bar{u}(p)\gamma_\mu(\not{p}+\not{q})\gamma_\nu u(p)] + (\mu\leftrightarrow\nu, q\to -q).
\end{aligned}
\tag{24}
$$

Exercise 3.3 Using this result prove that at tree level we have

$$
W_1 = \frac{Q^2}{4x^2}W_2, \tag{25}
$$

this is called the Callan Gross relation, which experimentally holds at the twenty percent level at $Q^2 \sim 50\ GeV$, which is consistent with the OPE results. This was the first evidence that quarks are spin 1/2.

Now the question arises as to whether or not we can Taylor expand the argument of the delta function. q is space-like and p is collinear and on shell, so we may take $p^\mu = \bar{n}\cdot p\frac{n^\mu}{2}$, and $q^\mu = \frac{Q}{2}(\bar{n}^\mu - n^\nu)$, this is the so-called "Breit" frame, and $|\,q^2\,| \gg \Lambda^2_{QCD}$. Since the invariant mass of the final state must be positive semi-definite we have the constraint

$$
1 \geq x \geq 0, \tag{26}
$$

Note that there is no limit in which we could write down a short distance expansion for this process. This implies that we can not match the operator product onto any finite number of local operators. We, will assume that x takes on values of order one, so that the invariant mass of the final state is much larger than Λ_{QCD}

$$
\frac{Q^2}{x}(1-x) \gg \Lambda_{QCD} \sim Q^2. \tag{27}
$$

The end point regimes where x approaches one and zero are called the "resonance" and "diffractive" regions respectively. These limits complicate the power counting and will not be discussed here. The reader is referred to [3] for pedagogical discussion of these limits.

Using our frame choice where $p^\mu = \bar{n}\cdot p\frac{n^\mu}{2}$, at tree level the hadronic tensor then matches onto

$$
ImW_{\mu\nu} \;=\; e^2 Q_f^2 \pi \sum_{spins} \delta(\bar{n}\cdot p n\cdot q - Q^2)[\bar{u}(p)\gamma_\mu(\not{n}\frac{\bar{n}\cdot p}{2} + \not{q})\gamma_\nu u(p)] + (\mu \leftrightarrow \nu, q \to -q), \tag{28}
$$

where we used the on-shell condition for the external quarks, and no further expansion can be performed. We may further simplify the Dirac structure using the on-shell condition $\not{p}u(p) = 0$ i.e. $\not{n}u(p) = 0$, and the decomposition

$$
\gamma_\mu = \not{n}\frac{\bar{n}_\mu}{2} + \not{\bar{n}}\frac{n_\mu}{2} + \gamma_\mu^\perp, \tag{29}
$$

then

$$
\begin{aligned}
ImW_{\mu\nu} \;&=\; e^2 Q_f^2 \pi \sum_{spins} \delta(\bar{n}\cdot p n\cdot q - Q^2)[\bar{n}\cdot(p+q) n_\mu n_\nu (\bar{u}(p)\frac{\not{\bar{n}}}{2}u(p)) + n\cdot q(\bar{u}(p)\gamma_\mu^\perp \frac{\not{\bar{n}}}{2}\gamma_\nu^\perp u(p))] \\
&+\; (\mu \leftrightarrow \nu, q \to -q) \\
&=\; -e^2 Q_f^2 \pi \sum_{spins} g_{\mu\nu}^\perp n\cdot q[\delta(\bar{n}\cdot p n\cdot q - Q^2)[(\bar{u}(p)\frac{\not{\bar{n}}}{2}u(p))] + (\mu \leftrightarrow \nu, q \to -q)].
\end{aligned} \tag{30}
$$

In the last line rotational symmetry was used to reduce the Dirac structure and the delta function constraint eliminated the first term.

Now we re-write this as

$$
\begin{aligned}
ImW_{\mu\nu} = -e^2 Q_f^2 \pi \sum_{spins} g_{\mu\nu}^\perp n\cdot q \quad & \int d\bar{n}\cdot s \left[\delta(\bar{n}\cdot s n\cdot q - Q^2)\right. \\
& \times[(\bar{u}(p)\delta(\bar{n}\cdot s - \bar{n}\cdot p)\frac{\not{\bar{n}}}{2}u(p))] + (\mu \leftrightarrow \nu, q \to -q)
\end{aligned} \tag{31}
$$

The reason for re-writing the result in this rather silly fashion is that it makes it clear how to do the matching. In particular, we can see that the matching needs to be in the form of a convolution of a Wilson coefficient and a non-local operator

$$O(\bar{n}\cdot s) = \frac{1}{2}\bar{\psi}(0)\not{n}\delta(\bar{n}\cdot(i\partial - s))\psi(0), \tag{32}$$

whose Feynman rule has the proper form to reproduce the square bracket on RHS of equation (31). From this expression we see the clear delineation between the light cone and short distance OPE. This expression is non-local, i.e. it involves and infinite number of derivatives. We are not free to expand in $\bar{n}\cdot i\partial$ as each term in such an expansion is of the same order.

The reader should be bothered by the gauge non-invariant nature of this result, due to the non-covariant derivatives, but but we elevate this operator to its proper gauge invariant form by covariantizing, i.e. $\partial_\mu \to D_\mu$. However, for the moment let us work in the light-cone gauge where $\bar{n}\cdot A = 0$, and when we are done it will be a simple step to covariantize the result. We can simplify the operator, by writing

$$\begin{aligned} O(\bar{n}\cdot s) &= \frac{1}{2}\bar{\psi}(0)\not{n}\delta(\bar{n}\cdot(i\partial - s))\psi(0) = \frac{1}{2}\int[dn\cdot x]\bar{\psi}(0)\not{n}e^{in\cdot x(\bar{n}\cdot(i\partial - s))}\psi(0) \\ &= \frac{1}{2}\int[dn\cdot x]e^{-in\cdot x\bar{n}\cdot s}\bar{\psi}(n\cdot x,0,0)\not{n}\psi(0). \end{aligned} \tag{33}$$

Now we can see clearly that we are matching onto a non-local operator in which the fields are separated by a light-like interval. Taking the hadronic matrix element of this operator yields the parton distribution function (PDF),

$$f(\zeta) = \frac{1}{2}\sum_{spins\ avg}\int[dn\cdot x]e^{-in\cdot x\bar{n}\cdot s}\langle P\mid\bar{\psi}(n\cdot x,0,0)\not{n}\psi(0)\mid P\rangle. \tag{34}$$

Boost invariance [16] implies that the dimensionless function f can only depend upon the ratio $\zeta = \bar{n}\cdot s/\bar{n}\cdot P$. Gauge invariance implies that the form the operator is uniquely fixed to be

$$f(\zeta) = \frac{1}{2}\sum_{spins}\int[dn\cdot x]e^{-in\cdot x\bar{n}\cdot s}\langle P\mid\bar{\psi}(n\cdot x)\not{n}W[n\cdot x,0]\psi(0)\mid P\rangle, \tag{35}$$

[16] Under a boost along the collinear direction parameterized by λ, $\bar{n}\cdot p$ simply rescales to $e^{\lambda}\bar{n}\cdot p$.

where W is the Wilson line defined by

$$W[0, n \cdot x] = e^{ig \int_0^{n\cdot x} \bar{n}\cdot A(\lambda n^\mu) d\lambda}. \tag{36}$$

Finally we may re-write the result for the hadronic tensor as

$$ImW_{\mu\nu} \;=\; -\pi g^{\perp}_{\mu\nu} \int d\zeta C(x/\zeta, Q^2/\mu^2) f(\zeta, \mu). \tag{37}$$

Note that C is a dimensionless function that can depend upon Q only through logarithms, this is called "Bjorken scaling". The tree level value for C is given by

$$C(x, \zeta) = 2e^2 Q_f^2 (\delta(1 - \zeta/x) - \delta(1 + \zeta/x)). \tag{38}$$

Note that the general result for the hadronic tensor can have other tensor structures. However, it just happens that their matching coefficients vanish at leading order in perturbation theory.

We will interpret $f(\zeta)$ as the parton distribution function (PDF), which we will now label as f_q. ζ may roughly be thought of as the momentum fraction (relative to the incoming proton) carried by the struck quark. When $\zeta < 0$ we interpret f as an anti-quark distribution function, and utilize the fact that $f_q(-\zeta) = -f_{\bar{q}}(\zeta)$ to re-write the result as

$$ImW_{\mu\nu} \;=\; -\pi g^{\perp}_{\mu\nu} \int_x^1 d\zeta C(\zeta/x)(f_q(\zeta) + f_{\bar{q}}(\zeta)) \tag{39}$$

and $C = 2e^2 Q_f^2(\delta(1-\zeta/x))$. x is interpreted as the parton momentum fraction carried by the quark which scatters off of the photon. Beyond tree level is not equal to ξ since momentum is carried away by radiation. C can only depend upon the fraction ζ/x as a consequence of scaling. The struck parton doesn't know about $n \cdot p$ only $\zeta n \cdot p$, so we make the replacement $x \to x/\zeta$. The lower limit in (39) comes from ensuring that the final state invariant mass is positive semi-definite [17].

We see that the OPE has factorized the calculable (C) from the incalculable ($f(\zeta)$). The predictive power of the result lies in this factorization. The function $f(\zeta)$ is *universal*. That is, it depends only upon the properties of the proton, and is independent of scattering

[17] Once hadronization is accounted for it must be positive definate.

process. Thus if we are able to extract $f(\zeta)$ from another process we may predict the DIS cross section. The result (39) is exactly the kind of convolution discussed in the introduction and shown in figure (2)[18].

Note that at higher loops in the matching (starting at one loop), we will run into both UV and IR divergences. The IR divergences will cancel in the matching, and the one loop corrections to $f(\zeta)$ will induce scale dependence (μ) due to the existence of logarithmic dependence on the hard scale Q^2. The subsequent running of the PDF follows the so-called "DGLAP" equation [19]. In the effective field theory approach the running will follow from standard operator renormalization. Given that the running is calculable in perturbation theory, this gives us additional predictive power in that, given the cross section at some Q^2, we can predict it at some other Q^2 despite our inability to calculate $f(\zeta)$. However, as we shall see in the next section, the OPE is not always sufficient to generate useful predictive results.

C. When the OPE is insufficient : The Drell Yan Process

The simplicity of the DIS result is a consequence of the fact that we have only one hadron in the initial state. But suppose we are interested in hadron-hadron scattering. In this case, as we shall see below, we need to go well beyond the OPE to make theoretical predictions.

To illustrate this point we consider the Drell-Yan process which corresponds to the following reaction

$$h + h' \to l^+(k) + l^-(k') + X \tag{40}$$

where X means that we sum over all final states, i.e. its inclusive. We may write the cross section for this process as

$$d\sigma = \frac{1}{2s}\int \frac{[d^3k]}{2E_k}\frac{[d^3k]'}{2E'_k}\int d^4x e^{-iq\cdot x}L^{\mu\nu}(k,k')\langle pp' \mid J_\mu(x)J_\nu(0) \mid pp'\rangle \tag{41}$$

and

$$L_{\mu\nu} = 4\frac{e^4}{q^4}(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k'). \tag{42}$$

[18] Albeit in the figure we have two incoming particles instead of one as in DIS.

[19] This acronym stands for the group of authors, Dokshitzer, Gribov, Lipatov, Altarelli and Parisi. The uninitiated reader can find a discussion of this topic in [7].

where $q_\mu = k_\mu - k'_\mu$. If the hadrons have spin, then we include a spin averaging. Let us perform an OPE on this correlator as we did in the case of DIS. As in the DIS case we define the hadronic current correlator as

$$H_{\mu\nu} \equiv \int d^4x e^{-iq\cdot x} \langle pp' \mid J_\mu(x) J_\nu(0) \mid pp' \rangle, \tag{43}$$

and for the purpose of illustration we will take the hadrons to be spin one half nucleons. We match using a quark a anti-quark pair as our external states with four momenta $n \cdot p \frac{\bar{n}^\mu}{2}$ and $\bar{n} \cdot p' \frac{n^\mu}{2}$, respectively. As in DIS the matrix element in (43) is not a time ordered product, and thus we do not have Wicks theorem at our disposal. This is no obstruction though. We simply insert a complete set of states between the currents and then use LSZ along with Wicks theorem in a standard fashion.

At leading order in the coupling the vacuum saturates the complete set of intermediate states and we find that partonically

$$\begin{aligned} H_{\mu\nu} &= (2\pi)^4 \delta^{(4)}(q-p-p') \bar{v}(p') \gamma_\mu u(p) \bar{u}(p) \gamma_\nu v(p') \\ &= -\frac{1}{2} (2\pi)^4 \delta^{(4)}(q-p-p') \bar{v}(p') \gamma_\mu v(p') \bar{u}(p) \gamma_\nu u(p) \\ &= -\frac{n_\mu \bar{n}_\nu}{2} (2\pi)^4 \int dn\cdot r d\bar{n}\cdot r \ \delta^{(2)}(q_\perp) \delta(n\cdot(q-r)) \delta(\bar{n}\cdot(q-r)) \\ &\qquad \times [\bar{v}(p') \delta(n\cdot(r-p')) \frac{\bar{n}\!\!\!/}{2} v(p')][\bar{u}(p) \frac{n\!\!\!/}{2} \delta(\bar{n}\cdot(r-p)) u(p)] \end{aligned} \tag{44}$$

Where we Fierz rearranged the result so as to collect momenta into individual bilinears. We have also dropped the bilinears which will have vanishing matrix elements between unpolarized nucleons. If we follow the same line of reasoning as in the DIS example, we would match onto the operator

$$O(n\cdot r, \bar{n}\cdot r) = \int [dn\cdot x][d\bar{n}\cdot x] e^{-in\cdot x \bar{n}\cdot r} e^{-i\bar{n}\cdot x n\cdot r} \bar{\psi}(n\cdot x) \frac{\bar{n}\!\!\!/}{2} \psi(0) \bar{\psi}(\bar{n}\cdot x) \frac{n\!\!\!/}{2} \psi(0) \tag{45}$$

and write the hadronic tensor, at leading order in the matching as,

$$H_{\mu\nu} \sim n_\mu \bar{n}_\nu \int d\zeta d\zeta' C(\zeta, \zeta', z, z') F(\zeta, \zeta') \tag{46}$$

where

$$F(\zeta, \zeta') \equiv \langle PP' \mid O(n \cdot r, \bar{n} \cdot r) \mid PP' \rangle \tag{47}$$

and $\zeta = n \cdot r / n \cdot P,\ \zeta' = \bar{n} \cdot r / \bar{n} \cdot P',\ z = n \cdot q / n \cdot P,\ z' = \bar{n} \cdot q / \bar{n} \cdot P'$. This result in itself is not terribly useful. The cross section will be a function of this unknown function F. Which in principle one could hope to extract, but in practice would be folly. More practically, if we can prove that we can factor the matrix element such that

$$\langle H(p)H'(p') \mid O(n \cdot s, \bar{n} \cdot s) \mid H(p)H'(p') \rangle = \langle H(p) \mid O(n \cdot s) \mid H(p) \rangle \langle H'(p') \mid O(\bar{n} \cdot s) \mid H'(p') \rangle \tag{48}$$

then we could write the cross section in terms of the *universal* functions $f_q(\zeta), f_{\bar{q}}(\zeta)$ and we would regain predictive power since these functions are process independent. i.e. the only depend upon the properties of the nucleons under study and not the hard underlying process. We will show in the next few sections that, by utilizing an EFT, we will be able to prove that (5) is correct up to corrections suppressed by powers pf Λ_{QCD}/Q..

D. A Cautionary Word About the OPE

Before moving onto understanding factorization using EFT, we must inject a short word of caution regarding the OPE. In the introduction to this chapter it was briefly mentioned that the OPE is truly only justified in Euclidean space. The reason for this is quite simple. We can think of the OPE as shrinking off-shell lines to a point as pictured in figure (1). Working in Euclidean space is equivalent to working with unphysical external momenta, which ensures that the intermediate states can never become real long lived particles with positive invariant mass. If we consider our example of $e^+e^- \to$ *hadrons* we could imagine working at a value of q^2 which sits near a resonance. In this case there is no reason to believe the OPE will give a sensible result. At scales well above the QCD scale this should not be a worry, since, save for the top quark threshold [20] the landscape is free of resonances, at least in the standard model. However, given the fact that QCD has multiple resonances above, but not too far from the Λ_{QCD}, we might worry that the OPE will give non-sense in this range of q^2. In fact, there is no reason to expect the OPE to work in this region [21]. After all, when we did the OPE we calculated the matching using quarks to get a result about hadrons. This is sometimes called "quark-hadron duality". We might expect this duality to hold away from resonances, but as we approach a resonance the production rate becomes sensitive to the details of hadronization. Nevertheless, there is a way to overcome this problem, or at least mitigate it, at the expense of losing some predicitive power.

At values of q^2 above but near the QCD scale, we instead only claim to have predictive power when we average (or smear) over values of q^2. That is, we can make predictions $e^+e^- \to hadrons$ cross section integrated (with some smooth weighting function) over a range of q^2. When we average in this way over resonance we regain quark-hadron duality. Physically this corresponds to the averaging in our wave mechanics example discussed (see eq. (2)). Equating the partonic and hadronic smeared rates is sometimes called "global" duality as opposed to "local" duality where one does not perform any averaging procedure. One expects local duality to only hold at q far above the QCD scale where we don't expect any resonances. To justify global duality one uses a dispersion relation to relate the physical Minkowski region to the unphysical Euclidean region. This can be done for both DIS as well as $e^+e^- \to hadrons$ (see [7, 11]). However, in some cases of interest, as in the Drell-Yan process under study here, it is not possible to completely justify duality assumptions, and determining the error incurred by these assumptions is challenging [8]. Furthermore, even if we rely on local duality (which should be a good approximation at scales $Q^2 \gg \Lambda_{QCD}^2$), in the case of Drell-Yan the OPE would still not lead to a useful prediction, because, as opposed to the case of DIS, the resulting matrix element is not a simple PDF since the external states include both hadrons. To overcome this we will have to show that the two hadrons factorize and to do so we will utilize the effective field theory SCET.

[20] At the top-anti-top threshold one can still calculate since the top is heavy enough that one can calculate in perturbation theory. Given the relatively large size of the mass, the system is Coulombic. Moreover, the lifetime is so short that the system never has time to hadronize. In this case one utilizes NRQCD to calculate systematically, see [12] for details.

[21] When studying inclusive B decays this is exactly the region probed, see [13] for details.

4. SOFT-COLLINEAR EFFECTIVE THEORY

We have already introduce the light cone vectors $(\bar{n}/n)^\mu$. Here we will set down the conventions we will be using the rest of the chapter.

$$\boxed{\begin{aligned}
&n^\mu = (1,0,0,1), \qquad \bar{n}^\mu = (1,0,0,-1),\\
&p_+ \equiv n\cdot p, \qquad p_- \equiv \bar{n}\cdot p,\\
&p^\mu = (n\cdot p,\, \bar{n}\cdot p,\, \vec{p}_\perp) = (n\cdot p)\,\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\,\frac{n^\mu}{2} + p_\perp^\mu,\\
&g_{\mu\nu} = \frac{n_\mu \bar{n}_\nu}{2} + \frac{n_\nu \bar{n}_\mu}{2} + g_{\mu\nu\perp}.
\end{aligned}} \tag{49}$$

We would now like to attack the problem of high energy scattering from the point of view of effective field theory. Consider the motion of a highly relativistic particle. Kinematically this scenario akin to a heavy quark in that the particle has a lot of inertia. Such a relativistic particle is not easily deterred from its path, just as a heavy quark sitting inside a meson won't budge when it is bombarded by soft gluons. As such, one might think that we could develop an effective field theory for energetic particles, integrating out all of the hard modes which cause its trajectory to deviate from a straight line, leaving behind the interactions which can be treated as small perturbations [22]. As shall this naive picture of a classical like source, is only partially true, due to collinear splittings of the line.

Consider a generic field which transforms irreducibly under a Lorentz transformation. Expanding it in terms of creation and annihilation operators we may write

$$\phi(x) = \int [d^4p](2\pi)\theta(p^0)\delta(p^2)(F(p,x)a_P + G(p,x)b_p^\dagger) \equiv \phi^+(x) + \phi^-(x), \tag{50}$$

where F and G carry the appropriate Lorentz indices and $b^\dagger$ creates anti-particles and is equal to $a^\dagger$ for real fields.

Suppose we are interested in describing a highly energetic particle such that $n\cdot p \gg (\bar{n}\cdot p, p_\perp)$. In the mode expansion we consider solutions to the equations of motion such

[22] This idea is very similar to looking at small fluctuations near the Fermi surface [14, 15].

that $n \cdot p\, \bar{n} \cdot p - p_\perp^2 = 0$, which implies that we have a hierarchy

$$n \cdot p \gg p_\perp \gg \bar{n} \cdot p. \tag{51}$$

Thus $\bar{n} \cdot p$ plays the role of the smallest scale in the problem. Let us define the dimensionless power counting parameter $\lambda \equiv p_\perp / n \cdot p$ such that the light cone momenta scale as $(1, \lambda, \lambda^2)$.

We would like the field to have support only over momenta of order this low scale. This will facilitate the power counting just as in HQET and NRQCD. As in those cases, we re-phase the field for the purpose of removing the large scales. Writing the momentum as

$$p = \mathbf{p} + k \qquad \mid k \mid \ll \mid \mathbf{p} \mid . \tag{52}$$

we remove the large momentum ($\mathbf{p}$) from the field via a re-phasing

$$\phi^+ \to \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot x} \phi^+_{\bar{n},\mathbf{p}}(x) \qquad \phi^- \to \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot x} \phi^-_{\bar{n},\mathbf{p}}(x) \tag{53}$$

where $\mathbf{p}^\mu = n \cdot \mathbf{p} \frac{\bar{n}^\mu}{2} + \mathbf{p}^\mu_\perp$ and k is the residual momentum. Once we have removed the large momentum we have

$$\partial \phi^\pm_{\bar{n},\mathbf{p}}(x) \sim \bar{n} \cdot p \sim \lambda^2, \tag{54}$$

for all the momentum components. That is the residual momenta p_r scale as

$$p^\mu_r \sim (\lambda^2, \lambda^2, \lambda^2). \tag{55}$$

The large momentum labels now act much like the mass in HQET, and can be treated as parameters in the Lagrangian so that it will make sense to have anomalous dimensions depend upon these labels. Furthermore, inverse powers of labels are NOT considered as avatars of non-locality.

It is then convenient to write

$$\phi^- \equiv \sum_{-\mathbf{p}} e^{-i\mathbf{p}\cdot x} \phi^-_{\bar{n},-\mathbf{p}}(x) \tag{56}$$

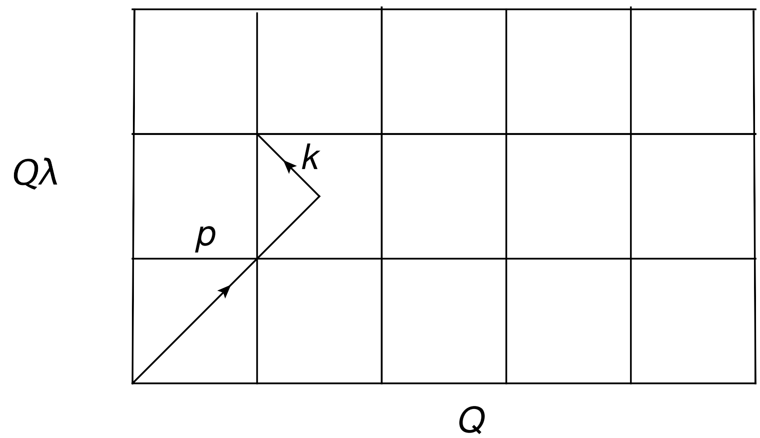


FIG. 3. The tessellation of momentum space into bins. The typical size of each bin is of order of the residual momentum λ^2. Lengths in the horizontal/vertical directions scale as $Q/\lambda Q$. Note that when we integrate over residual momenta there is no cut-off. When we integrate the residual integral is combined with the label sum.

such that [23]

$$\phi_{\bar{n},\mathbf{p}} = \phi^{+}_{\bar{n},\mathbf{p}}\theta(\mathbf{n}\cdot\mathbf{p}) + \phi^{-}_{\bar{n},-\mathbf{p}}\theta(-\mathbf{n}\cdot\mathbf{p}) \tag{57}$$

In this way the field destroys a particle when $\mathbf{n}\cdot\mathbf{p} > 0$ and creates an anti-particle $\mathbf{n}\cdot\mathbf{p} < 0$. In this way we can use one field for both particles and anti-particles.

Notice that since we have a sum over labels instead of an integral, $\phi_{n,\mathbf{p}}$ has the same units as ϕ. The sum implies that we are splitting the momenta into bins. We can think of these bins as tesselating momentum space, as shown in figure (3). The horizontal axis represents the large label $\mathbf{n}\cdot\mathbf{p}$ while the vertical axis the smaller label in the transverse momentum. The typical size of each box is the residual momentum $\bar{n}\cdot p$. Each box represents a bin, and just as in NRQCD, we will need to worry about what happens in the zero-bin. Indeed, strictly speaking we should write the label sum with an exclusionary condition such that we have

$$\phi(x) \to \sum_{\mathbf{p}\neq 0} e^{i\mathbf{p}\cdot x}\phi_{\bar{n},\mathbf{p}}(x). \tag{58}$$

We will return to the zero bin issue in SCET once we have developed the interacting theory.

Note that $\phi_{\mathbf{p}}$ is *not* the Fourier, or even a partial Fourier transform of the field. The discrete nature of the labels forbids this interpretation. Sometimes however, it will be useful to consider the full Fourier transform of the field by going to *continuous labels*. We do

[23] Note that the field is labelled by $\bar{n}$ which is the direction of the large momentum, despite the fact that the large label is $n\cdot\mathbf{p}$.

this by first Fourier transforming the residual x dependence and then combining residual momentum integral with the label sum such that

$$\sum_{\mathbf{p}\neq\mathbf{0}} e^{i\mathbf{p}\cdot x}\phi_{\mathbf{p}}(x) = \sum_{\mathbf{p}\neq\mathbf{0}} \int d^4k e^{i\mathbf{p}\cdot x} e^{ik\cdot x}\tilde{\phi}_{\mathbf{p}}(k) = \int d^4p\tilde{\phi}(p)e^{ip\cdot x}. \tag{59}$$

This Fourier transform must be used with care, since not all of the momenta components scale in the same way. Furthermore, in the last equality we have ignored the zero-bin exclusion which will automatically be taken care of when we subtract integrands, as we did in NRQCD (more on this later). Also, as in NRQCD, when we do loop integrals we will be combining label and residual momenta integrals which in effect is the same as going to continuous labels.

Note that it would seem a lot easier to use continuous labels and not have to worry about carrying around the sums all the time. So why do we bother with the label sums? The answer is, that the label sums make the power counting clear and inescapable. Once one has mastered the formalism however, one can indeed skip the sums and work directly with the Fourier transform. The sums are merely a scaffolding which will be removed in the end.

Now let us consider the propagator for a fermionic collinear field. We may expand the full QCD propagator in terms of labels and residual momenta such that

$$\frac{i\not{p}}{p^2+i\epsilon} = i\frac{\not{\bar{n}}}{2}\frac{1}{\bar{n}\cdot k - \frac{\mathbf{p}_\perp^2}{\mathbf{n}\cdot\mathbf{p}} + i\epsilon\ sgn(\mathbf{n}\cdot\mathbf{p})} + O(k) \tag{60}$$

where the perp ($\perp$) inner product is Euclidean. Note that propagator contains both quarks and anti-quarks poles, since the sum over labels includes $n\cdot\mathbf{p}<0$ and there will be poles on both sides of the real axis. If we wish we can avoid tis complication we can instead use conjugate spinors so that we can treat quarks and anti-quarks on the same footing.

We can arrive directly at the fermionic propagator (60) by taking full QCD and expanding the field according to (53).

$$S = \int d^4x\bar{\psi}(x)(i\not{\partial})\psi(x) = \int d^4x \sum_{\mathbf{p}} \bar{\psi}_{\bar{n},\mathbf{p}}(\not{\mathbf{p}} + i\not{\partial})\psi_{\bar{n},\mathbf{p}} \tag{61}$$

Notice that the phase is gone, and we have reduced the double sum over momenta to a single sum. Label conservation is implicitly enforced by the measure. However, we must be careful when we power count, as we will discuss shortly.

For a non-trivial representation of the Lorentz group, the components of the fields will not all scale in the same way. Thus to accomplish the goal of having each term scale homogeneously, we need to decompose the field into its large and small components just as we did in HQET and NRQCD. When we expand the field in plane wave solutions we have

$$\bar{n}\!\!\!/\psi_{\bar{n},\mathbf{p}} = (0 + O(\mathbf{p}_\perp/n\cdot\mathbf{p}))\psi_{\bar{n},\mathbf{p}}. \tag{62}$$

So to distinguish the large and small components of the field we utilize the projectors

$$\wp_{\bar{n}} = \frac{1}{4}\bar{n}\!\!\!/ n\!\!\!/ \qquad \wp_n = \frac{1}{4} n\!\!\!/ \bar{n}\!\!\!/. \tag{63}$$

$$\begin{aligned} \xi_{\bar{n},\mathbf{p}} &= \wp_{\bar{n}}\psi_{\bar{n},\mathbf{p}}, \qquad \rho_{\bar{n},\mathbf{p}} = \wp_n\psi_{\bar{n},\mathbf{p}}, \\ \bar{n}\!\!\!/\xi_{\bar{n},\mathbf{p}} &= 0, \qquad\qquad n\!\!\!/\rho_{\bar{n},\mathbf{p}} = 0. \end{aligned} \tag{64}$$

and we may write

$$\begin{aligned} \bar{\psi}_{\bar{n},\mathbf{p}}(\mathbf{p}\!\!\!/ + i\partial\!\!\!/)\psi_{\bar{n},\mathbf{p}} &= (\bar{\xi}_{\bar{n},\mathbf{p}} + \bar{\rho}_{\bar{n},\mathbf{p}})(n\cdot\mathbf{p}\frac{\bar{n}\!\!\!/}{2} + \mathbf{p}\!\!\!/_\perp + i\partial\!\!\!/)(\xi_{\bar{n},\mathbf{p}} + \rho_{\bar{n},\mathbf{p}}) \\ &= \bar{\xi}_{\bar{n},\mathbf{p}}(i\bar{n}\cdot\partial\frac{n\!\!\!/}{2})\xi_{\bar{n},\mathbf{p}} + \bar{\rho}_{\bar{n},\mathbf{p}}(n\cdot\mathbf{p}\frac{\bar{n}\!\!\!/}{2})\rho_{\bar{n},\mathbf{p}} + \bar{\xi}_{\bar{n},\mathbf{p}}\mathbf{p}\!\!\!/_\perp\,\rho_{\bar{n},\mathbf{p}} + \bar{\rho}_{\bar{n},\mathbf{p}}\mathbf{p}\!\!\!/_\perp\,\xi_{\bar{n},\mathbf{p}} \end{aligned}$$

where we have kept the only the leading order term for each type of bilinear. We will justify this approximation aposteriori. Now consider the equations of motion for ρ_n

$$(n\cdot\mathbf{p}\frac{\bar{n}\!\!\!/}{2})\rho_{\bar{n},\mathbf{p}} = -\mathbf{p}\!\!\!/_\perp\,\xi_{\bar{n},\mathbf{p}} \tag{65}$$

and we can now see explicitly that ρ_n is down compared to ξ_n by an anount of order $\mathbf{p}_\perp/\bar{n}\cdot\mathbf{p} \sim \lambda$. This does not however mean that we can drop it from the action since its the overal size of individual terms that matters. Also note that ρ_n has no kinetic term . So ρ_n is an auxilary field and we can integrate it out. i.e. we can replace it in the action using the equations of motion leaving

$$S = \int d^4x \sum_{\mathbf{p}} \bar{\xi}_{\bar{n},\mathbf{p}}(i\bar{n}\cdot\partial + \mathbf{P}\!\!\!/_\perp\frac{1}{n\cdot\mathbf{P}}\mathbf{P}\!\!\!/_\perp)\frac{n\!\!\!/}{2}\xi_{\bar{n},\mathbf{p}}. \tag{66}$$

Here we have defined the label operator[24] $\mathbf{P}$ [16] which picks out the relevant large momentum from the field and has the properties of a derivative operation on the space of labels such that

$$\mathbf{P}^\mu \left(\psi^\dagger_{\mathbf{k_1}}....\psi^\dagger_{\mathbf{k_n}}\psi_{\mathbf{p_1}}...\psi_{\mathbf{p_m}}\right) = (\mathbf{k_1}^\mu + ...\mathbf{k_n}^\mu - \mathbf{p}_1^\mu - .....\mathbf{p}_m^\mu)\left(\psi^\dagger_{\mathbf{k_1}}....\psi^\dagger_{\mathbf{k_n}}\psi_{\mathbf{p_1}}...\psi_{\mathbf{p_m}}\right). \quad (67)$$

This operator will simplify notation when writing down non-local operators.

As a side note, we have at this point established the following conventions for the collinear fields

$$\boxed{\begin{array}{ll} \psi_n : \bar{n}\cdot p \sim \mathcal{O}(1), & \not{n}\,\psi_n = 0 \\ \psi_{\bar{n}} : n\cdot p \sim \mathcal{O}(1), & \not{\bar{n}}\,\psi_{\bar{n}} = 0 \end{array}} \quad (68)$$

The inverse of the quadratic form in (66) reproduces the propagator (60). As mentioned above, the spacetime integral that remains will impose residual momentum conservation but we need to impose label conservation by hand. Physically this appears obvious, but mathematically seems unjustified. However, formally one can think of there being a Kronecker delta function in the matching from the full theory resulting in label conservation. We have also left the result in a rather bizarre form. The labels are just numbers, so there is no reason that we can not commute the transverse momenta through the large label denominator. However, we will leave it in this form for reasons which become clear once we include gauge fields, which we now address.

Consider a collinear gauge degree of freedom with momenta which scale as in (51). As for the case of the fermions, the components of the fields will scale differently. To determine how the individual components of the gauge field scale we may consider the two point function in a general covariant gauge

$$\langle A_\mu(x)A_\nu(0)\rangle \sim \int \frac{d^4p}{p^2}e^{ip\cdot x}(g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}), \quad (69)$$

where we have combined the label sum with the integral over residual momenta. Given the

[24] This sets the notation such that cap bold letters are operators on the space of labels which are small bold letters.

scaling of the momenta (e.g $n \cdot p \sim O(1)$), we can see that

$$\boxed{\begin{array}{llll} A_n : \bar{n} \cdot p \sim \mathcal{O}(1), & n \cdot A_n \sim \mathcal{O}(\lambda^2), & A^{\perp}_{n\mu} \sim \mathcal{O}(\lambda), & \bar{n} \cdot A_n \sim \mathcal{O}(1) \\ A_{\bar{n}} : n \cdot p \sim \mathcal{O}(1), & \bar{n} \cdot A_{\bar{n}} \sim \mathcal{O}(\lambda^2), & A^{\perp}_{\bar{n}\mu} \sim \mathcal{O}(\lambda), & n \cdot A_{\bar{n}} \sim \mathcal{O}(1) \end{array}} \tag{70}$$

A good mnemonic is $A_\mu \sim p^\mu$. One may be concerned by the apparent gauge dependence. One could certainly choose to work in non-covariant physical gauges, where the power counting will be different. However, the class of covariant gauges are chosen for the sake of convenience[25].

Consider now the action for the gauge field in the full theory

$$L = \frac{1}{2g^2} Tr \left([iD_\mu, iD_\nu] \right). \tag{71}$$

As in the quark case we remove the large momenta by a label rescaling and write the covariant derivative

$$iD_\mu \equiv id_\mu + gA^\mu_{n,q} \tag{72}$$

where

$$id_\mu = i\frac{n_\mu}{2}\bar{n} \cdot \partial + n \cdot \mathbf{P}\frac{\bar{n}^\mu}{2} + \mathbf{P}^\mu_\perp \tag{73}$$

Note that, given the scalings of the gluon components, each component of D_μ scales homogeneously and that the action scales homogeneously. In addition, we must include a gauge fixing term, as well as the action for the ghosts which are the same as in QCD with the derivative replaced as in (72).

5. THE MODAL DECOMPOSITION

So far all we have done is to formulate a field theory in a boosted frame. The non-trivial aspect of the effective theory only arises once we consider interactions with non-collinear modes. The question then becomes what are the relevant interacting modes? To answer this question we need to be more specific about the processes of interest, as the relevant modes

[25] Physical gauges can often complicate the power counting. For instance in HQET if chose the gauge $v \cdot A = 0$ it would seem that there are no interactions at leading order! However, this conclusion is erroneous as the power counting must be revamped to account for the fact that the heavy quark kinetic energy becomes a leading order effect (see for instance[13]).

depend upon the nature of the observable.

It should be emphasized that there is no exact algorithm for finding the relevant modes. Indeed, much like a choice of coordinates, the modal decomposition is not physical and can depend upon the choice of IR regulator [26] (not to mention the choice of reference frames). Of course what one seeks is the simplest description of the theory. Indeed, a bad choice of IR regulator can even obscure factorization [27]. It is important to remember that perturbative calculations in the effective theory need not reproduce the ultimate IR behavior of the theory, which, in general, may be strongly interacting (for asymptotically free theories). Factorization occurs at the matching scale and so should not depend upon the choice of IR regulator. However, as mentioned above, a poor IR regulator choice can lead to a realization of modes which does not manifestly factorize. Fortunately, the canonical choices of regulators does not lead to such problems. It is also important to point out that individual diagrams may include modes that cancel in the total amplitude. Sometimes however, the cancellation may not be manifest until a factorization theorem is written down.

A. What modes do we include?

How do we know when and how to break up our fields into sub-modes? As far as I am aware there is not definative associated algorithm. So instead here I will try to give a general set of guidelines for the process. A sufficient criterion for the inclusion of a mode is its existence as an external state, which are by definition "on-shell", meaning that the mode has the potential to have zero virtuality. This means that given a dispersion relation $E = f(k)$ then the scaling of E in the expansion parameter will be the same as $f(k)$. As an example of an off-shell mode we have the potential gluon mode in NRQCD which has $E \sim v^2$ while $k \sim v$ and its thus virtuality will *always* be space-like, i.e. it's off-shell $(E^2 - k^2) \sim -v^2$.

The choice of external kinematics thus picks us a subset of the necessary modes. As an example, if we are studying the scattering of high energy massless particles then we will need to include collinear modes in the theory. This seems clear given unitary (the optical theorem) implies that if they appear in final external states they must be in the loops. However, as alluded to above, this criterion is not a necessary condition as made clear by

[26] For example, Smirnov discusses forward scattering using a regulator where no separate Glauber mode appears; see [17] and the discussion in section 8.

[27] For a discussion of this issue in the context of DIS see [18].

the need for the potential mode. We will see that another example of such an exception arises in SCET when we consider forward scattering.

It is also possible to have on-shell modes which are not in X_i, such as soft modes in NRQCD [28]. So how do we know what modes to include beyond those in external states? The glib answer is, include whatever you need to reproduce all the non-analyticities (logs) and IR divergences contained the full theory. It is important to keep in mind that non-analyticites can be associated with both UV and IR divergences. That is, if we have large (UV) and small (IR) kinematic variables, an amplitude will, in general contain logs of both. The EFT need only reproduce the logs of the ladder type. The logs of the hard variables will be contained in matching coefficients. Schematically, we can see that logs will be accompanied by poles, by working in dim. reg., as dimensional analysis dictates that the kinematic variables be raised to fractional powers which leads to logs via

$$\Gamma[\epsilon](q^2)^{a\epsilon} \sim \frac{1}{\epsilon} + \log(q^2) + ... \tag{74}$$

Non-analyticities can arise at tree as well as at loop level. The famous tree example arises in NRQCD where gluon exchange in the t-channel gives rise to $1/\vec{q}^2$ poles, which corresponds to the forementioned off-shell potential mode. Usually we integrate out such modes as they can not be external states due to their off-shellness.

At the loop level the non-analyticities will be logs and roots which will be reproduced by the inclusion of the proper modes in the EFT. To determine what modes are are needed we need a mathematical criteria for the existence of a singluliarity. Such a criteria arises from what is known at the Landau equations [19, 20] for the existence of branch points in integrals. Landau showed that singularities (poles or branch cuts) arise if, *for a given value of the external invariants*, there is a pinch in the energy integration, i.e. when poles literally pinch or trap the integration contour. This imposes a set of conditions which are found by first Feynman parametrizing the denominators D_i into a single denomimator F equation $\mathrm{F}(\ell) = \sum_i \alpha_i D_i(\ell, p)$.

For there to be a singularity it is not enough that $F(\ell) = 0$. The zero must also be a point where the contour cannot be defomed away from the zero. That happens when the

[28] What about the optical theorem argument given above? The soft mode avoids being cut by contributing only to the running of the coupling in the potential and thus never contributes to a cut. There is an alternative formulation of NRQCD where the soft mode gets integrated out [**?**].

denominator is stationary with respect to the loop momentum:

$$\frac{\partial F}{\partial \ell^\mu} = 0. \tag{75}$$

Using Eq. (5 A), this gives

$$\sum_i \alpha_i \frac{\partial D_i}{\partial \ell^\mu} = 0. \tag{76}$$

The on-shell propagator poles approach the integration contour from different directions in such a way that there is no infinitesimal deformation of ℓ^μ that moves all denominators away from zero. In other words, the contour is trapped, or *pinched*[29].

A example of a pinch is found in the following integral

$$I(a) = \int dx \frac{1}{x-a+i\epsilon}\frac{1}{x-a-i\epsilon}, \tag{77}$$

since there is no way to deform the contour such that the poles are avoided. The reader can check for themselves that there is a non-trivial solution to the Landau equations [30].

Coleman and Norton (CN) [21] derived a theorem that paints a physical space-time picture of a process which leads to a pinch. Such a process corresponds to a physical (classical) picture of the scattering with energy and momentum conserved at each vertex, and the intermediate particles are on shell, moving forward in time. Note the CN theorem tells you about the existence of a branch point. This does not imply there is no non-analyticity for a given choice of kinematics if there is no corresponding physical space-time process. As mentioned above, in general we will also have UV logs. For example, consider ϕ^4 theory at one loop for the process $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$, with $s = (p_1+p_2)^2 = (p_3+p_4)^2$. The result for the bubble $B(s)$ has a square root branch cut (in the imaginary part) and is UV (but not IR) divergent

$$B(s) = \frac{1}{16\pi^2}\left[\Delta - \log\frac{m^2}{\mu^2} + 2 - \beta(s)\left(\log\frac{1+\beta(s)}{1-\beta(s)} - i\pi\right)\right], \qquad \beta(s) = \sqrt{1-\frac{4m^2}{s+i0}}. \tag{78}$$

$$\Delta = \frac{2}{4-d} - \gamma_E + \log(4\pi). \tag{79}$$

[29] We have assumed $\alpha_i > 0$, but it is also possible for there to be an endpoint singularity, i.e. $\alpha_i = 0$. In this case the associated propagator is off-shell and is shrunk to a point in the classical picture of CN.

[30] In the Landau equations it is assumed that the $i\epsilon$ is positive for as in time ordered products, thus to see the solution must re-write the second denominator as $-(-x+a+i\epsilon)$.

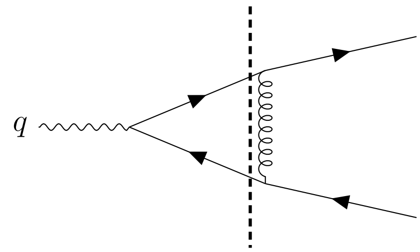


FIG. 4. The quark lines are on shell, with the angle between them fixed by q^2, while the gluon has virtuality q^2 and is not captured in the EFT.

This result, valid above threshold $s = 4m^2$, does not coresspond to a physical space-time diagram when $s > 4m^2$ since once the particles are produced they go off in different directions and can not recombine. The CN theorem tells us that there will be a branch point exactly at thereshold where the particles are produced at rest, sit there, and then recombine into the final state. The singularity arises only at threshold and the regulator will not make the integral finite at that point. Such a point singularities will be smoothed out by using a more physical wave packet description, though even then one would still need some sort of resummation to bring things under control. [31] A "perminant pinch", on the other hand, corresponds to a case where there is a pinch for ANY value of the external momentum. The classic example arises in the box diagram for massless particle scattering.

Libby and Sterman [1, 2] famously showed, using the CN picture, that perminant pinches, and their associated IR divergences, arise only from soft and collinear configurations in massless theories. So if we start off assuming we have collinear lines in external states, we know that in addition to collinear fields we will need soft fields. As we will see, depending upon the process, there can be various sub-types of soft and collinear modes responsible for such divergences. What type arises will depend upon the choice of observable. When we have massless theory IR divergences will always be associated with a log, as long as we are regulating using dimensional regularization. This follows from the fact that in dim. reg. scaleless integrals vanish. However, the reverese is not true, as the log can be associated with a UV divergence.

The CN picture is highly intuitive. As previously mentioned in this chapter, the source of IR divergences is the existence of degenerate states. We can see this in quantum mechanical perturbation theory where we recall that the first order wave function correction to the mth

[31] The existence of the singularity for this case is indeed telling us that we need a mode with a non-relativistic dispersion relation a la NRQCD.

energy eigenstate is given by

$$\delta \mid \psi^{(m)}\rangle = \sum_{n\neq m} \frac{\mid n\rangle\langle n \mid \delta H \mid \psi^{(m)}\rangle}{E_m - E_n}. \tag{80}$$

We see that the energy denominator can vanish when degeneracies exist. In quantum mechanics we avoid this problem by changing our basis of states. In field theoretic language this means that the states we are working with are not the correct asymptotic states. In QCD, for instance, we know that the correct asymptotic states are not massless quarks and gluons, but hadrons. Whereas in QED the true asymptotic states are coherent states of photons[22]. Since solving for the exact asymptotic states is not analytically tractable in QCD, we instead have a choice. Either we can choose observables which are not sensitive to the final state, such as totally inclusive processes, or we need to figure out how the long distance effects can be absorbed into *universal* matrix elements [32] which factorizes from the *non-universal* hard scattering, as discussed in the previous section.

While the effective theory must reproduce both the non-analyticities as well as the IR divergences, as mentioned above, not all non-analyticites will be matched in the EFT matrix elements, and may have to be part of Wilson coefficients. This also implies that the matching coefficients will, in general, be imaginary. For instance, consider cutting the one loop time like Sudakov form factor as in figure (4). This cut has two on shell matter lines but the gluon ican be hard (i.e. all momenta of order Q, and off shell, and thus can not be reproduced by the EFT as part of a matrix element, since the hard gluon is not in the EFT. The resulting $Log(Q^2) + i\pi$ from the loop will be captured in the matching cofficient as we shall see later on in the chapter. The matrix elements in the EFT only need capture the parts associated with the IR physics, i.e. the IR singularities and the non-analyticities associated with the small invariants. We can see explicitly that the $Log(Q)$ can not be coming from the matrix element in the low energy theory since when the gluon becomes soft the integrand looks like

$$\int \frac{[d^d k]}{k^2} \frac{1}{p_1 \cdot k} \frac{1}{p_2 \cdot k} \tag{81}$$

so that that large momenta (say p_1^+, p_2^-) are only multiplicative and the Q^2 dependence is

[32] Note that we are free to write down some formal non-local matrix element for any process, but if we don't factorize the long distance part into some universal factor which can be used in some other process, then we have not gained any predictive power.

captured by the Wilson coefficient. We see that while the IR divergence (coming from this soft piece) is acompanied by a log, as it must (see above), but that logs comes from the hard region. Also this soft integral is scaleless and vanishes in dim. reg. This does NOT necessarily mean that it can be ignored (see discussion in chapter 1).

We should pause here for a caveat regarding Coleman Norton. There are solutions to the Landau conditions which are not on the physical sheet. Famously, there are ''anomalous thresholds" [33] which can generate log branch points, as opposed to the square root banch points. Such thresholds may or may not correspond to physical processes (i.e. be on the physical Riemann sheet.)

Now we may use these results to determine which observables (as opposed to individual diagrams) are IR safe. Let us consider the current-current correlator

$$\Pi(q^2) = \int d^4x e^{iq\cdot x} \langle 0 \mid T(J(x)J(0)) \mid 0 \rangle \tag{82}$$

the imaginary part of which will be related to a cross section [34]. The physical picture for this correlator is depicted in figure (5). For massless partices there will be a physical process associated with a pinch at $q^2 = 0$ (all lines collinear, see below). However, away from that point, at finite q^2, we will generate a $Log(q^2)$ in the ampltiude. When $q^2 > 0$, two energetic quarks are produced going off in differing directions. There is no way for the quarks to come back and recombine unless there is some external current somewhere (which we assume there is not) to force them back leading to a collision. Thus there can be no IR singularity in this diagram since it can not be drawn as a space-time diagram. This is true to all orders in perturbation theory. As mentioned, the imaginary part of this correlator gives the inclusive cross section. So this correlator has a finite imaginary part, and this aligns with KLN theorem [10] which states that (aside from initial state singularities) totally inclusive processes are IR finite (since we are summing over all degeneracies). Notice that when taking the cuts we will be calculating physical on shell processes and thus wll find IR divergences, its just that the sum of the diagrams will be finite.

We can also look at this result from the perspective of the OPE (section 3). Since the identity operator is trivial there are no IR divergences on the RHS of the OPE and thus

[33] This misleading term is used to distinguish it from a "normal" threshold which turns on when s is large enough to produce a pair of the lowest lying excitations.

[34] Recall that when there are long range forces, cross sections can be IR divergent.

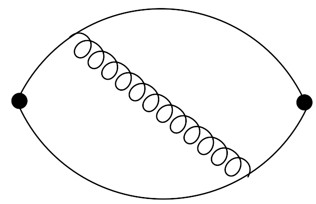


FIG. 5. A Feynman diagram contributions to the current-current correlator. Kinematics forbid the interpretation as a space-time diagram thus proving its IR insensitivity.

there can be none on the left. If we consider the higher dimensional operators (which vanish in perturbation theory) on the RHS of the OPE we find that, in perturbation theory, the matrix elements all vanish [35]. Thus the LHS must have no IR divergences to any power in q^2. That this is true can also be seen by noting that the current correlator has the same topology as the gluon self energy which has no IR divergences as long as the gluon is off-shell ($q^2 \neq 0$).

B. SCETI vs. SCETII

The question remains, what do we mean by "soft"? Generically, there are two cases of interest. When the soft modes have smaller invariant mass then the energetic collinear modes, the theory is called $SCETI$, and the soft modes are called Ultra-Soft (US), while the case of equal invariant masses is called $SCETII$[36]. The latter case is slightly more complicated due to the existence of a rapidity divergences, which are neither UV nor IR in nature and will be discussed later. Which modes are relevant for a particular observable depends upon the kinematics of the external lines and on what is being measured in the final state. Note that US modes leave collinear lines on shell but the soft do not.

If we are doing high energy scattering then the external states include, by necessity, at least one collinear mode. As already discussed these modes will have momenta which scale as $p^\mu \sim (1, \lambda^2, \lambda)$ where $\lambda = \Lambda/Q$. Here Λ is the lowest mass scale. Often $\Lambda \sim \Lambda_{QCD}$, but in general this need not be the case so it will be left as arbitrary unless otherwise stated. If we chose to work in the center of mass frame, we have two collinear directions in the initial state. If we work in the target frame then the other incoming mode must have soft momentum scaling as $k^\mu \sim (\lambda, \lambda, \lambda)$ [37] For an SCET discussion of DIS in various frames see [24]. In

[35] The OPE is defined with normal ordered operators, i.e. all destruction operators a_k sitting to the right.
[36] This distinction was first made in the context of heavy to light decays in [23].
[37] This scaling follows from boosting to the rest frame despite the fact that we are taking the on-shell state

order to keep the collinear lines on shell the US modes must scale as $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)$, and the soft mode will have $k^\mu \sim (\lambda, \lambda, \lambda)$. Given a collinear line, a soft emission will lead to an intermediate state with offshellness order order λ.

Notice that if $\lambda \sim \Lambda_{QCD}/Q$ then the invariant mass of the US modes will be order $k^2 \sim \Lambda^4_{QCD}/Q^2 \ll \Lambda^2_{QCD}$. This clearly can not be. We know that confinement of partons would never allow a state with such a long wavelength, it could not fit inside a hadron. This mode is outside the confinement radius, i.e. it's "hyper-confining". We conclude that for any physical, and hence gauge invariant, observable the US modes (whose invariant mass is order Λ^4_{QCD}/Q^2) must drop out of the final result. We will see how the US modes disappear in the explicit case of Drell-Yan. This will be the case whenever the we have a hyperconfining mode. Notice that this *does not* mean that generic US modes will never contribute, as they need not be hyperconfining. There are cases where $\lambda^2 \gg \Lambda^2_{QCD}/Q^2$ where the US can contribute. These are cases where there is an intermediate scale which is generated by the choice of observables (typically when probing the corners of phase space). In such cases the US modes are perturbative. Such a scenario will be discussed below in the context of quarkonia decay.

An immediate question arises. If we are interested in modes which leave collinear lines on shell, then should we consider modes with even smaller virtualities than the US? If the answer is yes, then where would we truncate this infinite set of modes [38]? The reason the answer is "no" becomes clear once we consider an example. Suppose we introduce a new mode with scalings $k^\mu \sim (\lambda^3, \lambda^3, \lambda^3)$. When we couple this k to a collinear line and mulitpole expand, the result would be independent of k, leading to vanishing scaleless integrals. Nonetheless we need to treat scaleless integrals with care, and can't simply discard them without reason. We have seen in our discussion of NRQCD, that scaleless integrals can play the role of "pull-up modes", flipping an IR divergence from a higher virtuality mode to a UV divergence. However, it is self consistent to simply take the US mode as the smallest virtuality mode in itself and ignore the modes with even smaller virtuality. All of the modes with smaller virtualities than the US are contained within the US (see for example [18]). In other words, you could add turtles upon turtles, but that's not the most economic approach. Finally, it is also important to note that we have not excluded the possibility that soft and US modes

to be light-like. That is, we are ignoring the effects of the proton mass, which will be power suppressed for all the cases of interest here.

[38] Notice that for a relativistic theory the scaling will always be integer in nature.

may exist in the same theory.

1. *Back to NRQCD* [39]

As a final note on the subject of modal reproduction, we shiould retunn to the questions of the NRQCD soft modes $(mv, m\vec{v})$ soft modes, which were necessary to reproduce non-analytic momentum dependence of full QCD. The soft mode in NRQCD neither keeps the quarks on shell nor exists as external states . However, the soft modes are in a sense not really part of the low energy theory. Indeed their loops need no IR regularization. They are there to reproduce logs via the velocity-renormalization group. The kinematics of the bound states are such that soft modes can never go on shell and thus can not contribute to physical cuts, however the soft loops will generate terms non-analytic in the spatial momenta, $Log(\vec{p}^2)$. One may think of the sole purpose of the soft mode as being the engine that generates/corrects potentials. As we saw, the soft mode is responsible for generating the runnings of the coupling in the potential. When we wrote down the Lagrangian for NRQCD we included the soft modes, but we really should think of NRQCD as a theory of quarks, radiation modes and non-local four quark operators which are the potentials. This is not to say that there wont be time ordered products of soft operators in low energy matrix elements. However renormalizing at the low scale they will only contribute analytic local contribution. For example, one could choose to work in the so-called "V-scheme" [25] where we choose our counter-terms such that soft matrix elements vanish when rernormalized at the low scale [40]. In this scheme the soft play no role at all in the low energy theory. Their role is simply to run the potentials.

C. Modes induced by special kinematics: SCETI

New modes can arise when we consider special kinematic configurations. Suppose that we were sitting at the threshold for top quark production, so that $q^2 \sim 4m_t^2$. Then, while there are still no IR singularities, at threshold the top quarks are moving non-relativistically and we should expect that long wavelength modes should become relevant to the production process.

[39] This section assumes the reader has studied the chapter on NRQCD and can be skipped by others without worry.

[40] In this scheme the coupling is defined in terms of the static potential as $V = -4\pi C_F \frac{\alpha_V(Q^2)}{Q^2}$.

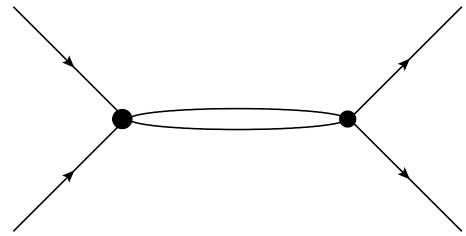


FIG. 6. Top quarks are produced at threshold such that the diagram can be realized as physical process in which all lines are long lived and thus generates singularities. The solid dots represent the photon lines which are highly virtual and are shrunk to a point.

Indeed, we will need potential modes to reproduce the IR behavior. This is consistent with the breakdown of the OPE near thresholds [41]. This is not to say that all is lost near the top threshold, it's just that, at threshold, the Wilson coefficients, which are supposed to be short distance dominated, will depend upon the scale $Q^2 \equiv q^2 - 4m_t^2$, and as long as this scale is larger than Λ^2_{QCD} the Wilson coefficients are still calculable in perturbation theory. Moreover, the top lifetime is short enough ($\Gamma_t \sim 1.5\ GeV$) that the true long distance effects (confinement) are suppressed . Indeed, one can systematically calculate the threshold production cross section by first matching onto NRQCD (with $v^2 = Q/(2m_t)$) and then performing the OPE. The non-perturbative effects contribute to the production cross section at the level of [12]

$$\frac{\delta\sigma}{\sigma} \sim \frac{\Lambda^4_{QCD} v^2}{m_t v^2 + i\Gamma_t} \leq 10^{-4}. \tag{83}$$

This is just one example of how corners of phase space can influence the IR modes. Indeed, SCET is very useful in such situation, as it allows for the resummation of the Logs of the new IR scales. We will consider an explicit example of this later in the chapter. But first we must build up the interacting theory.

D. Field Scalings, Interactions and Gauge Symmetry

To write down the interactions, we must first determine how the fields scale in our power counting parameter. We may fix the scaling using the the two point function. Consider the

[41] This breakdown is related to the fact that dispersion relation which justifies the use of the OPE can no longer be utilized, see footnote (7).

two point function for a collinear fermion

$$\langle 0 \mid T(\xi_n(x)\bar{\xi}_n(0) \mid 0\rangle = \int [d^4p] e^{ip\cdot x} \frac{i}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p}}, \tag{84}$$

where we have combined the label sum with the integration over the residual momentum. The measure scales as

$$(d^2p_\perp \sim \lambda^2)(d\bar{n}\cdot p \sim 1)(dn\cdot p \sim \lambda^2) \sim \lambda^4. \tag{85}$$

Which leads to the scaling $\xi_n(x) \sim \lambda$. Furthermore, rephasing the field does not change the fields' scaling so that $\xi_{n,p} \sim \lambda$.

Now let us consider collinear gluons. As with the fermions we scale out the large labels, but the gluons are self-conjugate, so that $A^\mu_{\mathbf{p}}$ destroys a gluon while $A^\mu_{-\mathbf{p}} = A^{\mu\star}_{\mathbf{p}}$ creates a gluon. The collinear gluon fields components scale inhomogenously as can be seen by studying the two point function in a general covariant gauge

$$\langle 0 \mid A^\mu_n(x) A^\nu_n(0) \mid 0\rangle) = -i \int \frac{[d^4p]}{p^2} e^{ip\cdot x} (g_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{p^2}). \tag{86}$$

where ξ is the gauge parameter.

For the n collinear field we have

$$p^\mu = n\cdot p \frac{\bar{n}^\mu}{2} + \bar{n}\cdot \mathbf{p} \frac{n^\mu}{2} + \mathbf{p}^\mu_\perp, \tag{87}$$

and by projecting onto the various components we may extract the scalings,

$$n\cdot A_n \sim \lambda^2,\ \bar{n}\cdot A_n \sim 1,\ A_\perp \sim \lambda. \tag{88}$$

A helpful mnemonic is that A^μ_n scales with λ in the same way as p^μ. An analogous analysis shows that for the US fields all the components scales homogenesouly $A_\mu \sim \lambda^2$.

Exercise 5.1 Altrernatively, one can determine the field scalings using the canonical commutation relations. Show that you get the same scalings 88 in this waty

Before going on to discuss interactions, let us next write down the action for gluon fields. The US fields have an action which is identical to the full QCD action, since both the field and momentum components scale homogeneously. The same can not be said for the collinear gluons. We could reconstruct the free action from the form of the propagator, but this would involve choosing the proper gauge fixing term. Instead, since we are for the moment ignoring interactions, we know that the action for the collinear gluons is just the boosted action for gluons in QCD which is Lorentz invariant. We can elevate the gluonic action to the interacting (within their respective sectors) level trivially by covariantizing the respective field strengths. For the US case we simply have the usual QCD action, but for the collinear gluon we should distinguish between large and small momenta. In the present case not all momenta are labels so instead we will utilize the previously introduced label operator ($\mathbf{P}$), which picks out the label of a field. It is thought of as a derivative operator that is only senstive the large part of the momentum. Now we can write the action for the purely collinear gluonic sector as

$$L_c = \frac{1}{2g^2} Tr[iD_\mu, iD^\mu]^2 \tag{89}$$

$$iD_\mu = i\frac{\bar{n}^\mu}{2} n \cdot \partial + \frac{n^\mu}{2} \bar{n} \cdot \mathbf{P} + \mathbf{P}_\perp^\mu + gA_n^\mu \tag{90}$$

where we have written the gluons in terms of generators in the fundamental representation. As the reader should verify, the individual terms in the contraction sum (89) each have leading order scaling. That is, even though individual pieces of the field strength tensor scale differently, the square of the tensor scales homogeneously as λ^4 as it must to cancel the scaling of the measure in the leading order action.

In addition we must include a gauge fixing term, which at this point is unspecified, as well as a similar action for the ghosts. The action for the collinear quarks follows in a similar fashion. We replace derivatives/labels with the covariant D

$$S = \int d^4x \; \bar{\xi}_n (in \cdot D + i\not{D}_\perp \frac{1}{i\bar{n} \cdot D} i\not{D}_\perp) \frac{\not{\bar{n}}}{2} \xi_n. \tag{91}$$

We have dropped the labels which will heretofore be implied whenever they are not made explicit. Label momentum conservation is assumed for all interaction terms. Notice that each component of D^μ scales homogeneously,

$$n \cdot D \sim \lambda^2 \quad \bar{n} \cdot D \sim 1 \quad D_\perp \sim \lambda. \tag{92}$$

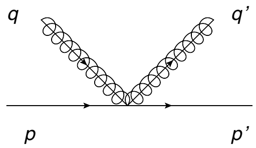
The action has a very unfamiliar form in that it is non-local, at least *naively*. That is, it involves an infinite number of derivatives. Another way of saying this is that the action is not written as an integral over a density. To see this re-write

$$\frac{1}{\bar{n} \cdot D} = \int_0^\infty d\lambda \; exp^{-\lambda \bar{n} \cdot D} \tag{93}$$

Now ignoring the gauge field for the moment in the covariant derivatives, if we treat the derivative as normal derivative operator (i.e. not as the label operator), we may use the exponential to shift the argument of the field and the fields are now light-cone separated and the action is non-local. However, the derivative operator is actually a *label* operator, and as such is just a number. Thus there is nothing at all non-local about the action (91). The action is perfectly local [42] since the (residual) fluctuations in the light cone momentum are much smaller then the inverse light-cone separation. This is just the statement that the residual momentum is hierarchally smaller than the label momentum. The integral representation of the inverse derivative is dominated by $\lambda \sim 0$ and the action does indeed look local. Of course when we use the complete covariant derivative we have this nagging issue of the gauge field in the denominator, but we shall see that gauge invariance will allow to clean this issue up shortly.

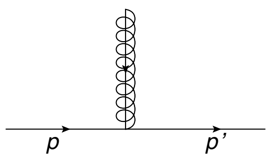
At this point we still have done nothing but formulated QCD in a boosted frame. In fact on the face of things it looks like have made our lives unecssarily complicated. The action in (91) generates a whole new set of Feynman rules for the quark-gluon interactions including interactions terms involving multiple gluons. The the two and one gluon Feynman rules are given by:

[42] This emphasizes the fact that locality is a scale dependent statement. We should really be saying "this theory is local on scales longer than ..."



$$
= \frac{igT^AT^B}{\bar{n}\cdot(p-q)}\left[\gamma_\nu^\perp\gamma_\mu^\perp - \frac{\gamma_\mu^\perp \not{p}_\perp}{\bar{n}\cdot p}\bar{n}_\nu - \frac{\not{p}'_\perp\gamma_\nu^\perp}{\bar{n}\cdot p'}\bar{n}_\mu + \frac{\not{p}'_\perp\not{p}_\perp}{\bar{n}\cdot p\bar{n}\cdot p'}\bar{n}_\mu\bar{n}_\nu\right]\frac{\not{\bar{n}}}{2}
$$

$$
+\frac{igT^BT^A}{\bar{n}\cdot(p'+q)}\left[\gamma_\nu^\perp\gamma_\mu^\perp - \frac{\gamma_\mu^\perp \not{p}_\perp}{\bar{n}\cdot p}\bar{n}_\nu - \frac{\not{p}'_\perp\gamma_\mu^\perp}{\bar{n}\cdot p'}\bar{n}_\nu + \frac{\not{p}'_\perp\not{p}_\perp}{\bar{n}\cdot p\bar{n}\cdot p'}\bar{n}_\mu\bar{n}_\nu\right]\frac{\not{\bar{n}}}{2}
$$



$$
= igT^A\left[n_\mu + \frac{\gamma_\mu^\perp \not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{p}'_\perp\gamma_\mu^\perp}{\bar{n}\cdot p'} - \frac{\not{p}'_\perp\not{p}_\perp}{\bar{n}\cdot p\bar{n}\cdot p'}\bar{n}_\mu\right]\frac{\not{\bar{n}}}{2}
$$

FIG. 7. Coupling of collinear gluons and quarks.

Exercise 5.2 Consider quark gluon scattering. We would like to see how the boosted QCD action in (91) reproduces the full theory result. Use the Feynman rules presented in the text to calculate the amplitude for the scattering of physically polarized gluons off of quarks. To show the equivalence at the level of the amplitude, you will need to expand the full theory spinor into large and small components as in eq. (64). Note that the equivalence must be exact.

It is only when we add interactions with non-collinear fields that we begin doing something non-trivial. However, we have positioned ourselves to add these modes in a very simple way. We utilized gauge invariance to fix the action for the collinear fields, but can we do the same for the US fields? The answer would be in the affirmative if we can implement distinct collinear and US gauge symmetries.

Let us examine the collinear gauge symmetry. Just as our fields scale homoegeously so should our gauge transformations. Collinear gauge transformations should not change the momentum support of the fields. So we should restrict our transformations to those which change the label momenta $\bar{n}\cdot\mathbf{p}$ and $\mathbf{p}_\perp$ only by order one and λ respectively, and furthermore the small momentum $n \cdot k$ should remain of order λ^2. Now given a gauge transformation

$$
U = e^{i\alpha^a(x)T^a}, \tag{94}
$$

we pull out the large phase from the gauge function such that

$$U(x) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot x} U_{\mathbf{k}}(x). \tag{95}$$

Then a gauge transformation on a collinear field is written as

$$\xi'_{n\mathbf{p}}(x) = \sum_{\mathbf{k}} U_{n\mathbf{k}}(x) \xi_{n\mathbf{p}-\mathbf{k}}(x). \tag{96}$$

Furthermore, given that we don't want the ultra-soft fields to carry large momenta they should not transform under collinear gauge transformations. If we consider US gauge transformations on the other hand, all field should transform.

Now to make the quark collinear action US gauge invariant we must include a US gauge field in (91). However, not all of the components of the covariant derivative should be completed by US gauge fields. We should include only those pieces which have commensurate scaling with the derivative. Given that A_{US} scales as λ^2, only $n \cdot D$[43] should be completed by the inclusion of the US gauge field.

To manifest US gauge invariance in the collinear gluon sector we utilize the background field formalism'whereby the US field is treated as a fixed background. In our case this is sensible since the US field varies on distance scales which are parametrically larger than those of the collinear fields. Note that even the collinear gluon gauge fixing term must be US gauge invariant. For a general covariant gauge, the collinear gauge fixing term is written as

$$L^n_{gf} = \frac{1}{\alpha} Tr[i d^\mu_{US}, A_{\mu n}]^2. \tag{97}$$

Here we have augmented the definition of the ordinary derivative to include the US gauge field

$$i d^\mu_{US} = \frac{\bar{n}^\mu}{2} n \cdot (i\partial + g A_{US}) + \frac{n^\mu}{2} \bar{n} \cdot \mathbf{P} + \mathbf{P}^\mu_\perp \tag{98}$$

We take the collinear field to transforms like matter fields (i.e. homogenously) since it is *not* the connection on the US fiber bundle but does carry the US gauge charge. Since the

[43] Recall that here we are considering only one collinear sector.

US gauge functions carry no large momenta, we have

$$\mathbf{P}U_{US}(x) = 0, \tag{99}$$

so that d_{US} transforms covariantly and the gauge fixing term is manifestly US gauge invariant. The existence of the US background field also affects the transformation of the collinear gluon under the collinear gauge transformation. To ensure that the transformed collinear field still transforms covariantly under US transformations the collinear field is transformed as follows

$$gA_n^\mu \to UgA_n^\mu U^\dagger + U(id_{US}^\mu)U^\dagger. \tag{100}$$

The collinear field strength is also modified by defining

$$G_n^{\mu\nu} = [id_{US}^\mu + A_n^\mu, id_{US}^\nu + A_n^\nu]. \tag{101}$$

such that the collinear gluon kinetic piece is both US and collinear gauge invariant. The total collinear action is then given by

$$\begin{aligned} S_c \; &= \; \sum_n \int d^4x \; \bar{\xi}_n (in\cdot D + \not{D}_n^\perp \frac{1}{\bar{n}\cdot D_n} \not{D}_n^\perp) \frac{\not{\bar{n}}}{2} \xi_n + \frac{1}{2g^2} Tr[G_n^{\mu\nu} G_{\mu\nu n}] \\ &+ \; \frac{1}{\alpha} Tr[id_\mu, A_n^\mu]^2. \end{aligned} \tag{102}$$

A subscript on a sub-set of covariant derivatives (D_n) has been included to remind ourselves that these terms only involve the collinear and not the US gauge field as well. Finally we must account for the ghosts in the theory. In the collinear sector we write

$$S_g = 2Tr\left(\bar{c}_n[id_\mu^{US}, [id^{US\mu} + gA_n^\mu, c_n]]\right). \tag{103}$$

The US gluonic action has the same form as the usual QCD Lagrangian. The final symmetry of the action is given by

$$G = \prod_{i=1}^{i=A} SU(3)_{n_i} \otimes SU(3)_{US}, \tag{104}$$

where we have allow for A possible collinear sectors.

This action (102) describes all modes whose virtuality is less then Q^2. In principle there

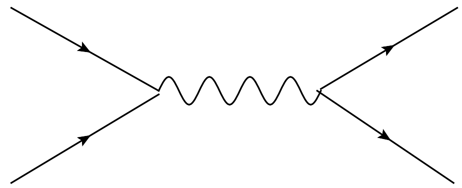


FIG. 8. Annihilation of two collinear quarks in the n and $\bar{n}$ direction, will generate an interaction which is local at scales smaller then the hard scale, Q^2. The resulting operator is power suppressed.

could be hard loops that could change the form of this action. Gauge symmetry however, forbids such corrections since the action we have written down is the unique invariant, at least up to leading order in λ, set of terms. In principle though, such hard modes could generate other operators which could connect various collinear sectors. For instance, in the full theory we will have interactions between collinear modes which arise due to exchange of a hard mode. Consider an n-collinear quark annihilating with an n'-collinear quark as shown in figure (5). This interaction will generate an operator of the form

$$O_{nn'} = C_{nn'}(n\cdot\mathbf{p}, n'\cdot\mathbf{k}', \tilde{n}\cdot\tilde{\mathbf{p}}, \tilde{n}'\cdot\tilde{\mathbf{p}}')(\bar{q}_{n\mathbf{p}}\gamma_\mu^\perp q_{n'\mathbf{k}'})(\bar{q}_{\tilde{n}\tilde{\mathbf{p}}}\gamma^{\mu\perp} q_{\tilde{n}'\tilde{\mathbf{k}}}), \tag{105}$$

Exercise 5.3

Calculate $C_{nn'}$ and show that the measure for this operator will scales like λ^2 and that therefore thss operators is power suppressed. This suppression assures us that at leading power we can treat this operator as external. i.e. we don't need to include it in the action.

1. Interactions Generated by External Currents

It would appear then that collinear sectors do not interact with each other. However, let us consider a quark bilinear current, such as the electromagnetic current which appears in DIS, and match it onto SCET. At leading order in both the coupling and power expansions we find [44]

$$\bar{\psi}(x)\gamma_\mu\psi(x) = \bar{\xi}_{n,\mathbf{p}}\gamma_{\mu\perp}\xi_{\bar{n},\mathbf{p}'}. \tag{106}$$

[44] Again the label sums are suppressed.

Given that the full theory current is gauge invariant, the SCET current should be invariant under both US and collinear gauge transformations.

Exercise 5.4 Match the full QCD electromagnetic current onto an SCET current, and then compute its one-loop matching and anomalous dimension.

The current clearly is not collinear gauge invariant since the two collinear fields, given that they have different collinear directions, *transform independently.* We are missing an important piece of physics.

To see what we are missing let us work in a frame where $\tilde{n} = \bar{n}$, so that $n \cdot \tilde{n} = 2$. Now consider an emission of a n collinear gluon off of the $\bar{n}$-collinear quark line. Such an emission throws the intermediate quark off-shell, by an amount of order Q^2, before it is annihilated by the current. Naivley one might think that this process is power suppressed because of the large scale in the denominator, but it is not. To see this we can power count the effective operator onto which we would match. Consider expanding the full theory result as follows

$$
\begin{aligned}
(ig)\bar{u}(p')\gamma_\mu T^a \frac{i(\not{p} - \not{k})}{(p-k)^2 + i\epsilon}\gamma^\nu \epsilon_\nu^{a\star}(k) u(p) &\approx (ig)\bar{\xi}_{n\mathbf{p}'}\gamma_\mu T^a \frac{i(n \cdot \mathbf{p}\frac{\not{\bar{n}}}{2} - \bar{n} \cdot \mathbf{k}\frac{\not{n}}{2})}{-n \cdot \mathbf{p}\bar{n} \cdot \mathbf{k} + i\epsilon} \bar{n} \cdot \epsilon^{a\star}(k)\frac{\not{n}}{2}\xi_{\bar{n}\mathbf{p}} \\
&= \bar{\xi}_{n\mathbf{p}'}\gamma_\mu^\perp g T^a \frac{\bar{n} \cdot \epsilon^{a\star}(k)}{\bar{n} \cdot \mathbf{k} - i\epsilon}\xi_{\bar{n}\mathbf{p}}
\end{aligned}
$$

In the first line we used the fact that for a collinear gluon with large momentum in the n^μ direction (so that its large component is $\bar{n} \cdot k$), the leading order component of the field is $\bar{n} \cdot A_n$. The intermediate propagator has been reduced to a very simply into what is called an "eikonal" form $1/\bar{n} \cdot k$. We can see that the net effect on the current operator is to multiply it by a factor which is order one

$$
\frac{\bar{n} \cdot A_{n\mathbf{k}}}{\bar{n} \cdot k} \sim 1. \tag{107}
$$

The eikonal propagator is a label and is thus part of the matching coefficient for the operator. Further emissions simply lead to more such order one factors, and thus when we match the current we must include an infinite number of such gluons emissions.

For the moment let us consider the Abelian case. For n photon emissions with momenta

k_n we will generate a matching coefficient of the form

$$\begin{aligned} C_n &= \frac{1}{n!}\sum_{perms} g^n \frac{1}{\bar{n}\cdot k_1}\frac{1}{\bar{n}\cdot(k_1+k_2)}\cdots\cdot\frac{1}{\bar{n}\cdot(k_1+k_2+...k_n)} \\ &= \frac{g^n}{n!}\prod_{i=1}^{n}\frac{1}{\bar{n}\cdot k_i} \end{aligned} \tag{108}$$

In the last line we used the "eikonal identity" which is easily proven by induction.

We may then write an effective current which includes arbitrary number of photon emissions as

$$J_\mu = \bar{\xi}_{n\mathbf{p}}\gamma^{\perp}_{\mu}\exp\left[\frac{-gn\cdot A_{\bar{n}\mathbf{k}'}}{n\cdot\mathbf{k}'}\right]\exp\left[\frac{g\bar{n}\cdot A_{n\mathbf{k}}}{\bar{n}\cdot\mathbf{k}}\right]\xi_{\bar{n}\mathbf{p}}. \tag{109}$$

where we have included emissions of off either side. Given that we have restricted ourselves to Abelian theories we are free to interchange the order of the exponentials so that fields living in the same sector live next to each other. We can now show that this form of the current is gauge invariant.

Consider how the exponentials transform under a collinear gauge transformations[45]

$$A^{\mu}_{\bar{n}\mathbf{k}}(x)\rightarrow A^{\mu}_{\bar{n}\mathbf{k}}(x)+\sum_{\mathbf{q}}\frac{1}{g}U^{\star}_{\mathbf{k}+\mathbf{q}}(x)(\bar{n}\cdot\mathbf{P}\frac{n^{\mu}}{2}+\mathbf{P}^{\mu\perp}-in\cdot\partial\frac{\bar{n}^{\mu}}{2})U_{\mathbf{q}}(x). \tag{110}$$

Now we will repristinate the exponential by reinserting the implied label sum and exponential

$$\exp\left[\frac{g\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot x}\bar{n}\cdot A_{n\mathbf{k}}}{\bar{n}\cdot\mathbf{k}}\right]\rightarrow\exp\left[\frac{g\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot x}\bar{n}\cdot A_{n\mathbf{k}}}{\bar{n}\cdot\mathbf{k}}+\sum_{\mathbf{k},\mathbf{q}}e^{-i\mathbf{k}\cdot x}U^{\star}_{\mathbf{k}+\mathbf{q}}U_{\mathbf{q}}(x)\frac{\bar{n}\cdot\mathbf{q}}{\bar{n}\cdot\mathbf{k}}\right]. \tag{111}$$

Showing that

$$\exp\left[\sum_{\mathbf{k},\mathbf{q}}e^{-i\mathbf{k}\cdot x}U^{\star}_{\mathbf{k}+\mathbf{q}}U_{\mathbf{q}}(x)\frac{\bar{n}\cdot\mathbf{q}}{\bar{n}\cdot\mathbf{k}}\right]=e^{-i\alpha(x)} \tag{112}$$

then makes the gauge invariance of the current (109) manifest. To prove (112) we use the fact that

$$-i\bar{n}\cdot\partial\alpha=\frac{1}{g}\sum_{\mathbf{k},\mathbf{q}}e^{-i\mathbf{k}\cdot x}U^{\star}_{\mathbf{k}+\mathbf{q}}U_{\mathbf{q}}\bar{n}\cdot\mathbf{q}, \tag{113}$$

which follow from (110) once the exponentials and label sums are made manifest. Inte-

[45] Working in background field gauge we should promote the partial derivative to covariant derivative including the US gauge field, but this does not affect the rest of these arguments.

grating (113) then leads to the result (112). The gauge invariance of the resulting action implies that this form receives no radiative corrections. That is, when we resummed the photonic interactions we assumed that each such operator in the sum is weighted identically. This seems like a strong assumption, however the power of gauge symmetry protects the exponential form to all orders. Of course this is not to say that there can not be corrections to the current in the matching. However, such corrections must correspond to an overall factor which is simply the statement that the hard corrections factorize. This also implies that there are no large logs that can set the scale for g in the exponential, and therefore the coupling g in the exponential is not evaluated at the hard scale[46].

The matching of the current does, in general, leads to label dependence. It is convenient to define a new gauge invariant "jet" field[47]

$$\chi_{n,\mathbf{P}} = \delta(\omega - \bar{n} \cdot \mathbf{P})(W_n \xi_n), \tag{114}$$

where

$$W_n = \exp\left[\frac{-g\bar{n} \cdot A_{n\mathbf{k}}}{\bar{n} \cdot \mathbf{k}}\right]. \tag{115}$$

This form implies that the matching coefficient can only depend on the sum of the total momentum in the jet. That this must be true is a consequence of collinear gauge invariance, since any dependence on individual momenta would destroy the exponential form.

The form of the current becomes more familiar looking once we go back to position space where

$$\chi_n(0) = \exp\left(ig\int_{-\infty}^{0} \bar{n} \cdot A_n(\lambda \bar{n}^\mu) d\lambda\right) \xi_n(0) \equiv W[0, -\infty]\xi_n(0). \tag{116}$$

The expression (115) is just the momentum space representation of a Wilson line. To see this explicitly let us consider the emission of an $\bar{n}$-collinear gluon off of an n incoming quark field

$$\begin{aligned} \exp\left(ig\int_{-\infty}^{0} d\lambda n \cdot A_{\bar{n}}(\lambda n^\mu) d\lambda\right) \xi_n(0) \mid p, -k\rangle \;&\approx\; ig\int_{-\infty}^{0} d\lambda n \cdot A_{\bar{n}}(\lambda n^\mu) d\lambda \;\xi_n(0) \mid p, -k\rangle \\ &=\; ig\int_{-\infty}^{0} d\lambda \bar{n} \cdot \epsilon(k) e^{i\lambda \bar{n}\cdot k} \xi_{\bar{n}} = g\frac{\bar{n} \cdot \epsilon(k)}{\bar{n} \cdot k}\xi_{\bar{n}} \end{aligned} \tag{117}$$

[46] Recall that the scale of the coupling is determined by minimizing large logs.
[47] The term "jet" is used to denote a collection of collinear lines.

Notice that in the last line we dropped the contribution from infinity. This corresponds to making the replacement

$$n \cdot k \to n \cdot k - i\epsilon. \tag{118}$$

In fact it is the $i\epsilon$ prescription which fixes whether or not the Wilson line goes to positive or negative infinity. This is also related to the fact that for the current to be independently collinear n and $\bar{n}$ gauge invariant the gauge field must vanish at infinity, since the Wilson line which transforms as

$$W(x,y) \to U(y)W(x,y)U^{\dagger}(x), \tag{119}$$

is "open" at that end where $x = 0$.

Notice that we made this assumption when we dropped the integration constant when solving (113). In covariant gauges the field vanishes at infinity[48]. We will see that our final expressions for physical observables must be independent of the gauge field at infinity.

In the non-Abelian case the order of the gluons emissions off of the lines matter, as we must keep track of the color and therefore we may no longer simply use the eikonal identity. So we must re-write the sum of diagrams as

$$W_n = \sum_{perms} \sum_{m=0}^{\infty} \frac{(-g)^m}{m!} \frac{\bar{n} \cdot A^{i_1}_{n,k_1} \ldots . \bar{n} \cdot A^{i_m}_{n,k_m}}{\bar{n} \cdot k_1 \bar{n} \cdot (k_1 + k_2) \ldots . \bar{n} \cdot (\sum_{i=1^m} k_i)} T^{a_1} \ldots . T^{a_m} \tag{120}$$

By inspection the Wilson line obeys the differential equation

$$\bar{n} \cdot (\mathbf{P} + gA_n) W_n = 0. \tag{121}$$

In the language of fiber bundles the Wilson line parallel transports an element of the associated vector bundle from one point on the manifold to another. A vector in the group representation ξ_n is "parallel", relative to the connection (A_μ) when it obeys $n \cdot D\xi_n(x^-) = 0$. That is ξ_n, under an infinitesmal displacemet of x^-, is not changing with respect to the connection. If we wish to parallel transport a vector along a finite path in space-time and keep it parallel, this defines W_n since the transported vector $W\xi_n$ also has vanishing covariant derivative and the covariant derivative obeys the Leibniz rule.

(121) is easily solved once we see that the analogous equation occurs when trying to solve

[48] Note that this is true even for instantons solutions which wind at infinity.

for the time evolution operator U which, in the interaction picture obeys,

$$i\frac{d}{dt}U(t,t_0) = H_{int}(t)U(t,t_0), \tag{122}$$

the solution of which is

$$U(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt' H_{int}(t')\right)\right], \tag{123}$$

and the time ordering accounts for the fact that the Hamiltonian does not necessarily commute with itself at differing times. Then the position space solution to (121) is found by defining an affine parameter λ analogous to time

$$W_n(x,y) = P\left[\exp\left(ig\int_{y}^{x} d\lambda \bar{n}\cdot A_n(\bar{n}^\mu\lambda)\right)\right], \tag{124}$$

where P stands for path ordering. Or back in momentum space (115). In the Feynman rules the $1/n!$ from expanding the exponent cancels with the number of possible ways of contracting the gluons.

> **Exercise 5.5** Given that W is the unique solution to the parallel transport equation show that under a gauge transformation the Wilson line must transform bi-locally, i.e. according to (119).

In the non-Abelian case the Wilson line associated with the emission of $(n,\bar{n})$ collinear gluons would appear to be sitting next to the $(\bar{n},n)$ collinear fields, which is not manifestly gauge invariant. However, this is not correct. Gauge invariance must somehow conspire to flip the order. So we made a mistake somewhere along the line. To see where we went wrong let's study the case of two gluon emission. First consider the diagram where an $(n,\bar{n})$

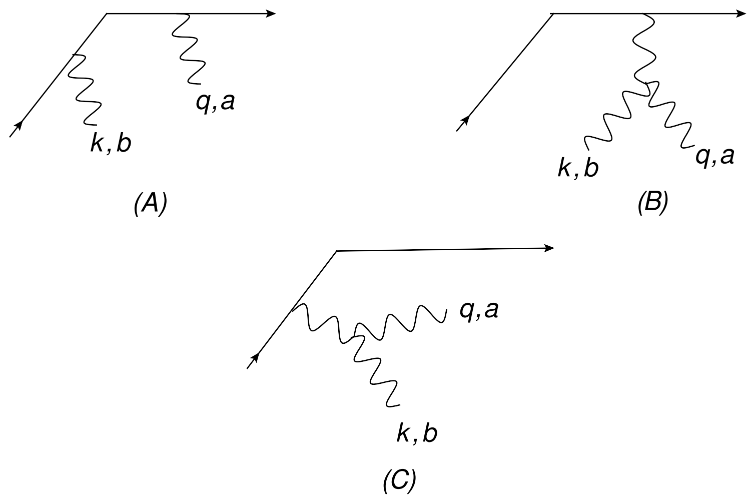


FIG. 9. Two gluon matching of a full theory current of the form $\bar{q}\Gamma q$. The gluon with momentum $q(k)$ is collinear to the in(out)going quark with momentum $\bar{p}(p)$

.

collinear gluon is emitted off of a $(\bar{n}, n)$ collinear quark line as shown in figure (6A).

$$
\begin{aligned}
Fig_{6A} &= g^2 \bar{u}(p) \not{\epsilon} T^a \frac{(\not{p} + \not{q})}{(p+q)^2 + i\epsilon} \Gamma \frac{(\not{\bar{p}} - \not{k})}{(\bar{p}-k)^2 + i\epsilon} \not{\epsilon} T^b u(\bar{p}) \\
&\approx g^2 n \cdot \epsilon^a(q) \bar{n} \cdot \epsilon^b(k) \bar{u}(p) \frac{\not{\bar{n}}}{2} T^a \frac{(\bar{n} \cdot p \frac{\not{n}}{2})}{(p+q)^2 + i\epsilon} \Gamma T^b \frac{(n \cdot \bar{p} \frac{\not{\bar{n}}}{2})}{(\bar{p}-k)^2 + i\epsilon} \frac{\not{n}}{2} u(\bar{p}) \\
&= -g^2 \frac{n \cdot \epsilon^a(q)}{(n \cdot q + i\epsilon)} \frac{\bar{n} \cdot \epsilon^b(k)}{(\bar{n} \cdot k - i\epsilon)} \bar{u}(p) T^a \Gamma T^b u(\bar{p})
\end{aligned}
\tag{125}
$$

Now consider the non-Abelian contribution coming from diagrams (6B,6C)

$$
\begin{aligned}
Fig_{6B} &= -ig^2 \bar{u}(p) \gamma_\mu T^c \frac{(\not{p} + \not{q} + \not{k})}{(p+k+q)^2 + i\epsilon} \Gamma u(\bar{p}) \frac{f^{abc}}{(k+q)^2 + i\epsilon} \times \\
&\quad \left[(2k+q) \cdot \epsilon^a(q) \epsilon(k)^{\mu b} + (q-k)^\mu \epsilon^b(k) \cdot \epsilon^a(q) + (k-2q) \cdot \epsilon^b(k) \epsilon^{a\mu}(q)\right] \\
&\approx i\frac{g^2}{2} \bar{u}(p) (f^{abc} T^c) \Gamma u(\bar{p}) \frac{n \cdot \epsilon^a(q)}{(n \cdot q + i\epsilon)} \frac{\bar{n} \cdot \epsilon^b(k)}{(\bar{n} \cdot k + i\epsilon)}
\end{aligned}
\tag{126}
$$

and

$$
Fig_{6C} \approx i\frac{g^2}{2} \bar{u}(p) (f^{abc} T^c) \Gamma u(\bar{p}) \frac{n \cdot \epsilon^a(q)}{(n \cdot q + i\epsilon)} \frac{\bar{n} \cdot \epsilon^b(k)}{(\bar{n} \cdot k - i\epsilon)} \tag{127}
$$

Adding the two non-Abelian diagrams, and using the the SU(3) commutation relations leads to the sum

$$Fig_{A+B+C} = -g^2 \frac{n \cdot \epsilon^a(q)}{(n \cdot q)} \frac{\bar{n} \cdot \epsilon^b(k)}{(\bar{n} \cdot k)} \bar{u}(p) T^b \Gamma T^a u(\bar{p})$$

which has flipped the color ordering. In this way the Wilson lines are in the correct position in the current operator to manifest gauge invariance. Notice that we had to ignore the $i\epsilon$ prescription to get a gauge invariant result[49]. Furthermore, also recall that the $i\epsilon$ was associated with the issue of the gauge field at infinity. To justify dropping the $i\epsilon$ we must ensure that in our final result we have an expression which is insensitive to whether or not the gauge field vanishes at infinity.

To prove that the matched current has the correct gauge invariant form to all orders in the coupling, we consider integrating back in the off-shell modes, and solving the classical equations of motion[26, 27]. Let us first define $\psi^Q_{n,\bar{n}}$ as off-shell $n(\bar{n})$ quark fields [50] which will interact with the on-shell $n(\bar{n})$ quark field and a $\bar{n}, n$ collinear gluon. $\psi^Q_{n,\bar{n}}$ represent the intermediate off-shell lines in figure (6) and still satisfy the conditions $\psi^Q_n = \frac{1}{4}\not{n}\not{\bar{n}}\psi^n_Q$ and $\psi^Q_{\bar{n}} = \frac{1}{4}\not{\bar{n}}\not{n}\psi^{\bar{n}}_Q$. We also know from our two gluon example that off-shell gluon fields play a crucial role in the generation of the proper Wilson lines, so we must also include an off-shell gluon fields $A^Q_{(n,\bar{n})}$ in both the n and $\bar{n}$ directions. A_n and $A_{\bar{n}}$ act as background fields for $A^Q_{(n,\bar{n})}$, since, for all intents and purposes, they are frozen over time scales on which $A^Q_{(n,\bar{n})}$ live. We may simplify matters by writing

$$A^\mu_Q = \frac{n^\mu}{2} \bar{n} \cdot A^Q_n + \frac{\bar{n}^\mu}{2} n \cdot A^Q_{\bar{n}}. \tag{128}$$

This expansion is sufficient because $A^Q_\perp$ gluons can only be pair produced (see (102)), and thus at tree level they can not contribute to any graph where the external states are on-shell [51].

The auxilary field action is given by

$$L_{aux} = \bar{\psi}^Q_n (gn \cdot (A_Q + A_{\bar{n}})) \frac{\not{\bar{n}}}{2} \xi_n + \bar{\psi}^Q_n (n \cdot \mathbf{P} + gn \cdot (A_Q + A_{\bar{n}}) \frac{\not{\bar{n}}}{2}) \psi^Q_n + (n \leftrightarrow \bar{n}) + h.c.. \tag{129}$$

---

49 The region of momentum space where the prescription matters is the soft region. Those who have read the section on the zero bin in the previous chapter can understand that this region should be irrelevant for the collinear mode anyway. This is consistent with the fact that collinear field (in SCETI) has an invariant mass which is parametrically larger then the US field and thus should have no support at infinity. We will discuss the zero bin in the context of SCET later in this chapter.

50 Technically we should introduce two distinct off-shell modes, one for incoming the other outgoing. However, the solution to outgoing is just the Hermitian conjugate of the incoming.

51 Recall that by "on-shell" we mean having virtuality at the lowest possible scale.

The labels have been dropped for notational simplicity. Using the usual power counting arguments it is straightforward to show that all these terms are leading order in λ.

The classical equation of motion for the ψ_n^Q field is

$$(n\cdot\mathbf{P}+gn\cdot(A_{\bar{n}}^Q+A_{\bar{n}}))\frac{\bar{\not{n}}}{2}\psi_n^Q=-gn\cdot(A_{\bar{n}}^Q+A_{\bar{n}})\frac{\bar{\not{n}}}{2}\xi_n \tag{130}$$

We can solve this equation by utilizing, in analogy to (121)

$$n\cdot(\mathbf{P}+g(A_{\bar{n}}+A_{\bar{n}}^Q))W_{\bar{n}}^Q=0, \tag{131}$$

it then follows that

$$\psi_n^Q=(W_{\bar{n}}^Q-1)\xi_n \tag{132}$$

where $W_{\bar{n}}^Q$ is a Wilson line defined with a gauge link composed of $A_{\bar{n}}^Q$.

Now if we consider matching the full theory currents onto the effective theory we find

$$(\bar{\psi}_n^Q+\bar{\xi}_n)\Gamma(\psi_{\bar{n}}^Q+\xi_{\bar{n}}^Q)=\bar{\xi}_nW_{\bar{n}}^{Q\dagger}\Gamma W_n^Q\xi_{\bar{n}}. \tag{133}$$

Our desired result will follow if we can show that

$$W_{\bar{n}}^{Q\dagger}W_n^Q=W_nW_{\bar{n}}^\dagger. \tag{134}$$

We begin by utilizing the equations of motion for the auxillary A_Q fields

$$[iD_\mu^Q,F^{Q\mu\nu}]=[iD_\mu^Q,[iD^{Q\mu},iD^{Q\nu}]]=0 \tag{135}$$

where

$$iD^{Q\mu}=\frac{n^\mu}{2}(\bar{n}\cdot\mathbf{P}+g\bar{n}\cdot(A_n+A_n^Q))+\frac{\bar{n}^\mu}{2}(n\cdot\mathbf{P}+gn\cdot(A_{\bar{n}}+A_{\bar{n}}^Q)). \tag{136}$$

Using the relation

$$iD_Q^\mu=\frac{n^\mu}{2}W_n^Q(\bar{n}\cdot\mathbf{P})W_n^{Q\dagger}+\frac{\bar{n}^\mu}{2}W_{\bar{n}}^Q(n\cdot\mathbf{P})W_{\bar{n}}^{Q\dagger} \tag{137}$$

which follows from (131) and accounting for the fact that $\mathbf{P}$ acts through the Wilson line onto whatever is standing to the right, the equations of motion reduce to

$$[W_{\bar n}^Q n\cdot\mathbf{P}W_{\bar n}^{Q\dagger}, [W_n^Q\bar n\cdot\mathbf{P}W_n^{Q\dagger}, W_{\bar n}^Q n\cdot\mathbf{P}W_{\bar n}^{Q\dagger}] = 0, \tag{138}$$

and a similar equation with $(n \leftrightarrow \bar n)$. Expanding out the commutator we find

$$\begin{aligned}2W_{\bar n}^Q n\cdot\mathbf{P}W_{\bar n}^{Q\dagger}W_n^Q\bar n\cdot\mathbf{P}W_n^{Q\dagger}W_{\bar n}^Q n\cdot\mathbf{P}W_{\bar n}^{Q\dagger} - W_{\bar n}^Q(n\cdot\mathbf{P})^2W_{\bar n}^{Q\dagger}W_n^Q\bar n\cdot\mathbf{P}W_n^{Q\dagger}\\ -W_n^Q\bar n\cdot\mathbf{P}W_n^{Q\dagger}W_{\bar n}^Q(n\cdot\mathbf{P})^2W_{\bar n}^{Q\dagger} = 0.\end{aligned} \tag{139}$$

By inspection we see that the ansatz (134) solves the equations of motion. To see this use the fact that our ansatz implies that $W_n^Q W_{\bar n} = W_{\bar n}^Q W_n$ and $W_n^\dagger W_{\bar n}^{Q\dagger} = W_{\bar n}^\dagger W_n^{Q\dagger}$. Plugging this result into (133) yields the desired gauge invariant form for the current in the effective theory

$$_{SCET}= \bar\xi_n W_n^\dagger\Gamma W_{\bar n}\xi_{\bar n}. \tag{140}$$

Note we took the matrix Γ to be color neutral in accordance with our assumption that we started with a full theory gauge invariant current. Also note that our calculation is completely classical, as no loops have been included. Loop corrections involving the off-shell modes can only affect the overall coefficient. Moreover, as previously mentioned gauge invariance, which is manifest at the level of the action, implies that the coefficients can only depend upon the total momentum of the collinear jet. As such, we can write the current as

$$J_{SCET} = C(\mu, n\cdot\mathbf{P}, \bar n\cdot\mathbf{P})\bar\chi_{n,P}\Gamma\chi_{\bar n,\bar P}. \tag{141}$$

It is interesting to ask, how the proper gauge invariant operator is built up in more complicated cases, such as those involving more then two jet directions. This is a valuable exercise which is left for the reader.

Exercise 5.6 Prove the following Wilson line identities.

$$W[a,b]^\dagger = W[b,a], \tag{142}$$

i.e., taking the Hermitian conjugate reverses the direction of the line. Furthermore, we have

$$W[a,b]W[b,c] = W[a,c] \tag{143}$$

which is sometimes called the "causality" condition, which is so named because of its similarity with time evolution. These two constraints then lead to the unitarity relation

$$W[a,b]W[a,b]^\dagger = W[a,b]W[b,a] = W[a,a] = \mathbb{1}. \tag{144}$$

In general when matching external currents we will simply write down the complete set of gauge invariant operators at any given order in the power expansion which are consistent with the quantum numbers of the current[52]. To do so we wish to construct an irreducible set of operator elements. Whereby an irreducible operator I mean one that can not be eliminated via the use of the equations of motion. We have already introduced the fermionic gauge invariant operator χ_n, and we may introduce a similar construction for the gauge boson field. Define

$$\mathcal{B}^\mu_{n\perp} = \frac{1}{g}\left[W_n^\dagger iD^\mu_{n\perp}W_n\right] = \frac{1}{g}\left[\frac{1}{\bar{n}\cdot\mathbf{P}}W_n^\dagger\left[i\bar{n}\cdot D_n\,,iD^\mu_{n\perp}\right]W_n\right] \tag{145}$$

and

$$B^\mu_{n\perp,\omega} = [B^\mu_{n\perp,\omega}\delta(\omega-\bar{n}\cdot\mathbf{P}^\dagger)]. \tag{146}$$

The square bracket notation indicates that $\mathbf{P}$ acts only within their purview. To prove (145) we write

$$\left[W_n^\dagger iD^\mu_{n\perp}W_n\right] = \left[\frac{1}{\bar{n}\cdot\mathbf{P}}\bar{n}\cdot\mathbf{P}W_n^\dagger iD^\mu_{n\perp}W_n\right] \tag{147}$$

then we use the fact that $\bar{n}\cdot\mathbf{P} = W_n^\dagger(\bar{n}\cdot D_n)W_n$. We can think of B^μ as a gauge invariant form of the connection.

[52] The the term "current" here means any operator thats not part of the action.

$B^\mu_{n\perp}$ scales as λ and its expansion starts with

$$B^\mu_{n\perp} = A^\mu_{n\perp} - \frac{k^\mu_\perp}{\bar{n}\cdot k}\bar{n}\cdot A_{n,k} + ... \tag{148}$$

Note that $B^\mu_{n\perp}$ is gauge invariant[53] despite the fact that its leading contribution is simply $A_\perp$. It is also important to keep in mind that $B^\mu_{n\perp}(x)$ is non-local/local as far as collinear/US gauge transformations is concerned. That is an US transformation acts at the point x since US does not effect the labels only the residual momentum which is carried by the x dependence.

What about the other components of B_μ? We could similarly define

$$n\cdot B_n = \frac{1}{g}\left(\frac{1}{\bar{n}\cdot\mathbf{P}}W_n^\dagger[i\bar{n}\cdot D_n, in\cdot D_n]W_n\right). \tag{149}$$

In principle we should include this in our list of gauge invariant building blocks. However, we can use the equations of motion to eliminate this block in favor of $B^\mu_{n\perp}$ and fermionic bilinears formed from χ_n. Similarly the object

$$B^{\mu\nu}_{n\perp\perp} = [\frac{g}{\bar{n}\cdot\mathbf{P}}W_n^\dagger[iD^\mu_{n\perp}, iD^\nu_{n\perp}]W_n] \tag{150}$$

can be traded for $B^\mu_{n\perp}$ using the equations of motion and the identity

[53] Collinear gauge transformations vanish at infinity in SCETI, where the IR scales is of order the US invariant mass. In SCETII this is no longer the case. This is discussed in section ().

Exercise 5.7 Show that $n \cdot B_n$ and $B_{\perp\perp}^{\mu\nu}$ defined in (149) and (150), respectively, can be eliminated using the equations of motion

$$[iD_\nu, [iD_\mu, iD_\nu]] = -g^2 T^a \bar{\chi}_n T^a \frac{\bar{\not{n}}}{2} \chi_n. \tag{151}$$

To eliminate $B_{\perp\perp}^{\mu\nu}$, first prove the identity

$$W_n^\dagger iD_{n\perp}^\mu W_n = \mathbf{P}_{n\perp}^\mu + gB_{n\perp}^\mu. \tag{152}$$

When proving this, account for the fact that the label-operator piece of the covariant derivative must be allowed to push through the full set of fields. That is, use

$$W_n^\dagger iD_{n\perp}^\mu W_n = [W_n^\dagger iD_{n\perp}^\mu W_n] + \mathbf{P}_{n\perp}^\mu, \tag{153}$$

where brackets indicate that the derivative does not act beyond the closing bracket.

$${}_n^\dagger iD_{n\perp}^\mu W_n = \mathbf{P}_{n\perp}^\mu + gB_{n\perp}^\mu. \tag{154}$$

The proof of which is left as an exercise for the reader.

Finally, we may further utilize the collinear Wilson lines to re-write the Lagrangian in (102) as

$$L \;=\; \bar{\xi}_n \left(in \cdot D + (\not{\mathbf{P}}_\perp + g\not{A}^\perp) W_n \frac{1}{\bar{n} \cdot \mathbf{P}} W_n^\dagger (\not{\mathbf{P}}_\perp + g\not{A}^\perp) \right) \frac{\bar{\not{n}}}{2}) \xi_n, \tag{155}$$

where we have used the identity

$$F[\bar{n} \cdot \mathbf{P} + g\bar{n} \cdot A_n] = WF[\bar{n} \cdot \mathbf{P}]W^\dagger. \tag{156}$$

This removes the cumbersome gauge field in the denominator in (102). Of course, in addition there is the piece of the action corresponding to the collinear fields in the $\bar{n}$ direction, which have been, and will be, suppressed to avoid trivial duplications.

E. Factorization of Modes

We have now seen that the collinear modes in differing directions [54] decouple from each other in the action. This implies that the stress-energy tensor can be written as

$$T_{\mu\nu} = T^{n}_{\mu\nu} + T^{\bar{n}}_{\mu\nu} + O(\lambda). \tag{157}$$

If we have a matrix element of a mixed set of collinear operators $O_n, O_{\bar{n}}$ with external states in differing collinear sectors it can be factored

$$\langle p_n p_{\bar{n}} \mid O_n O_{\bar{n}} \mid k_n k_{\bar{n}} \rangle = \langle p_n \mid O_n \mid k_n \rangle \langle p_{\bar{n}} \mid O_{\bar{n}} \mid k_{\bar{n}} \rangle. \tag{158}$$

We are not quite done factorizing *all* of the modes since the US fields still couple to the collinears in (102). However, we can can decouple the US fields by making a field redefinition [26] just as we did in the case of HQET. Define a new set of collinear fields [55]

$$\xi_{n,p} = Y_n \xi^{(0)}_{n,p} \quad , \tag{159}$$

where Y_n is a US Wilson line defined by

$$Y_n(x, -\infty) = P e^{ig \int_{-\infty}^{\bar{n}\cdot x} n\cdot A^{US}(n\lambda) d\lambda} \tag{160}$$

and plugging this into the action (102) the US gauge field in the $n \cdot D$ cancels. To complete this decoupling of modes we make a similar transformation on the gluonic fields

$$A^{a,\mu}_{n,p} = \mathcal{Y}^{ab}_n A^{(0)b,\mu}_{n,p} \tag{161}$$

where $\mathcal{Y}^{ab}$ is the Wilson line in the adjoint representation.

$$\mathcal{Y}^{ab}_n = \left[P e^{g \int_{-\infty}^{x} n\cdot A^{d,US}(n\lambda) f^{dab} d\lambda} \right]^{ab}. \tag{162}$$

[54] We have chosen only back to back collinear directions, but the procedure generalizes to an arbitrary number of directions.

[55] This is sometimes called the BPS field redefinition.

Then we define the decoupled gluon field as $A_n^{(0),\mu}$ via

$$A_n^\mu \equiv A_n^{\mu,b} T^b = A_n^{(0)a,\mu} \mathcal{Y}_n^{ba} T^b = A_n^{(0)a,\mu} Y_n T^a Y_n^\dagger \equiv Y_n A_n^{(0)\mu} Y_n^\dagger \tag{163}$$

where we used the Wilson line identity

$$\mathcal{Y}_n^{ba} T^b = Y_n T^a Y_n^\dagger \tag{164}$$

and the indices in the fundamental have been suppressed. (164) can be proven by first decomposing the Wilson lines into products of infinitesmals and then using the identity

$$e^{\xi A} B e^{-\xi A} = B + \xi[A,B] + \frac{\xi^2}{2!}[A,[A,B]] + \frac{\xi^2}{3!}[A,[A,[A,B]]] + ... \tag{165}$$

In this way one can see how to build up the adjoint Wilson line from the RHS of this equation.

Now consider how a collinear Wilson line changes under the field redefinition

$$W_n = \left[\sum_{perms} \exp\left(-g \frac{1}{\mathbf{P}} Y_n \bar{n} \cdot A_{n,q}^{(0)} Y_n^\dagger \right) \right] = Y_n W_n^{(0)} Y_n^\dagger \tag{166}$$

which follows from expanding the exponent and using the relations $\mathbf{P} Y = Y \mathbf{P}$ and $Y^\dagger Y = 1$.

Using the relation

$$Y_n^\dagger n \cdot (i\partial + g A^{(US)}) Y_n = i n \cdot \partial. \tag{167}$$

The US gauge field has now been eliminated, manifesting complete factorization of the modes. We will drop the (0) moniker from the fields henceforth leaving

$$L \;=\; \bar{\xi}_n (i n \cdot D_c + i \not{D}_{c\perp} \frac{1}{i \bar{n} \cdot D_c} i \not{D}_{c\perp}) \frac{\not{\bar{n}}}{2} \xi_n, \tag{168}$$

We have also returned to the simpler form of the action by writing it in terms of the collinear covariant derivative and used eq.(164).

The US fields reappear in the current (141), which now becomes

$$J_{SCET} = C(\mu, n \cdot \mathbf{P}, \bar{n} \cdot \mathbf{P}) \bar{\chi}_{n,P} Y_n^\dagger \Gamma Y_{\bar{n}} \chi_{\bar{n},\bar{P}}. \tag{169}$$

Before closing the section let us discuss a subtlety regarding the Wilson lines. We have fixed the direction of the path by inspecting the sign of the $i\epsilon$ term in the matching. The direction of the line depends upon whether or not the large light cone component is incoming or outgoing. But this can't be, the operator should unique, independent of the choice of external states. The solution to this dilemma lies in the interpolating fields for the external states [28]. GIVE AN EXAMPLE. (Doesnt zero bin solve this problem?)

6. REPARAMETERIZATION INVARIANCE (RPI)

We have derived the leading order action eq. (168) by doing a tree level matching calculating from full QCD. But how do we know that there are no other operators that say only arise at higher loop? In fact, consider the operator

$$O_2 \;=\; \bar{\xi}_n(iD^{\mu}_{c\perp}\frac{1}{i\bar{n}\cdot D_c}iD_{\mu c\perp})\frac{\bar{\not{n}}}{2}\xi_n, \tag{170}$$

this operator is leading order in the power counting and consistent with all the symmetries. In principle then it could show at higher orders in the loop expansion. However, we have yet to uncover all of the underlying invariances [56]

In particular, we note that the process of splitting up the collinear momenta into large and small pieces is ambigous. We are always free to shift the large light components by any amount of order λ^2 and shuffle it in to the other components. This is just shuffling part of a label into a residual momentum. The physics must remain invariant under these shifts, thus our action should be invariant. Let us explore the consequences of this invariance. What happens to our collinear fields under such a shift in our (repristinated) labels $p \to p+\kappa$.

$$\xi_{n,p}(x) \to e^{-i\kappa\cdot x}\xi_{n,p+\kappa}(x) \tag{171}$$

Invariance under this transformation is acheived by making the replacement

$$D^{\mu}_{c} \to D^{\mu}_{c} + D^{\mu}_{U} \tag{172}$$

the constraints comes from the fact that the combination $\mathcal{P}_\mu + i\partial_\mu$ acting a collinear field will

[56] There is a difference between a symmetry and invariance, as not all invariances correspond to symmetries.

be invariant under the action eq(89). Recall previously we learned that US gauge invariance forced us to complete one of the components of the collinear covariant derivative with the US gauge field as well see eq.(102). Here we are saying something more, since now we are adding terms which are sub-leading in the power expansion. Thus this reparameterization invariance has shown us the the Wilson coefficients of a subleading operators (with US fields) is tied the the leading order Wilson coefficient of the collinear operator.

In addition to the choice of momentum re-distribution, there is also freedom in choosing the light cone directions n and $\bar{n}$ (for a single jet). n and $\bar{n}$ must satisfy $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. So for any given choice n we can always deform is $n \to n + \delta n$ as well as for the barred case. We can consider three independent variations which preserve these conditions:

1. $\delta n^\mu = \Delta^\perp_\mu \quad : \quad \delta \bar{n}^\mu = 0$
2. $\delta \bar{n}^\mu = \epsilon^\perp_\mu \quad : \quad \delta n^\mu = 0$
3. $\delta n^\mu = \alpha n^\mu \quad : \quad \delta \bar{n}^\mu = -\alpha n^\mu$.

7. FACTORIZATION: EXAMPLES

It is important to understand that factorization is an *all orders* (both in α_s and power corrections) statement. That is, given any operator matrix element, we can always decompose it into a product (more generally a convolution) of matrix elements each of which contains only one species of fields. The reason for this is that the leading order action is factored, and that is all that really matters. Which is to say that, while higher order operators may involve both collinear and soft operators, when we take matrix elements we can always factorize them into a product of matrix elements because the states are always described by the eigenstates of the *leading order action*. Corrections to the states which arise as a consequence of time ordered products (see section 1.) can be factored as well. Thus *all* matrix elements factor, for any observable for which SCET applies! This seems like an very strong statement. But our enthusiasm is tempered by the fact that just because an observable factorizes does not mean that we can derive a practical theoretical prediction. That is, if we factorize a result into a set of long distance matrix elements which depend upon a handful of variables, then it seems unlikely we will have any predictive power, as one would have to somehow measure a set of multi-variable functions either in experiment or on the lattice.

The reason Drell-Yan and DIS are useful is because they only depend upon PDF's which are functions of one variable. Once one starts to look at more complicated observables, one generates more complicated matrix elements which are harder to extract from the data. Examples of such matrix elements are fragmentation functions [29] and transverse momentum dependent PDFs.

1. *Soft Theorems*

Weinberg famously proved that for a hard scattering process at the scale Q, the soft emissions ($k \ll Q$) factorize and are independent of the details of the scattering process. More precisely if we consider an $N+1$ point scattering amplitude in the limit where one of the particles becomes soft the result can be writte as

$$A(1,....N,s) \to A(1,....N)S(s) \tag{173}$$

where s is the soft momentum. $S(s)$ is a universal function that is independent of the details of the hard scattering amplitude, including the possibility of loops. Weinberg accomplished this by a careful study of the Feynman diagrams, but given our factorized action, this result comes as no surprise.

2. *Example: The Drell-Yan Process*

As we discussed in the first part of this chapter, the OPE is not always sufficient to generate a useful factorization formula for certain observables. In the case of Drell-Yan we saw that the rate was a function of the matrix element (47) which depends upon both incoming hadrons which is useless for all intents and purposes. We need to somehow factor (47) into a product of two matrix elements each of which depends upon only one of the incoming hadrons. Hopefully it is clear to the reader at this point how SCET will solve this problem. Taking the incoming hadrons to be light-like [57] and moving in the n and $\bar{n}$ directions, we have two factorized Hilbert spaces which will lead to the desired result. We begin by defining this observable in the full theory. Recall that in section (3 C) we wrote

[57] The masses are of order Λ_{QCD} and can be accounted for in power corrections.

down spin averaged cross section as

$$d\sigma = \frac{32\pi^2\alpha^2}{Q^4\,s}\,L_{\mu\nu}\,W^{\mu\nu}\frac{d^3k_1}{(2\pi)^3(2k_1^0)}\frac{d^3k_2}{(2\pi)^3(2k_2^0)}\,, \tag{174}$$

where

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{4}\sum_{\text{spins}}\sum_{X}(2\pi)^4\delta^{(4)}(p+\bar{p}-q-p_X)\langle pp'|J^\mu(0)|X\rangle\langle X|J^\nu(0)|pp'\rangle \\ &= \frac{1}{4}\sum_{\text{spins}}\int d^4x\, e^{-iq\cdot x}\langle pp'|J^\mu(x)\,J^\nu(0)|pp'\rangle\,, \end{aligned} \tag{175}$$

where $\bar{p}$ is the momentum of the p' proton. The sum over spins refers to the initial hadron spins (the sum over final hadron spins is included in the sum over X). Integrating Eq. (174) over the emission angles of the final leptons one obtains

$$\frac{d\sigma}{dQ^2} = \frac{2\alpha^2}{3\,Q^2\,s}\,\frac{1}{4}\sum_{\text{spins}}\langle pp'|\,\hat{W}\,|pp'\rangle\,, \tag{176}$$

where we have neglected the lepton masses and defined the operator

$$\hat{W}(\tau,Q^2) = -\int\frac{d^4q}{(2\pi)^3}\,\theta(q^0)\delta(q^2-Q^2)\int d^4x\, e^{-iq\cdot x}\,J^\mu(x)\,J_\mu(0)\,. \tag{177}$$

Proving (176) is left as an exercise for the reader.

Now we would like to match $\hat{W}$ onto the effective theory. We begin at the high scale where we integrate out all modes with virtuality Q^2. We match onto the set of operators with non-zero coefficients which scale as λ^n, minimizing n. At order λ^4, we have four operators. Here we write down only the operator involving quark fields, as it is the only one which has a non-zero matching coefficient at lowest order in the coupling.

$$\hat{W} \;=\; \frac{1}{Q^2}\int d\omega_i\, C(\omega_i,Q)\left[\overline{\chi}^{(i)}_{n,\omega_1}\,\bar{n}\!\!\!/\,\chi^{(i)}_{n,\omega_2}\right]\left[\overline{\chi}^{(i)}_{\bar{n},\omega_3}\,n\!\!\!/\,\chi^{(i)}_{\bar{n},\omega_4}\right] + \ldots.. \tag{178}$$

All the fields are localized to the same point since the measurement under consideration is not sensitive to the small momentum components of the fields. The sensitivity to the large components is captured by the label dependence in the Wilson coefficient. The dots represent operators which are higher order in λ, i.e. power corrections. We have introduced another piece of notation. Since we are now considering operators with more then one χ field in a given direction, we can no longer simply make the Wilson coefficient a function of the label operator and place it outside, as we did in the case of a simple current involving only one field in each the n and $\bar{n}$ directions. So instead we fix the label of each "jet" χ via

$$\chi_{n,\omega}^{(i)} \equiv \left[\delta(\omega - \mathbf{P})\, W_n^{\dagger} \xi_{n,p}^{(i)}\right], \tag{179}$$

and the label (ω) is now made continuous. Notice that χ has units of 1/2. This label is the *total* momentum of all the particles in the jet. Furthermore, we have Fierz rearranged the fields such that the bilinears are composed of fields which lie in the same sectors of the theory. The most general structure for such bilinear has the form

$$\bar{\chi}_n \Gamma \chi_n \qquad\qquad \Gamma = \{\bar{\not{n}}, \bar{\not{n}}\gamma_5, \bar{\not{n}}\gamma_\perp^\mu\}. \tag{180}$$

We can drop all bilinears which are not of the form $\bar{\not{n}} \otimes \mathbb{I}$ in spin-color space, since they will give vanishing contributions when we take the unpolarized hadronic (proton) matrix element.

Let us now re-consider (176). Given that the modes factorize at the level of the action we factorize the matrix element of $\hat{W}$ such that

$$\langle p_n p'_{\bar{n}}|\, \hat{W}\, |p_n p'_{\bar{n}}\rangle \;=\; \int\! d\omega_i\, \frac{C(\omega_i)}{Q^2} \langle p_n|\big[\overline{\chi}_{n,\omega_1}^{(i)}\; \bar{\not{n}}\; \chi_{n,\omega_2}^{(i)}\big]|p_n\rangle\langle p'_{\bar{n}}|\big[\overline{\chi}_{\bar{n},\omega_3}^{(i)}\; \not{n}\; \chi_{\bar{n},\omega_4}^{(i)}\big]|p'_{\bar{n}}\rangle. \tag{181}$$

Since the final and initial state are identical, the operator can not inject any momentum, thus $\omega_1 - \omega_2 = \omega_3 - \omega_4 = 0$ and the matrix elements are only a function of the sum of the labels. The coefficients $C(\omega_i)$ are dimensionless. These matrix elements are related to the parton distribution functions (PDF's)[58] $f_{q/p}(\xi)$ and $\bar{f}_{\bar{q}/p}(\xi)$ in the following way

$$\big\langle p_n\big|\overline{\chi}_{n,\omega}^{(i)}\; \bar{\not{n}}\; \chi_{n,\omega'}^{(i)}\big|p_n\big\rangle_{\text{spin avg.}} \;=\; 4\bar{n}\!\cdot\! p\,\delta(\omega_-) \int_0^1\! dz\, \big(\delta(\omega_+ - 2\xi\,\bar{n}\!\cdot\! p)\, f_{q/p}(\xi) - \,\delta(\omega_+ + 2\xi\,\bar{n}\!\cdot\! p)\, \bar{f}_{\bar{q}/p}(\xi)\big)\,, \tag{182}$$

[58] Loosely speaking, these functions can be thought of as the probability to find a parton $(q, \bar{q})$ in the nucleon which carries momentum fraction z. Note though that the integral over z is not normalized to one. On the other hand the total momentum is normed to one $\int d\xi \xi P(\xi) = 1$.

where we have defined $\omega_\pm = \omega \pm \omega'$ and the quark PDF is defined as (see [3] and references within)

$$f_{q/p}(\xi) \;=\; \frac{1}{2}\int\frac{dy}{2\pi}\, e^{-i\,\xi\bar{n}\cdot p\,y}\big\langle p\big|\,\bar{\psi}^{(i)}(\bar{n}^\mu y)W_n(y,0)\,\bar{\not{n}}\,\psi^{(i)}(0)\big|p\big\rangle\Big|_{\text{spin avg.}}. \tag{183}$$

The anti-quark PDF is given by

$$\bar{f}_{q/p}(\xi) \;=\; \frac{1}{2}\int\frac{dy}{2\pi}\, e^{-i\,\xi\bar{n}\cdot p\,y}\; Tr\;\bar{\not{n}}\big\langle p\big|\,\psi^{(i)}(\bar{n}^\mu y)W_n(y,0)\bar{\psi}^{(i)}(0)\big|p\big\rangle\Big|_{\text{spin avg.}} \tag{184}$$

W is the usual path ordered light-like Wilson line in the n direction[59]. The quark and anti-quark PDF's are related via

$$f_{i/p}(\xi) = -\bar{f}_{\bar{i}/p}(-\xi) \tag{185}$$

which can be seen by first going to light-cone gauge where the Wilson line becomes the identity. Then choosing the time coordinate to be null, i.e. quantizing on the light-cone [30] where the anti-commutation relations are

$$\{\psi(0,y_-,y_\perp),\bar{\psi}(0,0,0)\}\not{n} = \delta(y_-)\delta^2(y_\perp), \tag{186}$$

the required relation follows from a change of variables. Going from the bare to the renormalized PDF does not hinder this proof. The PDF only has support over the range $|\,z\,|< 1$. This can be seen by again going to the light-cone gauge, and inserting a complete set of states $|\;X\rangle$. Choosing $n\cdot x$ as the time coordinate (null quantization again) and imposing that the conjugate momentum (energy) be positive $\bar{n}\cdot P_X > 0$, leads to the desired result.

Given the its gauge invariance, if we can prove (182) in light-cone gauge, it must be true in all gauges. We start by using the relation (185) in the right hand side of (182) and then use the fact that f only has support in the range $|\,z\,|< 1$ to combine the two terms and extend the limits of integration to $(-\infty,\infty)$. In light-cone gauge, the right hand side of

[59] Recall that boosted QCD is identical to SCET when there is only one collinear sector involved.

(182) becomes,

$$4\bar{n}\cdot p\delta(\omega_-)\int_{-\infty}^{\infty}d\xi\,\delta(\omega_+ - 2\xi\,\bar{n}\cdot p)\,f_{q/p}(\xi) \;=\; \delta(\omega_-)\int[dy]\,e^{-i\omega_+ y/2}\big\langle p\big|\,\bar{\psi}(\bar{n}^\mu y)\,\bar{n}\!\!\!/\,\psi(0)\big|p\big\rangle \tag{187}$$

where we have dropped the quark flavor index.

Now consider the LHS of (182). In light cone gauge we may write

$$\big\langle p_n\big|\overline{\chi}_{n,\omega}\,\bar{n}\!\!\!/\,\chi_{n,\omega'}\big|p_n\big\rangle \;=\; \big\langle p_n\big|\bar{\xi}_{n,p}\delta(\omega-\mathbf{P}^\dagger)\,\bar{n}\!\!\!/\,\delta(\omega'-\mathbf{P})\xi_{n,p'}\big|p_n\big\rangle \tag{188}$$

where the dagger on the label operator is to remind us that it acts to the left. Defining

$$\mathbf{P}_\pm = \frac{1}{2}(\mathbf{P}^\dagger \pm \mathbf{P}) \tag{189}$$

we can write

$$\delta(\omega-\mathbf{P}^\dagger)\delta(\omega'-\mathbf{P}) = 2\delta(\omega-\omega')\delta(\omega+\omega'-2\mathbf{P}_+) \tag{190}$$

where use has been made of the identity $\mathbf{P}_- = 0$, when placed within our forward matrix element. We are then left with [60]

$$\begin{aligned}
\big\langle p_n\big|\overline{\chi}_{n,\omega}\,\bar{n}\!\!\!/\,\chi_{n,\omega'}\big|p_n\big\rangle \;&=\; 2\delta(\omega_-)\big\langle p_n\big|\bar{\xi}_{n,p}\,\bar{n}\!\!\!/\,\delta(\omega_+-2\mathbf{P}_+)\xi_{n,p'}\big|p_n\big\rangle\\
&=\; 2\delta(\omega_-)\int[dy_-]\big\langle p_n\big|\bar{\xi}_{n,p}\,\bar{n}\!\!\!/\,e^{-iy_-(\omega_+-2\mathbf{P}_+)}\xi_{n,p'}\big|p_n\big\rangle\\
&=\; 2\delta(\omega_-)\int[dy_-]e^{-i\omega_+ y_-}\big\langle p_n\big|\bar{\xi}_n(y_-)\,\bar{n}\!\!\!/\,\xi_n(-y_-)\big|p_n\big\rangle\\
&=\; \delta(\omega_-)\int[dy_-]e^{-i\omega_+ y_-/2}\big\langle p_n\big|\bar{\xi}_n(y_-)\,\bar{n}\!\!\!/\,\xi_n(0)\big|p_n\big\rangle, \qquad (191)
\end{aligned} \tag{192}$$

utilizing $\chi_n = \psi$ justifies (182).

We can now write down an expression for the cross section that takes on a more canonical form. First we relate the labels to the more standard convolution variables via

$$\omega_+ \equiv \omega_1+\omega_2 = 2\sqrt{s}\xi_1 \qquad \omega'_+ \equiv \omega_3+\omega_4 = 2\sqrt{s}\xi_2, \tag{193}$$

[60] If a coordinate is not made explicit as an argument then in our notation this implies that coordinate vanishes.

where $\sqrt{s}$ is the hadronic center of mass energy and $\xi_{1,2}$ represent the partonic momentum fractions. At tree level the hard matching coefficient depends only upon $\xi_{1,2}$ and $\sqrt{s}$. We will thus write

$$H^{qq}(\xi_1\xi_2) = C^{qq}(\omega_+\omega'_+ = 4s\xi_1\xi_2). \tag{194}$$

At loop level the matching coefficient will involve logs of $\mu/\sqrt{\xi_1\xi_2 s}$. The tree level matching coefficient comes from the annihilation of a quark anti-quark pair into a virtual photon [61] which then decays into two leptons. For quarks with charge q_i a simple calculation gives

$$H^{qq} = -\frac{2\pi\tau}{3}q_i^2(\tau - \xi_1\xi_2) \tag{195}$$

where $\tau = Q^2/s$ is the square of the partonic center of mass energy. The final result may then be written as

$$\frac{1}{4}\langle p_n p'_{\bar{n}}|\,\hat{W}\,|p_n p'_{\bar{n}}\rangle \;=\; \frac{1}{\tau}\int_0^1 d\xi_1 d\xi_2 H^{qq}(\xi_1\xi_2)(f_{q/p}(\xi_1)\bar{f}_{q/p'}(\xi_2) + f_{q/p}(\xi_2)\bar{f}_{q/p'}(\xi_1)) \tag{196}$$

Examples of other canonical factorization proofs, including cases which are less inclusive then Drell-Yan, can be found in [27].

3. *The Running of the PDF's*

The PDF's are low energy non-perturbative matrix elements, which by dimensional analysis, depend upon the factorization scale μ through the ratio of μ to the QCD scale Λ, there are no other scales around [62]. Whereas the hard matching coefficients depends upon the ratio μ/Q. Thus to minimize the logs we should choose $\mu \sim Q$. This, of course, generates large logs in the PDF. As should be familiar by now to the reader, these logs are re-summed by running the PDF from the high scale Q to a low scale near Λ. This running is canonically accomplished using the DGLAP equations (see for instance [7]). The calculation in the EFT is effectively identical to these calculations. As such, we will not go into all of the details of the calculation, but instead will focus on the salient aspects which will be relevant to the next section where we will show how to re-sum logs in the threshold region.

[61] The weak interaction is being ignored as its inclusion is not illuminating.
[62] The proton mass is taken to scale as Λ_{QCD}

The operator we wish to renormalize (183) is [63] ,

$$O(\bar{n}\cdot r) \equiv \frac{1}{2}\int [dy]\, e^{-i\,\bar{n}\cdot r\, y}\bar{\psi}^{(i)}(y\bar{n}^{\mu})W_n(y,0)\,\not{\bar{n}}\,\psi^{(i)}(0). \tag{197}$$

Moreover, we are interested in the forward matrix elements of this operator. To renormalize the operator we need to know the form of the tree level matrix element which is given by

$$\begin{aligned}
\langle p \mid O(\bar{n}\cdot r) \mid p\rangle &= \frac{1}{4}\sum_{spin}\int [dy]\, e^{-i\,\bar{n}\cdot r\, y}\langle p \mid \bar{\psi}^{(i)}(y\bar{n}^{\mu})W_n(y,0)\,\not{\bar{n}}\,\psi^{(i)}(0)\mid p\rangle \\
&= \frac{1}{4}\sum_{spin}\int [dy]\, e^{-i\,\bar{n}\cdot r\, y}\langle p \mid \bar{\psi}^{(i)}(y\bar{n}^{\mu})\,\not{\bar{n}}\,\psi^{(i)}(0)\mid p\rangle \\
&= \frac{1}{8}\int [dy]\, e^{-i\,\bar{n}\cdot(r-p)\, y}(\bar{n}\cdot p)Tr(\not{n}\not{\bar{n}}) \\
&= \delta(1-\xi)
\end{aligned} \tag{198}$$

where $\xi \equiv \bar{n}\cdot r/\bar{n}\cdot p$. As previously discussed, the matrix element must be a function only of this ratio as a consequence of boost invariance. When we calculate radiative corrections, the general form of the UV divergence will not correspond to (198), implying operator mixing as we shall now demonstrate.

We begin by noticing that the PDF is not a time ordered product of local fields. As such, we can not apply Wicks theorem with impunity. Instead we insert a complete of (out) states and then use LSZ reduction. This amounts writing down all of the cut Feynman diagrams. Setting aside for the moment the Wilson line contributions, at $O(g^2)$ contribution to the matrix element we have two contributions. A self energy contribution where the intermediate state is the vacuum, and a diagram where a gluon is exchanged between the quarks. These two diagrams are shown in figure (7). Let us first consider the diagram with a gluon in the intermediate state. In the diagram the dotted line implies that the gluon is on shell so that we truncate its propagator and insert a factor of $(2\pi)\delta(k^2)$. The external state injects the momentum $\bar{n}\cdot p$ and the operator injects $-\bar{n}\cdot r$. Thus the diagram with real emission is given by

$$I_{real} = \frac{g^2}{2}C_F\int [d^dk][dy](2\pi)\delta(k^2)\theta(k_0)e^{iy(\bar{n}\cdot p(1-\xi)-\bar{n}\cdot k)}Tr[\not{p}\gamma^{\mu}\frac{i(\not{p}-\not{k})}{(p-k)^2-i\epsilon}\frac{\not{\bar{n}}}{2}\frac{-i(\not{p}-\not{k})}{(p-k)^2+i\epsilon}\gamma_{\mu}] \tag{199}$$

[63] When we take the matrix elements of this operator the spin averaging is implied.

FIG. 10. Contributions to the quark PDF at $O(g^2)$. The dotted line (referred to as the "cut", denotes an on-shell particle. The arrow directions denotes the direction fermion number flow. The dot represent the field operator insertion which terminates the external line.

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where the trace arises because we must sum over spins. Notice that the contribution coming from the left side of the "cut" (dotted) line is the complex conjugate of right hand side as expected given that after inserting the complete set of states the PDF is the square of an emission amplitude[64].

$$\begin{aligned} I_{real} &= g^2 C_F \int [d^d k](2\pi)\delta(k^2)\theta(k_0)\delta(\bar{n}\cdot p(1-\xi) - \bar{n}\cdot k)\frac{\bar{n}\cdot k}{(k\cdot p)} \\ &= g^2 \frac{C_F}{(2\pi)}(1-\xi)\theta(1-\xi)\int \frac{[d^{2-2\epsilon}k_\perp]}{(k_\perp^2)}. \end{aligned} \tag{200}$$

This integral is both IR and UV divergent. The IR divergence is of no consequence as we are only interested in the anomalous dimension (we can't claim to be able to reliably calculate the IR piece of the PDF). Splitting this integral into UV and IR divergent pieces, as we did in the first chapter, leads to

$$I_{real}^{UV} = \frac{\alpha}{2\pi} C_F (1-\xi)\theta(1-\xi)\frac{1}{\epsilon}. \tag{201}$$

The virtual self energy graph is identical to the QCD massless self energy and is also scaleless. It contributes half the wave function renormalization (of which there are two)

$$I_{self} = -\frac{\alpha}{4\pi} C_F \frac{\delta(1-\xi)}{\epsilon}. \tag{202}$$

[64] Note that the time evolution operator is conjugated on the RHS of the cut because what was an initial state now becomes a final state. The rules for cut diagrams are sometimes called the "Cutkowsky rules" and can be found in [7]. However, it is perhaps simpler to think of just squaring the gluon emission amplitude as mentioned in the text. Also note that one must include a factor of 1/2 for the Jacobian induced when going to light cone coordinates.

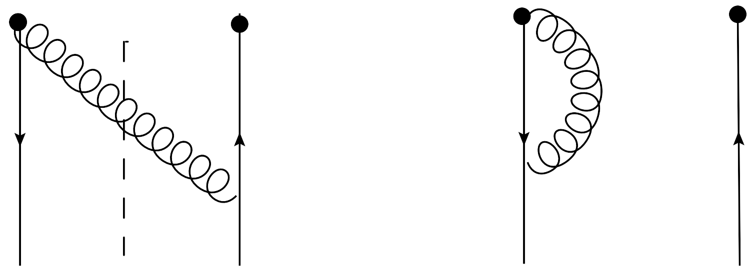

FIG. 11. Contributions to the quark PDF at $O(g^2)$ due to the Wilson line.

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Again note that this diagram is both UV and IR divergence (its scaleless), and we keep only the UV divergent part.

Returning now to the Wilson line contribution[65]. At $O(g)$ we have

$$W_n(y,\infty) = 1 + ig\int_y^\infty \bar{n}\cdot A^a T^a(\bar{n}^\mu\lambda)d\lambda. \tag{203}$$

Notice that in Feynman gauge there are no diagrams in which the two gluons emitted from the Wilson line vertex contract with each other since $\bar{n}^2 = 0$. The real emission graph on the LHS of figure (8) gives

$$I^R_{Wilson} \;=\; \frac{g^2}{4}C_F\int[dy]\sum_{pol}[d^dk](2\pi)\delta(k^2)\theta(k_0)\exp^{iy(\bar{n}\cdot p(1-\xi)-\bar{n}\cdot k)}\frac{\bar{n}\cdot\epsilon^\star}{\bar{n}\cdot k}Tr[\not{p}\not{\bar{n}}\frac{(\not{k}-\not{p})\not{\epsilon}}{(k-p)^2}] \tag{204}$$

where we have explicitly left in the sum over polarizations of the gluon. Performing this sum as well as the trace leaves

$$I^R_{Wilson} \;=\; \frac{g^2}{2\pi}C_F\theta(1-\xi)\frac{\xi}{1-\xi}\int[d^{2-2\epsilon}k_\perp]\frac{1}{k_\perp^2}. \tag{205}$$

$$\tag{206}$$

Extracting the UV divergence gives

$$I^R_{Wilson} = 2\times\frac{\alpha}{2\pi}\frac{1}{\epsilon}C_F\theta(1-\xi)\frac{\xi}{1-\xi}, \tag{207}$$

65 We could choose to work in the light-cone gauge in which case these diagrams will not be there. But the light-cone gauge introduces a new set of technical issues [31] that are preferably avoided.

where we have multiplied by a factor of two to account for the symmetrized diagram. Now let us consider the diagram where the gluon reconnects to its source line.

Now consider virtual Wilson line graph from the RHS of figure (8). Its result is given by

$$I^V_{Wilson} = 2ig^2 C_F \delta(1-\xi) \int [d^d k] \frac{\bar{n}\cdot(p+k)}{(k^2+i\epsilon)((p+k)^2+i\epsilon)(\bar{n}\cdot k)}. \tag{208}$$

Performing the $n \cdot k$ integral by contours leaves

$$I^V_{Wilson} = \frac{\alpha}{\pi} C_F \delta(1-\xi) \int \frac{d^{2-2\epsilon}k}{k_\perp^2} \int_{-1}^{0} (1+\frac{1}{y}) dy, \tag{209}$$

where a change of variables has been made such that at $y = \frac{\bar{n}\cdot k}{\bar{n}\cdot p}$. Notice that the integral over y is divergent an unregulated. This type of divergence in a light cone momentum integral is called a “rapidity divergence” and will be discussed extensively later in this chapter. For the moment let us regulate this divergence by going back to the eikonal propagator and make the replacement

$$\bar{n}\cdot k \to \bar{n}\cdot k - \delta. \tag{210}$$

This result in the shift $y \to y + \hat{\delta}$, where $\hat{\delta} = \delta/\bar{n}\cdot p$ and now we have

$$I^V_{Wilson} = 2 \times \frac{\alpha}{2\pi} C_F \delta(1-\xi) \frac{1}{\epsilon}(1+\log\hat{\delta}) \tag{211}$$

where again a factor of two has now been included to account for the symmetric diagram which is not shown in the figure. We have kept track of the $\hat{\delta}$ despite the fact that it's UV finite, to make sure that it cancels in our final result. This cancellation will be manifest once we return to the real Wilson line diagram, the result of which is singular upon integration over ξ. To regulate this divergence let us return to (204) and shift the eikonal propagator as we did for the virtual diagram such that

$$I^R_{Wilson} = \frac{\alpha}{\pi}\frac{1}{\epsilon} C_F \frac{\xi}{1-\xi+\hat{\delta}} \theta(1-\xi). \tag{212}$$

Notice that if we sum over the real and virtual Wilson line diagrams and integrate over ξ we find a result which is independent of the rapidity regulator. Also note that away from $\xi = 1$ we can ignore the rapidity regulator. So we should be able to explicitly write down

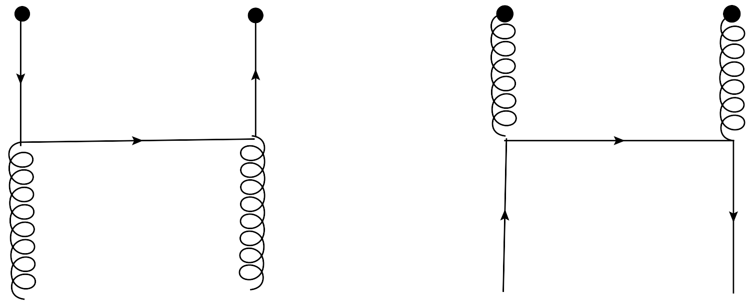

FIG. 12. Diagrams which mix the quark and gluon PDFs, which are avoided when working with non-flavor singlet PDFs.

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a result which is independent of δ if we define the result in a distributional sense. To do so we write

$$\frac{1}{1-\xi+\hat{\delta}} = \frac{1}{(1-\xi)_+} - \log(\hat{\delta}) \tag{213}$$

where the plus distribution is defined as

$$\int_0^1 d\xi \frac{f(\xi)}{(1-\xi)_+} = \int_0^1 d\xi \frac{f(\xi)-f(1)}{(1-\xi)_+} \tag{214}$$

such that

$$I^R_{Wilson} = \frac{\alpha}{\pi}\frac{1}{\epsilon}C_F(\theta(1-\xi)\frac{\xi}{(1-\xi)_+} - \log(\hat{\delta})\delta(1-\xi)). \tag{215}$$

Summing the real and virtual Wilson line graphs leaves a result which is manifestly independent of the $\hat{\delta}$ regulator. The reader should recall that the PDF has support over the range $\mid \xi \mid < 1$, as previously discussed. Thus from the point of view of the low energy matrix element we need to justify the lower limit of integration in (214). This bound follows from the fact that the the Wilson coefficient vanishes when the parton momentum fraction is less then zero. Physically, this requirement is self evident.

Finally we also have graphs which mix the gluon and quark PDFs which arise from the diagrams shown in figure (7 3), however, we can avoid this issue by considering quark bilinears which are not isospin singlet quark PDFs. If the quark bilinear is an isotripet (or any non-singlet representation)[66] it can not mix into a gluonic PDF which is an isosinglet.

Putting all of the diagrams together we find that the UV singular (i.e. bare) part of the

[66] The only non-vanishing matrix element of an isotriplet between proton states is the diagonal third (by convention) generator.

forward (non-singlet) matrix element of $O(\xi)$ is given by

$$
\begin{aligned}
O^B(\xi) &= \frac{\alpha_s(\mu) C_F}{2\pi} \frac{1}{\epsilon} \left(\frac{3}{2}\delta(1-\xi) + \theta(1-\xi)(1-\xi) + 2\theta(1-\xi)\frac{\xi}{(1-\xi)_+} \right) \\
&= \frac{\alpha_s(\mu) C_F}{2\pi} \frac{1}{\epsilon} \left(\frac{3}{2}\delta(1-\xi) + \theta(1-\xi)\frac{1+\xi^2}{(1-\xi)_+} \right)
\end{aligned}
\tag{216}
$$

The term in the parenthesis (times a factor of $C_F\alpha/\pi$) is commonly known as the quark splitting function and is denoted by $P_{q\to q}(\xi)$.

As a consistency check of our calculation we note that

$$
\int_0^1 d\xi \langle p \mid O(\xi) \mid p \rangle \propto \langle p \mid \bar{\psi}(0) \not{n} \psi(0) \mid p \rangle \tag{217}
$$

which is just the expectation value of the conserved quark current[67], and does not get renormalized. As such all the UV divergences in the expression (216) must vanish upon integration. A quick calculation shows that this is indeed the case.

Exercise 7.1 Include quark-mass effects and determine how they modify the relevant current conservation and matching relations in this section.

How do we renormalize the PDF given that the one loop corrections, except for the self energy contributions, don't have counter-terms which are the same as the form of the tree level result (198)? This situation is reminiscent of operator mixing, where renormalization is no longer multiplicative. That is, the counter-term needed to renormalize an operator is proportional to another operator. But in our case what are these other operators? Well the label formalism makes this clear, as the PDF operator (197) has a large label $(n \cdot r)$, which can change under radiative corrections. The emission of a collinear gluon off of a quark line changes the label of that line, so a quark operator with label $\mathbf{p}$ will change to an operator with label[68] $\mathbf{p}'$. Furthermore, given that the labels are continuous, the renormalization

[67] Strictly speaking since we are looking at the non-singlet operator this current is only conserved up to quark masses. However, this does not change the conclusion that the current is not renormalized, as the masses are IR parameters and have no effect on the UV divergences.

[68] Note that the perp label is irrelevant at leading order in the power expansion.

involves an integral rather than a sum, such that [69]

$$O^B(\xi) = \int \frac{d\xi'}{\xi'} Z(\xi,\xi') O^R(\xi'), \tag{218}$$

which is a continuous version of the operator mixing statement

$$O_i^B = Z_{ij} O_j^R. \tag{219}$$

Using our result (216) we find

$$Z(\xi,\xi') = \delta(1-\xi/\xi') + \frac{\alpha_s(\mu) C_F}{2\pi}\frac{1}{\epsilon}\left(\theta(1-\xi/\xi')\frac{1+(\xi/\xi')^2}{(1-\xi/\xi')_+} + \frac{3}{2}\delta(1-\xi/\xi')\right). \tag{220}$$

Z must be a function of the ratio ξ/ξ'. To see this note that Z knows nothing about $n \cdot p$, since that is the large momentum in the matrix element.

Now using the fact that the bare operator is independent of μ and

$$\frac{\partial \alpha}{\partial \log \mu} = -2\epsilon\alpha_s + \beta(\alpha_s). \tag{221}$$

we have

$$\mu \frac{d}{d\mu} O^R(\xi) = \int_\xi^1 \frac{d\xi'}{\xi'} P_{q\to q}(\xi/\xi') O_R(\xi'), \tag{222}$$

which is the DGLAP equation. The upper limit of integration is a reflection of the fact that the PDF has support in the range zero to one. The lower limit is a consequence of the theta function in eq. (220). (**it seems like there should be a factor of 2 difference between the Pqq in the bare calc and the result since $d/dlog\mu\ \alpha = -2\epsilon\alpha$?)**

The nature of the integral can be easily discerned by thinking of ξ as a matrix index, and can be solved my matrix diagonalization. This is accomplished by going to moment (Mellin) space, i.e considering

$$\int_0^1 \xi^{N-1} d\xi \ \mu \frac{d}{d\mu} O^R(\xi) = \int_0^1 d\xi' \xi'^{N-1} O_R(\xi') \int_0^1 dx P_{q\to q}(x) x^{N-1}, \tag{223}$$

[69] Again, if we were interested in the singlet operator then the quark would mix with the gluon and we would have a two by two coupled system of equations.

or

$$\mu \frac{d}{d\mu} O^R(N) = P_{q\to q}(N) O_R(N). \tag{224}$$

Where we first reversed the order of integration and then performed a change of variables to reach the final form. While the solution to this equation is straight-forward the inverse Mellin transform is typically performed numerically. What logs do these solutions resum? First we can see that going to moment space will not only allows us to simply solve the RG equation but also deconvolves the hard scattering function, which depends upon the large label momenta as well as the hard scale Q (which is implicit in (196)), from the PDF's. Thus if we run the PDF's up to the hard scale from the QCD scale (Λ), then we will have summed all of the $log(Q/\Lambda)$ contributions at any given order in $\alpha_s log(Q/\Lambda)$.

A. Hybrid EFTs: Combining SCETI and NRQCD[70] and non-Canonical Resummations

In the previous chapter we attacked the problem of onium decay utilizing NRQCD. Now we will consider an observable which necessitates a merger of NRQCD with SCET. Recall that in the case of the total width of onia the only sensitivity to IR physics was relegated to the initial state. By considering a fully inclusive observable we have effectively integrated out the final states. This allowed us to write the squared matrix element in terms of a finite number of local operators. However, suppose we choose to measure a final state kinematic variable. For simplicity let us consider a radiative decay, so that we can search for a final state photon. By considering electromagnetic decays we will have an external parameter (namely the photon momentum q) to treat as an experimental knob [71]. If we consider the region where the photon is sufficiently energetic, then we will no longer be able to simply match onto a finite set of local operators at the level of the amplitude squared. Let us see why this is true.

From a field theoretic standpoint the end point spectrum (where q is approaching its maximum) is particulary compelling as we can choose to study a region of parameter space where the long distance physics is no longer captured by a set of numbers (the local four

[70] This section requires the readers to have gone through the NRQCD chapter.

[71] The analogy with weak decay is not exact, as the photon can also be considered a hadron (the rho meson has the same quantum numbers as the photon). So the photon is not always produced during the hard short distance/time piece of the factorized reaction.

quark operators), but instead by functions of one variable (as we will see below). Moreover, the sensitivity to the end point region will generate a set of logs that need to be resummed.

There are two types of mechanisms which contribute to the quarkonium decay $H \to \gamma + X$: Direct production, where the photon is produced as part of the hard interaction, and fragmentation, whereby the photon is produced as part of the long distance hadronization process. Both processes are important phenomenologically, but we will concentrate on direct production for pedagogical purposes. The reader interested in the complete result may consult [32]. Also we will concentrate on the decay of the $^1S_0$ state (the η) since it has the minimal annihilation channel, namely a photon and one gluon.

In the photonic end-point spectrum the pure NRQCD calculation will no longer suffice as can be seen by studying the kinematics. Defining the dimensionless photon energy

$$z = \frac{E_\gamma}{m}, \tag{225}$$

the *partonic* kinematics (m is the heavy quark mass and $M = 2m$[72] is the onium mass to leading order in the relative velocity) are such that $0 \le z \le 1$. Hadronically the end point will be shifted by an amount of order the binding energy, i.e. mv^2. We will see how this correction is systematically accounted for in the EFT. We write the light-like photon momentum as $q^\mu = mz\bar{n}^\mu$. Near the endpoint ($z \sim 1$) the photon recoils against a jet with momentum

$$p_X^\mu = mn^\nu + m(1-z)\bar{n}^\mu + k^\mu, \tag{226}$$

where k^μ is the small residual momentum. Note that $\bar{n} \cdot p_X \sim m$ and $p_X^2 \sim M^2(1-z)$ and thus we are not free to integrate out the final state jet at the scale m since near the end point $M^2(1-z) \ll m^2$.

What are the relevant modes for this process? We have the potentials, soft and ultra-soft modes (mv^2, mv^2) that determine the bound state dynamics. In addition, we have energetic collinear modes which form the final state jet. The jet will be described by fields with a large light cone component of order the hard scale m, and invariant mass $\sim m\sqrt{(1-z)}$. Given the scaling for collinear modes, namely $m(1, \lambda^2, \lambda)$, we fix the $\lambda \sim \sqrt{1-z}$. Now we must decide on a lower bound on z (call it z_L). If we take z_L too close to one then the

[72] Whether or not we distinguish between the scales m and M depends upon the order of accuracy of interest. If we are matching at tree level, then whether we match at scales proportional to m or M is irrelevant as the difference of $\log(2)$ would be sub-leading.

jet invariant mass will start becoming too small. Once the jet invariant mass reaches order Λ_{QCD}, we reach what is known as the "resonance" region, where the final state can only be populated by one or two hadrons. In this regime the operator product expansion will then no longer be useful, as we would lose quark hadron duality (see the discussion on e^+e^- to hadrons). Said less technically, once we force ourselves into single particle final state it's clear that we start to probe intimate details about hadron formation.

On the other hand, if we take z_L too small, then the invariant mass of the jet will approach m and we would have no parametrically large ratios. In this case we would return to what is effectively the same calculation that was done in the last chapter. Away from the end point we can match onto local four quark operators even when studying the photon spectrum, but now the Wilson coefficients will depend upon z[73].

In this section we will consider values of z such that $(1-z) \sim v^2 \sim \frac{\Lambda_{QCD}}{m}$ where the last $\sim$ is what is assumed to be (numerically) true in bottomonia [74]. With this scaling we have the following hierarchy

$$M \gg M\sqrt{1-z} \gg Mv^2 \sim M(1-z). \tag{227}$$

Given our reasoning regarding the necessary modes in the EFT, in addition to the NRQCD modes which account for the bound state dynamics, we will also need collinear modes with invariant masses $\sim M\sqrt{1-z}$ as well as US modes with virtuality $\sim M(1-z)$[75]. There is no distinction between the SCETI US modes and the NRQCD US modes since they have the same scaling for their momenta.

Is it possible that the theory also needs soft modes with momentum scaling $(\lambda, \lambda, \lambda)$? If such modes existed they could not cross the cut as $(P_c + P_s)^2 \sim \lambda \gg \lambda^2$, which is kinematically forbidden once we have chosen to look at the end point. Such modes could exist in virtual diagrams and indeed they do[76], as the usual soft modes in NRCQD.

With this hierarchy we see that we must work in two stages [77]. First we match the full

[73] The reader may be wonder why we should trust the total width calculation given that it is equal to the integral of the differential width which we do not trust near the end point. The answer is that while we do not trust the end point calculation point by point, the integral over the end point region (given a sufficiently large, greater than Λ_{QCD}, region) is trustworthy for exactly the same reason we can calculate the total cross section of $e^+e^- \to hadrons$. See the footnote on page

[74] For charmonia it is unclear that this scaling will apply, and a new power counting might be more appropriate [33].

[75] A bit of nomenclature. Typically when there is only one soft type mode in a problem the mode is called Soft, even though technically its Ultra-Soft. This is common practice in the SCET literature.

[76] It is interesting to note that had we been looking at B meson decay, the soft mode and the US mode would yield identical diagrams order by order. This is a telltale sign that we are over-counting, and that the zero bin will eliminate this extraneous mode.

[77] Why don't we have to do one stage as in NRQCD? Because in this example there is no dispersion relation

theory onto an effective theory where the low energy scale is $M\sqrt{1-z}$ by integrating out all of the hard modes with invariant mass of order m. In this theory the final state jet is considered as being "long-distance" (i.e. on shell with virtuality must less than M^2) and $\lambda \sim \sqrt{1-z}$. We then run down to this intermediate scale and integrate out the collinear modes, while retaining the US modes whose virtuality is of order Λ_{QCD}. In doing so we integrate out the jet by performing an OPE, and match onto a set of non-local four quark operators. The reason for the non-locality will become clear when the matching is done. We then will run these operators down to the lowest scale in the theory $mv^2 \sim \Lambda_{QCD}$ at which point there are no more hierarchies around and all the large logs will have been vanquished.

We can see explicitly that the existence of the hierarchy of scales ruins the perturbative expansion in the end point spectrum. The one loop expansion for the decay of a onium in the ${}^1S_0$ octet state is given by [34]

$$\left(\frac{d\Gamma}{dz}\right)_{{}^1S_0} \sim \Gamma_0 \frac{\alpha_s}{2\pi}\left(-2C_A\left(\frac{Log(1-z)}{(1-z)}\right)_+ - (\frac{23}{6}C_A - \frac{n_f}{3})\frac{1}{(1-z)_+}\right), \tag{228}$$

where only the singular terms have been kept. The first/second terms are counted as double and single logs respectively as is demonstrated by taking moments of this spectrum

$$\begin{aligned}\int_0^1 dz z^{N-1}\frac{\alpha_s}{2\pi}\left(\frac{Log(1-z)}{(1-z)}\right)_+ &= \int_0^1 dz(z^{N-1}-1)\frac{\alpha_s}{2\pi}\left(\frac{Log(1-z)}{(1-z)}\right)_+ \\ &= \frac{\alpha_s}{4\pi}\left(Log^2 N + 2\gamma_E Log N + \frac{\pi^2}{6} + \gamma_E^2\right) + O(1/N),\end{aligned} \tag{229}$$

where $N \to \infty$ as $z \to 1$. The existence of these double logs will change the power counting. That is, the leading log resummation now only includes the double logs. These are called "Sudakov logs" and arise as a consequence of loop momentum regions where we have simultaneous soft and collinear singularities. We will have much more to say about this when we discuss SCETII. Our goal here will be to re-sum the perturbative series to regain systematic control over our predictions. Working in the EFT allows us to perform this resummation via renormalization group evolution.

which ties the scales together.

1. Matching at the scale M

We begin by integrating out the hard modes and matching onto a set of operators which contain both NRQCD and SCET fields. Notice that the off-shellness of the final state is much less than m, so when we match we will match at the level of the amplitude (and not the amplitude squared as in the case of Drell Yan) by generating a set of operators which interpolate for the state of interest, ${}^1S_0$ and allow for the tree level annihilation into a photon and a gluon. Thus the first operator we should write down is a fermion bilinear, with the appropriate quantum numbers, and two field strengths (recall that we insist upon manifest gauge invariance), one for the gluon and one for the photon. Thus we are interested in the matching to

$$O_8^{BB}({}^1S_0) \equiv O_8^a({}^1S_0)\frac{(ge_b)}{m}B_n^{a\mu\perp}B_{\bar{n}\mu}^{\perp} = \frac{ge_b}{m}\chi^\dagger T^a\psi B_n^{a\mu\perp}B_{\bar{n}\mu}^{\perp} \tag{230}$$

where the photons' field strength is differentiated from the gluons' by the fact that it does not carry a color index a and we have included a factor of the strong and electromagnetic coupling for later convenience[78]. Recall that the NRQCD fields are defined within a preferred frame while the SCET fields are described relativistically. However, since we are interested in decays where we work in the rest frame of the onium this is not an issue. If we were interested, on the other hand, in onium production then we would need to introduce a set of boost matrices which allow us to contract the Lorentz indices between the NRQCD and SCET fields.

The leading order matching diagrams are shown in figure (11), in which case a short calculation determines the Wilson coefficient $C_8({}^1S_0) = 1$.

Exercise 7.2 Suppose we were interested in the decay of a P-wave (${}^3S_1$) state. Write down the appropriate operator and calculate its matching coefficient at tree level.

2. Running from M to $M\sqrt{1-z}$

To perform the runnings we must calculate the anomalous dimensions of the operator (230). Aside from wave function renormalization there are two diagrams that contribute

[78] e_b accounts for the charge of the b quark.

to the "SCET piece" of the process which are shown in figure (12). Let us consider the collinear loop correction. The three gluon Feynman rule is purely in the collinear sector, as such, it is identical to the QCD Feynman rule. To generate the Feynman rule for the two gluon vertex we need to expand the field strength building block ($B_\perp^{a\mu}$) to second order in the gauge field.

> **Exercise 7.3** Show that the two gluon Feynman rule for the collinear field strength operator is given by
>
> $$\Gamma^{A^2}_{B^c_\mu}[(p_1,a),(p_2,b)] = \frac{i}{2} g_s^2 f^{abc} (\epsilon^b_{\mu\perp} \frac{\bar{n}\cdot\epsilon^a(p_1)}{\bar{n}\cdot p_1} - \epsilon^a_{\mu\perp} \frac{\bar{n}\cdot\epsilon^b(p_2)}{\bar{n}\cdot p_2}). \tag{231}$$
>
> not including terms proportional to $(\bar{n}\cdot A)^2$.

We will now use the result of the exercise and note that terms of the form $(\bar{n}\cdot A\ \bar{n}\cdot A)$ give a vanishing contribution. The three gluon vertex is identical to the canonical vertex in QCD. Performing the contraction leaves

$$iA_c = -g^2 C_A O_8^{BB}(^1S_0) \int \frac{d^dk}{k^2+i\epsilon} \frac{1}{(p-k)^2+i\epsilon} \left[\frac{n\cdot(2p-k)}{(n\cdot k)} + \frac{n\cdot(p+k)}{n\cdot(p-k)} \right] \tag{232}$$

Notice that the $i\epsilon$ in the eikonal propagators has purposefully been left out for reasons which will be discussed below. We can simplify the integrand by shifting the momentum and making appropriate changes of variables to find

$$iA_c = -2g^2 C_A O_8^{BB}(^1S_0) \int \frac{[dn\cdot k][d\bar{n}\cdot k][d^{2-2\epsilon}k_\perp]}{(k^2+i\epsilon)((p-k)^2+i\epsilon)} \left[2\frac{n\cdot p}{n\cdot k} - 1 \right]. \tag{233}$$

The non-trivial scalar integral that needs to be evaluated is

$$I_c = \mu^{2\epsilon} \int \frac{[dn\cdot k][d\bar{n}\cdot k][d^{2-2\epsilon}k_\perp]}{(n\cdot k)(k^2+i\epsilon)} \frac{1}{(k+p)^2+i\epsilon} = \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{-(p^2+i\epsilon)} \right)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon^2 n\cdot p \Gamma[1-2\epsilon]}. \tag{234}$$

This integral is most simply performed by doing the $\bar{n}\cdot k$ integral by contours, thus avoiding the issue of the pole prescription for the eikonal propagator. The external momentum has been kept off shell for the purposes of extracting the UV divergences, otherwise the integral

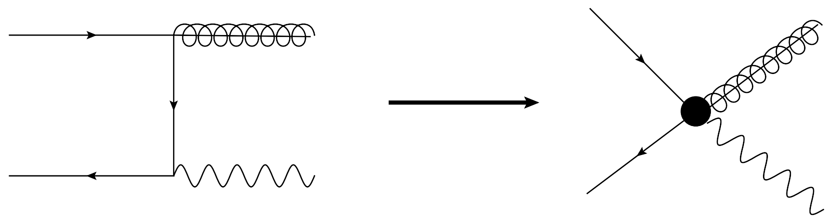


FIG. 13. Leading order contribution to the matching onto the ${}^1S_0$ octet operator. There is also the crossed diagram in the full theory which is not shown. The curly line with a line through it denotes a collinear gluon.

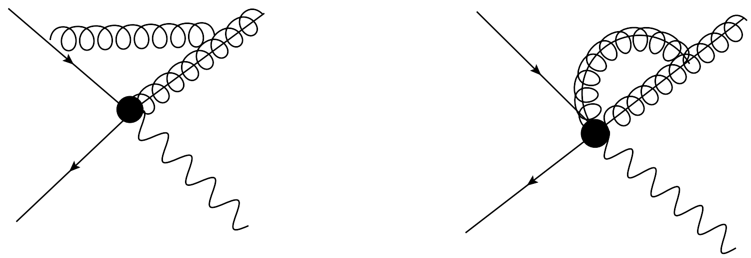


FIG. 14. Diagrams which contribute to the running of the opertor (230). The curly line without the line through it is a US gluon. The diagram on the left has a sister diagram with the gluon connecting to the incoming anti-quark.

would vanish in dim. reg. The result for the diagram is then given by

$$A_c = -\frac{g^2 C_A}{8\pi^2} O_8^{BB}({}^1S_0) \left(\frac{4\pi\mu^2}{-(p^2+i\epsilon)}\right)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon^2\Gamma[1-2\epsilon]}. \tag{235}$$

To calculate the US diagram, one could determine the Feynman rules by direct extraction from the SCET Lagrangian, but in simple cases, such as this, it is sometimes easier to draw the diagram in the full theory and then perform the appropriate multipole expansion. The result of the sum of the two US diagrams in figure (12) is given by

$$iA_{US} = O_8^{BB}({}^1S_0) g^2 C_A \int \frac{[d^dk]}{(k^2+i\epsilon)} \frac{1}{(n\cdot k + \bar{n}\cdot k - i\epsilon)} \frac{\bar{n}\cdot p}{(p^2 - n\cdot k \bar{n}\cdot p + i\epsilon)} \tag{236}$$

The integral is most easily performed by first doing the $\bar{n}\cdot k$ integral by contours, with the final result

$$A_{US} = -\frac{\alpha C_A}{(4\pi)} O_8^{BB}({}^1S_0) \left(4\pi \frac{(\bar{n}\cdot p)^2\mu^2}{p^4}\right)^\epsilon (-1-i\epsilon)^{-2\epsilon}\Gamma(1-2\epsilon)\Gamma(\epsilon+1)\Gamma(2\epsilon+1)\frac{1}{\epsilon^2} \tag{237}$$

where the gluon has been kept off-shell ($p^2 \neq 0$) to regulate the IR divergence. We also have the NRQCD vertex corrections arising some the exchange of a US gluon between the quark and anti-quark. This diagrams yields

$$A_V = \frac{\alpha}{4\pi}(C_F - \frac{C_A}{2}\left(\frac{1}{\epsilon} - Log(M^2/\mu^2))\right). \tag{238}$$

In addition we must consider the renormalization of the collinear gluon field as well as the octet non-relativistic current. The renormalized operator is then given by [79]

$$(O_8^{BB}(^1S_0))_{ren} = Z_\psi Z_3^{1/2} Z_O^{-1}(O_8(^1S_0))_{bare}, \tag{239}$$

with

$$\begin{aligned} Z_\psi &= 1 + \frac{\alpha_s C_A}{4\pi}\frac{1}{\epsilon} \\ Z_3^{1/2} &= 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{2}{3}n_f\right). \end{aligned} \tag{240}$$

Electromagnetic counter-terms are ignored since we are dropping these corrections so there is no Z factor for the elecromagnetic gauge potential. Soft NRQCD loops can not contribute until higher orders in α_s.

Expanding out all of the integrals in ϵ we find

$$Z_O - 1 = \frac{\alpha_s}{4\pi}\left[C_A\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\log\left(\frac{\mu^2}{(n\cdot p)^2}\right)\right) + \frac{17}{6\epsilon} - \frac{n_f}{3\epsilon}\right] \tag{241}$$

Notice that the dependence of the counter-term on the off-shellness has cancelled as it must, since it is an IR quantity which should not appear in the counter-term. Even so, this result is still non-canonical in the sense that there is a log in the counter-term, moreover it depends upon a kinematic variable. What's pleasing about the EFT is that this logarithmic dependence on this external momentum should not be considered as non-analyticity since this (label) momentum can never go to zero, after zero binning. Thus this log is just a number. From this result we can calculate the one loop anomalous dimension for the

[79] Recall that $O_R = Z_O^{-1}O_B = Z_O^{-1}O(\phi_B) = Z_O^{-1}O(Z^{1/2}\phi_R)$.

operator

$$\gamma_{O_8(^1S_0)} \approx \lim_{\epsilon\to 0}(-\epsilon g\frac{d}{dg}+\mu\frac{\partial}{\partial\mu})Z_O^{-1}=\frac{\alpha_s}{2\pi}(C_A\log\frac{\mu^2}{M^2}-\frac{n_f}{3}+\frac{17}{6}C_A), \tag{242}$$

where use of the relation $n\cdot p=M$ has been made. The log in the anomalous dimensions will be responsible for generating double logs. The coefficient of the logs is called the "cusp anomalous dimension" and we will have more to say about it below.

3. Running from M to $M\sqrt{1-z}$

Before we solve the RG equation let us pause to discuss the power counting. The existence of the log in the anomalous dimension seems troubling. If we suppose that $\alpha_s\log(\mu/M)\sim 1$, then we should worry that at n-th loop we might generate $\alpha_s^n\log^n(\mu/M)$ leading to the need to perform a resummation of the anomalous dimensions. However, it is easy to see that the cusp anomalous dimension can have no more than a single log by noting that the collinear integrals can only yield logs which are functions of p^2/μ^2, where as the logs in the soft sector will depend upon $(\bar{n}\cdot p\mu)/p^2$ (see eq.(237)). Since we know that the IR regulator (p^2) dependence must cancel in the anomalous dimensions we are assured that there is a most one log since higher order logs will leave cross terms left behind.

The log does however change the nature in which we power count. We will define the power counting of the anomalous dimension as follows

$$\gamma\sim(\alpha_s\log(\mu/M))_{LL}+(\alpha_s^2\log(\mu/M)+\alpha_s))_{NLL}+.... \tag{243}$$

where "LL" stands for leading-log and "NLL" next-to leading-log. Here we will work at LL level where $\alpha_s\log^2(M/\mu)\sim 1$ (but $\alpha_s\log(M/\mu)<1$) which will restrict the range of z over which our calculation is valid. Thus we will only keep the one loop cusp piece of the anomalous dimension (242). Notice that contrary to more canonical RG resummations, tree level matching will suffice until NNLL, since at NLL we are taking $\alpha_s\log(M/\mu)\sim 1$ and one loop matching starts at $O(\alpha_s)$. It is conventional to run the Wilson coefficient (as opposed to the operator), whose anomalous dimension is, by definition, minus γ_O. To solve the RG

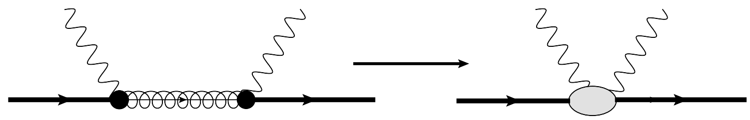


FIG. 15. Matching at the intermediate scale $m\sqrt{1-z}$. The thick lines stand for the heavy quark bilinear.

equation at leading order, we may use

$$\log(\mu/M) = \frac{2\pi}{\beta_0}\left(\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(M)}\right) \tag{244}$$

to write the RG equation in the form

$$\frac{dC^8_{^1S_0}}{C^8_{^1S_0}} = \frac{C_A}{\pi}\left(\frac{2\pi}{\beta_0}\right)^2 \frac{d\alpha_s(\mu)}{\alpha_s(\mu)}\left[\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(M)}\right]. \tag{245}$$

which is easily solved for

$$C^8_{^1S_0}(\mu) = C^8_{^1S_0}(M)\exp\left[-\left(\frac{4\pi C_A}{\beta_0^2\alpha_s(M)}\right)\left[\log\left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right) - 1 + \frac{\alpha_s(M)}{\alpha_s(\mu)}\right]\right], \tag{246}$$

where we choose to run down from the hard scale M to the intermediate scale $\mu = M\sqrt{1-z}$. If we expand out this result in a series in $\alpha_s(M)$ we find

$$C^8_{^1S_0}(\mu) = C^8_{^1S_0}(M)\exp\left[-\frac{C_A\alpha_s(M)}{8\pi}\log^2(1-z) - \frac{C_A\beta_0\alpha_s^2(M)}{48\pi^2}\log^3(1-z) + O(\alpha_s(M)^3) + ...\right] \tag{247}$$

Since we are only working at leading log order, keeping the terms beyond the double log does not improve the accuracy of the prediction. The general form of the exponent is

$$\log(C^8_{^1S_0}/C^8_{^1S_0}) = F(\alpha_s\log(M/\mu))\log(M/\mu) + G(\alpha_s\log(M/\mu)) + \alpha_s H(\alpha_s\log(M/\mu)) + ... \tag{248}$$

Notice that the power counting becomes clear once when we consider the exponential form. If we wished to go beyond leading log we would need the two loop cusp-anomalous dimension and we would need to keep the full one loop non-cusp piece as well, but as previously mentioned the tree level matching would still suffice.

4. Matching at the scale $M\sqrt{1-z}$

At the scale $M\sqrt{1-z}$ we must integrate out the collinear modes leaving a purely soft theory. We do so by performing an OPE, as illustrated in figure (13). The intermediate state $|\,X\rangle$, which recoils against the photon, carries momentum p_X with $p_X^2 \sim (1-z)M^2 \sim \lambda^2 M^2$. p_X is parameterized as in (226)

$$p_X^\mu = mn^\mu + m(1-z)\bar{n}^\mu + k^\mu, \tag{249}$$

k^μ is the residual momentum of the heavy quark pair and is of order $\lambda^2 m \sim mv^2$. Its value is determined by the bound state dynamics as well as the possible soft radiation in the decay process. This matching procedure is very similar to the case of DIS. By the optical theorem the decay rate is proportional to the imaginary part of the time ordered product of two insertions of $O_8^{BB}(^1S_0)$. As usual to perform the OPE we match by using quarks, and for our case of interest we choose them to be in the state $|^1S_0^8\rangle$. We may thus match onto the partonic decay rate to calculate the matching coefficient.

$$\Gamma = (\frac{ge_b}{m})^2 C^8_{^1S_0}(\mu)^2 \quad Im\left[i\sum_{\mathbf{p},\mathbf{p}'}\int [d^4q](2\pi)\delta(q^2)\sum_\lambda \epsilon_{\lambda\mu}(q)\epsilon^\star_{\lambda\nu}(-q)\int \; d^4y e^{i\frac{n\cdot q\bar{n}\cdot y}{2}} e^{-2imv\cdot y} e^{i\frac{\bar{n}\cdot\mathbf{p}n\cdot y}{2}-i\mathbf{p}_\perp\cdot y_\perp}\right.$$
$$\left. \times\; \langle^1S_0^8(k)\mid T[\chi^\dagger_\mathbf{v} T^a\psi_\mathbf{v}(y)B^{a\mu\perp}_{n,\mathbf{p}}(y)][\psi^\dagger_\mathbf{v} T^b\chi_\mathbf{v} B^{b\nu\perp}_{n,\mathbf{p}'}(0)]\mid^1S_0^8(k)\rangle\right], \tag{250}$$

where the labels as well the dependence on the coordinates conjugate to residual momenta have been repristinated. The photon has been reduced out and the exponentials come from the field labels.

The coordinate system is chosen such that the transverse photon momentum $q_\perp$ vanishes and $\bar{n}\cdot q = 0$ as well due to the on-shell condition. In addition, there are also the soft (in NRQCD, as opposed to Ultra-soft), order mv, labels on the heavy quark. In the rest frame the net large three momentum carried by the heavy quarks is zero. We have dropped these labels to keep the notation a little more manageable.

The leading order contribution comes from one gluon exchange, i.e. the simple Wick contraction of the two gluon fields in the time ordered product. The contraction leads to a

Kronecker delta function which kills the $\mathbf{p}'$ label sum. The multipole expansion ensures that the momentum conserving delta function arising from the integration over d^4y will have an argument which scales homogeneously. Using this delta function leads to the gluonic propagator denominator

$$\frac{1}{p_X^2 + i\epsilon} = \frac{1}{M^2(1-z) + Mn\cdot k + i\epsilon}, \tag{251}$$

where $M = 2m$ is twice the quark mass which equals hadron mass (M_η) up to binding corrections of order mv^2. Taking the imaginary generates the partonic result for the lifetime

$$\frac{1}{z}\frac{d\Gamma}{dz} = \frac{(eg_s C^8_{^1S_0}(\mu_c = M\sqrt{1-z}))^2}{4\pi m^2}\langle {}^1S_0^8 \mid \chi^\dagger T^a \psi \delta((1-z) + i\frac{n\cdot\partial}{M})\psi^\dagger T^a \chi \mid {}^1 S_0^8\rangle, \tag{252}$$

the labels have been dropped from the quark fields as they are now superfluous. μ_c is the collinear matching scale. The derivative inside the delta function may automatically be lifted to a covariant derivative on symmetry grounds

$$n\cdot\partial \to n\cdot D(A_{US}). \tag{253}$$

Technically one can derive this replacement by performing the matching including US gluon emissions.

We then define the "shape function"

$$f_\eta^8(n\cdot\hat{k}) = \frac{\langle \eta \mid \chi^\dagger T^a \psi \delta(n\cdot\hat{k} - in\cdot\hat{D})_{ab}\psi^\dagger T^b \chi \mid \eta\rangle}{\langle \eta \mid \chi^\dagger T^a \psi\psi^\dagger T^a \chi \mid \eta\rangle}, \tag{254}$$

where hatted quantities are now in units of M.

Exercise 7.4 Show that

$$f^8_\eta(n\cdot\hat{k}) \;=\; \int [dt] e^{-itn\cdot\hat{k}} \frac{\langle \eta \mid \chi^\dagger T^a \psi(tn^\mu)(e^{-ig\int_0^t d\lambda\; n\cdot A(n^\mu\lambda)})_{ab}\psi^\dagger T^b \chi(0)\mid \eta\rangle}{\langle \eta \mid \chi^\dagger T^a \psi\psi^\dagger T^a \chi \mid \eta\rangle}. \tag{255}$$

The soft derivative has been gauge completed into a covariant derivative, as required by US gauge invariance. The normalization has been chosen so that the first [80] moment is one. $f^8(n\cdot\hat{k})$ is a measure of the quark anti-quark pair residual momentum k_+ in the bound state, it has support over the interval $(1-\hat{M}_\eta) < n\cdot\hat{k} < \infty$ [81].

The general form of the matching, i.e. beyond tree level, can be written as a convolution

$$\frac{1}{z}\frac{d\Gamma}{dz} \;=\; \frac{(ge_b C^8_{^1S_0}(\mu_c))^2}{4\pi M^2}\int dx f^8_\eta(1-x,\mu) J(\mu, x-z)\langle \eta \mid \chi^\dagger T^a\psi\psi^\dagger T^a\chi \mid \eta\rangle, \tag{256}$$

where for convenience the change variables has been made to $x = 1-n\cdot\hat{k}$. Note that the matching coefficient (which represents a "jet") can only be a function of $x-z$ as can be seen from eq. (251). To show that this form is maintained to all orders we can write down a "factorization theorem"[82], which proves that the most general all orders form for the rate is given by the convolution of soft and collinear functions (256).

5. *Going to All Orders: the Factorization Theorem*

To accomplish this we start with eq. (250) which is reproduced here for convenience [83]

$$\Gamma = (\frac{ge_b}{m})^2 C^8_{^1S_0}(\mu)^2 \quad Im\Bigg[i\sum_{\mathbf{p},\mathbf{p}'}\int [d^4q](2\pi)\delta(q^2)\sum_\lambda \epsilon^\lambda_\mu(q)\epsilon^{\lambda\star}_\nu(-q)\int \; d^4y e^{i\frac{n\cdot q\bar{n}\cdot y}{2}} e^{-2im\mathbf{v}\cdot y} e^{i\frac{\bar{n}\cdot\mathbf{p}n\cdot y}{2} - i\mathbf{p}_\perp\cdot y_\perp}$$
$$\times\; \langle \eta \mid T[\chi^\dagger_\mathbf{v} T^a\psi_\mathbf{v}(\bar{n}\cdot y, n\cdot y, y_\perp) B^{a\mu\perp}_{n,\mathbf{p}}(\bar{n}\cdot y, n\cdot y, y_\perp)][\psi^\dagger_\mathbf{v} T^b\chi_\mathbf{v} B^{b\nu\perp}_{n,\mathbf{p}'}(0)] \mid \eta\rangle\Bigg]. \tag{257}$$

[80] The moment transform is taken with weight z^{N-1}.

[81] This can be seen by noting that in the effective theory the energy of the η state is $M_\eta - M$. Notice that there is no upper limit since the heavy quark mass is taken to infinity in the effective theory.

[82] The factorization theorem for the end point can be found in [35]. An SCET a derivation, including power corrections, can be found in [36].

[83] Note that the external states here are taken to be leading order eigenstates of the effective theory Hamiltonian (see section). In (250) partonic states were used because we were discussing the matching.

Let us now peform the requisite multipole expansion by first Fourier transforming the residual dependence on the coordinates. The phase of the exponential is given by

$$i\phi = \bar{n}\cdot y(\frac{n\cdot r}{2}+\frac{n\cdot q}{2}-m)+n\cdot y(\frac{\bar{n}\cdot r}{2}+\frac{\bar{n}\cdot \mathbf{p}}{2}-m)-y_\perp\cdot(r_\perp+\mathbf{p}_\perp), \tag{258}$$

where r is conjugate to y. We then combine the label sum and residual momentum integral of the collinear quark

$$\sum_{\mathbf{p}}\int[d^4r]\to\int[d^4p] \tag{259}$$

as discussed in section (). Since we have chosen to study the endpoint where $\frac{n\cdot q}{2}-m\sim m\lambda^2$ which is of the same order as $n\cdot r$ we must keep this dependence on the residual momentum in the first term. The residual momentum carried by the heavy quarks conjugate to $n\cdot y$ and $y_\perp$ (not shown in (258)) are subleading compared to the label momentum of the gluon and can be dropped everywhere except in the first term.

The rate is then given by

$$\begin{aligned}\Gamma \;=\; & \frac{(ge_b)^2}{16\pi^2}C^8_{^1S_0}(\mu)^2\sum_{\mathbf{p}'}\int\; d\bar{n}\cdot y(dn\cdot q)(n\cdot q)e^{i\bar{n}\cdot y(\frac{n\cdot q}{2}-m)}\\ \times\; & Im[(-i)\langle 0\mid B^{a\mu\perp}(\bar{n}\cdot y,\bar{n}\cdot p=M,p_\perp=0)B^{b\perp}_{n,\mu}(0)\mid 0\rangle]\\ \times\; & \langle\eta\mid T[\chi^\dagger T^A\psi(\bar{n}\cdot y,0,0)](Y_n^\dagger(\bar{n}\cdot y)^{Aa}Y_n(0))^{bB}[\psi^\dagger T^B\chi]\mid\eta\rangle\end{aligned} \tag{260}$$

where $B^{a\mu\perp}(\bar{n}\cdot y,\bar{n}\cdot p=M,p_\perp=0)$ is partial Fourier transform. The residual momentum of the heavy quarks has been reabsorbed into the operator $\chi^\dagger T^a\psi(\bar{n}\cdot y,0,0)$. In the last line we have performed the BPS field redefinition on the collinear fields and used the fact NRQCD and collinear SCET Hilbert spaces are decoupled. Use has been made of the fact that soft matrix element is purely real[84], as one might expect from its physical interpretation, in analogy to the PDF. We then define a soft function S

$$S(\bar{n}\cdot y)\;=\;\langle\eta\mid[\chi^\dagger T^c\psi(\bar{n}\cdot y)][Y_n^\dagger(\bar{n}\cdot y)Y_n(0)]^{cd}[\psi^\dagger T^d\chi(0)]]\mid\eta\rangle \tag{261}$$

[84] The proof of which is left as an exercise for the reader.

and a collinear function J^{ab}

$$J(\bar{n}\cdot y) \;=\; Im[(-i)\frac{1}{(N_c^2-1)}\langle 0 \mid T(B_n^{\mu c\perp}(\bar{n}\cdot y, n\cdot p = M, p_\perp = 0)B^{c\perp}_{n,,\mu}(0)) \mid 0\rangle]. \tag{262}$$

Use has been made of color singlet nature of the state to simplify the color indices. Furthermore, as will be discussed when we renormalize it, S is automatically time ordered. J is a function of the collinear fields and is sometimes called the "jet function". This jet of collinear fields has invariant mass of order $p^2 \sim M^2(1-z) \sim M\Lambda_{QCD}$. Fourier transforming S and J to $\tilde{S}$ and $\tilde{J}$ respectively, leads to the promised convolution form of (256)

$$\frac{1}{z}\frac{d\Gamma}{dz} \;=\; \frac{(ge_b)^2}{4\pi^2 M}C^8_{^1S_0}(\mu)^2\int dx \tilde{J}(z-x)\tilde{S}(1-x). \tag{263}$$

Comparing (255) with (261) we make the identification

$$\tilde{S}(1-x) = \langle \eta \mid \chi^\dagger T^a \psi \psi^\dagger T^a \chi \mid \eta\rangle f^8_\eta(\hat{k}_+ \equiv 1-x). \tag{264}$$

J may be calculated in perturbation theory, since its invariant mass is large compared to Λ_{QCD}.

Exercise 7.5 Show that at tree level

$$J(x-z, M, \mu_c = M\sqrt{1-z}) \;=\; \delta(x-z), \tag{265}$$

where $x \equiv 1-\hat{l}_+$. As expected, with no soft radiation $x = z$. When we include radiative correction we expect that $x < z$ since the invariant mass of the intermediate stay will grow (see eq.(249)), which will be born out in our calculations below. The μ dependence has been made explicit reminding us that beyond tree level there will be logs that need to be resummed when running down from $M\sqrt{(1-z)}$ to $M(1-z)$. Then use this result in conjunction with the tree level result for the shape function to show that the tree level result for the width is given by

$$\Gamma^0(\eta) = \frac{(g_s e_b)^2}{4\pi M^2}\langle \eta \mid \chi^\dagger T^a \psi \psi^\dagger T^a \chi \mid \eta \rangle \tag{266}$$

Exercise 7.6 Calculate the matching coefficient needed to prove (266).

$f_\eta^8(n\cdot\hat{k})$ is a non-perturbative distribution function over which we have no analytic control. It must either be extracted from the data or measured using lattice QCD. However, if we were to consider color singlet channels (which are subleading for the η states) then we can calculate the structure function. To see this we note that the for a singlet operator we may use the so-called "vacuum saturation" approximation whereby the matrix element of singlet matrix element can be written as

$$\langle \eta \mid (\psi^\dagger \chi)\delta(1-x+in\cdot\hat{k})(\chi^\dagger\psi) \mid \eta\rangle \approx \langle \eta \mid (\psi^\dagger\chi)\delta(1-x+in\cdot\hat{k}) \mid 0\rangle\langle 0 \mid (\chi^\dagger\psi) \mid \eta\rangle. \tag{267}$$

where we have chosen to work in the light-cone gauge to eliminate the gauge field from the covariant derivative.

Exercise 7.7 Determine the size of the corrections to the vacuum saturation approximation. Furthermore, explain why this approximation is not valid for color octet matrix elements.

Now expand the delta function as a power series such that

$$\begin{aligned}
\langle \eta \mid O^1({}^1S_0)(x,\mu) \mid \eta\rangle &= \sum_{n=1}^{\infty} \theta^{(n)}(1-x)\langle \eta \mid (\psi^\dagger\chi) \mid 0\rangle\langle 0 \mid (in\cdot\hat{\partial})^n(\chi^\dagger\psi) \mid \eta\rangle \\
&= \sum_{n=1}^{\infty} \frac{\theta^{(n)}(1-x)}{M^n}\langle \eta \mid (\psi^\dagger\chi) \mid 0\rangle\langle 0 \mid [H,....[H,(\chi^\dagger\psi)]....] \mid \eta\rangle.
\end{aligned} \tag{268}$$

In going from the first to the second line we have used the fact that the hadron is at rest to remove the spatial part of the derivatives. The insertion of .... corresponds to n nested commutators. Then using [85] $H \mid \eta\rangle = (M_\eta - M) \mid \eta\rangle$, we are left with

$$\langle \eta \mid O^1({}^1S_0)(x,\mu) \mid \eta\rangle = \delta(\frac{M_\eta}{M} - x)\langle \eta \mid (\psi^\dagger\chi) \mid (\chi^\dagger\psi) \mid \eta\rangle. \tag{269}$$

Using this result we see that the end-point is now shifted to the hadronic end-point,

$$\delta(\frac{M_\eta}{M} - x) \sim \delta(M_\eta - 2E_\gamma). \tag{270}$$

The matrix element of the octet operator, on the otherhand is not-calcuable and can be treated using a model [32, 37].

6. Running from the scale $M\sqrt{1-z}$ to $M(1-z)$.

Below the scale $M\sqrt{1-z}$ we have a purely soft (US) theory which we will use to renormalize the structure function. The running of the operator $f^8_\eta(\hat{k}_+)$ is done using partonic external states in much the same way the running of the PDF discussed in section (7 3), but with some key distinctions. For the PDF we have only light-like Wilson lines, as there are no soft fields in the problem. In the present case if we redefine the heavy quark fields in $f^8_\eta(\hat{k}_+)$, such that there are no longer any leading order US gluon interactions with the heavy quarks in the action (see section), then we get a product of three Wilson lines, with the first and the third along the time-like vector v^μ. Also notice, that the Wilson line generated by the heavy quark interactions transforms in the octet representation, as can be seen using

[85] Recall that in the effective theory the rest masses have been removed.

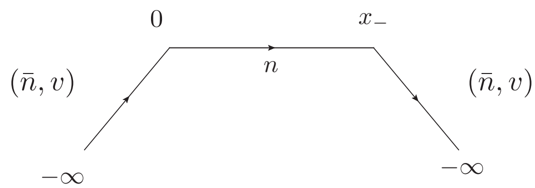
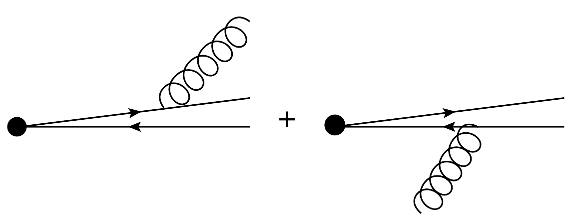


FIG. 16. Comparing the Wilson lines in DIS (at $x \sim 1$) to those of the onium structure functions. The distinction lies in the directions of the incoming and out going lines, as well as the fact that for our ${}^1S_0$ case the quarks generate octet Wilson lines.

the identity (164). We can make the DIS case more like the present case by considering the threshold region, i.e. $x \sim 1$. In this case (U)soft modes no longer cancel and one generates a product of three Wilson lines. The Wilson line configurations for the two cases are shown in figure (7 A 6). For a discussion of DIS in the region $x \sim 1$ within SCET see [24].

It is illuminating to see how the adjoint Wilson lines arise at the diagrammatic level, i.e without performing the field redefinition. Consider US emission from the heavy quarks. We begin by noting that the gluon should couple to the quark/anti-quark pair at the same point in space-time once we have performed the multipole expansion. The long wavelength gluon interacts with incoming quark-anti-quark pair as one particle, we just need to make sure we include the correct group theory factor. That is, since the vertex is a color octet ($\chi^\dagger T^a \psi$) the sum of the two soft emission diagrams gives



$$= -i \frac{T^a T^b - T^b T^a}{E + q_0 - \frac{(\vec{p}+\vec{q})^2}{2m} + i\epsilon} \approx \frac{f^{abc} T^c}{q_0 + i\epsilon}. \qquad (271)$$

The absorbed gluon has color index/momentum b/q and and the US limit has been used such that $\vec{p} \cdot \vec{k}/m \sim v^3 \ll k_0$, where $(E, \vec{p})$ is the external, on shell, energy and momentum. The solid dot represents an insertion of T^a stemming from the ${}^1S_0$ operator. This is exactly the Feynman rule generated by an adjoint Wilson (with direction set by the sign of the $i\epsilon$), as expected from our arguments via field redefinitions.

FIG. 17. The underlying Feynman diagrams, with their possible cuts, that contribute to the one loop running. The dark circles are insertions of the currents. The gluon that runs between the currents is collinear and is responsible for the light-like intermediate Wilson line. The other gluon is US. The figure on the left has a symmetric partner which is not shown.

FIG. 18. The underlying Feynman diagrams drawn when we consider the non-local operator directly. The haevy dot correponds to the delta function in eq. (254). Symmetric diagrams and wave function renormalizations are not shown.

To calculate these diagrams we will first perform the BPS field redefinition to put all of the US gluons into the matrix element. Thus we wish to calculate the anomalous dimensions of the product of adjoint Wilson lines as discussed above,

$$f_\eta^8(\hat{k}_+) = \int_{-\infty}^{\infty} [dt] e^{i\hat{k}_+ t} \langle \eta \mid \psi^\dagger T^a \chi(0) (Y_v^{Aa}(-\infty, 0))^\dagger Y_n^{AB}(t, 0) Y_v^{Bb}(-\infty, t) \chi^\dagger T^b \psi(t) \mid \eta \rangle. \tag{272}$$

where

$$(Y_v^{Aa}(-\infty, 0))^\dagger = (\bar{P} exp^{-g \int_{-\infty}^0 v \cdot A(\lambda v) d\lambda})^{aA}, \tag{273}$$

where $\bar{P}$ stands for anti-path ordered. Notice that $Y_v^{Bb}(t, -\infty)$ has a gluon field evaluated at $v^\mu(\rho - t) + tn^\mu$, where ρ parameterizes the path along the time-like direction.

As with the case of the PDF, this is not a time ordered product, so it would seem that we are not free to use Wicks' theorem. We could use the optical theorem as we did in the case of the PDF in which case, at one loop, we would draw the two diagrams in figure (15). But to develop our calculational flexibility let us take a different approach here. In general it is certainly true that we may not treat a non-local operator as a time ordered product and proceed to use the LSZ reduction [86] formalism to calculate. However, recalling the fact

[86] For an introduction see e.g. [7]

that Wilson lines act as classical sources (which traverses the path given in figure (14)), the time ordering is automatically fixed by the interaction vertices.

Consider first the diagram on the LHS of the figure. Expanding out the Wilson lines to order g^2 and working with vanishing residual momentum leads to the integral (all momentum here are hatted, i..e normalized to the onium mass)

$$I_{LHS} = -ig^2(\mu/M)^{2\epsilon} C_A O^8(^3S_1) \int \frac{[d^{2-2\epsilon}q_\perp]}{(q^2+i\epsilon)} \frac{[dq_+][dq_-]}{(q_+ + q_- + i\epsilon)} \frac{1}{q_+} (\delta(q_+ + k_+) - \delta(k_+)) \tag{274}$$

where k_+ is the momentum injected buy the operator. The second delta function leads to a scaleless integral and may be set to zero as a consequence of the fact that we know the matrix element, being an inclusive cut amplitude is IR finite. As such we know that the sum of the diagrams leads to a net vanishing of the IR poles. As long as all of the IR poles are coming from scaleless integrals then we can be assured that the sum of these integrals wriiten as $\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$ vanishes (see section). The first integral is easily performed by doing the dq_- integral by contours leading to

$$I_{LHS} = \frac{g^2}{8\pi}(\mu/M)^{2\epsilon} C_A O^8(^3S_1) \frac{\theta(k_+)}{(k_+)} (2^{2\epsilon}\pi^\epsilon) \frac{Csc[\pi\epsilon]}{\Gamma[1-\epsilon]} (k_+)^{-2\epsilon}, \tag{275}$$

Exercise 7.8 Show that the results of this diagram are independent of whether or not one chooses to use Wicks theorem or not. Note however that one must be careful with the direction of momentum flow in the gluon propagator due to the fixed ordering.

The diagram on the RHS is given by

$$I_{RHS} = \frac{g^2}{4\pi} C_A O^8(^3S_1)(\mu/M)^{2\epsilon} \frac{(k_+)^{-2\epsilon}\theta(k_+)}{(k_+)} (2^{2\epsilon}\pi^\epsilon) \frac{Csc[\pi\epsilon]}{\Gamma[-\epsilon]}, \tag{276}$$

The results for these integrals must treated as distribution. Using our definition of the + distribution discussed in (214) (trading the log divergence for $\frac{1}{\epsilon}$) we may utilize the result

$$\frac{\theta(k_+)}{(k_+)^{1+2\epsilon}} = -\frac{1}{2\epsilon}\delta(k_+) + \theta(k_+)\left[\frac{1}{(k_+)_+} - 2\epsilon\left(\frac{\log k_+}{k_+}\right)_+ + O(\epsilon^2)\right]. \tag{277}$$

Summing all the diagrams (including mirror partners) leaves for divergent part of the bare one loop structure function

$$f^8_\eta(\hat{k}_+)_0 = \delta(\hat{k}_+)\left(1 - \frac{\alpha_s C_A}{2\pi}\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\log\frac{\mu^2}{M^2}\right)\right) + \frac{\alpha_s C_A}{4\pi}\frac{\theta(\hat{k}_+)}{(\hat{k}_+)_+}(-\frac{4}{\epsilon}) \tag{278}$$

Factors of γ and $Log(4\pi)$ have been absorbed into μ. As previously mentioned at leading log we may match a tree level, so we may ignore the finite parts of this result.

The renormalized and bare structure functions are related via a convolution

$$f^8_\eta(\hat{k}_+)_B = \int_{-\infty}^{\infty} dl^+ Z(\hat{k}_+ - \hat{l}_+) f^8_\eta(\hat{l}_+)_R, \tag{279}$$

with

$$Z(\hat{k}_+ - \hat{l}_+) = \delta(\hat{l}_+ - \hat{k}_+)\left(1 - \frac{\alpha_s(\mu) C_A}{2\pi}\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\log\frac{\mu^2}{M^2}\right)\right) - \frac{\alpha_s(\mu) C_A}{\pi}\frac{\theta(\hat{k}_+ - \hat{l}_+)}{(\hat{k}_+ - \hat{l}_+)_+}(\frac{1}{\epsilon}). \tag{280}$$

This equation should be thought of as a mixing relation with the second term mixing operators with $k_+ > l_+$.

It is important to re-iterate that, unlike the case of the PDF whose discontinuity has support over a finite interval, this structure function, in perturbation theory, has support over the range $0 \leq k_+ \leq \infty$, since in the EFT the heavy quark mass is taken to infinity. Thus the result (280) should be treated as distributions over the interval $(-\infty, \infty)$ not the interval $(0, 1)$, which defines the + distributions. Thus strictly speaking we should not be using the relation (277). While for our leading log resummation this fact will not be relevant, the distinction in the support interval has important consequences. In particular [38], notice that while the moments of the *bare* structure function are related to local operators via

$$\int_{-\infty}^{\infty} (-\hat{k}_+)^n d\hat{k}_+ f^8_B(\hat{k}_+) = \langle \eta \mid \psi^\dagger T^a \chi (in \cdot \hat{D})^n \chi^\dagger T^a \psi \mid \eta \rangle, \tag{281}$$

the same can not be said about the renormalized moments since these moments will diverge and need to be regulated. This is as opposed to the case of DIS where the range of integration is finite (see eq. (217)).

Using (280) we see that the renormalized (heneforth dropping the R subscript) shape function obeys the renormalization group equation

$$\mu \frac{d}{d\mu} f^8(\hat{k}_+) = -\int_{-\infty}^{\infty} \gamma(\hat{k}_+ - \hat{l}_+) f^8(\hat{l}_+) d\hat{l}_+, \tag{282}$$

with the anomalous dimension is given by

$$\gamma(\hat{k}_+ - \hat{l}_+) = -\frac{\alpha_s C_A}{\pi}\left[\delta(\hat{k}_+ - \hat{l}_+) + 2\left[\frac{\theta(\hat{k}_+ - \hat{l}_+)}{\hat{k}_+ - \hat{l}_+}\right]_+\right]. \tag{283}$$

As in the case of the PDF it is simplest to run the system by first deconvolving via a Mellin space transformation of the decay

$$\int_0^{M_\eta/M} z^{N-1}\frac{d\Gamma}{dz} = \int_0^{M_\eta/M} z^{N-1}\frac{(eg_s C^8_{{}^1S_0}(\mu_c))^2}{4\pi m^2}\langle \eta \mid \chi^\dagger T^a \psi \psi^\dagger T^a \chi \mid \eta\rangle \int_z^{M_\eta/M} dx f^8_\eta(1-x,\mu)\tilde{C}(\mu, x-z) \tag{284}$$

Recall that $\mu_c(z) \sim M\sqrt{1-z} \sim \sqrt{M\Lambda_{QCD}}$ and μ is the low scale, the natural scale for the structure function $\mu \sim M(1-z) \sim Mv^2 \sim \Lambda_{QCD}$. The limits on the x integral arise from the fact that the matching coefficients vanish when $(\hat{l}_+ \equiv 1-x) < (\hat{k}_+ \equiv 1-z)$. Roughly we should think of $\hat{k}_+$ as the residual soft momentum thats comes out of the bound state and l_+ as the momentum that is carried into the vertex. $\hat{k}_+/\hat{l}_+$ would be analogous to ξ/x in DIS.

Changing the order of integration and making a change of variables gives

$$\int_0^{M_\eta/M} z^{N-1}\frac{d\Gamma}{dz} = \frac{(eg_s C^8_{{}^1S_0}(\mu_c))^2}{4\pi m^2}\langle \eta \mid \chi^\dagger T^a \psi \psi^\dagger T^a \chi \mid \eta\rangle f^8_\eta(N,\mu)\tilde{C}(\mu, N) \tag{285}$$

where

$$\begin{aligned} f^8_\eta(N,\mu) &= \int_0^{M_\eta/M} f^8(1-y)y^N \\ \tilde{C}(\mu,N) &= \int_0^{M_\eta/M} \tilde{C}(y(1-w))w^{N-1} = \int_0^{M_\eta/M} \tilde{C}((1-w))w^{N-1} + O(1/N), \end{aligned} \tag{286}$$

and in the last line we have used the fact that the end-point limit $z \to 1$ is equivalent to $N \to \infty$. This also allows us to drop the distinction between N and $N-1$. Notice that when we took the moments we held μ_c fixed even though it parametrically depends upon z.

Let us now complete the resummation by solving the RG equation for the coefficients $\tilde{C}(\mu, N)$. Using the relation (282) and making the same change of variables as above we have

$$\mu \frac{d}{d\mu} \int_0^{M_\eta/M} y^N f^8(1-y) = - \int_0^{M_\eta/M} z^N f^8(1-z) dz \int_0^1 w^N \gamma((1-w)) dw + O(1/N). \tag{287}$$

In deriving this result use has been made of the fact that the structure function has no support below $1 - M_\eta/M$ and that the anomalous dimension only mixes down due to the theta function in (283). The moments of the anomalous dimension matrix for the WIlson coefficient are given by

$$\gamma(N, \mu) = \frac{2\alpha_s}{\pi} C_A \log \frac{\mu N}{M}, \tag{288}$$

where we have again only kept the leading log piece. We may use this result to solve the RG equation and sum the logs of the ratio of scales $(\sqrt{(1-z)}M)/((1-z)M)$, or equivalently in moment space $N/\sqrt{N}$.

The LL solution to the RG equation is then given by

$$C^8_{{}^1S_0}(M/N) = C^8_{{}^1S_0}(M/\sqrt{N}) \left(\frac{\alpha_s(M/N)}{\alpha_s(M/\sqrt{N})} \right)^{\frac{8\pi C_A}{\alpha_s(M/N)\beta_0^2}} \tag{289}$$

7. *Putting it all together*

We may then use the result (246) to write down the final resummed result in moment space.

$$\begin{aligned} \Gamma[N] \;=\; & (\frac{g^2 e_b^2}{4\pi^2 M}) \left(\exp\left[- \left(\frac{4\pi C_A}{\beta_0^2 \alpha_s(M)} \right) \left[\log\left(\frac{\alpha_s(\mu_c)}{\alpha_s(M)} \right) - 1 + \frac{\alpha_s(M)}{\alpha_s(\mu_c)} \right] \right] \right)^2 \left(\frac{\alpha_s(M/N)}{\alpha_s(M/\sqrt{N})} \right)^{\frac{8\pi C_A}{\alpha_s(M/N)\beta_0^2}} \times \\ & f^8_\eta(N; \mu = M/N) J(N, \mu_c), \end{aligned}$$

The spectrum in momentum space follows via an inverse Mellin transform [87].

[87] The inverse Mellon transform can be done anaytically [39]. In performing the transform one integrates over regions of N where the resummation is not valid. However, this does not pose a practical problem (see [39] and section (XA) of [40]). For a discussion of reummations directly in momentum space see [41].

The RG scales for the soft and collinear functions have been chosen to be at their respective "natural" scales. As such, they contain no large logs, which have all been captured (to leading log accuracy) in the coefficient. Notice that the intermediate RG scale μ_c is chosen to have the fixed value $\mu_c = M\sqrt{1-z_c}$ where z_c is the cut below which the result is no longer valid. Recall that is this resummed formula is only accurate up to corrections of order $(1-z_c) \sim N^{-1}$. We could keep terms of order $(1-z)$ in the matching but that would not improve the accuracy of the calculation unless we also resummed the leading (double) logs associated with those corrections as well. That is there will be terms of order

$$\delta\Gamma \sim \alpha_s^n (1-z) Log^{2n}(1-z) \tag{290}$$

which we would need to resum that would necessitate looking at power corrections to the soft and jet functions [88].

The effect of the resummation is to cut off the end point spectrum which begins to grow as z approaches one. This effect is called "Sudakov suppression" and is a result of the fact that it is highly improbable that the accelerating color charges wont radiate. Of course the effects of the resummation are intertwined with the non-perturbative structure function[89].

B. Resumming Logs in semi-exclusive Processes

Let us now use our technology to streamline the very old problem of the IR catastrophe in scattering (see for example [7]). Suppose we were interested in the exclusive scattering of an electron off of a potential in electrodynamics. Calculating the one loop correction we would find an IR divergence stemming from the virtual photon exchange graph. We know that this IR divergence would cancel with the real radiation if we were to make the process inclusive, but that's not what we're interested in. Clearly what we are trying to calculate is unphysical and the reason is that any real detector (the detector can at most be the size of the horizon) has a finite energy resolution E_c, and as such, we are forced to calculate semi-inclusively, including all soft radiation up to energy E_c. In so doing we will now have a finite result. However, we know that in the limit where E_c vanishes the amplitude will contain powers of $\log(E_c^2/Q^2)$, which is a large log that needs to be resummed. In fact, we should

[88] Such a study has been performed for B meson decays [36].
[89] For a phenomenological analysis see [32].

expect things to be even worse because in the limit where the electron mass (m) vanishes we expect additional singularities to arise from collinear divergences. In the real world these divergences are again cut-off by the need to include a detector resolution since a detector can not distinguish between the degenerate states one of which being an electron and the other composed of an electron and exactly collinear photon. However, for the sake of simplicity we will avoid this additional complication by working with a finite electron mass which will cut off these logs and replace them with $Log(m^2/Q^2)$ in perturbation theory which we will not bother to resum as we will be taking $m \sim Q$. So we will be working in HQET and not SCET for this demonstration. The two massive lines will be taken to have four-velocities v and v' respectively.

The resummation of these logs was originally accomplished by summing an infinite set of real and virtual diagrams (see [7, 11] and references therein). The leading result, in an expansion in E_c/Q, which we will be trying to reproduce is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\log\left(\frac{m^2}{E_c^2}\right)\left(w\frac{\log[w+\sqrt{w^2-1}]}{\sqrt{w^2-1}}-1\right),\right] \tag{291}$$

where $w = \frac{p \cdot p'}{m^2}$.

Here is where the EFT shows its teeth, as we will we now reach this result by calculating the one loop anomalous dimensions and solving a trivial RG equation.

We begin by determining the relevant scales. The largest scale is the hard scattering scale Q, which we will assume is larger but of order m, and the low scale is the detector resolution E_c. We will assume the following hierarchy

$$Q \sim m \gg E_c. \tag{292}$$

Exercise 7.9 Show that at leading power in E_c/Q the cross section factorizes into the form

$$\Gamma = H(Q,m)S(E_c) \tag{293}$$

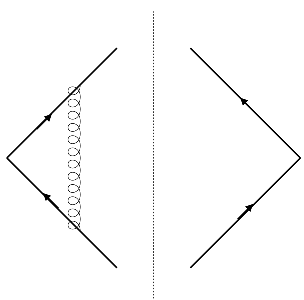
and

$$S(E_c) = \langle 0 \mid S_v(0)S^\dagger_{v'}(0)\sum_X \theta(E_c - E_{X_i}) \mid X\rangle\langle X \mid S_{v'}(0)S^\dagger_v(0) \mid 0\rangle \tag{294}$$

where E_{X_i} is the energy of the $i'th$ photon in the final state.

A loop correction to the Born level cross section will generate logs of the ratio $(m^2, Q^2)/E_c^2$. Thus to resum the logs we will run the soft function from $\mu = E_c$ to $\mu = m$. In this way the hard function will have no large logs, and all the logs will sit in the RG factor, which we will now determine. Of course, we could alternatively run the hard function down to the scale E_c.

To run the soft function we need only calculate virtual graphs since the real graphs are UV finite as we are only integrating over $E < E_c$. The vertex correction is given by



$$= -ig^2(v\cdot v')\int [d^dk]\frac{1}{k^2+i\epsilon}\frac{1}{v\cdot k+i\epsilon}\frac{1}{v'\cdot k+i\epsilon}.$$

For this integral it is useful to use "HQET parameters" via the indentity

$$\frac{1}{a^r b^s} = 2^s\frac{\Gamma[r+s]}{\Gamma[r]\Gamma[s]}\int_0^\infty d\lambda\frac{\lambda^{s-1}}{(a+2b\lambda)^{r+s}}, \tag{295}$$

in conjunction with Feynman parameters. We then extract the UV divergence by regulating the IR. Including the factor of two for the symmetric graph as well as the external line renormalizations leads to the RG equation for the renormalized soft factor

$$\frac{dS_R}{d\log\mu} = \gamma_s S_R = 2\frac{\alpha}{\pi}(wr(w)-1)S_R \tag{296}$$

where

$$r(w) = \frac{1}{\sqrt{w^2-1}}\log(w+\sqrt{w^2-1}). \tag{297}$$

Readers familiar with HQET will recognize γ_s as the "cusp" anomalous dimension [90] relevant for renormalizing the heavy-heavy form factor.

More generally (i.e. for tangents not normed to one) a Wilson lines anomalous dimension depends only upon its "cusp angle" defined via

$$\frac{w}{\sqrt{(v^2)(v')^2}} = \cosh\theta \tag{298}$$

as it is the kink in the Wilson line that induces the UV singularity. Notice that in the limit where either of the lines is light-like an additional singularity arises. Our caclulation is not valid in that limit since we've assumed $m \sim Q$. We will return to this limit in the next section.

Since $E_c < m$, and below the scale m the electron is integrated out, the coupling is fixed, and result of solving the RG equation leads to the result (291). Extending this result to QCD is simple at one loop as it entails the inclusion of a factor of C_F in the anomalous dimensions. Note that in QED this result is exact (i.e. the anomalous dimensions does not get corrected since there are no electron loops and all higher order diagrams are cancelled by the one loop counter-term), but in QCD it is not. In QCD, the exponentiation found here continues to all orders [91] and is called "non-Abelian exponentiation" [42, 43], which can be elegantly proved using the "replica trick" [44]. This result has been extended to the case with an arbitrary number of Wilson line directions [45].

Finally, we may consider the introduction of an additional hierarchy, $Q \gg m$. In this limit we find $w = Q^2/m^2 + O(1)$ and the anomalous dimensions becomes

$$\gamma_s = \frac{2\alpha}{\pi}Log[\frac{Q^2}{m^2}], \tag{299}$$

[90] This is actually twice the cusp since we are looking at the squared matrix element (294).

[91] Exponentiation here has a well defined meaning in that only certain color factors show up in the exponent.

and the result for the cross section becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\log(\frac{m^2}{E_c^2})\log(\frac{Q^2}{m^2})\right]. \tag{300}$$

Note that sometimes the term "cusp anomalous dimensions" is used for the coefficient of the logarithm. We see that we run into a log in the anomalous dimensions just we did in (242). One should worry at this point that we may need to resum logs in the anomalous dimensions. However, as we will prove in section (7 D), there is at most a single logs in γ_s [46].

Of course formally we have no right to take the limit $Q \gg m$ since electrons would become collinear. Thus in this limit we must include the resummation of the Logs of Q/m by running the SCET currents down from Q to m much as we did when we considered the end point spectrum in onium decays. The currents leading anomalous dimension is just the cusp as will be derived in the next section. The equality of the anomalous dimensions at first seems rather odd, given that at the scale Q we match onto a current which will contain both soft and collinear Wilson lines

$$J \sim \bar{\xi}_n W_n S_n^\dagger \gamma_\Gamma^\perp S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n}}, \tag{301}$$

and thus its running will get contributions from both soft and collinear graphs. Whereas we were able to get the right answer (at leading double log accuracy) directly from just the soft Wilson line (294). However, as we will see that the soft piece of the calculation in the high energy theory is sufficient to reproduce the leading double log result for slightly subtle reasons.

Running the currents from Q to m, matching at the scale m onto the soft function and running down to the scale E_c simply reproduces (neglecting the running of the coupling) (300). Note that this procedure will only work to leading (double) log accuracy. To properly set up the EFT for the case of massive fermions with a hierarchy $Q \gg m \gg E_c$ we have to introduce more modes. This scenario arises in when studying top quark production [47].

We may ask what happens as we take $m \sim E_c$? In this limit we find a complication. Namely, that in addition to an energy resolution in the detector, we must also worry about angular resolution. That is, when the electron mass goes to zero the photon can split into a collinear electron positron pair which then radiates collinear photons and so on, forming a jet. So the notion of an exclusive scattering cross section makes sense only when we coarse grain over angular resolutions, and ask questions about the number of jets in the final state instead of the number of electrons. These calculations touch upon the field of jet physics, for which, SCET is a very sharp tool (for a discussion see [48],[49]).

Note that we have come back full circle to the KLN theorem. We always run into problems when we consider S matrix elements of (nearly) massless particles because the asymptotic states manifest a degeneracy and we must question whether or not our observables make physical sense. In practice we handle this by coarse graining and not asking questions we can't answer. Formally there is an alternative way to approach the problem, which is to redefine our asymptotic states to *include* soft radiation[50] forming coherent states. However, from a calculational standpoint this method becomes challenging.

C. The role of the zero bin in SCETI

In the chapter on NRQCD we saw that the zero bin subtraction [51] was essential to make certain integrals well defined, and to convert IR divergences into UV divergences, the so-called "pull-up" mechanism. In NRQCD we wanted to make sure that all the IR divergences lived in the U-soft sector as the other modes are short distance modes. This is similarly true in SCETI.

This not to say that the results from the previous section are wrong, however not enough care was given to interpreting the intermediate stages of the result. In particular, we need to make sure when we are using dimensional regularization that we are distinguishing between UV and IR poles. More specifically, the collinear graph should not have any soft IR divergences. Note that the off-shellness cuts off the collinear divergence, but the soft divergence is being regulated by the dimensional continuation. The soft momentum region in those graphs should be accounted for in the U-soft diagrams. That is, when we sum over the discrete labels, the "zero bin" should be excluded, since that bin is accounted for by U-soft diagrams. Of course in the end, the sum is changed into an integral, so what this really means is that we need to subtract the U-soft limit of the collinear diagrams. Upon doing so we should find that the poles in epsilon arising from the soft region of phase space are no longer present.

To see this we must return to the collinear integral (233) and separate the IR and UV poles. The integral in the transverese momentum is pure UV pole, but the result of the integral regulates the IR pole as well by leaving over a factor of $(n \cdot k)^{-\epsilon}$. The final integral over $n \cdot k$ is IR divergent. We keep track of this fact by changing the exponent to ϵ_{IR}. Expanding out the result, and keeping only the pole terms gives

$$iA_C = \frac{iC_A g^3}{4\pi^2}\left(\frac{1}{\epsilon_{UV}\epsilon_{IR}} + \frac{2}{\epsilon_{UV}} + \log[\frac{\mu^2}{-p^2}]\frac{1}{\epsilon_{IR}}\right) \tag{302}$$

Consider the US (zerobin) limits of the collinear integral (233)

$$I_c^{zb} = \int \frac{[dn \cdot k][d\bar{n} \cdot k][d^{2-2\epsilon}k_\perp]}{(n \cdot k\bar{n} \cdot k - k_\perp^2 + i\epsilon)(p^2 - n \cdot p\bar{n} \cdot k - n \cdot k\bar{n} \cdot p + i\epsilon)}\left[2\frac{n \cdot p}{n \cdot k}\right]. \tag{303}$$

The integrals are nearly identical except that now the $n \cdot k$ integral is unbounded from above leaving a scaleless integral

$$1/(n \cdot k)^{1+\epsilon} = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}. \tag{304}$$

Using this result and pulling out the pole piece leaves the collinear zero bin

$$iA_C^{zb} = \frac{iC_A g^3}{4\pi^2}\left(-\frac{1}{\epsilon_{UV}^2} + \frac{1}{\epsilon_{UV}\epsilon_{IR}} + \log[\frac{\mu^2}{-p^2}]\frac{1}{\epsilon_{IR}} - \log[\frac{\mu^2}{-p^2}]\frac{1}{\epsilon_{UV}}\right) \tag{305}$$

We see that upon subtraction the zero bin eliminates the soft divergences from the collinear contribution. Furthermore, since there is no zero bin for the soft contribution (no need to worry about double counting[92], the sum of the UV divergences in the subtracted collinear contribution should be the same as the pole piece of (311). That is to say, if you ignored the zero bin and naively set $\epsilon_{IR} = \epsilon_{UV}$, as we did in the previous section, we would still get the correct answer. In this case the use of the zero bin was not terribly illuminating. However, there are cases where, despite the vanishing of the zero bin, understanding its effects can streamline calculations. Such is the case in NRGR at higher orders [52]. In glauber theory, to be discussed below, ignoring the zero-bin would leave the theory completely muddled [53]. Finally, as we saw in NRQCD, the zero bin is sometimes necessary to make sense of

[92] Collinear modes can't overlap with soft modes because the soft integrals are independent of the large scales/rapidities. There are exceptions to this rule when there are soft momenta in external states (see [51]) .

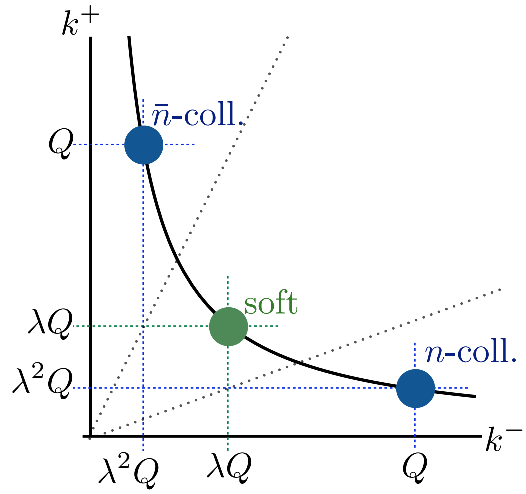


FIG. 19. The mass-shell hyperbolae showing the distinction between the different sectors. The separation between soft and collinear modes is arbitrary and leads to rapidity divergences. The soft sector has two distinct rapidity (UV) divergences that must cancel with rapidity (IR) divergences arising from the collinear sector. λ is the ratio of Q to the IR scale.

integrals.

D. SCETII

In SCETII things get a little more complicated. This theory arises when the requisite fields are such that the "soft" and the collinear modes have the same invariant mass. Here the term soft is being used in a generic sense. Modes which have invariant masses of the same order as the collinear modes will be called soft, as opposed to U-soft as in the SCETI case. This equality in invariant masses introduces new intricacies into the theory as the distinction between soft and collinear modes now becomes frame dependent. These two modes sit on the same invariant mass hyperbola as shown in figure (17), and we must introduce some factorization parameter to distinguish them. Usually we factorize using the invariant mass, here instead we use the rapidity. In doing so we must break boost invariance, by construction. Of course the symmetry will be restored once we include the contributions from all sectors since the result will be independent of the rapidity factorization scale.

An important distinction between SCETI and SCETII is that in the latter the collinear fields do not transform under soft gauge transformations, as such a transformation would inject an order λ momentum which would throw the collinear line off shell just as a soft emission off of a collinear line would do. However, as the reader should verify, soft emission off of a collinear line is leading order in the power counting and we must resum these emissions when we match. As one would expect the resummation leads to soft Wilson lines. However, now, since the collinear fields no longer transform under soft gauge transformation, we may no longer fix the place of the Wilson lines by imposing gauge invariance. Instead, we may fix the position of the Wilson lines (for most cases) by using what is known as the "method of descent" which will be discussed below.

1. *Resummations in SCETII: The Massive Sudakov Form Factor*

The canonical SCETII scenario is the massive Sudakov form factor which is the process of the scattering[93] of a high energy collinear particle via a hard interaction with an external potential in a gauge theory with massive bosons [94] . We will assume that the hard scattering scale Q is parametrically larger than the mass of the gauge boson M and that these are the only two scales in the problem. That is, we will assume that the coupling freezes below the scale M so that there is no dynamically generated scales below M. How do we argue that this process is properly described by SCETII? We begin with the gauge boson which has both soft and collinear components. Since it's massive we have no choice but to make its soft momentum scaling $k^\mu \sim (M, M, M)$. As to the collinear momenta, the only IR scale is M, so the invariant mass (in the theory above M) is the next lowest scale i.e. $p^2 \sim M^2$. We see that in general when there are only two scales in a theory there is no choice but SCETII.

The Sudakov form factor is defined as in terms of the Fourier transform of the current transition matrix element

$$\int d^4x e^{-iQ\cdot x}\langle p_n|J_\mu(x)|p_{\bar{n}}\rangle \equiv \bar{u}(p_n)\gamma_\mu^\perp u(p_{\bar{n}})\, F(Q^2, M^2)\delta^{(4)}(p_n - p_{\bar{n}} - Q) \tag{306}$$

where we work in the "brick wall" frame such that the jet directions are back-to-back.

At that scale Q we match the current onto

$$J^\mu_{SCET} = \bar{\xi}_n W_n \gamma_\mu^\perp C(n\cdot\mathcal{P}, \bar{n}\cdot\mathcal{P}) W^\dagger_{\bar{n}}\xi_{\bar{n}}. \tag{307}$$

We know that the soft emissions are order one and that they should be resummed in the

[93] More generally one may consider the space-like ($Q^2 < 0$) or time-like ($Q^2 > 0$) form factor.

[94] We will assume that this theory is the low energy limit of a theory in the Higgs phase at scales below the Higgs mass (m_H). So the hard scattering scale obeys $Q < m_H$.

matching. We may fix their positions by using the aforementioned "method of descent" where at the scale Q we match onto SCETI instead of SCETII. That is, we take the virtuality of the collinear modes to be of order $p^2 \sim MQ$ instead of M^2. Why would we do this? We are running down from Q to M in virtuality so there is no reason we can't break up the process into two steps, by introducing the intermediate scale MQ. Note that in this particular problem this intermediate scale is not physical in the sense that it does not show up anywhere in the external lines[95]. Thus we normally would simply "run through" this scale without thought. However, we are certainly free to stop and this scale, match and then run down with no change in the final results. Thinking in this way, when we match at Q what we called softs are now US since now $\lambda = \sqrt{M/Q}$. Thus, as per usual in SCETI the US couplings in the Lagrangian can be removed via a BPS field redefinition. So when we match the current becomes

$$J_\mu \; = \; \bar{\xi}_n W_n Y_n^\dagger \gamma_\mu^\perp C(n \cdot \mathcal{P}, \bar{n} \cdot \mathcal{P}) Y_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n}}. \tag{308}$$

At the scale $\sqrt{MQ}$ we reduce the virtuality of the collinear lines to M^2, i.e. we lower λ to M/Q. Note the distinction with the true SCETI theory (end-point decays) where there were no collinear modes below the intermediate scale. When we lower λ from $\sqrt{M/Q}$ to M/Q the soft Wilson lines match onto US, with their positions are fixed. Thus the SCET current becomes

$$J_\mu \; = \; \bar{\xi}_n W_n S_n^\dagger \gamma_\mu^\perp C(n \cdot \mathcal{P}, \bar{n} \cdot \mathcal{P}) S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n}}. \tag{309}$$

In this simple case we might have guessed the position of the Wilson lines, but this method saved the work of doing the matching. We will see, when we discuss Glauber modes, that sometimes this method lacks the necessary discriminating power to fix the Wilson line positions. Notice that despite, the fact that the collinear Wilson lines don't transform under soft transformations, the operator is still soft gauge invariant since the soft Wilson lines transform as

$$S_{n,\bar{n}} \rightarrow U_s S_{n,\bar{n}}. \tag{310}$$

[95] The reader may be puzzled by why the same can not be said of the soft modes in NRQCD. But one must recall that the soft scale mv does exist in the bound state as the inverse Bohr radius.

Why is this true for soft and not ultra softs? Because the soft momentum swamps the residual momentum of the collinear mode. Recall that the collinear field coordinate dependent is purely residual, i.e. its conjugate momenutm scales as $O(\lambda^2)$.

We now factorize the current as follows

$$F(Q^2, M^2) = H(Q,\mu)S(M,\mu)J_n(Q,M,\mu)J_{\bar{n}}(Q,M,\mu) \tag{311}$$

where

$$\begin{aligned} S &= \langle 0 \mid S_n^\dagger S_{\bar{n}} \mid 0\rangle \\ J_n &= \langle p_n|\bar{\xi}_n W_n|0\rangle \\ J_{\bar{n}} &= \langle 0|\bar{W}_{\bar{n}}^\dagger \xi_{\bar{n}} \mid p_{\bar{n}}\rangle. \end{aligned} \tag{312}$$

and H is the matching coefficient. Notice that the collinear J functions depends upon Q even though their invariant masses scale as M, because, as shall see, they are sensitive to the energy of the jet.

Consider the one loop vertex correction to the soft function.

$$I_S = \int [d^n k] \frac{1}{(k^2 - M^2)} \frac{1}{(-n\cdot k + i\epsilon)} \frac{1}{(-\bar{n}\cdot k + i\epsilon)}, \tag{313}$$

integrating over $k_\perp$.

$$I_S \sim \int [d^2 k](n\cdot k\, \bar{n}\cdot k - M^2)^{-2\epsilon} \frac{1}{(-n\cdot k + i\epsilon)} \frac{1}{(-\bar{n}\cdot k + i\epsilon)} \tag{314}$$

This integral is not regulated by dim. reg. To understand it let's consider its pathological regions. The integral diverges when the rapidity $(n \cdot k/\bar{n}\cdot k)$ approaches infinity or zero. In these limits the soft integral overlaps with the two collinear rapidity regions. We need some way to regulate these rapidity divergences. Before doing so let us consider the collinear contribution, as it too should have a rapidity divergence. In fact, we know that the rapidity divergences in the soft sector must cancel those in the collinear sectors. After all, these divergences arise as an artifact of our factorization of modes with the same invariant mass. One could ask why the same argument does not apply when considering the additional UV divergences which exist in the EFT relative to the full theory? The reason is that in the

EFT we have removed degrees of freedom which live above the factorization scale, whereas in the case of rapidity divergences the full and effective theory both posses the same range in rapidity.

Let us see how this cancellation occurs at one loop. The contribution from one collinear sector is given by

$$I_n = \int [d^n k] \frac{1}{(k^2 - M^2)} \frac{1}{(k^2 - n \cdot k\, \bar{n} \cdot p_1 + i\epsilon)} \frac{1}{(-\bar{n} \cdot k + i\epsilon)}. \tag{315}$$

We see that it only has divergences associated with the limit where $(n \cdot k/\bar{n} \cdot k)$ approaches infinity, and similarly with $(n \to \bar{n})$ for the $I_{\bar{n}}$ collinear integral. Since there is only one border between a collinear sector and the neighboring soft sector this is exactly what we would expect. To explicitly see the cancellation we need to introduce a rapidity regulator.

There are multiple ways in which to regulate the rapidity divergences, and the formalism developed here can be applied using any sensible choice, such as the delta regulator [54]. Here we will concentrate on the regularization introduced in [55, 56], which is closely related to dimensional regularization and is implemented by modifying the momentum space Wilson lines in the following fashion.

$$W_n = \sum_{\text{perms}} \exp\left[-\frac{g w^2}{\bar{n} \cdot \mathcal{P}} \frac{|\, \bar{n} \cdot \mathcal{P}_g \,|^{-\eta}}{\nu^{-\eta}} \bar{n} \cdot A_n \right] \tag{316}$$

$$S_n = \sum_{\text{perms}} \exp\left[-\frac{g w}{n \cdot \mathcal{P}} \frac{|\, 2\mathcal{P}_3 \,|^{-\eta/2}}{\nu^{-\eta/2}} n \cdot A_s \right] \tag{317}$$

We have introduced a new dimensionful parameter ν which will play the role of an effective rapidity cut-off. The soft Wilson line is regulated using $|2\mathcal{P}_3|$ since in the collinear limit(s) it becomes $|\bar{n}(n) \cdot \mathcal{P}|$. Note the differing powers of η in the soft and collinear Wilson lines. The appropriate power of η is fixed by ensuring that the rapidity divergences cancel to all orders which we shall show below. Alternatively, the power can be fixed by regulating the full theory diagram and taking limits of the integrand. The relative factor of two comes from that fact that for a given gluon line in the full theory there are two soft eikonal vertices (connecting the two eikonal lines) relative to the one collinear eikonal vertex. A book-keeping parameter w, will play a role when we derive RG equations, has been introduced for convenience, and will eventually be set to one. Here we will calculate to one loop. At higher orders one has to modify the label operator in the Wilson line to ensure gauge invariance and the readers interested in details should consult [56]. There are modifications of this regulator that have been introduced [57] that make higher loop integrals more tractable.

Regulated in this way, the effective theory will have divergences when η or ϵ approach zero. The order of the limits is crucial to sensibly renormalize the theory. Given our physical arguments regarding the nature of the rapidity divergences, the proper order of limits must be: $\eta \to 0$, then $\epsilon \to 0$ with $\eta/\epsilon^n \to 0$ for all $n > 0$. The physical reason for this ordering is clear since we must remain on the invariant mass hyperbola when we take the rapidity cut-off to its limit. To see how this works in practice let us evaluate the integrals I_S and I_n using this regulator.

The I_S integral is most simply evaluated by first doing the k_0 integral by contours. The result, after repristinating the expression with the coupling, group theory factor and the relevant numerator for the Sudakov form factor, in Feynman gauge, is given by [96]

$$I_S = -g^2 C_F (e^{\gamma_E \epsilon} 2^{-\eta-2} \pi^{-5/2}) \left(\frac{\mu}{M}\right)^{2\epsilon} \left(\frac{\nu}{M}\right)^{\eta} \frac{\Gamma(1/2 - \eta/2)\Gamma(\epsilon + \eta/2)}{\eta} \tag{318}$$

Expanding first in η and then in ϵ we find

$$I_S = g^2 C_F \left[-\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon)\left(\frac{\mu}{M}\right)^{2\epsilon}}{4\pi^2 \eta} + \frac{1}{4\pi^2}\left(\frac{\ln(\frac{\mu}{\nu})}{\epsilon} + \ln^2(\frac{\mu}{M}) - 2\ln(\frac{\mu}{M})\ln(\frac{\nu}{M}) + \frac{1}{2\epsilon^2}\right) - \frac{1}{96}\right] \tag{319}$$

Similarly, the collinear integral I_n is given by

$$I_n = g^2 C_F \left[\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon)\left(\frac{\mu}{M}\right)^{2\epsilon}}{8\pi^2 \eta} + \frac{1}{4\pi^2}\left(\ln(\frac{\mu}{M})\ln(\frac{\nu}{\bar{n}\cdot p_1}) + \ln(\frac{\mu}{M}) + \frac{1}{2\epsilon}\left(1 + \ln(\frac{\nu}{\bar{n}\cdot p_1})\right) + \frac{1}{2}\right) - \frac{1}{48}\right], \tag{320}$$

and $I_{\bar{n}}$ by replacing $\bar{n}\cdot p_1$ with $n \cdot p_2$. Summing the sectors we find

$$I_S + I_{\bar{n}} + I_n = g^2 C_F \left[\frac{1}{4\pi^2}\left(\frac{1}{2\epsilon^2} + \frac{\ln(\frac{\mu}{Q})}{\epsilon} + \frac{1}{\epsilon} + \ln^2(\frac{\mu}{M}) + 2\ln(\frac{\mu}{M}) + 2\ln\frac{M}{\mu}\ln\frac{Q}{M} + 1\right) - \frac{5}{96}\right], \tag{321}$$

where we have used $\bar{n}\cdot p_1 = n\cdot p_2 = Q$. We see that in the sum the η (rapidity) divergences vanish, there is no dependence on the scale ν and the answer is boost invariant.

---

[96] w has been set to one, and is utilized below when we derive the renormalization group equation. We have also absorbed the $\overline{MS}$ factor into μ to simplify the expressions.

Let us pause a moment to study the logs which appear in the various sectors. The soft sector integrals only involve one scale, M and thus the rapidity logs and UV (invariant mass) logs must be of the form $Log(M/\mu)$ and $Log(M/\nu)$ respectively. This fixes the "natural" (the scale at which there is no large log) rapidity and mass scales, to both be M. This is as opposed to the collinear sectors whose rapidity scale is Q and whose mass scale is M.

Finally, there is something else that is odd about this result. The EFT result (i.e. not including the hard pieces which are not shown here) depends upon Q, the hard scale, which we had purportedly integrated out. So how did this scale show up in the EFT? We can track that down by looking at the individual pieces, none of which depend upon Q. However when we combine all the pieces the light cone momenta from the two jets combined into the scale Q in the log. Normally, we would control the logs of this scale by choosing $\mu = M$ and then running the hard matching coefficient down to this scale which would then sum all of the logs of the ration Q/M. But now this will no longer work since choosing $\mu = M$ in the EFT does not get rid of all the large logs. So we need more than just the usual RG to control the logs in this problem.

2. *The Rapidity Renormalization Group*

We can now write down a renormalization group equation in a rather straightforward manner. We begin by examining the Sudakov form factor of the space-like current in terms of the SCETII fields,

$$F(Q) \ = \ H(Q^2,\mu)J_n(M,\mu,\nu,Q)J_{\bar{n}}(M,\mu,\nu,Q)S(M,\mu,\nu,M). \tag{322}$$

Note that we have now added the dependence upon the rapidity scale ν which was absent in (235).

The one loop values of matrix elements $J_n, J_{\bar{n}}, S$ defined in (312), are given by (319,320). The renormalizaton group follows from the set of equations

$$\frac{d}{d\ln[\mu]}(J_n, S)^{\text{bare}} = \frac{d}{d\ln[\nu]}(J_n, S)^{\text{bare}} = 0. \tag{323}$$

Moreover the independence of μ and ν leads to

$$[\frac{d}{d\ln[\mu]},\frac{d}{d\ln[\nu]}]=0\,, \tag{324}$$

which is of course true for any observable not just the Sudakov form factor.

Defining the anomalous dimension under μ and ν variations as (γ_μ,γ_ν) respectively, such that

$$\gamma_\mu^{n,S}=-Z_{n,S}^{-1}(\frac{\partial}{\partial\ln[\mu]}+\beta\frac{\partial}{\partial g})Z_{n,S}\,, \tag{325}$$

$$\gamma_\nu^{n,S}=-Z_{n,S}^{-1}\frac{\partial}{\partial\ln\nu}Z_{n,S}\,, \tag{326}$$

equation (324) imposes the constraint

$$(\frac{\partial}{\partial\ln[\mu]}+\beta\frac{\partial}{\partial g})\gamma_\nu=\frac{d}{d\ln[\nu]}\gamma_\mu\ =\mathbb{Z}\Gamma_{\rm cusp}\,, \tag{327}$$

which holds for any observable of interest. $\Gamma_{\rm cusp}$ is the aforementioned cusp anomalous dimension. $\mathbb{Z}$ is an integer whose value depends upon whether we are considering an amplitude or the square of an amplitude. For the Sudakov form factor $\mathbb{Z}$ is either 1 or 2 (see below). The last equality comes from the consistency of μ-anomalous dimension with the hard anomalous dimension which is linear in the logarithm with coefficient $\Gamma_{\rm cusp}$. The universal relation between the collinear c and soft S anomalous dimension

$$-2\mathbb{Z}_c=\mathbb{Z}_S \tag{328}$$

follows automatically from the ν independence of the hard function, as will be discussed below.

Let us now apply the RRG to the Sudakov case we studied above. Since our regulator allows us to define the jet and soft functions independently we may renormalize them in standard fashion by absorbing $\frac{1}{\epsilon}$ and $\frac{1}{\eta}$ divergences in the renormalization constants, and then run renormalized quantities individually. We define the renormalization factor $Z_n,\ Z_S$ via

$$J_n^R=Z_\psi^{1/2}Z_n^{-1}J_n^B\quad S^R=Z_S^{-1}S^B \tag{329}$$

where I^B corresponds to bare quantities and I^R to renormalized. Then using our result from above, at one loop we have

$$\begin{aligned} Z_S &= 1 - \frac{g(\mu)^2 w^2 C_F}{4\pi^2}\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{M}\right)^{2\epsilon}}{\eta} - \frac{1}{2\epsilon^2} - \frac{\ln\frac{\mu}{\nu}}{\epsilon}\right], \\ Z_n &= 1 + \frac{g(\mu)^2 w^2 C_F}{4\pi^2}\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{M}\right)^{2\epsilon}}{2\eta} + \frac{1}{2\epsilon}\left(1+\ln\frac{\nu}{\bar{n}\cdot p_1}\right)\right], \end{aligned} \tag{330}$$

where Z_ψ is wave function renormalization which is the same as in full QCD.

$$Z_\psi = 1 - \frac{g(\mu)^2 C_F}{16\pi^2\epsilon}. \tag{331}$$

The μ anomalous dimensions are given by

$$\begin{aligned} \gamma_\mu^n &= \frac{g^2(\mu)C_F}{4\pi^2}\left(\frac{3}{4}+\ln\frac{\nu}{\bar{n}\cdot p_1}\right), \\ \gamma_\mu^{\bar{n}} &= \frac{g^2(\mu)C_F}{4\pi^2}\left(\frac{3}{4}+\ln\frac{\nu}{n\cdot p_2}\right), \\ \gamma_\mu^S &= \frac{g^2(\mu)C_F}{4\pi^2}\ln\frac{\mu^2}{\nu^2}. \end{aligned} \tag{332}$$

As a consistency check see that

$$\gamma_\mu^n + \gamma_\mu^{\bar{n}} + \gamma_\mu^S = -\gamma_H = \frac{g^2(\mu)C_F}{4\pi^2}\left(\ln\frac{\mu^2}{Q^2}+\frac{3}{2}\right), \tag{333}$$

where γ_H is the anomalous dimension of the hard matching coefficient. As a side note we see that the leading piece of the anomalous dimension for the running of the current could have been read off directly from the soft function as discussed below eq.(301). The collinear contribution serves to remove the rapidity regulator. Finally, we can also see from this result that the cusp anomalous dimensions is at most linear in logarithms [97] , since higher powers in $Log(\mu/\nu)$ would have cross terms that could not be cancelled by the collinear contribution.

Both the large logarithms, due to large invariant mass ratio and large rapidity ratio, can

[97] This statement excludes logs that can be absorbed into the renormalized coupling constant.

be resummed by the RG equations

$$\mu \frac{d}{d\mu}(J_n, S) = \gamma_\mu^{n,S}(J_n, S)\,,$$
$$\nu \frac{d}{d\nu}(J_n, S) = \gamma_\nu^{n,S}(J_n, S)\,. \tag{334}$$

Using the fact that bookkeeping parameter (which gets set to one at the end of the calculation) obeys [98]

$$\nu \frac{dw}{d\nu} = -\frac{\eta}{2} w. \tag{335}$$

The result for the rapidity anomalous dimension is

$$\begin{aligned}\gamma_\nu^S &= \frac{\alpha}{2\pi} C_F Log(M^2/\mu^2)\\ \gamma_\nu^n &= -\frac{\alpha}{\pi} C_F Log(M^2/\mu^2).\end{aligned} \tag{336}$$

The relation (327) guarantees that the μ and ν evolutions commute, hence, the evolution in μ-ν plane is path independent. However, care must be taken when solving the ν-RG equation. γ_ν contains terms of form $\alpha_s^n(\mu)\ln^m(\mu/M)$ with $m \leq n$. For instance, one can see from Fig. (20) that the one loop result will be multiplied by a series of logarithms of the form $\sum_n [\beta_0 \alpha_s \ln(\mu/M)]^n$. These logarithms can be large if $\mu \gg M$, for example, and would require resummation. This is easily obtained by solving the consistency relation (327) up to the required order in perturbation theory,

$$\begin{aligned}\gamma_\nu &= \int^{\ln\mu} d\ln(\mu') \frac{d}{d\ln(\nu)} \gamma_\mu(\mu') + \text{const.}\\ &\propto \int^{\ln\mu} d\ln(\mu') \Gamma_{\text{cusp}}(\mu') + \text{const.}\,,\end{aligned} \tag{337}$$

where integration constant is fixed by the fixed order calculation of anomalous dimension and corresponds to its non-cusp piece. Eqn. (337) completely fixes the logarithmic (μ) structure of γ_ν to all orders in perturbation theory when expanded in $\alpha_s(\mu)$. If we had calculated γ_ν to higher orders we would see these logarithms explicitly. Thus, it constitutes a check on the higher order calculations. In its integrated form γ_ν resums the set of diagrams which renormalize the coupling, which in the Abelian case, arise from the bubble chain shown in

[98] This is analogue of $\mu\frac{d}{d\mu}(\mu^{-2\epsilon}g) = -2\epsilon\mu^{-2\epsilon}g + ...$ Only here the coupling does not depend upon ν.

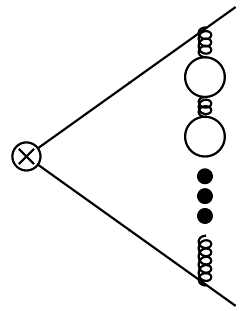


FIG. 20. Coupling renormalization (Abelian) contributes to γ_ν and is missed in the fixed order one loop result.

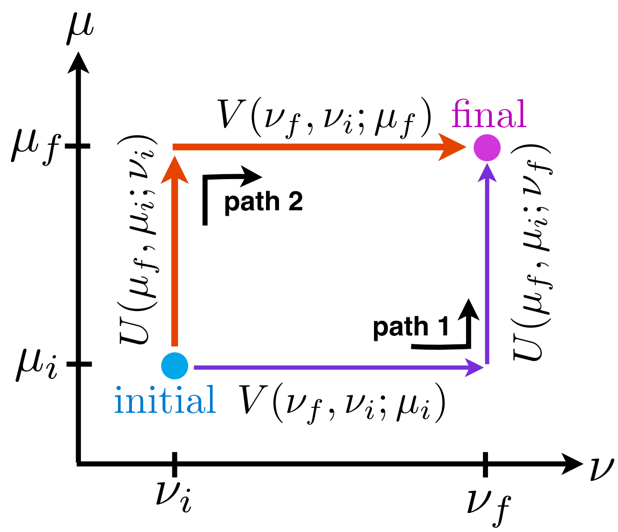


FIG. 21. Two alternate paths are shown for evolution in μ-ν plane. Due to independence of μ and ν scales evolution along either path will yield the same result.

figure 20, thus taking into account the running of α_s. The fixed order form of γ_ν suffices when evolution is done along path 1 shown in figure (21) with $\mu_i \sim \nu_i \sim M \ll \mu_f \sim \nu_f$. However, the integrated form (337) is required when evolution is done along path 2. Since $\mu_f \gg M$ there are large logarithms in γ_ν that require resummation in addition to the rapidity logs. In figure (21), U and V are the evolution factors in μ and ν respectively and μ_i, ν_i are the scales for the initial conditions. The notation $U(\mu_f, \mu_i; \nu_a)$ implies running μ from μ_i to μ_f at fixed $\nu = \nu_a$; similarly for $V(\nu_f, \nu_i; \mu_a)$. Along path 1, we have chosen to run first in ν and then in μ. Path 2 shows the alternate choice and should yield the same result, thus

$$V(\nu_f, \nu_i; \mu_f)U(\mu_f, \mu_i; \nu_i) = U(\mu_f, \mu_i; \nu_f)V(\nu_f, \nu_i; \mu_i)\,. \tag{338}$$

To ensure this in practice, we must use the resummed form of γ_ν when calculating

$V(\nu_f, \nu_i; \mu_f)$.

Notice that these anomalous dimensions depend upon the "low" energy parameter, M, which normally would, and should, not show up in the expression for an anomalous dimension. However, we must recall as far as the rapidity divergences are concerned M is not a low energy parameter, but just the invariant mass of the hyperbola along which the rapidity renormalization group flows.

To sum the large logarithms we first identify the natural scales for the Hard, Soft and Jet Function which are given by $(\mu_H), (\mu_S, \nu_S)$ and (μ_J, ν_J) respectively. Numerically they can be read off from (319,320)

$$\mu_H \sim Q, \mu_S \sim \nu_S \sim \mu_J \sim M, \nu_J \sim Q. \tag{339}$$

To eliminate the large logarithms we may run in both μ and ν to some fixed scale, while evaluating the fixed order functions at their natural scales. That is, we may write

$$\begin{aligned} S(\mu,\nu) &= V_S(\nu,\nu_S;\mu)(U_S(\mu,\mu_S;\nu_S)S(\mu_S,\nu_S)) \\ J_n(\mu,\nu) &= V_J(\nu,\nu_J;\mu)(U_J(\mu,\mu_J;\nu_J)J_n(\mu_J,\nu_J)) \\ H(\mu) &= H(\mu_H)U(\mu,\mu_H)\,, \end{aligned} \tag{340}$$

where $U_{n,S}$ and $V_{n,S}$ are respectively μ and ν evolution factors for jet and soft functions. In (340) we have chosen to run first in μ and then in ν. We could equally well have switched the order leading to the same result. Note that in the ordering of eqn. (340) we are required to use the integrated form of γ_ν of eqn. (337) in order to resum all the large logs due to the running coupling. We get,

$$U_S(\mu,\mu_S;\nu_S) = exp\left[-\frac{8\pi C_F}{\beta_0^2}\left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu_S)} - \frac{1}{\alpha(\nu_S)}\ln\frac{\alpha(\mu)}{\alpha(\mu_S)}\right)\right] \tag{341}$$

$$V_S(\nu,\nu_S;\mu) = exp\left[\frac{2C_F}{\beta_0}\ln\left(\frac{\alpha(\mu)}{\alpha(M)}\right)\ln\left(\frac{\nu^2}{\nu_S^2}\right)\right] \tag{342}$$

$$U_J(\mu,\mu_J;\nu_J) = exp\left[-\frac{2C_F}{\beta_0}\left(\frac{3}{4}+\frac{1}{2}\ln\left(\frac{\nu_J^2}{Q^2}\right)\right)\ln\frac{\alpha(\mu)}{\alpha(\mu_J)}\right] \tag{343}$$

$$V_J(\nu,\nu_J;\mu) = exp\left[-\frac{C_F}{\beta_0}\ln\left(\frac{\alpha(\mu)}{\alpha(M)}\right)\ln\left(\frac{\nu^2}{\nu_J^2}\right)\right] \tag{344}$$

$$U_H(\mu,\mu_H) = exp\left[-\frac{8\pi C_F}{\beta_0^2}\left(\frac{1}{\alpha(\mu_H)} - \frac{1}{\alpha(\mu)} - \frac{1}{\alpha(Q)}\ln\frac{\alpha(\mu)}{\alpha(\mu_H)}\right)\right] \tag{345}$$

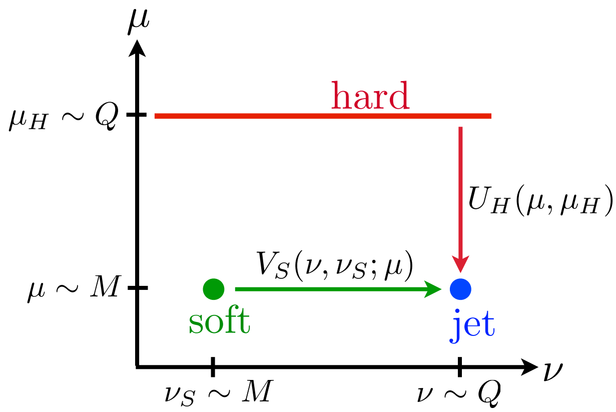


FIG. 22. Simplest running strategy to resum all the large logarithms in the Sudakov Form Factor, following Refs. [55, 56].

with

$$S(\mu_S, \nu_S) = 1 + \frac{\alpha(\mu_S) C_F}{\pi} \left[\ln^2(\frac{\mu_S}{M}) - 2\ln(\frac{\mu_S}{M})\ln(\frac{\nu_S}{M}) - \frac{\pi^2}{24} \right] \quad (346)$$

$$J_n(\mu_J, \nu_J) = 1 + \frac{\alpha(\mu_J) C_F}{\pi} \left[\ln(\frac{\mu_J}{M})\ln(\frac{\nu_J}{n \cdot p_1}) + \frac{3}{4}\ln(\frac{\mu_J}{M}) - \frac{\pi^2}{12} + \frac{1}{2} \right]. \quad (347)$$

Using relations (341) to (344) we can explicitly verify the commutation relation (338) at the order we are working. Equations (340) to (346) give the resummation for the most general choice of scales μ and ν. However, in order to resum all the logarithms, the most convenient choice of scales is $\mu = \mu_J = \mu_S \sim M$ and $\nu = \nu_J \sim Q$. Running with this choice of scales only requires running the hard function in μ and soft function in ν to the natural scales of the jet function. This strategy is shown in figure (22). With this strategy, it is not required to use the integrated form (337) and the fixed order form of γ_ν suffices.

The physics of the RRG flow can be understood from figure (23). A change in the scale ν corresponds to a flow between the soft and collinear regions. The natural scale for the soft function is $n \cdot k \sim \bar{n} \cdot k \sim M$ whereas the collinear functions sit at the scale Q. To sum the logarithms we may slide the cut-off(s) of the soft function up the hyperbola, such that the scale ν minimizes the logarithms in the collinear sectors.

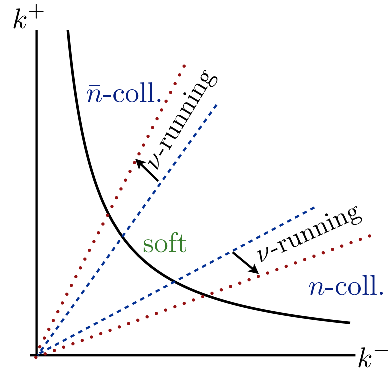


FIG. 23. Running in ν corresponds to flow along the mass-shell hyperbola.

3. The Role of the Zero Bin in SCETII

Notiice that in resumming the logs in the massive Sudakov form factor we made no mention of zero bins. As previously emphasized, the key difference between the case of SCETI this case and is that in SCETII the soft and collinear modes have the same invariant mass. This fact is what forced us into introducing a rapidity regulator and factorization scale. Note that this is not to say that rapidity divergent integrals can't show up in SCETI. Indeed, we saw in the case of the running of the PDF that they do. However, in SCETI these divergences cancel sector by sector, as they must, since there is no need for a rapidity factorization scale.

In SCETII the role of the subtraction is to ensure that the rapidity cutoff takes its appropriate value in each sector. When using the η regulator all of the subtractions are scaleless and only play a formal role. But with other rapidity regulators the subtractions can be non-zero and play a role in ensuring gauge invariance [54].

8. GLAUBERS FACTORIZATION VIOLATION AND THE REGGE LIMIT

So far we have designed an EFT fit to describe hard interactions between massless particles. We have implicitly assumed that all of the invariants are of the same order. i.e, in two to two scattering, is theory will properly describe kinematicswhere $s \sim |t|$, but if we consider $s \gg |t|$ the theory would break down. Such a limit corresponds to near forward scattering and is one of the most vexing problems in quantum field theory. This topic is vast (see, for instance, [58, 59]) and we will not have space to do it justice. However, it is

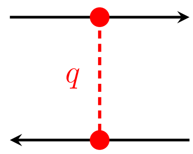


FIG. 24. A leading order contribution to near forward scattering arises through the exchange of a Glauber gluon which is not a dynamical field, much as the potential in NRQCD. The vertical gluon line is shown as a dashed red Glauber exchange.

incumbent upon us to understand how to include such interactions in the EFT formalism. We will be following the treatment in [53], and the reader interested in details should consult this reference.

How can we see that our theory is incomplete? If we follow our guidelines we would conclude that the only modes needed [99] are collinear and (U)soft. However, this is clearly wrong as can be seen by considering the tree level diagram shown in Fig. (24). This process takes place via the exchange of a t-channel gluon whose momentum scales as

$$k^\mu \sim (\lambda^2, \lambda^2, \lambda). \tag{348}$$

This mode keeps both collinear fields on-shell, but is itself an off-shell mode, much like the potential mode in NRQCD, and is called the "Glauber" mode for historical reasons [100]. A simple matching calculation leads to the amplitude (for quark-anti-quark scattering)

$$O_n^{qq} = -i(8\pi\alpha)[\bar{u}_n T^a \frac{\bar{\not{n}}}{2} u_n] \frac{1}{q_\perp^2} [\bar{v}_{\bar{n}} \bar{T}^a \frac{\not{n}}{2} v_{\bar{n}}]. \tag{349}$$

The anti-quark spinors have been taken to be in the $\bar{3}$ representation, to put quarks and anti-quarks on the same footing[101] That is, $\bar{v}/v$ creates/destroys an anti-quark. Following our usual power counting rules we find that the quark-anti-quark Glauber action generated by this matching scales as

$$S_G^{qq} = \int (d^4x \sim \lambda^{-2})(\bar{\psi}_{n,\omega} T^a \frac{\bar{\not{n}}}{2} \psi_{n,\omega} \sim \lambda^2)(\frac{1}{\mathcal{P}_\perp^2} \sim \lambda^{-2})(\bar{\psi}_{\bar{n},\omega'} \bar{T}^a \frac{\not{n}}{2} \psi_{\bar{n},\omega'} \sim \lambda^2) \sim 1. \tag{350}$$

One might ask why there is only one measure for a presumably non-local operators, i.e.

[99] Assuming we stay away from exceptional configurations such as thresholds.

[100] The mode is named after Roy Glauber and his 1959 work on high-energy, eikonal scattering [60]. He later won the Nobel Prize for his work on quantum optics in 2005.

[101] Compared to the usual (e.g. as in [7]) spinor bilinears $\bar{v}(p)T^a v(k)$ we have $-\bar{v}(k)\bar{T}^a v(p)$ after the replacement $v \to v^\star$.

why not d^4xd^4y? The answer is that we must recall that as far as the residual momentum dependence through x is concerned, the operator is completely local. The denominator is a number (label) so it does not lead to non-locality in the space of residual x dependence.

In addition, we have other operators which account for glue-glue and glue-quark forward scattering as well, which is discussed below.

Given that this interaction is leading order,[102] it poses a threat to all factorization theorems since it couples modes (transferring $k_\perp$) which we previously assumed to factorize. Indeed, if the Glaubers contribute to a cross section, then the PDF's become insufficient since the Glauber operators can couple to spectator partons in the hadron. That is, when we take the hadronic matrix element, the Glauber fields need not contract with the ("active") fields which constitute the hard scattering process. Moreover, the limit $s \gg t$ (called the "Regge" limit) will introduce large logs of the ratio t/s which need to be resummed. Note that to resum the logs we do not need any non-perturbative information, but to properly account for the spectator effects in the matrix element would necessitate some non-perturbative information that goes beyond the PDFs'. Thus even for Drell-Yan we must prove that the Glauber interaction does not contribute if we expect to be able to make systematic predictions using only the PDF's as non-perturbative inputs. The cancellation of the Glauber effects in canonical scattering observables was first proven in [61]. We will not discuss how these cancellations occur in EFT, as this would take us too far afield. We will however, now address the resummation of the large logs in the Regge limit.

A. The Operator Basis

The generalization of the result in (350) to include all possible external states, with varying spin and collinear directions (which we choose to be limited to n and $\bar{n}$) is straightforward.

[102] Even if the coupling is taken to be small and the Glauber is treated perturbatively, it only takes one Glauber exchange to lead to convolutions between different collinear sectors.

Exercise 8.1

Show that near forward anti-quark-gluon, quark-gluon and gluon-gluon scattering match onto the amplitudes

$$
\begin{aligned}
iM_{\bar{q}g} =& i\Big[if^{BA_3A_2}g_\perp^{\mu_2\mu_3}\,\bar{n}\cdot p_2\Big]\Big[\frac{-8\pi\alpha_s(\mu)\delta^{BC}}{\vec{q}_\perp^{\,2}}\Big]\Big[\bar{v}_{\bar{n}}\frac{\not{n}}{2}\bar{T}^C v_{\bar{n}}\Big],\\
iM_{qg} =& i\Big[\bar{u}_n\frac{\not{\bar{n}}}{2}T^B u_n\Big]\Big[\frac{-8\pi\alpha_s(\mu)\delta^{BC}}{\vec{q}_\perp^{\,2}}\Big]\Big[if^{CA_4A_1}g_\perp^{\mu_1\mu_4}\,n\cdot p_1\Big],\\
iM_{gg} =& i\Big[if^{BA_3A_2}g_\perp^{\mu_2\mu_3}\,\bar{n}\cdot p_2\Big]\Big[\frac{-8\pi\alpha_s(\mu)\delta^{BC}}{\vec{q}_\perp^{\,2}}\Big]\Big[if^{CA_4A_1}g_\perp^{\mu_1\mu_4}\,n\cdot p_1\Big]
\end{aligned}
\tag{351}
$$

respectively. The gluon color/polarization are labeled by A_i/μ_i.

Using the results of this exercise we may match onto the proper operators in the EFT by utilizing our gauge invariant building blocks. We generate a series of quartic operator composed of the product of two bilinears, for instance,

$$
O_{,}^{qB} = \overline{\chi}_n T^B \frac{\not{\bar{n}}}{2}\,\chi_n \qquad\qquad \mathcal{O}_n^{gB} = \frac{i}{2} f^{BCD}\mathcal{B}^C_{n\perp\mu}\,\frac{\bar{n}}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{D\mu}_{n\perp}. \tag{352}
$$

Notice by using gauge symmetry we have been able to fix the full operator for the gluonic case only matching to the perpendicular components of the polarization, recall eq. (148). We might think that at this point we can combine these bilinears with a factor of $1/\mathcal{P}_\perp{}^2$ to form our Glauber operators, but that scenario is clearly incomplete, as we can also have Glauber interactions between collinear lines and soft lines since such an interaction is related to $n-\bar{n}$ forward scattering by a boost. In this case, the full theory diagrams match onto the same amplitudes (351), but now with soft external fields. Thus we have additional bilinear operators where the collinear field are replaced by softs (e.g. for fermions $\psi_s^{n,\bar{n}}$). Notice that the soft fields have inherited collinear subscripts $(n,\bar{n})$ for reasons which will become clear below. The full list of bilinears is shown in the table. The fields $\psi_S^{n,\bar{n}}$, are soft invariant building blocks ($S_n^\dagger\psi$) defined with a soft Wilson line whose direction is picked out by the corresponding collinear bilinear with which it is paired. The sensitivity to the collinear directions can be seen to arise when doing the matching. Notice that when we pair these bilinears to form Glauber operators, we will have three types of Glauber modes since exchanges between soft and collinear bilinears lead to momentum scalings as $(\lambda,\lambda^2,\lambda)$ and

$\mathcal{O}_n^{qB} = \overline{\chi}_n T^B \frac{\bar{\not{n}}}{2}\,\chi_n$	$\mathcal{O}_n^{gB} = \frac{i}{2} f^{BCD} \mathcal{B}^C_{n\perp\mu}\,\frac{\bar{n}}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{D\mu}_{n\perp}$
$\mathcal{O}_{\bar{n}}^{qB} = \overline{\chi}_{\bar{n}} T^B \frac{\not{n}}{2}\,\chi_{\bar{n}}$	$\mathcal{O}_{\bar{n}}^{gB} = \frac{i}{2} f^{BCD} \mathcal{B}^C_{\bar{n}\perp\mu}\,\frac{n}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{D\mu}_{\bar{n}\perp}$
$\mathcal{O}_s^{q_n B} = 8\pi\alpha_s\left(\bar{\psi}^n_S T^B \frac{\not{n}}{2}\psi^n_S\right)$	$\mathcal{O}_s^{g_n B} = 8\pi\alpha_s\left(\frac{i}{2} f^{BCD} \mathcal{B}^{nC}_{S\perp\mu}\,\frac{n}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{nD\mu}_{S\perp}\right)$
$\mathcal{O}_s^{q_{\bar{n}} B} = 8\pi\alpha_s\left(\bar{\psi}^{\bar{n}}_S T^B \frac{\bar{\not{n}}}{2}\psi^{\bar{n}}_S\right)$	$\mathcal{O}_s^{g_{\bar{n}} B} = 8\pi\alpha_s\left(\frac{i}{2} f^{BCD} \mathcal{B}^{\bar{n}C}_{S\perp\mu}\,\frac{\bar{n}}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{\bar{n}D\mu}_{S\perp}\right)$

TABLE I. Summary of bilinear which will be used in the matching. Notice that the soft fields have inherited collinear subscripts.

$(\lambda^2,\lambda,\lambda)$ respectively[103]. The relatively large scaling of order λ will always flow into the collinear sector whose corresponding momentum scales as λ^0, and thus keeps the collinear line on-shell.

So far we have seen that we will generate operators which couple $n-\bar{n}$ as well as $s-(n,\bar{n})$ sectors, but we should expect operator which couple all three as well. The bottom/top collinear lines in these contributions carry momentum scaling as $(1,\lambda^2,\lambda)/(\lambda^2,1,\lambda)$.

To account for such contributions, we must augment our operator basis. However, as opposed to the previously discussed cases, the position of the Wilson lines is not easily determined using the method of descent discussed in section (7 D 1). The reason for this can be seen from the fact that at leading order in λ soft radiation can be emitted from the Glauber line itself.

The leading order Glauber action can then be written as

$$\mathcal{L}_G = \sum_{i,j=q,g} \mathcal{O}_n^{iB}\frac{1}{\mathcal{P}_\perp^2}\mathcal{O}_s^{BC}\frac{1}{\mathcal{P}_\perp^2}\mathcal{O}_{\bar{n}}^{jC} + \sum_{i,j=q,g}(\mathcal{O}_n^{iB}\frac{1}{\mathcal{P}_\perp^2}\mathcal{O}_s^{j_n B} + \mathcal{O}_{\bar{n}}^{iB}\frac{1}{\mathcal{P}_\perp^2}\mathcal{O}_s^{j_{\bar{n}} B})\,. \tag{353}$$

Illustrate how the Ward identity is satisfied, and give Feynman rules for the operators in RS.

Figure (25) shows the matching between the full theory and the EFT for one soft gluon emission. The diagrams on top are the full theory diagrams that build up the soft Wilson lines in $\mathcal{O}_\int$ in the three sector operator. The lower diagram is how we draw the EFT result. Note that the vertex is not local and reproduces the sum of the full diagrams. The Feynman

[103] Here we will concentrate on SCETII processes which only involve soft modes. Interestingly enough though there can be scenarios where both soft and Usoft modes exist in the theory simultaneously [53].

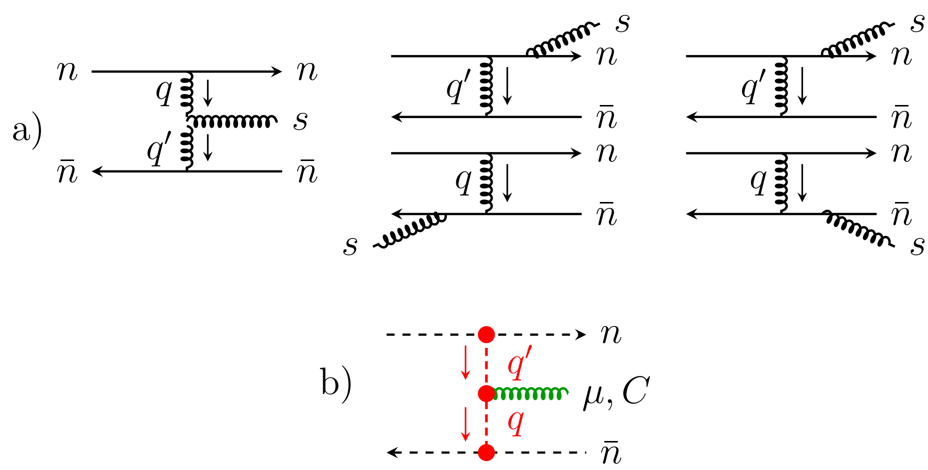


FIG. 25. Matching graphs with one soft emission in the presence of Glauber exchange. In the full-theory diagrams the emitted gluons are shown as black wavy lines; in the EFT diagram the soft emission is shown as a green coil and the red dashed line denotes Glauber exchange.

rule for this "Lipatov vertex" [62] is given by

$$\text{Fig. 25} = \left[\bar u_n \frac{\bar{n}\!\!\!/}{2} T^A u_n\right]\left[\bar v_{\bar n}\frac{n\!\!\!/}{2}T^B v_{\bar n}\right] \\ \times \frac{8\pi\alpha_s}{\vec q_{1\perp}^{\,2}\vec q_{1\perp}^{\,\prime 2}}\, igf^{ABC}\left[q_\perp^\mu + q_\perp^{\prime\mu} - n\cdot q'\frac{\bar n^\mu}{2} - \bar n\cdot q\frac{n^\mu}{2} - \frac{\bar n^\mu \vec q_\perp^{\,\prime 2}}{\bar n\cdot q} - \frac{n^\mu \vec q_\perp^{\,2}}{n\cdot q'}\right]. \tag{354}$$

Notice the distinction between q and q'. The soft light-cone momenta scales as λ and thus should be routed into the appropriate collinear line. That is, $q^\mu \sim (\lambda, \lambda^2, \lambda)$ while $q^{\mu\prime} \sim (\lambda^2, \lambda, \lambda)$.

B. The One Loop Glauber Box and Cross Box

Let us start with the one-loop Glauber box and cross box shown in Fig. 26.

We decompose the loop momentum as

$$\int d^d k \equiv \int \frac{d^d k}{(2\pi)^d} = \frac{1}{2}\int d(n\cdot k)\, d(\bar n\cdot k)\, d^{d-2}k_\perp, \tag{355}$$

where $d = 4-2\epsilon$, and recall the forward conditions $n\cdot p_4 = n\cdot p_1$ and $\bar n\cdot p_3 = \bar n\cdot p_2$. The box and cross-box loop integrals involve two Glauber denominators and two propagators from

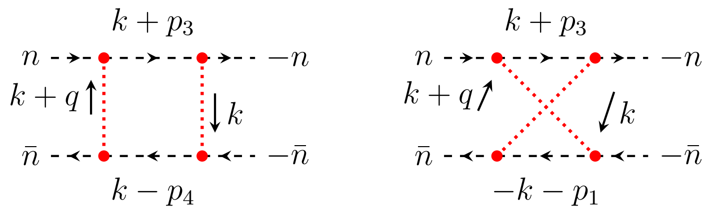


FIG. 26. One-loop Glauber box and crossed-box diagrams. The red dotted lines denote Glauber exchanges between the n and $\bar{n}$ collinear lines.

the collinear quarks. They are

$$I_{\rm Gbox} = \int \frac{d^{d-2}k_\perp\, d(n\cdot k)\, d(\bar{n}\cdot k)}{2\,\vec{k}_\perp^2(\vec{k}_\perp+\vec{q}_\perp)^2 \Big[n\cdot(k+p_3) - \frac{(\vec{k}_\perp+\vec{q}_\perp/2)^2}{\bar{n}\cdot p_2} + i0\Big]\Big[-\bar{n}\cdot(k+p_4) - \frac{(\vec{k}_\perp+\vec{q}_\perp/2)^2}{n\cdot p_1} + i0\Big]},$$
$$I_{\rm Gcbox} = \int \frac{d^{d-2}k_\perp\, d(n\cdot k)\, d(\bar{n}\cdot k)}{2\,\vec{k}_\perp^2(\vec{k}_\perp+\vec{q}_\perp)^2 \Big[n\cdot(k+p_3) - \frac{(\vec{k}_\perp+\vec{q}_\perp/2)^2}{\bar{n}\cdot p_2} + i0\Big]\Big[\bar{n}\cdot(k+p_1) - \frac{(\vec{k}_\perp+\vec{q}_\perp/2)^2}{n\cdot p_1} + i0\Big]}. \tag{356}$$

These graphs involve logarithmically divergent integrals of the form $\int d(n\cdot k)/(n\cdot k+\Delta\pm i0)$ and $\int d(\bar{n}\cdot k)/(\bar{n}\cdot k+\Delta\pm i0)$ that are not regulated by dimensional regularization. In QED we can add the two integrands to get a well defined integral, but in QCD the color factors obstruct this resolution and thus we must introduce a rapidity regulator, as discussed in Sec. 7 D. However, as pointed out in [63] it behooves us to keep the Glauber regulator distinct from the Wilson line regulator.

When we regulated the soft and collinear rapidity divergences in SCETII it was crucial that we use the same regulator since collinear and soft loops individually have 1/ divergences that cancel in their sum. No such cancellation is required in the case of Glauber loops. In fact, as we will see below the Glauber box has no rapidity divergence. Thus for each Glauber propagator with momentum k, we will introduce a factor of $|\,2k_z/\nu'\,|^{-\eta'}$. The scale ν' is there pro-forma as all Glauber loops will be finite when η' goes to zero. Nonetheless is imperative that we take the $\eta'\to 0$ limit prior to taking $\eta\to 0$. This is only an issue starting at two loops and details can be found in [63].

Consider the single box diagram. After performing the Glauber energy integral by con-

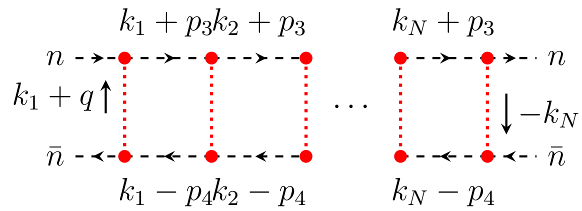


FIG. 27. Multi-Glauber box topology. The red dotted lines denote Glauber exchanges between the n and $\bar{n}$ collinear lines.

tours we then use the result,

$$\int \frac{[dk_z]}{-2k_z + A + i\epsilon} \mid \frac{2k_z}{\nu} \mid^{-\eta} = -\frac{1}{4} + O(\eta). \tag{357}$$

and find

$$I_{\text{Gbox}} = -\frac{i}{4\pi} \int \frac{d^{d-2}k_\perp}{\vec{k}_\perp^2 (\vec{k}_\perp + \vec{q}_\perp)^2} \left[-i\pi + O(\eta)\right]. \tag{358}$$

We see, as promised that the result is finite. We also notice that the result is independent of A which depends upon the transverse momenta. Thus the matter line is *effectively* eikonal and thus has a Wilson line description. However, at higher loops this will no longer be the case. This eikonalization allows one to resum all of the boxes into a phase, as dicussed in the problem below. What about the cross-box? Once we regulate the integral we can do the energy integral by contours with the poles being on the same side and thus the integral vanishes. Note that w/o the regulator this conclusion is in doubt.

C. The Eikonal Phase

Now we mnight expect tha the Glaubers will resum into a phase since since high energy scattering is a semi-classical process. Let us show that this is indeed the case by consider the N Glauber box diagram shown in figure (**??**). Given the result of this exercise, we can resum the Glauber boxes shown in Fig. 27 by going to impact-parameter space.

Using the integral representations

$$|2k_z|^{-\eta} = \int_{-\infty}^{\infty} d\alpha \, \frac{\eta}{2} \mid x \mid^{-1+\eta} \, e^{ik_z \alpha}, \tag{359}$$

$$\frac{1}{k^z+\Delta+i0} = \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha k^z}(-i)\theta(\alpha)e^{i\alpha\Delta}, \tag{360}$$

and doing the energy integrals by contours, the N-Glauber box contribution can then be written as

$$\begin{aligned}
&= -i(2g^2)^{N+1}\mathcal{S}^{n\bar{n}}_{(N+1)} I^{(N)}(q_\perp) \int \frac{dk_1^z \cdots dk_N^z \left|2k_1^z(2k_1^z-2k_2^z)\cdots(2k_{N-1}^z-2k_N^z)2k_N^z\right|^{-\eta}\nu^{N\eta}}{2^N(-k_1^z+\Delta_1+i0)\cdots(-k_N^z+\Delta_N+i0)} \\
&= -2i(g^2)^{N+1}(-i)^N \mathcal{S}^{n\bar{n}}_{(N+1)} I^{(N)}(q_\perp)\left(\frac{\eta}{2}\right)^{N+1}\int_{-\infty}^{+\infty}\left[\prod_{i=1}^{N} dk_i^z\, d\alpha_i\,\theta(\alpha_i)\right]\left[\prod_{j=1}^{N+1} dx_j\,|x_j|^{-1+\eta}\right] \\
&\quad\times \exp\big[ik_1^z x_1 + i(k_2^z-k_1^z)x_2+\cdots+i(k_N^z-k_{N-1}^z)x_N - ik_N^z x_{N+1}\big]\exp\left[\sum_{m=1}^{N} i\alpha_m(k_m^z+\Delta_m)\right] \\
&= 2(-ig^2)^{N+1}\mathcal{S}^{n\bar{n}}_{(N+1)} I_\perp^{(N)}(q_\perp)\left(\frac{\eta}{2}\right)^{N+1}\int_{-\infty}^{+\infty}\left[\prod_{j=1}^{N+1} dx_j\,|x_j|^{-1+\eta}\right] \\
&\quad\times \theta(x_2-x_1)\theta(x_3-x_2)\cdots\theta(x_{N+1}-x_N)\exp\left[\sum_{m=1}^{N} i\Delta_m(x_{m+1}-x_m)\right] \\
&= 2(-ig^2)^{N+1}\mathcal{S}^{n\bar{n}}_{(N+1)} I_\perp^{(N)}(q_\perp)\frac{1}{(N+1)!}\left[1+O(\eta)\right] \\
&\Longrightarrow \frac{1}{(N+1)!}\left[i\phi(b_\perp)\right]^{N+1} 2\mathcal{S}^{n\bar{n}}.
\end{aligned} \tag{361}$$

The color-spin factor for quark (T^A) anti-quark ($\bar{T}^a$) scattering appearing above is defined by

$$\left(T^{A_1}\cdots T^{A_N}\right)\otimes\left(\bar{T}^{A_1}\cdots\bar{T}^{A_N}\right)\mathcal{S}^{n\bar{n}} = \left[\bar{u}_n\frac{\bar{n}\!\!\!/}{2}T^{A_1}\cdots T^{A_N}u_n\right]\left[\bar{v}_{\bar{n}}\frac{n\!\!\!/}{2}\bar{T}^{A_1}\cdots\bar{T}^{A_N}v_{\bar{n}}\right] \equiv \mathcal{S}^{n\bar{n}}_{(N)}. \tag{362}$$

and

$$\mathcal{S}^{n\bar{n}} = \bar{u}_n\frac{n\!\!\!/}{2}u_n\bar{v}_{\bar{n}}\frac{\bar{n}\!\!\!/}{2}v_{\bar{n}.} \tag{363}$$

The impact-parameter-space Glauber phase is

$$\begin{aligned}
\phi(b_\perp) &= -T^A\otimes\bar{T}^A\, g^2(\mu)\int\frac{d^{d-2}q_\perp}{\vec{q}_\perp^{\,2}}\left(\iota^\epsilon\mu^{2\epsilon}\right)e^{i\vec{q}_\perp\cdot\vec{b}_\perp} \\
&= -T^A\otimes\bar{T}^A\, g^2(\mu)\,\frac{\Gamma(-\epsilon)}{4\pi}\left(\frac{\mu|\vec{b}_\perp|e^{\gamma_E}/2}{2}\right)^{2\epsilon}.
\end{aligned} \tag{364}$$

D. The Factorization Theorem

The Glauber obstruct naive factorization. However, aside from Glauber interactions the soft and collinear modes do factorize. Thus we can still write down a genearlized factorization theorem which captures the leading order (in t/s) contribution to the near forward scattering amplitude. Whether or not the Glauber can be treated perutbatively will, in general, depend upon the specific choice of observable, but for the moment we will concentrate on an all orders theorem in the coupling at the scale of the glauber virtuality.

We start with the time evolution operator responsible for Glauber interactions bewteen projectiles a and b moving in the n and $\bar{n}$ collinear directions.

$$U(a,b;T) = \lim_{T\to\infty(1-i0)} \int \left[\mathcal{D}\phi\right] \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}_G\right] = 1 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} U_{(i,j)}. \tag{365}$$

For any number of Glauber potential insertions, one can then factorize the soft and collinear operators to give a factorized expression [64] for the amplitude for scattering of projectile κ with κ' is

$$\begin{aligned} M^{\kappa\kappa'} &= i\sum_{MN}\int\int_{\perp(N,M)} J_{\kappa N}^{A_1\ldots A_N}(l_{1\perp}\ldots l_{N\perp},\epsilon,\eta) S_{(N,M)}^{A_1..A_N;B_1..B_M}(l_{1\perp}\ldots l_{N\perp};l'_{1\perp}\ldots l'_{M\perp}) \bar{J}_{\kappa' M}^{B_1\ldots B_N}(l'_{1\perp}\ldots l'_{M\perp},\epsilon,\eta) \\ &\equiv i\sum_{MN} J_{\kappa N}\otimes S_{MN}\otimes J_{\kappa' N} \end{aligned} \tag{366}$$

where, following the notation in [64], we defined

$$\int\int_{\perp(N,M)} = \frac{(-i)^{N+M}}{N!M!}\int\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{d^{d'}l_{i\perp}}{l_{i\perp}^2}\frac{d^{d'}l'_{j\perp}}{l'^2_{j\perp}}\delta^{d'}(\sum l_{i\perp}-q_\perp)\delta^{d'}(\sum l'_{j\perp}-q_\perp)) \tag{367}$$

where κ and κ' label the external states, i.e. quarks or gluons. The subscripts (N,M) denote the number of Glauber lines going (in/out). Note that in Eq. (366) all of the Glauber light cone momentum integrals have been performed. since these momenta components scale as λ^2 they are dropped in the soft loops according to the multipole expansion necessary to maintain manifest power counting. Thus all of the Glauber loops correspond to box integrals which are rapidity finite and give a result independent of the perp momenta. This

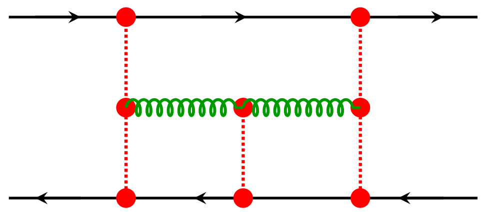


FIG. 28. Tennis-court topology for soft exchange between Glauber interactions. The solid black lines denote the n and $\bar{n}$ collinear sectors, the red dotted lines denote Glauber exchanges, and the green coil denotes the soft exchange.

fact is conceptually important as it implies that the collinear line effectively eikonalizes, and thus can be treated as a Wilson line, which connects the EFT method to the Wilson line methodology used to study forward scattering [65, 66].

The jet function are defined as time order products, e.g. at the one and two Glauber gluon level

$$
\begin{aligned}
J^{A_1}(k_\perp) &= \int dx_1^\pm \langle p \mid T((O_n^{qA_1} + O_n^{gA_1})(k_\perp, x_1^\pm) \mid p' \rangle \\
J^{A_1A_2}(k_\perp, k'_\perp) &= \int dx_1^\pm dx_2^\pm \langle p \mid T((O_n^{qA_1} + O_n^{gA_1})(k_\perp, x_2^\pm)(O_n^{qA_2} + O_n^{gA_2})(k_\perp, x_1^\pm) \mid p' \rangle, \qquad (368)
\end{aligned}
$$

(369)

the jets are written in this way because the combination $(O_n^{qA_1} + O_n^{gA_1})$ is an eigenvector of $\nu\frac{d}{d\nu}$ as shown in [53], if we constrain ourselves to collinear corrections. Soft fields do not couple directly to the collinear lines but nonetheless can lead to mixing between jets with different numbers of Glaubers. As an example, we have the "tennis court" topology shown in Fig. (28), which mixes J_2 and J_3.

At tree level the Jet function is given by

$$
\begin{aligned}
J_q^{(0)A_1\ldots A_N} &= g^N \bar{u}_n T^{A_1} \ldots T^{A_N} \frac{\not{\bar{n}}}{2} u_n \\
J_g^{(0)A_1\ldots A_N} &= g^N \epsilon^{\star\mu}\epsilon^{\star\nu} b_{\mu\nu} \mathcal{T}^{A_1} \ldots \mathcal{T}^{A_N}. \qquad (370)
\end{aligned}
$$

where $b^{\mu\nu} = g_\perp^{\mu\nu}\bar{n}\cdot p_1 - \bar{n}^\mu p_1^\nu - \bar{n}^\mu p_4^\nu + \frac{p_1^\perp \cdot p_4^\perp}{\bar{n}\cdot p_4}\bar{n}^\mu\bar{n}^\nu$ and $\bar{n}\cdot p_1 = \bar{n}\cdot p_4$. $T^A/\mathcal{T}^A$ are generates in the fundamental/adjoint representation. Note that the intermediate propagtor is gone because it is eikonal and is replaced by unity once the light-cone momentum integral has been performed using Eq. (357).

While the tree level soft function is given by

$$S_{(i,j)}^{(0)A_1\ldots A_i;B_1\ldots B_j}(l_{i\perp};l'_{i\perp}) = 2\delta_{ij}i^j j!\delta^{A_1B_1}....\delta^{A_NB_N}\prod_{a=1}^{j}l'^2_{i\perp}\prod_{n=1}^{j-1}\delta^{2-2\epsilon}(l_{n\perp}-l'_{n\perp}). \quad (371)$$

E. Rapidity Renormalization and the Mixing Matrices

To renormalize this theory we define the bare jet function $J_{\kappa(i)}^{A_1\cdots A_i}(\ell_{1\perp},\ldots,\ell_{i\perp},\epsilon,\eta)$ in terms of the renormalized jet function $J_{\kappa(j)}^{B_1\cdots B_j}(k_{1\perp},\ldots,k_{j\perp},\epsilon,\nu)$ via a convolution over $Z_{J(j,i)}^{B_1\cdots B_jA_1\cdots A_i}$,

$$J_{\kappa(i)}^{A_1\cdots A_i}(\ell_{1\perp},\ldots,\ell_{i\perp},\epsilon,\eta) = \sum_j\int_{\perp(j)}J_{\kappa(j)}^{\{B_j\}}(\{k_{j\perp}\},\epsilon,\nu)Z_{J(j,i)}^{\{B_j\}\{A_i\}}(\{k_{j\perp}\},\{\ell_{i\perp}\},\epsilon,\eta,\nu), \quad (372)$$

i labels the number Glaubers and κ is the parton identity.

$\int_{\perp(j)}$ is given by

$$\int_{\perp(j)} \equiv \frac{(-i)^j}{j!}\int\left[\prod_{m=1}^{j}\frac{d^{d'}k_{m\perp}}{k^2_{m\perp}}\right]\delta^{d'}\left(\sum_{m=1}^{j}k_{m\perp}-q_\perp\right).$$

(373)

In Eq. (372) the counterterm Z_J cancels the $1/\eta$ divergences in $J_{\kappa(i)}(\eta)$ that arise from collinear momenta approaching the soft phase-space region. The ϵ dependence denotes the potential presence of infrared divergences. Due to the $n\leftrightarrow\bar{n}$ symmetry of the rapidity regulator used here, we have the same Z_J renormalization factor for $\bar{J}$

The bare soft function is renormalized by Z_S factors, however, unlike the collinear case, the Z_S factors appear on both sides, since they absorb $1/\eta$ poles from soft momenta approaching the n-collinear or $\bar{n}$-collinear limits, respectively,

$$S_{(i,j)}^{\{A_i\}\{A'_j\}}(\{\ell_{i\perp}\},\{\ell'_{j\perp}\},\eta) = \sum_{r,r'}\int\int_{\perp(r,r')}Z_{S(i,r)}^{\{A_i\}\{B_r\}}(\{\ell_{i\perp}\},\{k_{r\perp}\},\eta,\nu)S_{(r,r')}^{\{B_r\}\{B'_{r'}\}}(\{k_{r\perp}\},\{k'_{r'\perp}\},\nu)$$
$$\times Z_{S(r',j)}^{\{B'_{r'}\}\{A'_j\}}(\{k'_{r'\perp}\},\{\ell'_{j\perp}\},\eta,\nu). \quad (374)$$

Here we have omitted ϵ arguments for simplicity. By symmetry, the same Z_S appears on the left and right hand side of $S_{(r,r')}$. Imposing the fact that the bare functions are independent of ν give the RRG equations

$$\begin{aligned}
\nu\frac{\partial}{\partial\nu}J_{\kappa(i)} &= \sum_{j=1}^{\infty} J_{\kappa(j)}\otimes\gamma^{J}_{(j,i)},\\
\nu\frac{\partial}{\partial\nu}S_{(i,j)} &= -\sum_{k=1}^{\infty}\gamma^{S}_{(i,k)}\otimes S_{(k,j)} - \sum_{k=1}^{\infty} S_{(i,k)}\gamma^{S}_{(k,j)},\\
\nu\frac{\partial}{\partial\nu}\bar{J}_{\kappa'(i)} &= \sum_{j=1}^{\infty}\gamma^{\bar{J}}_{(i,j)}\otimes\bar{J}_{\kappa'(j)}.
\end{aligned} \tag{375}$$

Each J_i and S_{ij} is decomposed into irreducible representations of the $SU(N)$ symmetry.

In general, operators with different numbers of Glaubers, but in the same color irrep, will mix (for a discussion of the general structure see [64]). The single Glauber exchange is pure 8_A (so there is no need to label the color irrep for $\gamma_{(1,1)}$) renormalizes multiplicativley, as opposed to via convolution, as in the case of Multi-Glauber perators.

The full RRG system can be truncated is we assume that the couple at the scale of the exchange virtuality is peturbative since ach Glauber comes with a factor of $\alpha_s(t)$.

Note that, in general, $i \neq j$ in the above factorization, i.e. there are transitions from i to j Glauber exchanges through the soft function. While the rapidity running of this system is quite nettlesome, there are significant simplifications. Not surprisingly, by symmetry $\gamma_J = \gamma_{\bar{J}}$. Furthermore, the anomalous dimensions for the collinear and soft pieces are not independent as there is a consistency condition [56] which follows from the fact that the full amplitude is free of rapidity divergences, which are an artifact of factorization. In the present case, due to the mixing structure, this condition can be written in terms of a matrix equality [64] $(Z_S)_{ij} = (Z_J^{-1})_{ij}$. In general the mixing matrix will be quite complicated, however, there are simplifications. In particular, there is no mixing between the one Glauber operator and multy-Glauber operators, i.e. $\gamma_{1m} = 0$ for $m \neq 1$. This was proven in [64],

F. A Note on Rapidity Regulartization

In our discussion of SCETII we introduced a rapidity regulator for each Glauber exchange we include a factor of $\mid k_z \mid^{-\eta}$ as well as factor of $\mid 2k_z \mid^{-\eta}$ for soft Wilson emissions.

G. Color Decomposition

It proves useful to decompose the tensor product of generators into irreducible reps of $SU(N)$ which is accomplished by introducing color projectors $P_{m,R}^{AB}$, which projects a tensor product of m generators into the representation R. Explicit expressions for these porjectors can be found in [67]. We can then write the amplitude as

$$iM_{MN} = \sum_{RR'} J_M^{\alpha\ R} \otimes S_{MN}^{RR'AB} \otimes J_N^{R'B} \tag{376}$$

Note that irreps. of different dimensions are always orthogonal. Single gluon exchange is the "octet". Two gluon exchange is decomposed as

$$8 \otimes 8 = 1 \oplus 8_A \oplus 8_S \oplus 10 \oplus \oplus \bar{1}0 \oplus 27. \tag{377}$$

A and S stand for anti-symmetric and symmetric tensor products. The singlet is the famous "Pomeron" which can be exchanged between hadrons without changing their external quantum numbers. So a proton can scatter elastically via Pomeron exchange.

H. The Gluon Regge Trajectory

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