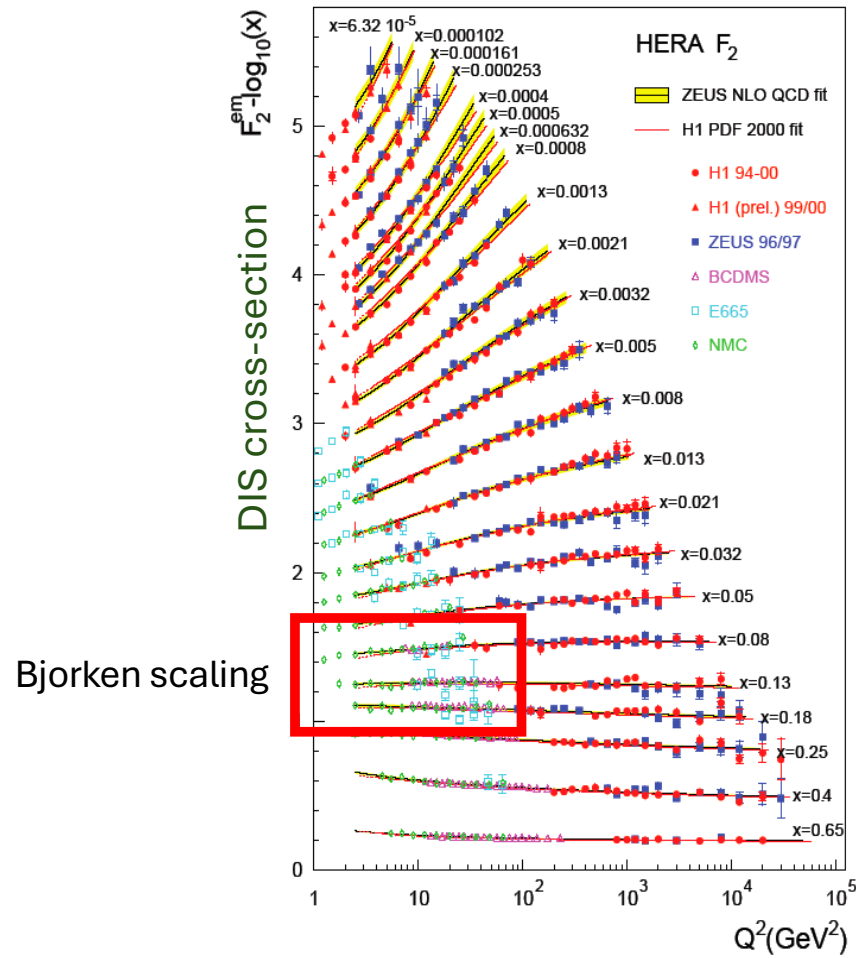


# Higgs School Lecture V

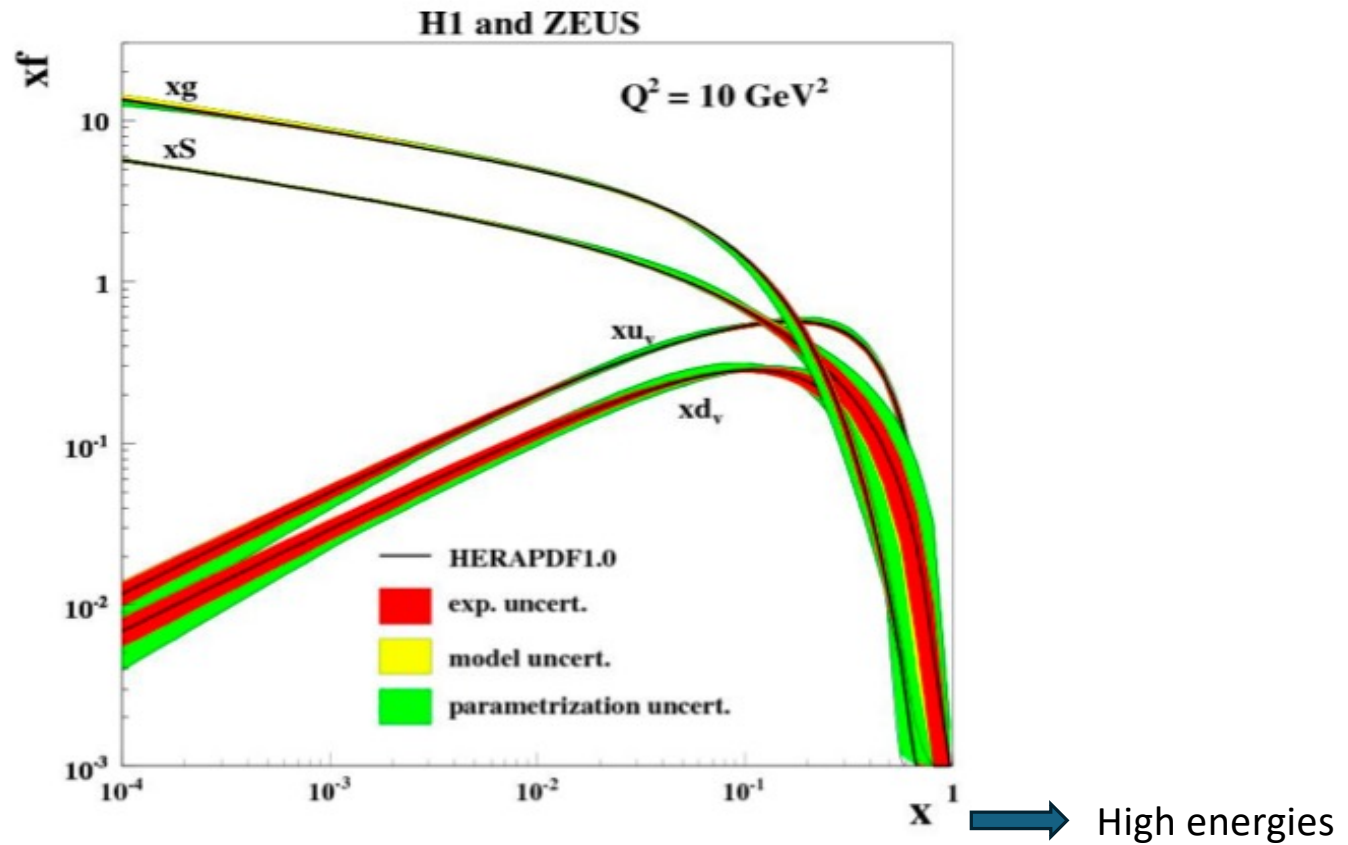
So far...

- 1) DIS and Bjorken scaling, parton model
- 2) Free field theory and Light Cone OPE
- 3) Scaling Violations and Callan-Szymanski Eqns.
- 4) Anomalous Dimensions, Splitting Functions, The DGLAP equations
- 5) Ladder Structure of high order contributions – resumming leading logarithms
- 6) BFKL RG

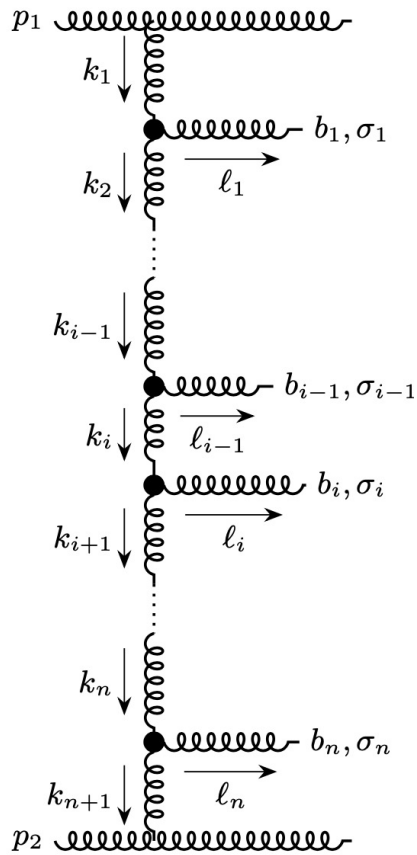
# DIS and scaling violations



## Growth of parton distributions at small x



## BFKL: $2 \rightarrow N$ QCD amplitudes in Regge asymptotics



Compute multiparticle in multi-Regge kinematics (MRK) of QCD:

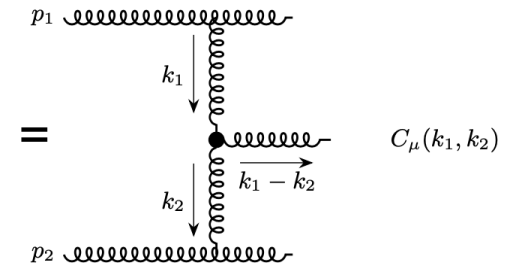
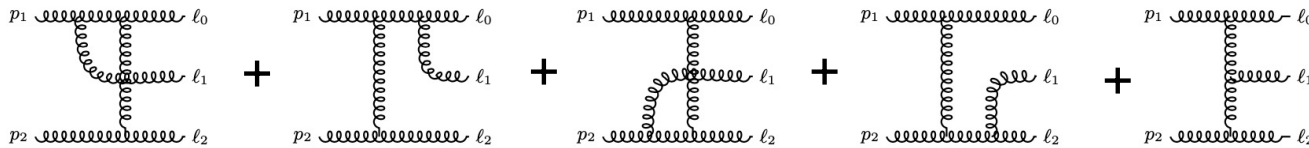
$$y_0^+ \gg y_1^+ \gg y_2^+ \gg \dots \gg y_N^+ \gg y_{N+1}^+ \quad \text{with} \quad \mathbf{k}_i \simeq \mathbf{k}$$

BFKL ladder is **ordered in rapidity**. Produced partons are wee in longitudinal momentum ("slow") but hard in transverse momentum  
 – weak coupling Regge regime of QCD

BFKL: Balitsky-Fadin-Kuraev-Lipatov (1976-1978)

# BFKL: Building blocks

Lipatov effective vertex:

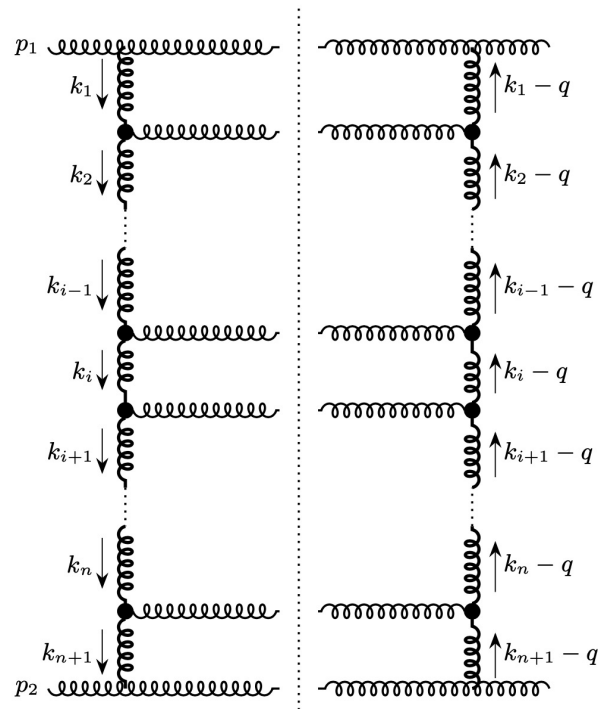


$$C_\mu(\mathbf{q}_1, \mathbf{q}_2) \simeq -\mathbf{q}_{1\mu} + \mathbf{q}_{2\mu} + p_{1\mu} \left( \frac{p_2 \cdot k}{p_1 \cdot p_2} - \frac{q_1^2}{p_1 \cdot k} \right) - p_{2\mu} \left( \frac{p_1 \cdot k}{p_1 \cdot p_2} - \frac{q_2^2}{p_2 \cdot k} \right)$$

Gauge covariant, satisfies  $k_\mu C^\mu = 0$



## 2 → N + 2 amplitude in Regge asymptotics

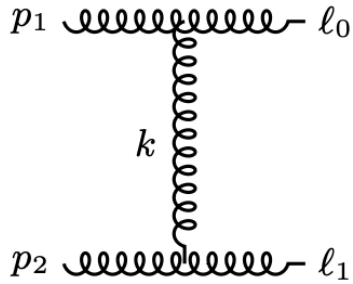


Optical theorem:

$$\text{Im } \mathcal{A}_{2 \rightarrow 2} = \sum_{n=0}^{\infty} \frac{1}{2} \int d(\text{P.S.}^{n+2}) \sum_{\text{color, polarizations}} \mathcal{A}_{2 \rightarrow n+2}(\{k\}) \mathcal{A}_{2 \rightarrow n+2}^{\dagger}(\{k\} - q)$$

Phase space Amplitude × c.c

## Start with the Born amplitude



Start with the three gluon amplitude we introduced in the first lecture

where

$$\Gamma_{\mu\mu'\rho}^{\alpha\alpha'a} = -igf^{\alpha\alpha'a} \left( \eta_{\mu\mu'} (-p_1 - \ell_0)_\rho + \eta_{\mu\rho} (k + p_1)_{\mu'} + \eta_{\mu'\rho} (\ell_0 - k)_\mu \right)$$

$$= -igf^{\alpha\alpha'a} \left( \eta_{\mu\mu'} (-p_1 - \ell_0)_\rho + \eta_{\mu\rho} (2k + \ell_0)_{\mu'} + \eta_{\mu'\rho} (p_1 - 2k)_\mu \right)$$

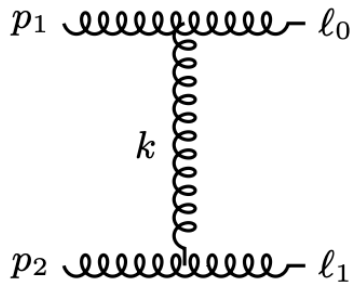
In the eikonal limit, great simplification

$$\Gamma_{\mu\mu'\rho}^{\alpha\alpha'a} \approx 2igf^{\alpha\alpha'c} \eta_{\mu\mu'} p_{1,\rho}$$

The external momenta, being nearly on-shell, vanish upon contraction with polarization tensors, here  $e_\mu(p_1)$  and  $e_{\mu'}(\ell_0)$

$$s = 2p_1 \cdot p_2, \quad t = -k^2 = -(p_1 - \ell_0)^2, \quad T_{\alpha\alpha'}^a = if_{a\alpha\alpha'},$$

## Start with the Born amplitude



Two-body phase space:

$$\begin{aligned} \int d(\text{P.S.}^2) &= \int \frac{d^4 \ell_0}{(2\pi)^3} \delta(\ell_0^2) \frac{d^4 \ell_1}{(2\pi)^3} \delta(\ell_1^2) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \ell_0 - \ell_1) \\ &= \frac{1}{(2\pi)^2} \int d^4 k \delta[(p_1 - k)^2] \delta[(p_2 + k)^2] \end{aligned}$$

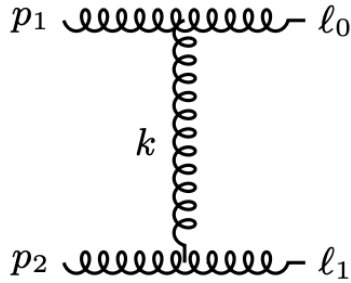
Change of variables:  $\ell_0 = p_1 - k$ ,  $\ell_1 = p_2 + k$

*Introduce Sudakov variables*  $k = \rho p_1 + \lambda p_2 + k_\perp$   $0 \leq \rho \leq 1$ ;  $-1 \leq \lambda \leq 0$

which gives  $d^4 k = \frac{s}{2} d\rho d\lambda d^2 \mathbf{k}$  where we used  $p_1 = (p_1^+, 0, 0, 0)$  and  $p_2 = (0, p_2^-, 0, 0)$ , with  $s = 2 p_1^+ p_2^-$

$$\begin{aligned} \longrightarrow \int d(\text{P.S.}^2) &= \frac{s}{2(2\pi)^2} \int d\rho d\lambda d^2 \mathbf{k} \delta[-s\lambda(1-\rho) - \mathbf{k}^2] \delta[s\rho(\lambda+1) - \mathbf{k}^2] \longrightarrow \frac{s}{2(2\pi)^2} \int d\rho d\lambda d^2 \mathbf{k} \delta(-s\lambda - \mathbf{k}^2) \delta(s\rho - \mathbf{k}^2) \\ & \qquad \qquad \qquad 1 \gg \rho \sim \frac{\mathbf{k}^2}{s}, \quad 1 \gg |\lambda| \sim \frac{\mathbf{k}^2}{s} \end{aligned}$$

## Start with the Born amplitude



Impose gauge condition:  $\epsilon(p_1) \cdot p_2 = \epsilon'(\ell_0) \cdot p_2 = 0$

Manifest with the replacement  $\epsilon(p_1) \rightarrow \tilde{\epsilon}(p_1) = \epsilon - \frac{p_2 \cdot \epsilon}{p_2 \cdot p_1} p_1$   
 $\epsilon'(\ell_0) \rightarrow \tilde{\epsilon}'(\ell_0) = \epsilon' - \frac{p_2 \cdot \epsilon'}{p_2 \cdot \ell_0} \ell_0$

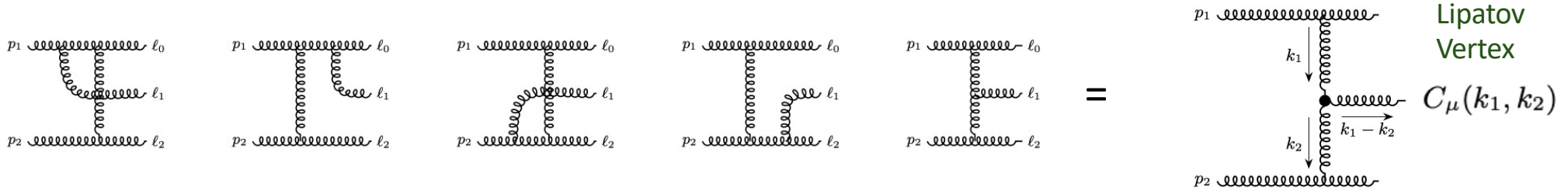
$$\Gamma_{p_1 \ell_0}^{abc} \equiv \frac{1}{g} \Gamma_{\mu \mu' \rho}^{abc} \tilde{\epsilon}^\mu \tilde{\epsilon}'^{\mu'} \frac{\sqrt{2} p_2^\rho}{s} = -\sqrt{2} i f^{abc} \left( -g_{\mu \mu'} + \frac{p_{2\mu} p_{1\mu'} + p_{2\mu'} \ell_{0\mu}}{p_2 \cdot p_1} + (p_1 - \ell_0)^2 \frac{p_{2\mu} p_{2\mu'}}{2(p_2 \cdot p_1)^2} \right) \epsilon^\mu(p_1) \epsilon'^{\mu'}(\ell_0)$$

A particular axial gauge tailored to high energy asymptotics...

$$\mathcal{A}_{2 \rightarrow 2, p_1 + p_2 \rightarrow \ell_0 + \ell_1}^{\alpha \alpha' \beta \beta'} = \Gamma_{p_1 \ell_0}^{\alpha \alpha' c} \frac{g^2 s}{t} \Gamma_{p_2 \ell_1}^{\beta \beta' c}$$

Born amplitude in high energy asymptotics

## 2→3 (5 point) tree amplitude



Computing the sum of these amplitudes in MRK kinematics gives  $\mathcal{A}_{2\rightarrow 3,b}^{\mu\mu'\nu\nu'\sigma}(k_1, k_2) = -\frac{2ig^3 s}{k_1^2 k_2^2} \eta^{\mu\mu'} \eta^{\nu\nu'} f_{acb} T^a \otimes T^c C^\sigma(k_1, k_2)$

(see for instance, DelDuca, hep-ph/9503226)

Three-body phase space:

$$\int d(\text{P.S.}^3) = \int \prod_{i=0}^1 \left[ \frac{d^4 \ell_i}{(2\pi)^3} \delta(\ell_i^2) \right] 2\pi \delta \left( \left[ p_1 + p_2 - \sum_{i=0}^1 \ell_i \right]^2 \right) \quad \rightarrow \quad \frac{s^2}{4(2\pi)^5} \int \prod_{i=1}^2 d\rho_i d\lambda_i d^2 \mathbf{k}_i \delta[-s\lambda_1 - \mathbf{k}^2] \delta[s\rho_2 - \mathbf{k}^2] \delta[-\rho_1 \lambda_2 s - \mathbf{k}^2]$$

employing a change of variables  $\ell_0 = p_1 - k_1$

$\ell_1 = k_1 - k_2$

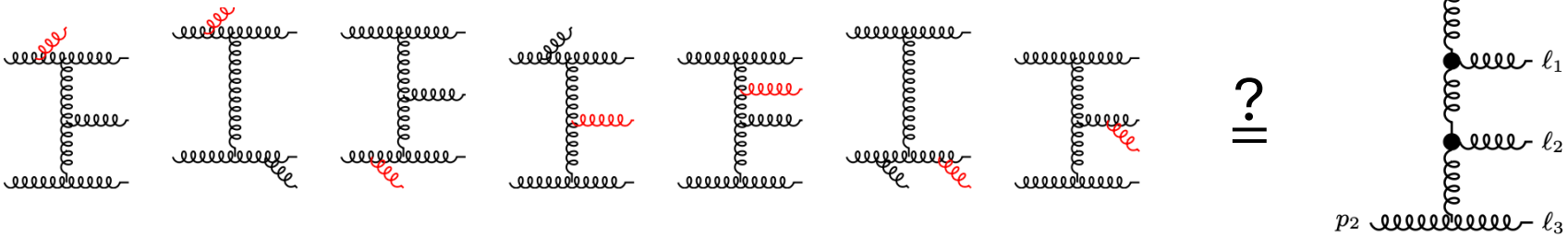
$\ell_2 = p_2 + k_2$

and MRK kinematics

$1 \gg \rho_1 \gg \rho_2 \sim \mathbf{k}^2/s$

$1 \gg |\lambda_2| \gg |\lambda_1| \sim \mathbf{k}^2/s$

## 2 → 4 (6 point) tree amplitude

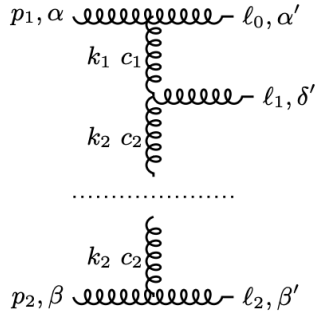


Subset of graphs in MRK kinematics

While the explicit computation is challenging but doable, going beyond becomes increasingly cumbersome...

We will make use of dispersive techniques instead

## Back to the 5 point amplitude



Residue of  $k_2$  pole  $P_{k_2^2} \mathcal{A}_{2 \rightarrow 2+1}^{\alpha\alpha'\beta\beta'\delta'} = \underbrace{\mathcal{A}_{p_1+(-k_2) \rightarrow l_0+l_1}^{\alpha\alpha'c_2\delta'}}_{\text{Born amplitude}} sg^2 \Gamma_{p_2 l_2}^{\beta\beta'c_2}$

$$\mathcal{A}_{p_1+(-k_2) \rightarrow l_0+l_1}^{\alpha\alpha'c_2\delta'} = g \Gamma_{p_1 l_0}^{\alpha\alpha'c_1} \frac{(p_1 - k_2)^2}{k_1^2} \left( -g_{\mu\nu} + \frac{-p_{1,\mu} k_{2,\nu} + p_{1,\nu} l_{1,\mu}}{-p_1 \cdot k_2} + (-k_2 - l_1)^2 \frac{p_{1,\mu} p_{1,\nu}}{2(p_1 \cdot k_2)^2} \right) \epsilon^\mu(-k_2) \epsilon^\nu(l_1)$$

Then (after taking advantage of MRK ordering)

$$P_{k_2^2} \mathcal{A}_{2 \rightarrow 2+1}^{\alpha\alpha'\beta\beta'\delta'} = -ig^3 \frac{s}{k_1^2} \Gamma_{p_1 l_0}^{\alpha\alpha'c_1} \Gamma_{p_2 l_2}^{\beta\beta'c_2} f^{c_2 \delta' c_1} \left( -2p_{2,\nu} \frac{p_1 \cdot l_1}{p_1 \cdot p_2} - (k_1 + k_2)_\nu + 2 \frac{l_1 \cdot p_2}{p_1 \cdot p_2} p_{1,\nu} + \frac{k_1^2}{p_1 \cdot l_1} p_{1,\nu} \right) \epsilon^\nu(l_1)$$

Likewise, can take the residue of  $k_1$  pole

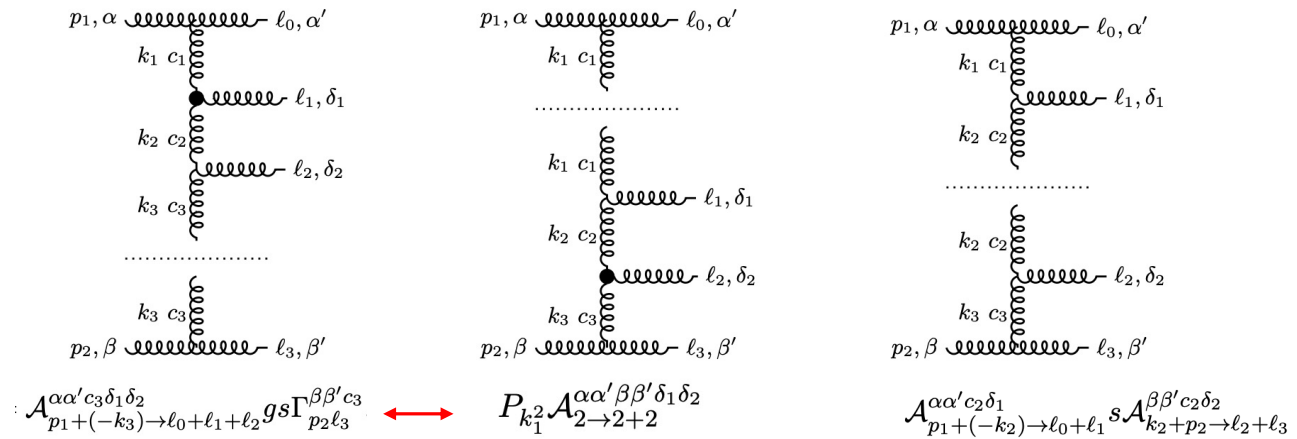
$$P_{k_1^2} \mathcal{A}_{2 \rightarrow 2+1}^{\alpha\alpha'\beta\beta'\delta'} = ig^3 \frac{s}{k_2^2} \Gamma_{p_1 l_0}^{\alpha\alpha'c_1} \Gamma_{p_2 l_2}^{\beta\beta'c_2} f^{c_1 \delta' c_2} \left( 2p_{1,\nu} \frac{p_2 \cdot l_1}{p_1 \cdot p_2} - (k_1 + k_2)_\nu - 2 \frac{l_1 \cdot p_1}{p_1 \cdot p_2} p_{2,\nu} - \frac{k_2^2}{p_2 \cdot l_1} p_{2,\nu} \right) \epsilon^\nu(l_1)$$

Compare and extract simultaneous residue: Recover result of explicit computation

$$\mathcal{A}_{2 \rightarrow 2+1}^{\alpha\alpha'\beta\beta'\delta'} = ig^3 \frac{s}{k_1^2 k_2^2} \Gamma_{p_1 l_0}^{\alpha\alpha'c_1} \Gamma_{p_2 l_2}^{\beta\beta'c_2} f^{c_1 \delta' c_2} C_\nu(k_1, k_2) \epsilon^\nu(l_1)$$

↑  
Lipatov vertex

# The 6 point amplitude



For more detail, see arXiv:2507.21252, pages 21-22

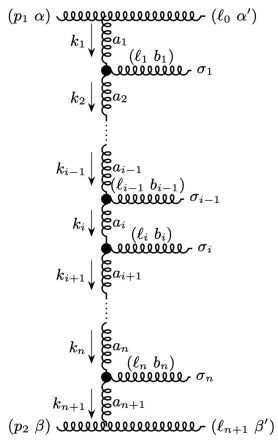
$$\mathcal{A}_{2\rightarrow 2+2}^{\alpha\alpha'\beta\beta'\delta_1\delta_2} = \frac{sg^4}{k_1^2 k_2^2 k_3^2} \Gamma_{p_1 l_0}^{\alpha\alpha'c_1} \gamma^{c_1\delta_1c_2}(k_1, k_2) \gamma^{c_2\delta_2c_3}(k_2, k_3) \Gamma_{p_2 l_3}^{\beta\beta'c_3}$$

$\uparrow \qquad \qquad \qquad \uparrow$

$$\gamma^{c_2\delta_1c_1}(k_1, k_2) \equiv i f^{c_2\delta_1c_1} C_\nu(k_1, k_2) \epsilon^\nu(l_1)$$

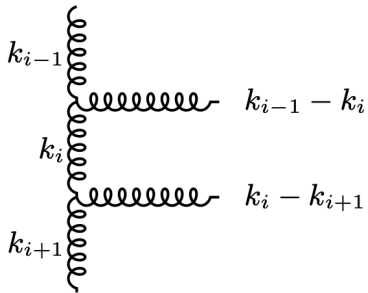
This procedure can be generalized to the 2 → n MRK amplitude

## 2 → n MRK amplitude

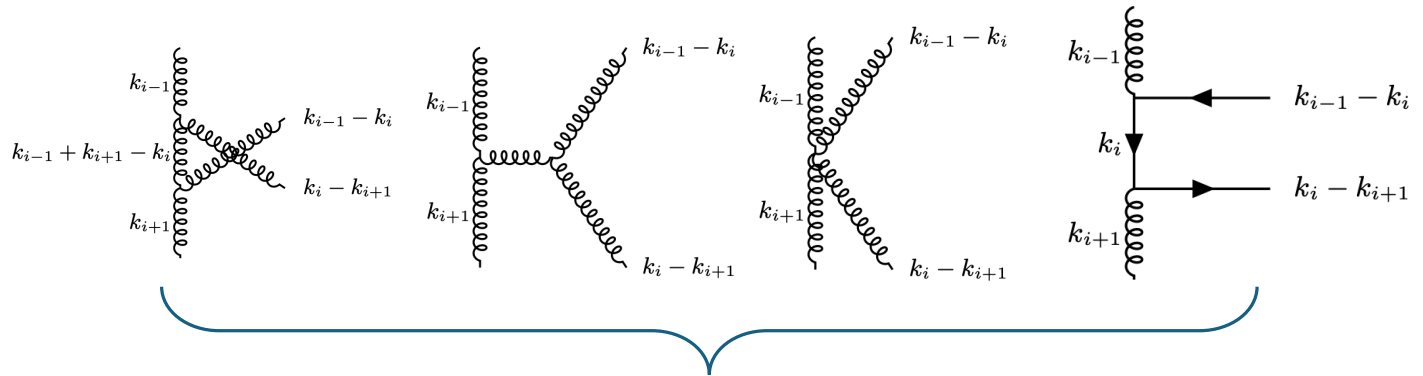


$$A_{2 \rightarrow n+2}^{\alpha\alpha'\beta\beta'\delta_1 \dots \delta_n} = \frac{sg^{n+2}}{\prod_{i=1}^{n+1} k_i^2} \Gamma_{p_1 \ell_0}^{\alpha\alpha' c_1} \left( \prod_{i=1}^n \gamma^{c_i \delta_i c_{i+1}}(k_i, k_{i+1}) \right) \Gamma_{p_2 \ell_{n+1}}^{\beta\beta' c_{n+1}}$$

Comments:

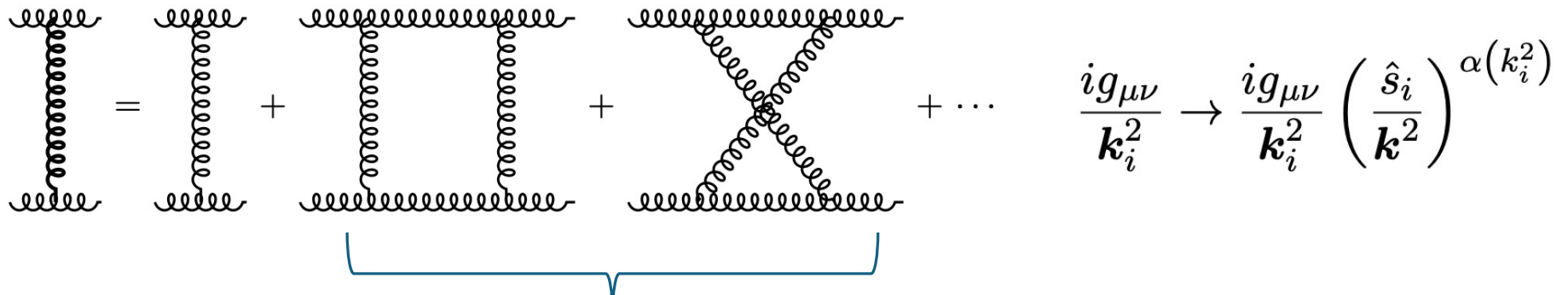


Leading in MRK



suppressed by factors of  $\rho, \lambda$   
Sub-leading logarithmic contributions

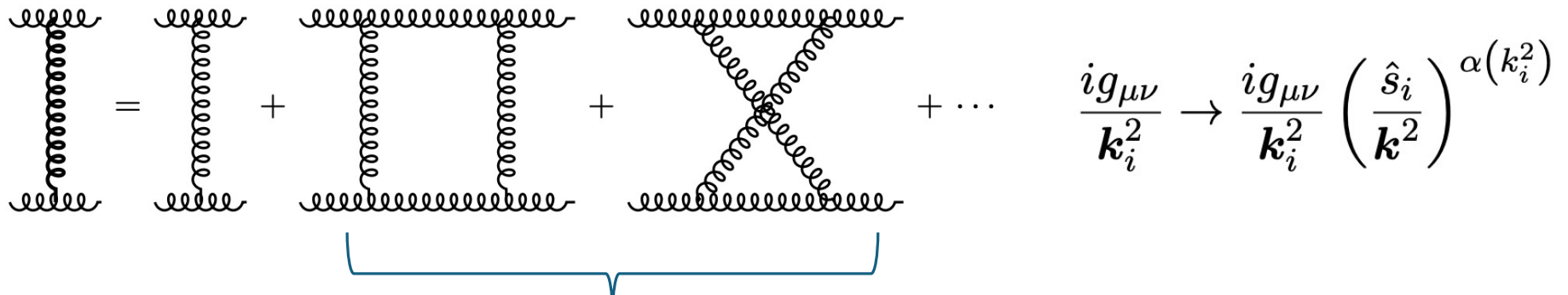
## Virtual graphs: Reggeization



One loop explicit computation in MRK

$$\alpha(t) = \alpha_S N_c t \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{1}{\mathbf{q}^2 (\mathbf{k} - \mathbf{q})^2} = \frac{\alpha_S N_c}{2\pi} \log \left( \frac{-t}{\lambda^2} \right)$$

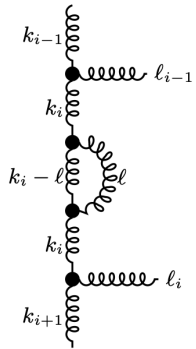
# Virtual graphs: Reggeization



One loop explicit computation in MRK

$$\alpha(t) = \alpha_S N_c t \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{1}{\mathbf{q}^2 (\mathbf{k} - \mathbf{q})^2} = \frac{\alpha_S N_c}{2\pi} \log \left( \frac{-t}{\lambda^2} \right)$$

Insert loop in the middle of ladder



$$l = \rho p_1 + \lambda p_2 + l,$$

$$\rho_{i-1} \gg \rho \gg \rho_i$$

$$\lambda_{i-1} \ll \lambda \ll \lambda_i$$

$$\sigma(\mathbf{k}_i) = \frac{g^2 N_c}{2} \frac{i}{k_i^2} \int \frac{d^4 \ell}{(2\pi)^4} \frac{i}{\ell^2} \frac{i}{(k - \ell)^2} C^\mu(k_i, k_i - \ell) C_\mu(k_i - \ell, k_i)$$

$$= \frac{g^2 N_c s}{4} \frac{i}{k_i^2} \int \frac{d^2 \ell}{(2\pi)^2} \int \frac{d\rho d\lambda}{(2\pi)^2} \frac{i}{s\lambda\rho - \ell^2 - i\epsilon} \frac{i}{-s\rho(\lambda_i - \lambda) - (k_i - \ell)^2 - i\epsilon}$$

Leading contribution from product of Lipatov vertices

$$\longrightarrow \times \frac{2s\mathbf{k}_i^4}{((\lambda_i - \lambda)s + i\epsilon)((\rho_i - \rho)s - i\epsilon)}$$

## Virtual graphs: Reggeization

$$\begin{aligned}
 & \text{Diagrammatic equation: } \text{Gluon line} = \text{Gluon line} + \text{One-loop box} + \text{Two-loop X} + \dots \\
 & \frac{ig_{\mu\nu}}{k_i^2} \rightarrow \frac{ig_{\mu\nu}}{k_i^2} \left( \frac{\hat{s}_i}{k^2} \right)^{\alpha(k_i^2)}
 \end{aligned}$$

One loop explicit computation in MRK

$$\alpha(t) = \alpha_S N_c t \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{1}{\mathbf{q}^2 (\mathbf{k} - \mathbf{q})^2} = \frac{\alpha_S N_c}{2\pi} \log \left( \frac{-t}{\lambda^2} \right)$$

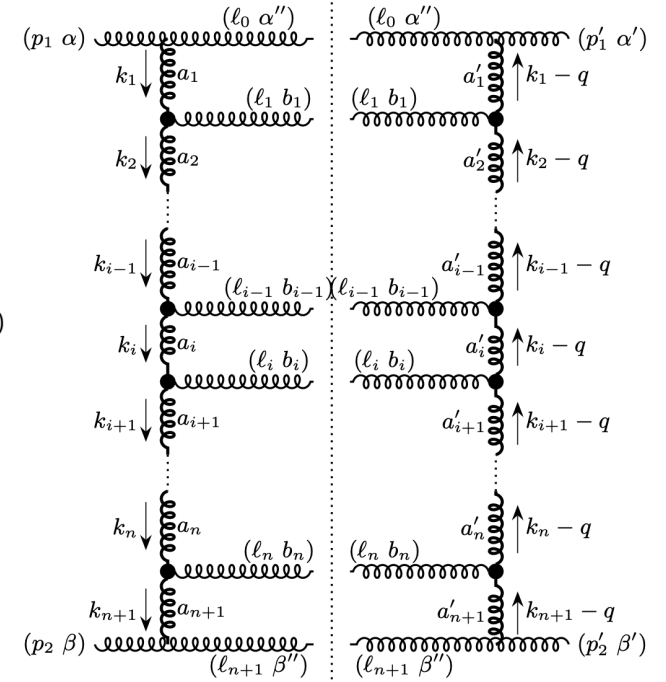
Integrating over  $\lambda$  using Cauchy's theorem,

$$\begin{aligned}
 \sigma(\mathbf{k}_i) &= -\frac{g^2 N_c}{2} \int \frac{d^2 \ell}{(2\pi)^2} \frac{\mathbf{k}_i^2}{\ell^2 (\mathbf{k}_i - \ell)^2} \int \frac{d\rho}{2\pi} \frac{1}{\rho} \\
 &= \log \left( \frac{\rho_{i-1}}{\rho_i} \right) \alpha(\mathbf{k}_i^2) \approx \log \left( \frac{\hat{s}_i}{k^2} \right) \alpha(\mathbf{k}_i^2)
 \end{aligned}$$

Exponentiation (reggeization ansatz) results in the desired form

## BFKL derivation

$$\begin{aligned}
 \text{Im } \mathcal{A}_{2 \rightarrow 2}^{\mu\mu'\nu\nu'} &= \eta^{\mu\mu'} \eta^{\nu\nu'} \sum_{n=0}^{\infty} \frac{4s^2 g^{2n+4} (-1)^n}{2^{n+2} (2\pi)^{3n+2}} \int \prod_{i=1}^n \left( \frac{d\rho_i}{\rho_i} d^2 \mathbf{k}_i \right) d\rho_{n+1} d^2 \mathbf{k}_{n+1} \delta [s\rho_{n+1} - \mathbf{k}_{n+1}^2] \\
 &\times P_S \left[ \sum_{b_i} G_n(b_1, \dots, b_i) G_n(b_1, \dots, b_i) \right] \frac{1}{\mathbf{k}_1^2 (\mathbf{q} - \mathbf{k}_1)^2} \left( \frac{1}{\rho_1} \right)^{\alpha(\mathbf{k}_1^2) + \alpha((\mathbf{q} - \mathbf{k}_1)^2)} \\
 &\times \prod_{i=1}^n C^{\mu_i}(k_i, k_{i+1}) C_{\mu_i}(q - k_i, q - k_{i+1}) \frac{1}{\mathbf{k}_{i+1}^2 (\mathbf{q} - \mathbf{k}_{i+1})^2} \left( \frac{\rho_i}{\rho_{i+1}} \right)^{\alpha(\mathbf{k}_{i+1}^2) + \alpha((\mathbf{q} - \mathbf{k}_{i+1})^2)}
 \end{aligned}$$



# BFKL derivation

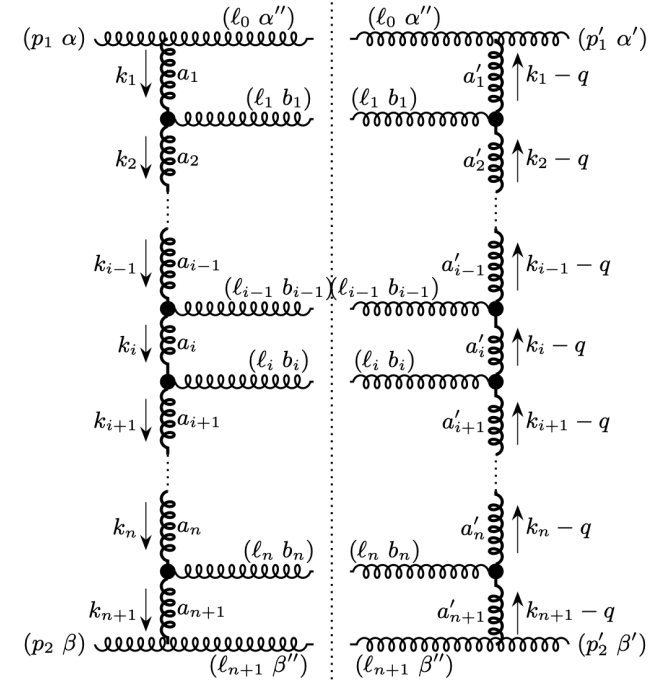
Singlet projection of color factors

$$P_S \left[ \sum_{b_i} G_n(b_1, \dots, b_i) G_n(b_1, \dots, b_i) \right] ;$$

A) From sum over color factors:

$$\begin{aligned} & \sum_{\alpha'' \beta'' b_1, \dots, b_n} f_{\alpha a_1 \alpha''} f_{\beta'' a_{n+1} \beta} f_{\alpha'' a'_1 \alpha'} f_{\beta' a'_{n+1} \beta'} \prod_{i=1}^n f_{a_i a_{i+1} b_i} f_{a'_i a'_{i+1} b_i} \\ &= \sum_{\alpha'' \beta'' b_1, \dots, b_n} \left( T^{a_1} T^{a'_1} \right)_{\alpha \alpha'} \left( T^{a'_{n+1}} T^{a_{n+1}} \right)_{\beta' \beta} \prod_{i=1}^n f_{a_i a_{i+1} b_i} f_{a'_i a'_{i+1} b_i} \end{aligned}$$

Tensor product of two adjoint representations: for  $N_c=3$ ,  $\mathbf{8} \otimes \mathbf{8} = (\mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27})_S \oplus (\mathbf{8} \oplus \mathbf{10} \oplus \mathbf{10})_A$



## BFKL derivation

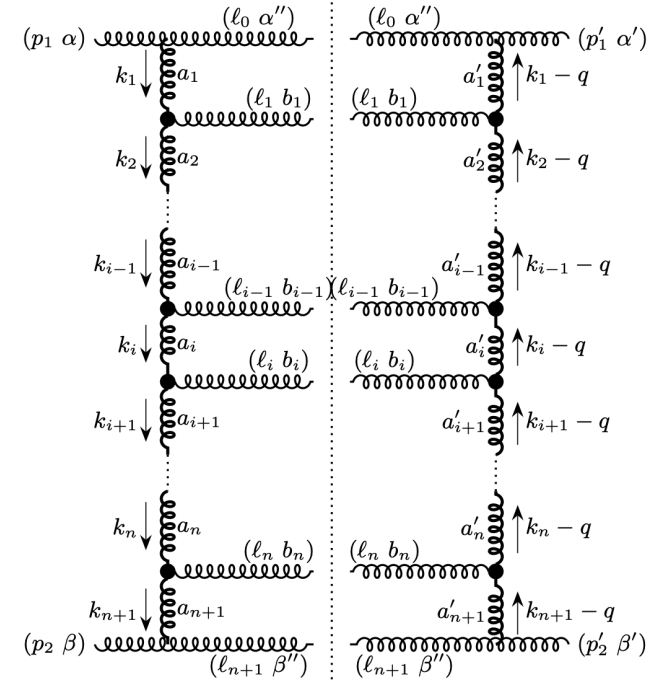
Singlet projection of color factors

$$P_S \left[ \sum_{b_i} G_n(b_1, \dots, b_i) G_n(b_1, \dots, b_i) \right];$$

B) To obtain singlet representation contract  $\alpha$  with  $\alpha'$  and  $\beta$  with  $\beta'$ :

$$\sum_{\alpha'' \beta'' b_1, \dots, b_n} \left( T^{a_1} T^{a'_1} \right)_{\alpha \alpha'} \left( T^{a_{n+1}} T^{a'_{n+1}} \right)_{\beta' \beta} \prod_{i=1}^n f_{a_i a_{i+1} b_i} f_{a'_i a'_{i+1} b_i}$$

$$\begin{aligned} \longrightarrow &= \text{Tr}(T^{a_1} T^{a'_1}) \text{Tr}(T^{a_{n+1}} T^{a'_{n+1}}) \prod_{i=1}^n f_{a_i a_{i+1} b_i} f_{a'_i a'_{i+1} b_i} \\ &= N_c^2 \delta^{a_1, a'_1} \delta^{a_{n+1}, a'_{n+1}} \prod_{i=1}^n f_{a_i a_{i+1} b_i} f_{a'_i a'_{i+1} b_i} = N_c^{n+2} (N_c^2 - 1) \end{aligned}$$

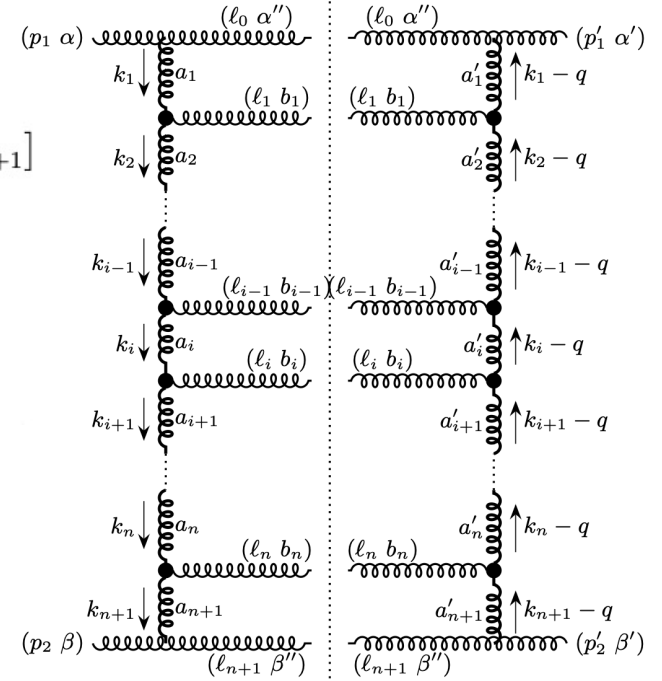


## BFKL derivation

$$\begin{aligned}
 \frac{\text{Im } \mathcal{A}_{2 \rightarrow 2}^{\mu\mu'\nu\nu'}}{\mathcal{A}_0^{\mu\mu'\nu\nu'}(s, t)} &= \frac{s t}{2} \sum_{n=0}^{\infty} \frac{g^{2n+2} N_c^{n+2} (N_c^2 - 1) (-1)^{n+1}}{(2\pi)^{3n+2}} \int \prod_{i=1}^n \left( \frac{d\rho_i}{\rho_i} d^2 \mathbf{k}_i \right) d\rho_{n+1} d^2 \mathbf{k}_{n+1} \delta [s\rho_{n+1} - \mathbf{k}_{n+1}^2] \\
 &\times \frac{1}{\mathbf{k}_1^2 (\mathbf{q} - \mathbf{k}_1)^2} \left( \frac{1}{\rho_1} \right)^{\alpha(\mathbf{k}_1^2) + \alpha((\mathbf{q} - \mathbf{k}_1)^2)} \prod_{i=1}^n \left( \mathbf{q}^2 - \frac{\mathbf{k}_i^2 (\mathbf{q} - \mathbf{k}_{i+1})^2 + \mathbf{k}_{i+1}^2 (\mathbf{q} - \mathbf{k}_i)^2}{(\mathbf{k}_i - \mathbf{k}_{i+1})^2} \right) \\
 &\times \frac{1}{\mathbf{k}_{i+1}^2 (\mathbf{q} - \mathbf{k}_{i+1})^2} \left( \frac{\rho_i}{\rho_{i+1}} \right)^{\alpha(\mathbf{k}_{i+1}^2) + \alpha((\mathbf{q} - \mathbf{k}_{i+1})^2)}.
 \end{aligned}$$

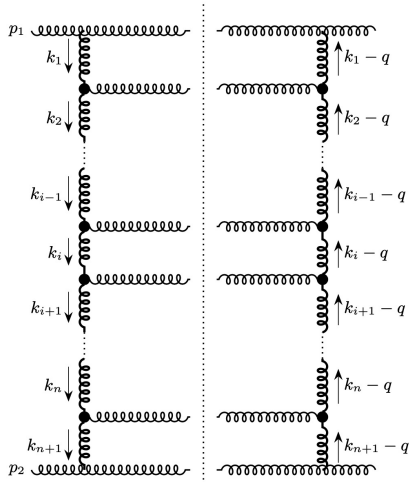


$$C^{\mu_i}(k_i, k_{i+1}) C_{\mu_i}(q - k_i, q - k_{i+1})$$



## 2 → N + 2 amplitude in the Regge limit: the BFKL equation

BFKL Pomeron: compound color singlet state of two reggeized gluons

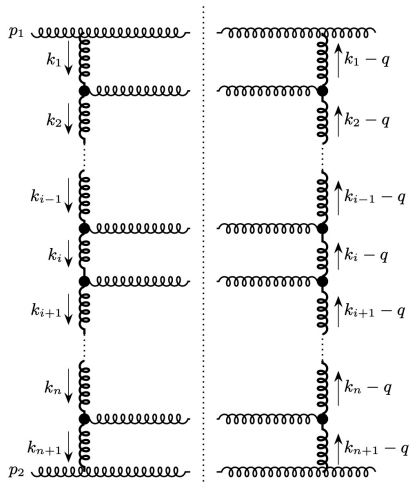


The imaginary part of this 2 → N + 2 amplitude simplifies greatly in Mellin space

$$\begin{aligned}
 \mathcal{M}_\ell(\mathbf{q}^2) &\equiv \int_1^\infty d\left(\frac{s}{\mathbf{k}^2}\right) \frac{\text{Im} \mathcal{A}_{2 \rightarrow 2}^{\mu\mu'\nu\nu'}(s, t)}{\mathcal{A}_0^{\mu\mu'\nu\nu'}(s, t)} \left(\frac{s}{\mathbf{k}^2}\right)^{-\ell-1} \\
 &= 2\pi\mathbf{q}^2 \alpha_s N_c^2 (N_c^2 - 1) \sum_{n=0}^\infty (-2\alpha_s N)^n \int \prod_{i=1}^{n+1} \frac{d^2\mathbf{k}_i}{(2\pi)^2} \frac{1}{\mathbf{k}_i^2 (\mathbf{q} - \mathbf{k}_i)^2} \frac{1}{\ell - \alpha(\mathbf{k}_i^2) - \alpha((\mathbf{q} - \mathbf{k}_i)^2)} \\
 &\quad \times \prod_{i=1}^n \left( \mathbf{q}^2 - \frac{\mathbf{k}_i^2 (\mathbf{q} - \mathbf{k}_{i+1})^2 + \mathbf{k}_{i+1}^2 (\mathbf{q} - \mathbf{k}_i)^2}{(\mathbf{k}_i - \mathbf{k}_{i+1})^2} \right)
 \end{aligned}$$

## 2 → N + 2 amplitude in the Regge limit: the BFKL equation

BFKL Pomeron: compound color singlet state of two reggeized gluons



Writing

$$\mathcal{M}_\ell(\mathbf{q}^2) \equiv \int_1^\infty d\left(\frac{s}{\mathbf{k}^2}\right) \frac{\text{Im} \mathcal{A}_{2 \rightarrow 2}^{\mu\mu'\nu\nu'}(s, t)}{\mathcal{A}_0^{\mu\mu'\nu\nu'}(s, t)} \left(\frac{s}{\mathbf{k}^2}\right)^{-\ell-1}$$

$$\text{with } \mathcal{M}_\ell(\mathbf{q}^2) = 2\pi \mathbf{q}^2 \alpha_s N_c^2 (N_c^2 - 1) \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2 (\mathbf{q} - \mathbf{k})^2} f_\ell(\mathbf{k}, \mathbf{q})$$

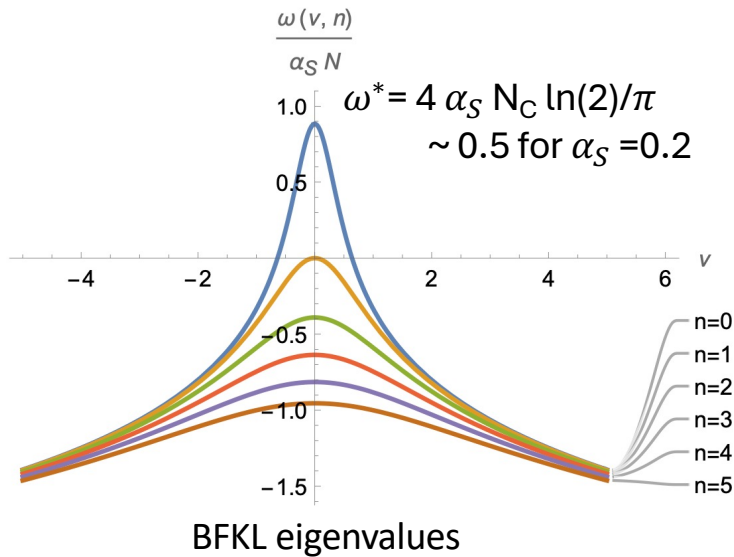
$f_\ell(\mathbf{k}, \mathbf{q})$  remarkably satisfies a simple BFKL integral equation

$$(\ell - \alpha(\mathbf{k}^2) - \alpha((\mathbf{q} - \mathbf{k})^2)) f_\ell(\mathbf{k}, \mathbf{q}) = 1 - 2\alpha_s N_c \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} \frac{f_\ell(\mathbf{k}', \mathbf{q})}{\mathbf{k}'^2 (\mathbf{q} - \mathbf{k}')^2} \left( \mathbf{q}^2 - \frac{\mathbf{k}^2 (\mathbf{q} - \mathbf{k}')^2 + \mathbf{k}'^2 (\mathbf{q} - \mathbf{k})^2}{(\mathbf{k} - \mathbf{k}')^2} \right)$$

➔  
for  $\mathbf{q}=0$

$$\ell f_\ell(\mathbf{k}) - 1 = 4\alpha_s N_c \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} \frac{1}{(\mathbf{k} - \mathbf{k}')^2} \left[ \frac{\mathbf{k}^2}{\mathbf{k}'^2} f_\ell(\mathbf{k}') - \frac{\mathbf{k}^2}{\mathbf{k}'^2 + (\mathbf{k} - \mathbf{k}')^2} f_\ell(\mathbf{k}) \right]$$

## 2 → N + 2 amplitude in the Regge limit: the BFKL equation



Solved as a simple eigenvalue equation as a function of Fourier conjugate variables -conformal spin  $\nu$  and azimuthal variable  $n$

Performing the inverse Mellin transform, one obtains

$$\sigma(s) \sim s^{\omega^*} = s^{0.5}$$

a much faster growth than the  $\text{Ln}^2(s)$  predicted by Froissart...

The fine print:

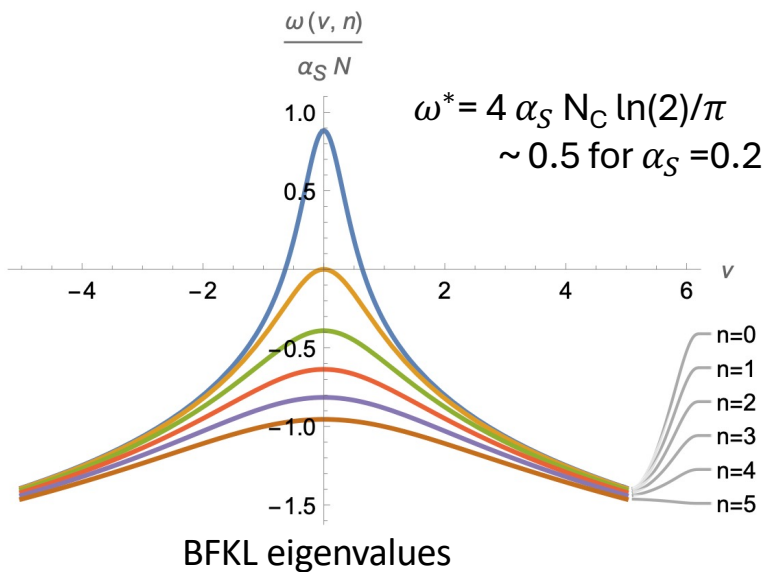
$$f_\ell(\mathbf{k}) \equiv \int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{\mathbf{k}^2}{\mathbf{p}^2} g_\ell(\mathbf{k}, \mathbf{p}) \quad \longrightarrow \quad \ell g_\ell(\mathbf{k}, \mathbf{p}) - (2\pi)^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}) = 4\alpha_S N_c \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \frac{1}{(\mathbf{k} - \mathbf{k}')^2} \left[ g_\ell(\mathbf{k}', \mathbf{p}) - \frac{\mathbf{k}^2}{\mathbf{k}'^2 + (\mathbf{k} - \mathbf{k}')^2} g_\ell(\mathbf{k}, \mathbf{p}) \right]$$

$$g_\ell(\mathbf{k}, \mathbf{p}) = \frac{1}{\sqrt{\mathbf{p}^2 \mathbf{k}^2}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu a_\ell(\nu, n) e^{i\nu(\lambda_k - \lambda_p)} e^{in(\phi_k - \phi_p)} \quad \frac{1}{\sqrt{\mathbf{p}^2 \mathbf{k}^2}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu a_\ell(\nu, n) \omega(\nu, n) e^{i\nu(\lambda_k - \lambda_p)} e^{in(\phi_k - \phi_p)}$$

$\lambda_k = \log(\mathbf{k}^2/\mu^2), \lambda_p = \log(\mathbf{p}^2/\mu^2)$

$$\ell a_\ell(\nu, n) - 2 = a_\ell(\nu, n) \omega(\nu, n) \implies a_\ell(\nu, n) = \frac{2}{\ell - \omega(\nu, n)}$$

## 2 → N + 2 amplitude in the Regge limit: the BFKL equation



This so-called LLx (leading log in x) result has been extended to NLLx accuracy.

Excellent review of state-of-the art:  
 Del Duca, Dixon, arXiv:2203.13026

After much sophisticated analysis, this gives

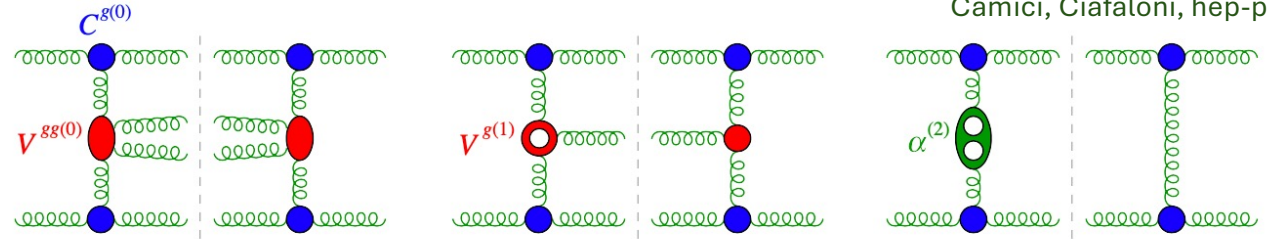
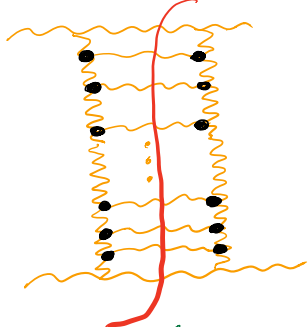
$$\sigma(s) \sim s^{0.3} \text{ - in reasonable agreement with HERA data}$$

Not the full story...

and only a preview to a richer, many-body picture

# 2 → N + 2 amplitude in the Regge limit: the NLL BFKL equation

BFKL Pomeron

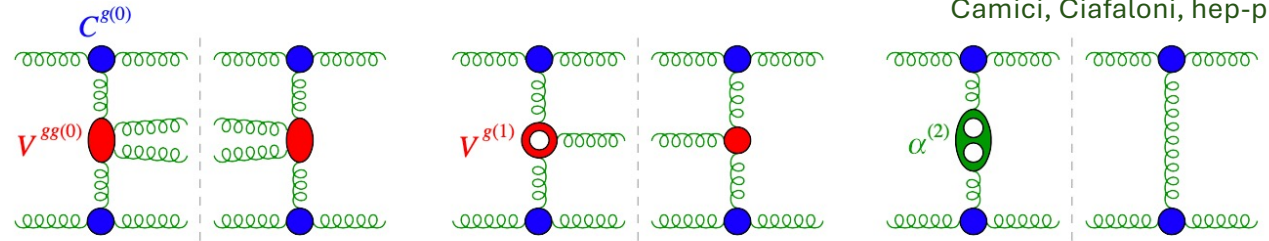
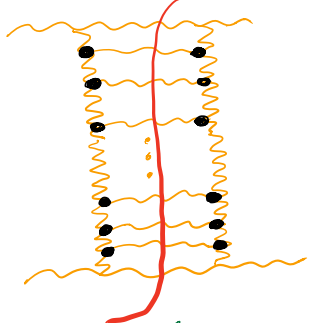


Fadin, Lipatov, hep-ph/9802290  
Camici, Ciafaloni, hep-ph/8903389

Regge factorization at NLL [ $\alpha_S(\alpha_S \ln(s/t))^n$ ]: Includes one loop corrections to the Lipatov vertex ( $V^{g(1)}$ ) and two loop corrections to the Regge trajectory ( $\alpha^{(2)}$ )

# 2 → N + 2 amplitude in the Regge limit: the NLL BFKL equation

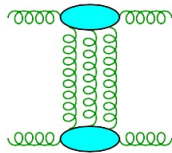
BFKL Pomeron



Fadin, Lipatov, hep-ph/9802290  
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Beyond NLL:

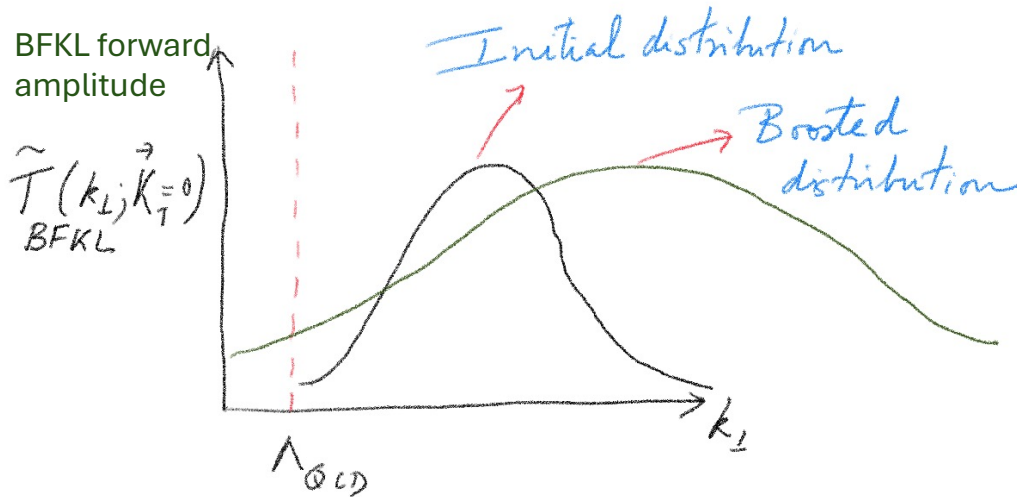


Three reggeized gluon exchange corresponds to Regge cut in angular momentum plane – this can be computed

Falcioni et al., arXiv: 2111.10664,  
arXiv:2112.11098

Figures from excellent review of state-of-the art:  
Del Duca, Dixon, arXiv:2203.13026

## BFKL: infrared diffusion and gluon saturation



For a fixed large  $Q^2$  there is an  $x_0(Q^2)$  such that below  $x_0$  the OPE breaks down...

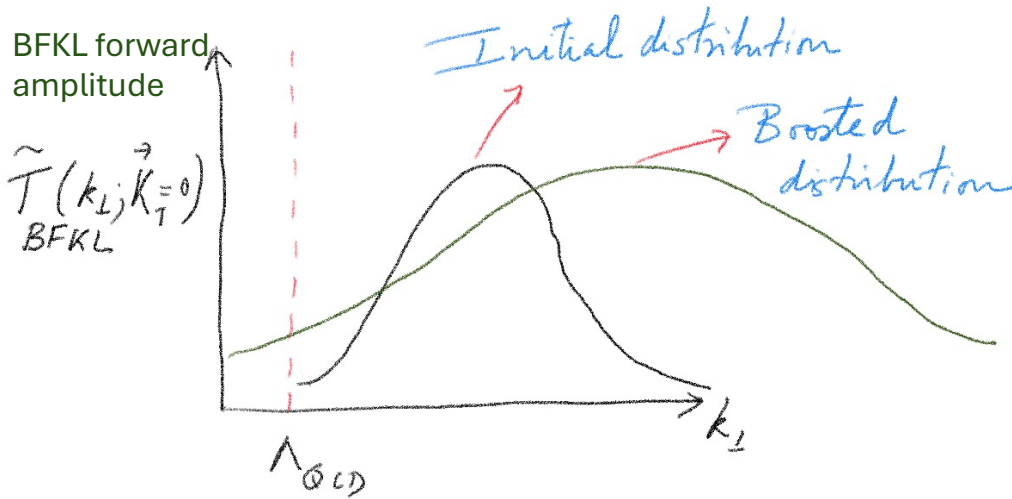
significant nonperturbative corrections in the leading twist coefficient and anomalous dimension functions due to diffusion of gluons to small values of transverse momentum.

A. H. Mueller, PLB 396 (1997) 251

NLLx BFKL does not cure infrared diffusion

Gluon saturation cures infrared diffusion

# BFKL: infrared diffusion and gluon saturation



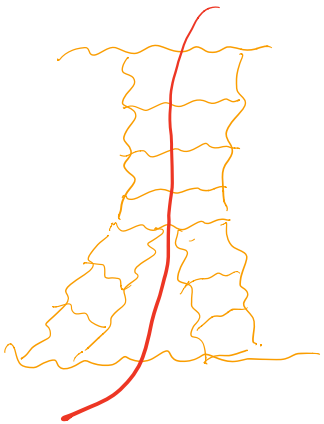
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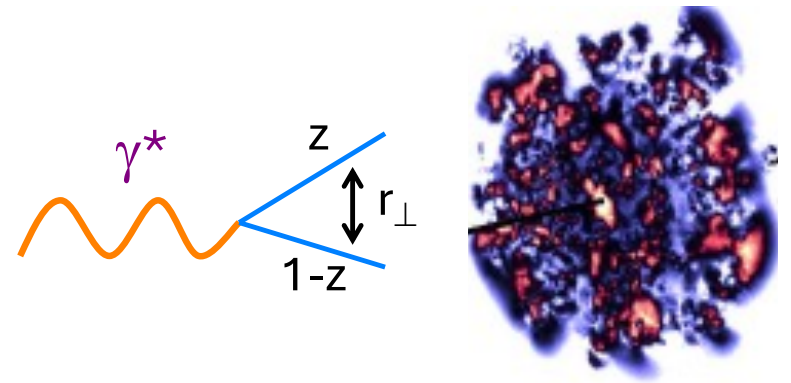
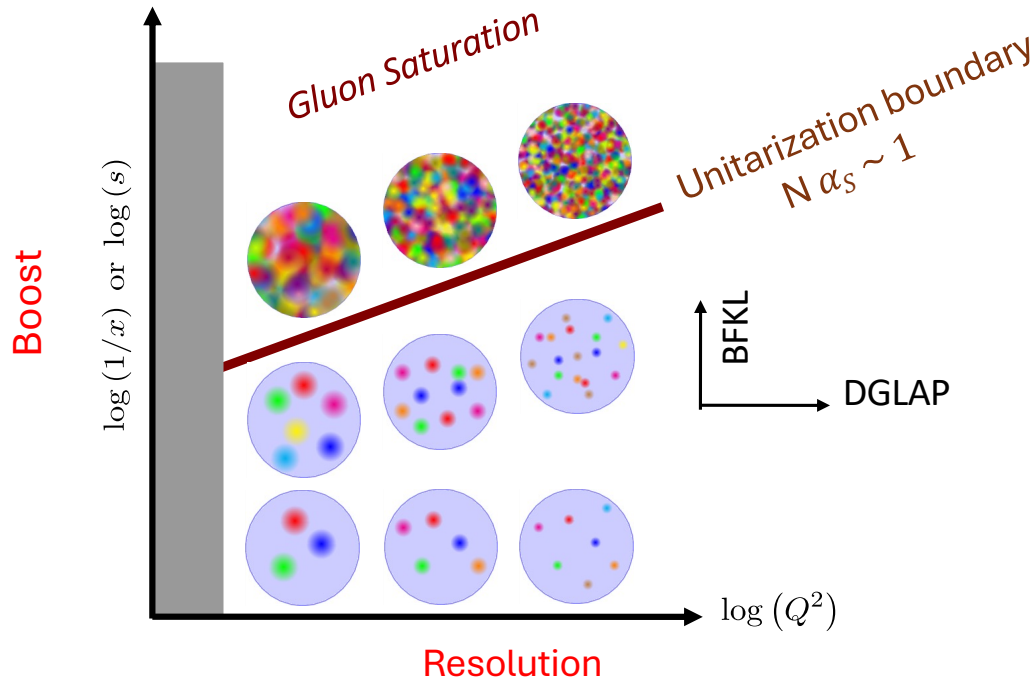
Gluon saturation cures infrared diffusion



+ other higher twist cuts of  $O(1)$  when gluon occupancy  $N \equiv \frac{xG_A(x, Q_S^2)}{2(N_c^2 - 1)\pi R_A^2 Q_S^2} = \frac{1}{\alpha_S(Q_S)}$

Classicalization when  $\alpha_S(Q_S) \ll 1$  for saturation scale  $Q_S \gg \Lambda_{QCD}$

# Maximal packing of gluons: Gluon saturation



Color screened wee partons  
 live on surface of sphere of radius  
 $1/k^+ \sim 1/k_{\perp} \sim 1/Q_S(x, b)$

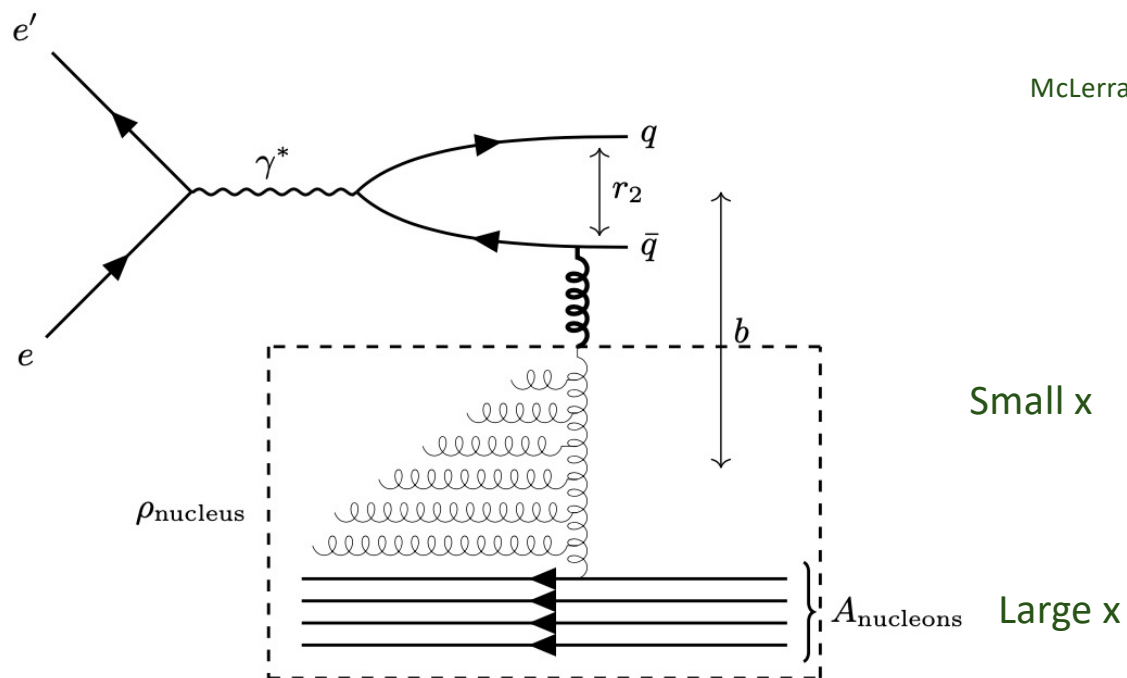
Emergent dynamics of semi-classical lumps that unitarize the cross-section described by the saturation scale  $Q_S(x, b)$ .

Due to asymptotic freedom, the many-body strong field dynamics of these lumps can be computed in weak coupling

The Color Glass Condensate: An EFT of gluon saturation

# Color Glass EFT

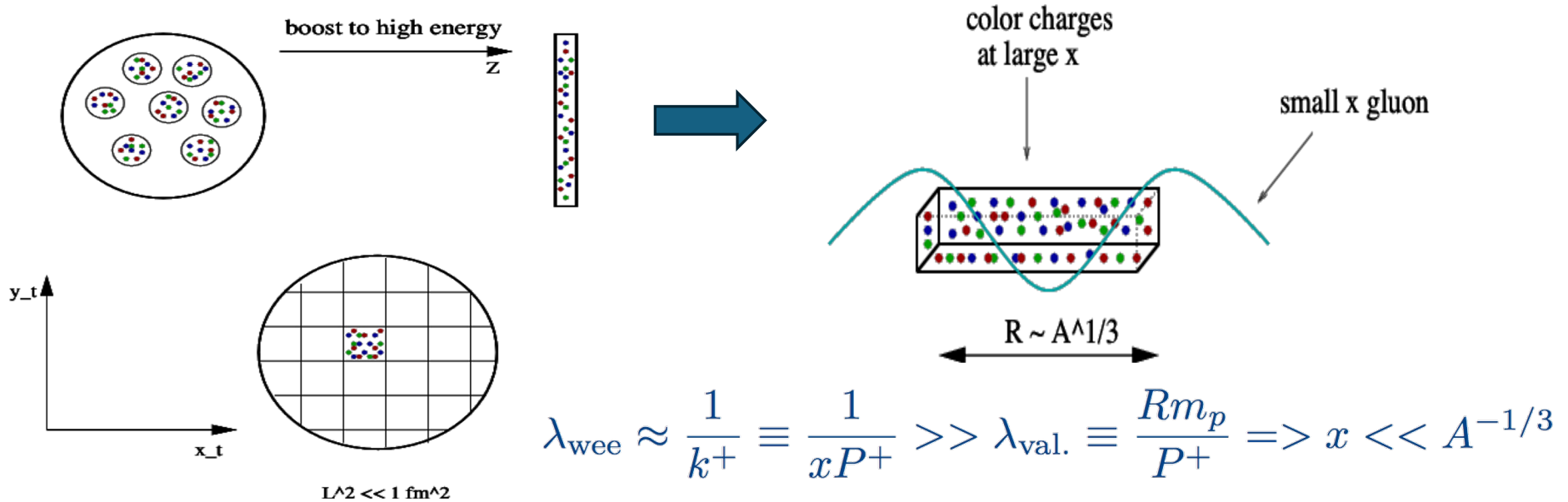
McLerran, RV (1994)



Fundamental basis for CGC EFT: large  $x$  modes are static on the light cone.  
small  $x$  "wee" modes are dynamical

This "Born-Oppenheimer" separation of time scales, allows one to focus on the dynamics of the highly occupied wee modes coupled to large  $x$  modes

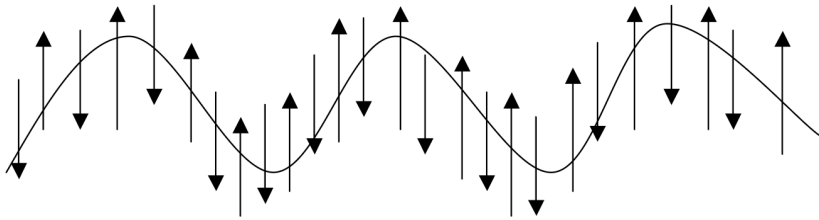
# Color Glass EFT-classical color charges



Coarse grained field theory:

# of random quarks in box of size  $\Delta x_{\perp} \sim \frac{1}{p_{\perp}}$  ;  $p_{\perp} \gg \Lambda_{\text{QCD}}$  =  $k_{\Delta x_{\perp}} = \frac{N_{\text{val}}}{\pi R^2} (\Delta x_{\perp})^2$

PATH INTEGRAL: SU(2):



Coarse graining -> Box of size  $1/p_t$  in transverse plane

Sum over SU(2) spins in box:

$$\sum_l v_l^{(k)} \sum_{m=-l}^l |l, m\rangle \langle l, m| \rightarrow \int d^3l e^{-2l^2/k}$$

Classical color/spin density:

$$l^a = (\Delta x_\perp)^2 \frac{1}{g} \rho^a(x_\perp) \Rightarrow$$

$$2 \frac{l^2}{k} = \frac{\pi R^2}{g^2 A} (\Delta x_\perp)^2 \rho^a \rho^a$$

Random walk of spins

$$2 \otimes 2 = 1 \oplus 3$$

$$(2 \otimes 2) \otimes 2 = (1 \oplus 3) \otimes 2 = 2 \oplus 2 + 4$$

$v_s^{(k)}$  Multiplicity of representation  $s$   
when  $k$  fundamental reps. are multiplied

$$v_s^{(k)} = v_{s-1}^{(k-1)} + v_{s+1}^{(k-1)} \rightarrow v_s^{(k)} \approx \frac{2^{k+1/2}}{k\sqrt{k\pi}} s e^{-s^2/2k}$$

Jeon, RV. PRD (2004)

Denote SU(3) representations by  $(m, n)$  :

$$3 = (1, 0) \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$$

$$\bar{3} = (0, 1) \quad \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Recursion relation for SU(3) :

$$\begin{array}{c}
 \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \\
 (m, n) \quad (1, 0) \quad = \quad (m+1, n) \\
 + \\
 \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \\
 (m-1, n+1) \quad (m, n-1)
 \end{array}$$

$N_{m,n}^{(k)}$  = multiplicity of  $(m, n)$  state in the  $k$ th iteration

$$N_{m,n}^{(k+1)} = N_{m-1,n}^{(k)} + N_{m+1,n-1}^{(k)} + N_{m,n-1}^{(k)}$$

Trinomial coefficients:  $G_{k;m,n} = \frac{k!}{\left(\frac{k+2m+n}{3}\right)! \left(\frac{k-m+n}{3}\right)! \left(\frac{k-m-2n}{3}\right)!}$

**Solution:**  $N_{m,n}^{(k)} = G_{k;m,n} + G_{k;m+3,n} + G_{k;m,n+3}$

$$-G_{k;m+2,n-1} - G_{k;m-1,n+2} - G_{k;m+2,n+2}$$

Use Stirling's formula...

$$D_2^{m,n} = \frac{(m^2 + mn + n^2)}{3} + (m + n) \quad \text{Quadratic Casimir}$$

$$D_3^{m,n} = \frac{1}{18}(m + 2n + 3)(n + 2m + 3)(m - n) \quad \text{Cubic Casimir}$$

$$N_{m,n}^{(k)} \approx \frac{27mn(m+n)}{k^3} \frac{3^{3/2+k}}{2k\pi} \exp(-3 D_2^{m,n}) (1 + 3 D_3^{m,n}/k^2)$$

Normalization:

$$\mathcal{N} \int dm dn d_{mn} N_{m,n}^{(k)} = 1$$

Dimension of representation

$$d_{mn} = \frac{1}{2}(m+1)(n+1)(m+n+2) \approx \frac{mn(m+n)}{2}$$

Prove:  $1 \approx \left(\frac{N_c}{k\pi}\right)^4 \int d^8\mathbf{Q} e^{-N_c\mathbf{Q}^2/k + 3D_3(\mathbf{Q})/k^2}$

Classical color charge:  $\mathbf{Q} = (Q_1, Q_2, \dots, Q_8)$ ;  $|\mathbf{Q}| = \sqrt{Q^a Q^a} = \sqrt{D_2^{m,n}}$   
 $D_3(\mathbf{Q}) = d_{abc} Q^a Q^b Q^c$

Proof: For any SU(3) representation,

$$d^8Q = d\phi_1 d\phi_2 d\phi_3 d\pi_1 d\pi_2 d\pi_3 dm dn \left( mn(m+n) \frac{\sqrt{3}}{48} \right)$$

Canonically conjugate "Darboux" variables

Canonical phase space volume of SU(3):

Johnson; Marinov;  
Alexeev, Fadeev, Shatashvili

$$\int \prod_{i=1}^3 d\phi_i d\pi_i = \frac{(2\pi)^3}{2} m n (m + n)$$

Hence, 
$$\int d^8 Q = \frac{(2\pi)^3}{32\sqrt{3}} \int dm dn (m^2 n^2 (m + n)^2)$$

Argument of probability integral has identical argument in m & n to RHS- hence can express in terms of LHS.

End of Proof.


## Color Glass- path integral over replicas


$$\mathcal{Z}[j] = \int [d\rho] W_{\Lambda^+}[\rho] \left\{ \frac{\int^{\Lambda^+} [dA] \delta(A^+) e^{iS_{\Lambda^+}[A, \rho] - \int j \cdot A}}{\int^{\Lambda^+} [dA] \delta(A^+) e^{iS_{\Lambda^+}[A, \rho]}} \right\}$$

For large nuclei,  $A \gg 1$ ,

Random walk in color space of SU(3)

$$W_{\Lambda^+} = \exp \left( - \int d^2 x_{\perp} \left[ \frac{\rho^a \rho^a}{2\mu_A^2} - \frac{d_{abc} \rho^a \rho^b \rho^c}{\kappa_A} \right] \right)$$

  
 Pomeron excitations

  
 Odderon excitations

Jeon, RV, hep-ph/0406169

$W_{\Lambda^+}[\rho]$  : nonpert. gauge inv. weight functional defined at initial  $x_0 = \Lambda^+ / P^+$

$S_{\Lambda^+}[A, \rho]$ : Yang-Mills action + gauge-inv. coupling of sources to fields (Wilson line)

## Saddle point of CGC action: Weizsäcker-Williams and color memory

$$A_i = 0 \quad \Big| \quad A_i = -\frac{1}{ig} U \partial_i U^\dagger$$

$x^- = 0$

$$D_i \frac{dA^{i,a}}{dy} = g\rho^a(x_t, y)$$

$$U = P \exp \left( i \int_y^\infty dy' \frac{\rho(x_t, y')}{\nabla_t^2} \right) \quad y = \ln(x^-/x_0^-)$$

solution of the YM-eqns: two **pure gauges (zero field strength)** separated by shockwave discontinuity

U represents the **color memory** effect

- a color rotation and  $p_T \sim Q_S$  kick experienced by quark-antiquark pair traversing the shock wave

Pate, Raclariu, Strominger, PRL (2017)

Ball, Pate, Raclariu, Strominger, RV,  
Annals of Physics (407 2019) 15

# Saddle point of CGC action: Weizsäcker-Williams and color memory

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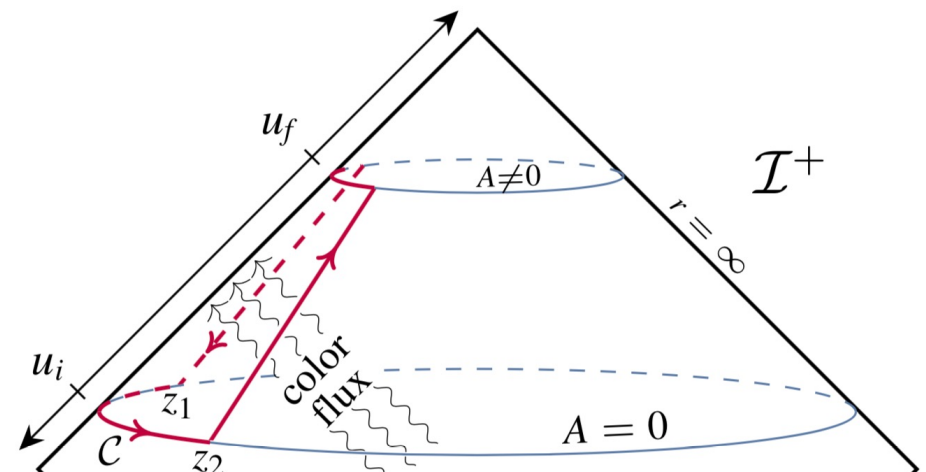
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solution of the YM-eqs: two **pure gauges (zero field strength)** separated by shockwave discontinuity

Stereographic projection of dynamics of glue to celestial sphere at “null infinity”

Mathematically analogous to an observable “gravitational memory effect” in GR

Deep connections to asymptotic BMS extension of Poincare group in gravity and to soft theorems



## Renormalization group evolution of color charges with x

$$\mathcal{O}_{\text{NLO}} = \left( \text{diagram 1} + \chi(x_{\perp}, y_{\perp}) \text{diagram 2} \right) \mathcal{O}_{\text{LO}}$$

$$\langle \mathcal{O}_{\text{LO}} + \mathcal{O}_{\text{NLO}} \rangle = \int [d\tilde{\rho}] W[\tilde{\rho}] [\mathcal{O}_{\text{LO}} + \mathcal{O}_{\text{NLO}}] = \int [d\tilde{\rho}] \left\{ \left[ 1 + \ln \left( \frac{\Lambda^+}{p^+} \right) \mathcal{H} \right] W_{\Lambda^+} \right\} \mathcal{O}_{\text{LO}}$$

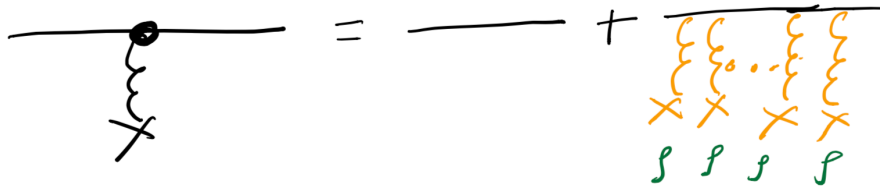
Independence of l.h.s on  $\Lambda^+ \Rightarrow$

$$\frac{\partial W[\tilde{\rho}]}{\partial Y} = \mathcal{H} W[\tilde{\rho}]$$

JIMWLK\* Hamiltonian –see Iancu lectures

\* JIMWLK: Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

## Shock wave propagators in the CGC



$$= S_0(p) T_q(p, q) S_0(q)$$

$$T_q(p, q) = (2\pi) \delta(p^- - q^-) \gamma^{\text{sign}(p^-)} \int d^2 z_\perp e^{i(p_\perp - q_\perp) z_\perp} V^{\text{sign}(p^-)}(z_\perp)$$



$$= G_0(p) T_g G_0(q)$$

$$T_g^{uv}(p, q) = (2\pi) \delta(p^- - q^-) (2p^-)^{\text{sign}(p^-)} \int d^2 z_\perp e^{-i(p_\perp - q_\perp) z_\perp} U^{\text{sign}(p^-)}(z_\perp)$$

McLerran, RV (1994, 1998)

Balitsky (1995)

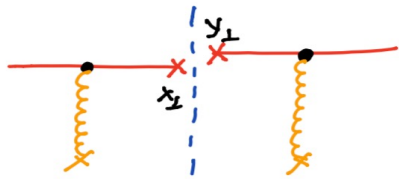
Ayala, Jalilian-Marian, McLerran, Venugopalan (1995)

Hebecker, Weigert (1997)

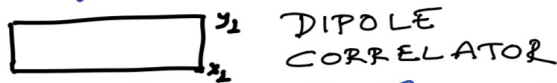
Balitsky, Belitsky (2001)

# An example: quark scattering on dense target (forward p+A)

LO



Amp.  $\times$  c.c.  $\rightarrow 1 - \frac{1}{N_c} \langle \text{Tr} (V(x_2) V^\dagger(y_1)) \rangle_{x_2}$



DIPOLE CORRELATOR  
BUILDING BLOCK OF HIGH ENERGY QCD

Balitsky-Kovchegov equation:  
RG evolution of dipole correlator  
(JIMWLK for  $N_c \gg 1$  and  $A \gg 1$ )

$$\frac{ds}{dy} = K_{\text{BFKL}} \otimes (s - ss)$$

NLO

if  $x_1 \gg x_2 \gg x_3$



These soft bremsstrahlung contributions absorbed in RG evolution of DIPOLE CORRELATOR

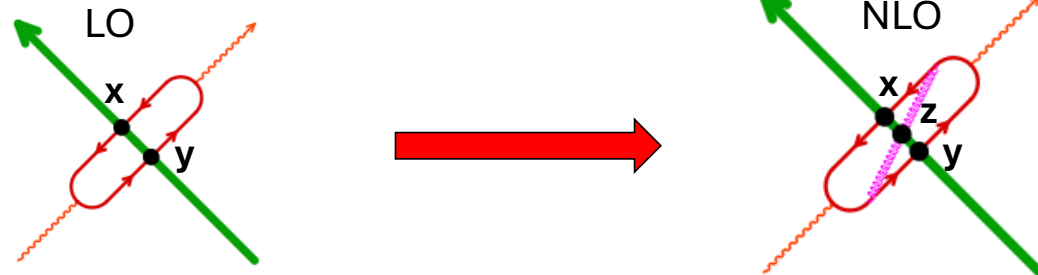
$$S_{y_1} \equiv \frac{1}{N_c} \langle \text{Tr} (V_{x_2} V_{y_1}^\dagger) \rangle_{y_1}$$

$$\rightarrow \frac{1}{N_c} \langle \text{Tr} (V_{x_2} V_{y_1}^\dagger) \rangle_{y_2}$$

SOFT BREMSSTRAHLUNG  
+ MULTIPLE SCATTERING = SHADOWING

# DIS: dipole evolution in gluon shockwave background

Example: 2-point “dipole” correlator:



Cross-section free rapidity divergence → RG equation for Wilson line correlators sourced by shockwave

$$\frac{\partial}{\partial Y} \langle \text{Tr}(V_x V_y^\dagger) \rangle_Y = -\frac{\alpha_S N_c}{2\pi^2} \int_{z_\perp} \underbrace{\frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2}}_{\text{BFKL kernel}} \langle \text{Tr}(V_x V_y^\dagger) - \frac{1}{N_c} \text{Tr}(V_x V_z^\dagger) \text{Tr}(V_z V_y^\dagger) \rangle_Y$$

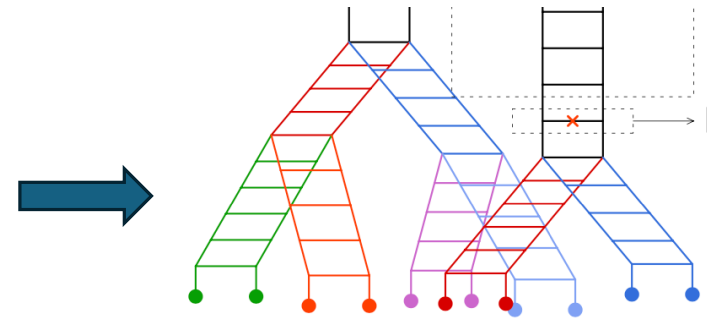
$Y = \text{Ln}(1/x)$

Closed form expression ( $A \gg 1, N_c \rightarrow \infty$ ):  
non-linear **Balitsky-Kovchegov (BK)** eqn.

Evolved shockwave scattering off dipole probe  
contains all-twist multi-pomeron “fan” diagrams

BFKL obtained as the leading twist result...

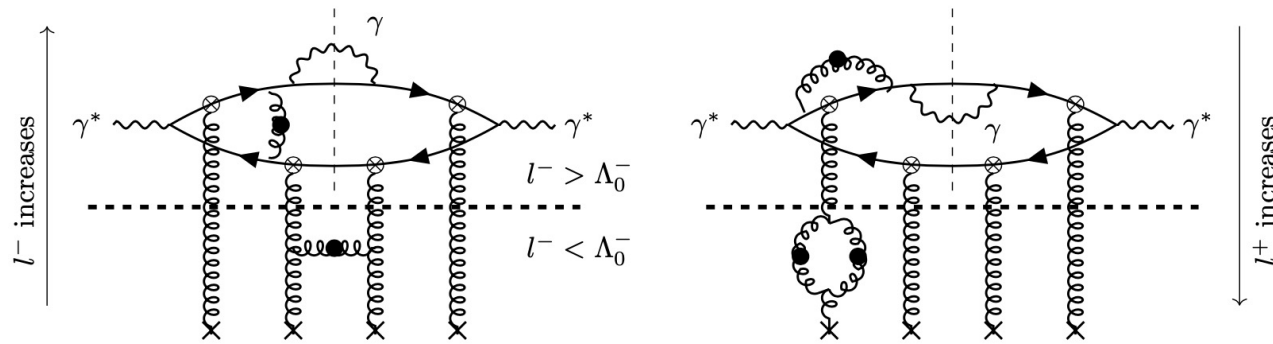
RG for n-point correlators – **JIMWLK** eqn



JIMWLK: Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

# CGC EFT state-of-the art: NLO+NLLx accuracy

Examples: l) photon+dijets in DIS

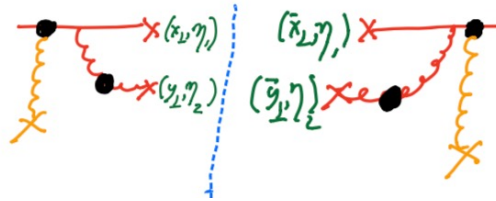


Roy, RV, arXiv:1911.04530

A large number of NLO computations by multiple groups using different methods

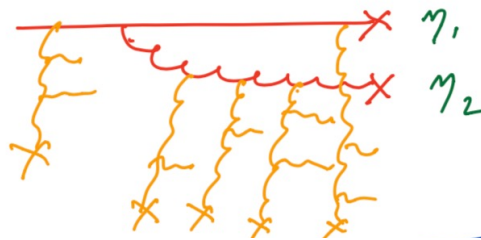
# Semi-inclusive final states

$PA \rightarrow qg + X$  CROSS-SECTION



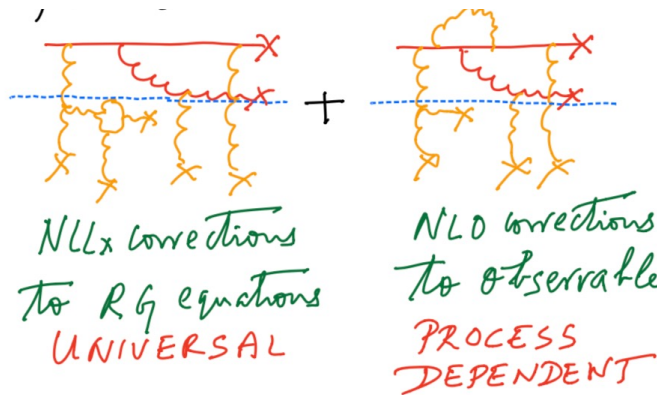
SENSITIVE TO BOTH 2-POINT  
DIPOLE WILSON LINE CORRELATOR  
& NLO 4-POINT QUADRUPOLE CORRELATOR

SOFT BREMSSTRAHLUNG ALSO  
SHADOWS SUCH CORRELATORS



JIMWLK RG equations  
describe bremsstrahlung of  
DIPLES, QUADRUPOLES, SEXTUPOLES, ...  
in an infinite hierarchy

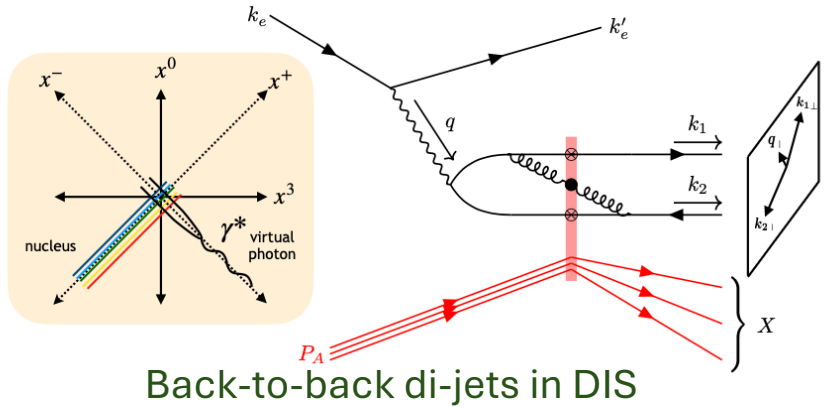
Sadly, LLx resummation  
not good enough for  
precision studies



- \* PARTS OF THIS ARE COMPUTED  
→ NLO JIMWLK "available"  
NLO BK more accessible
- \* PARTS IMPOSED: Eg, RUNNING  
COUPLING



# Gluon Weizsäcker-Williams distribution: complete NLO results



## Factorization of small-x TMDs to NLO accuracy

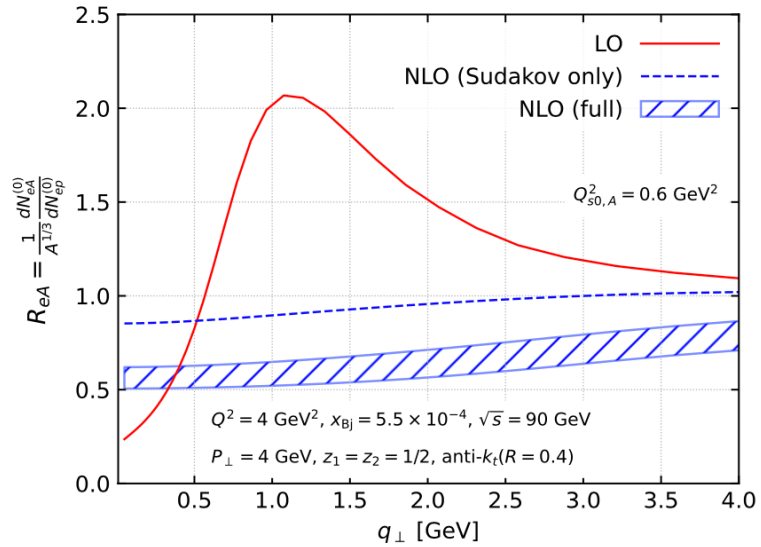
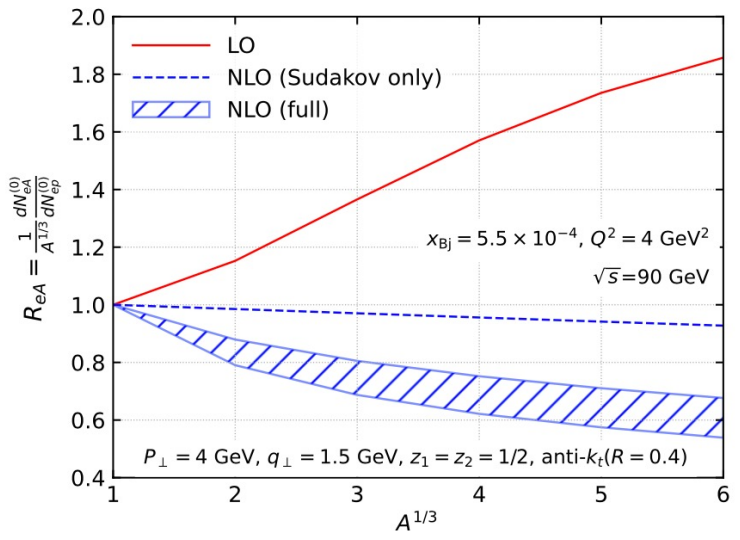
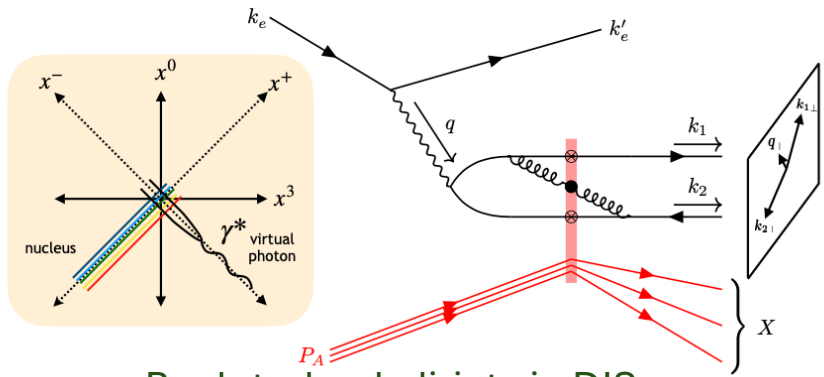
$$\begin{aligned}
 d\sigma^{(0),\lambda=\Gamma} &= \mathcal{H}_{\text{LO}}^{0,\lambda=\Gamma} \int \frac{d^2\mathbf{B}_\perp}{(2\pi)^2} \int \frac{d^2\mathbf{r}_{bb'}}{(2\pi)^2} e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_{bb'}} \hat{G}_{\eta_c}^0(\mathbf{r}_{bb'}, \mu_0) \mathcal{S}(\mathbf{P}_\perp^2, \mu_0^2) \\
 &\times \left\{ 1 + \frac{\alpha_s(\mu_R) N_c}{2\pi} f_1^{\lambda=\Gamma}(\chi, z_1, R) + \frac{\alpha_s(\mu_R)}{2\pi N_c} f_2^{\lambda=\Gamma}(\chi, z_1, R) + \alpha_s(\mu_R) \beta_0 \ln\left(\frac{\mu_R^2}{P_\perp^2}\right) \right\} \\
 &+ \mathcal{H}_{\text{LO}}^{0,\lambda=\Gamma} \int \frac{d^2\mathbf{B}_\perp}{(2\pi)^2} \int \frac{d^2\mathbf{r}_{bb'}}{(2\pi)^2} e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_{bb'}} \hat{h}_{\eta_c}^0(\mathbf{r}_{bb'}, \mu_0) \mathcal{S}(\mathbf{P}_\perp^2, \mu_0^2) \\
 &\times \frac{-2\chi^2}{1+\chi^4} \left\{ \frac{\alpha_s(\mu_R) N_c}{2\pi} [1 + \ln(R^2)] + \frac{\alpha_s(\mu_R)}{2\pi N_c} [-\ln(z_1 z_2 R^2)] \right\} + \mathcal{O}\left(\frac{q_\perp}{P_\perp}, \frac{Q_s}{P_\perp}, \alpha_s R^2, \alpha_s^2\right)
 \end{aligned}$$

$\hat{G}^0$  and  $\hat{h}^0$  respectively are unpolarized and linearly polarized **WW distributions**,

$\mathcal{S}$  the Sudakov soft factor resumming double+single logs in  $P_\perp/q_\perp$

$f_1$  and  $f_2$  are finite pure  $\mathcal{O}(\alpha_s)$  contributions

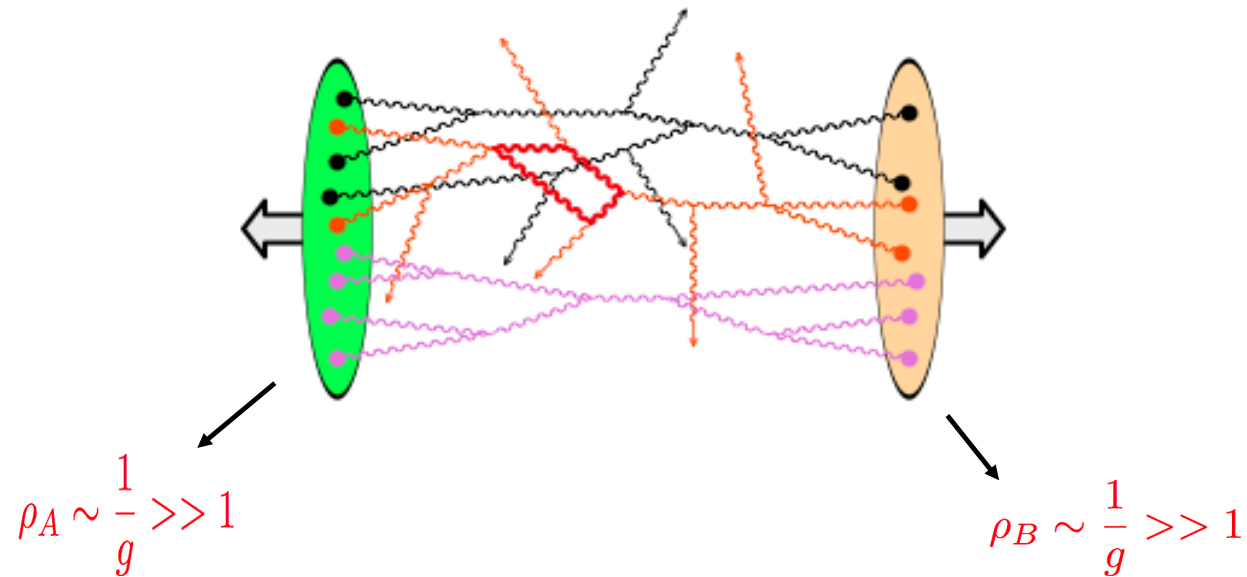
# Gluon Weizsäcker-Williams distribution: complete NLO results



Global analyses to extract “universal” TMDs from p+A collisions at the LHC and e+A collisions from the EIC

A long ways to go – since such NLO (NNLO in usual pQCD counting) analyses in p+A at the LHC are not available

## Particle production in presence of strong time-dependent sources




$P_n$  obtained from cut vacuum graphs in field theories with strong time dependent sources

# Probability of producing n particles in theory with sources

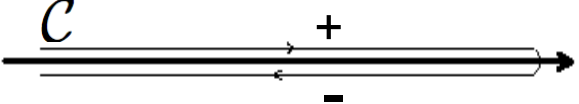
Lehmann-Symanzik-Zimmerman (LSZ)

$$\langle p_1 \cdots p_n \text{out} | 0_{\text{in}} \rangle = \frac{1}{Z^{n/2}} \int \left[ \prod_{i=1}^n d^4 x_i e^{ip_i \cdot x_i} (\partial_{x_i}^2 + m^2) \frac{\delta}{i\delta\rho(x_i)} \right] e^{i\mathcal{V}[\rho]}$$

n-particle probability  $P_n = \frac{1}{n!} \prod_{i=1}^n \int \frac{d^3 p_i}{2E_{p_i}} |\langle p_1 \cdots p_n \text{out} | 0_{\text{in}} \rangle|^2$


 $P_n = \frac{1}{n!} \mathcal{D}^n [j_+, j_-] \exp(iV[j_+] - iV^*[j_-]) |_{j_+ = j_- = j}$

Schwinger-Keldysh contour



$$D[j_+, j_-] \equiv \frac{1}{Z} \int_{x,y} G_{+-}^0(x,y) (\partial_x^2 + m^2) (\partial_y^2 + m^2) \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

$$\int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} e^{ip \cdot (x-y)}$$

$$G_{++}^0 = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$G_{--}^0 = \frac{-i}{p^2 - m^2 - i\epsilon}$$

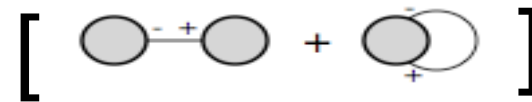
$$G_{+-}^0 = 2\pi\theta(-p^0) \delta(p^2 - m^2)$$

## N-particle distributions: inclusive multiplicity

$$\langle n \rangle = \sum_n n P_n \equiv D \left[ \underbrace{e^D e^{iV} e^{-iV}}_{e^{iV_{SK}}} \right]$$

Gelis, RV ; NPA776 (2006)135  
NPA 779 (2006)177

$$\langle n \rangle = \int_{x,y} Z G_{+-}^0(x,y) [\Gamma_+(x) \Gamma_-(y) + \Gamma_{+-}(x,y)]$$



$$\Gamma_{\pm}(x) = \frac{\partial_x^2 + m^2}{Z} \frac{\delta iV_{SK}}{\delta j_{\pm}(x)} \Big|_{j_+ = j_- = j}$$

One-point function in the background field

$$\Gamma_{+-}(x,y) = \frac{(\partial_x^2 + m^2)(\partial_y^2 + m^2)}{Z} \frac{\delta^2 iV_{SK}}{\delta j_{\pm}(x) \delta j_{\mp}(y)} \Big|_{j_+ = j_- = j}$$

Two-point function  
in the background field

Inclusive multiplicity at NLO in strong fields:  $O(g^0)$

$$\langle n \rangle = \int_{x,y} Z G_{+-}^0(x,y) [\Gamma_+(x) \Gamma_-(y) + \Gamma_{+-}(x,y)]$$

$$[ \text{diagram 1} + \text{diagram 2} ]$$

$$\langle n \rangle_{\text{NLO}} = \text{diagram 1} + \text{diagram 2}$$

$$\phi_c^{(0)} \phi_c^{(1)} \quad \langle \delta\phi_{\text{quant.}} \delta\phi_{\text{quant.}} \rangle$$

Product of classical field and 1-loop correction to classical field

Small fluctuation propagator in classical background field

Inclusive multiplicity to LO in strong fields:  $O(1/g^2)$

$$\langle n \rangle_{\text{LO}} = \text{[tree diagram with three nodes and a central line labeled } \frac{y}{+} \text{]} + \text{[tree diagram with three nodes and a central line labeled } \frac{x}{-} \text{]} \quad \text{but } (g\rho)^\infty \text{ means arbitrary number of insertions of sources } \rho$$

In the Schwinger Keldysh formalism, each node of a tree includes a sum over  $\pm$

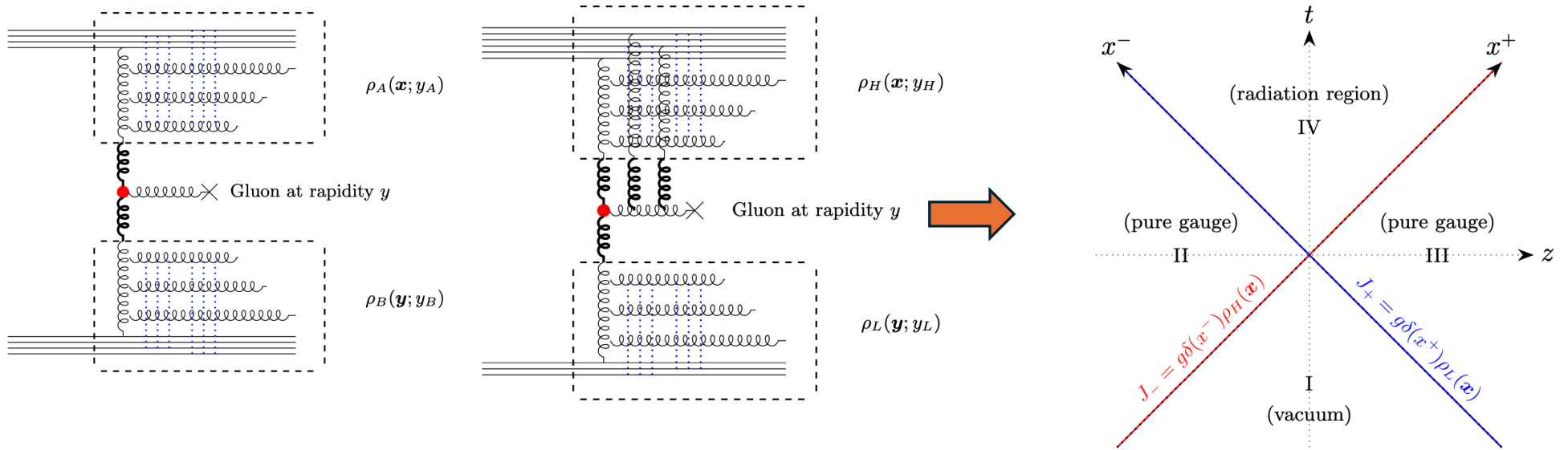
$$G_{i+}^0(x, y) - G_{i-}^0(x, y) = G_R^0(x, y); \quad i = \pm$$

Recursive use of this identity shows that sum of all tree diagrams is the *retarded* solution of classical equations of motion with

$$\lim_{x^0 \rightarrow -\infty} \phi_c(x) = 0$$

Hence, leading order result for the inclusive multiplicity in the strong fields in a heavy-ion collision given by solutions of the QCD Yang-Mills equations !

# Gluon shockwave collisions: Lipatov vertex and reggeization



Weizsäcker-Williams gluon radiation field in light cone gauge

$$a_i(k) = -\frac{2ig}{k^2 + i\epsilon} \int \frac{d^2\mathbf{q}_2}{(2\pi)^2} \left( q_{2i} - k_i \frac{\mathbf{q}_2^2}{\mathbf{k}^2} \right) \frac{\rho_L(\mathbf{q}_2)}{\mathbf{q}_2^2} \left( U(\mathbf{k} + \mathbf{q}_2) - (2\pi)^2 \delta^2(\mathbf{k} + \mathbf{q}_2) \right)$$

Lipatov vertex  
in  $A^- = 0$  gauge

reggeized gluons from  
semi-classical source dists.

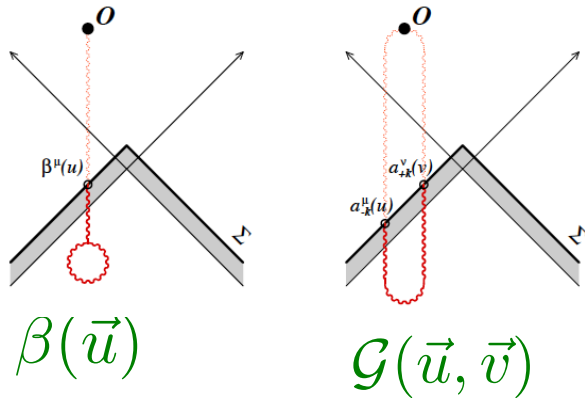
$$U(x^-, \mathbf{x}) \delta(x^+) = \exp \left( ig \int_{-\infty}^{x^-} dz^- \bar{A}_-(z^-, \mathbf{x}) \cdot T \right)$$

$$\bar{A}_\mu(x^-, \mathbf{x}) = -g \delta_{\mu-} \delta(x^-) \frac{\rho_H(\mathbf{x})}{\square_\perp}$$

$\ln(U) \rightarrow$  reggeized gluon

# QCD factorization of wee gluon distributions of the nuclei

Gelis, Lappi, RV (2008)

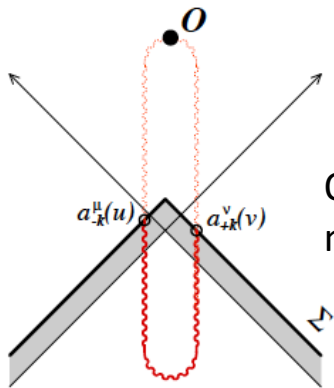


$$\mathcal{O}_{\text{NLO}} = \left[ \frac{1}{2} \int_{\vec{u}, \vec{v}} \mathcal{G}(\vec{u}, \vec{v}) \mathcal{T}_u \mathcal{T}_v + \int_{\vec{u}} \beta(\vec{u}) \mathcal{T}_u \right] \mathcal{O}_{\text{LO}}$$

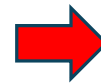
$$\mathcal{T}_u = \frac{\delta}{\delta A(\vec{u})} \quad \text{linear operator of source on initial "Cauchy" surface}$$

These 1- and 2-point have logs  
- are resummed to all orders by  
the JIMWLK Hamiltonian

$$\mathcal{O}_{\text{NLO}} = \left[ \ln \left( \frac{\Lambda^+}{p^+} \right) \mathcal{H}_1 + \ln \left( \frac{\Lambda^-}{p^-} \right) \mathcal{H}_2 \right] \mathcal{O}_{\text{LO}}$$



Quantum fluctuations that cross-talk between  
nuclei before the collision are suppressed

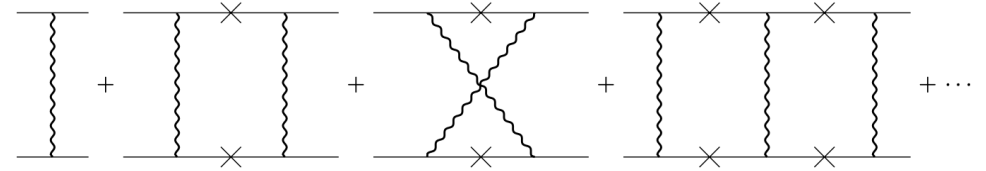


QCD factorization !

QCD-gravity double copy in Regge asymptotics

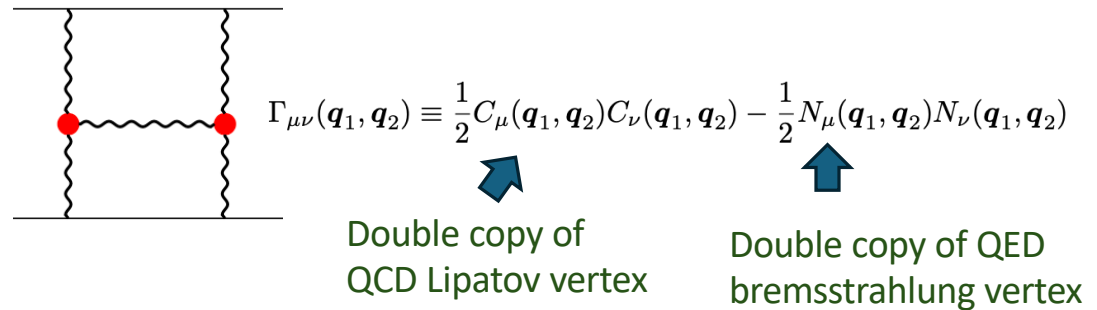
# Multiparticle production in gravity: amplitudes to shockwave collisions

A) In gravity, the dominant contribution at large impact parameters is Eikonal scattering (analogously to the CGC)



B) Gravitational Lipatov vertex is dominant radiative correction in Regge kinematics

Lipatov, PLB 116B (1982); JETP 82 (1982)



$$S = e^{2i(\delta_0 + \delta_1 + \delta_2 + \dots)}$$

$$\delta_0 = Gs \log\left(\frac{L}{b}\right),$$

↑  
Leading Eikonal term (real)

$$\delta_1 = \frac{6G^2 s}{\pi b^2} \log s,$$

↑  
Sub-leading quantum gravity correction  $\sim \frac{l_P^2}{b^2}$

$$\delta_2 = \frac{2G^3 s^2}{b^2} \left[ 1 + \frac{i}{\pi} \log s \left( \log \frac{L^2}{b^2} + 2 \right) \right]$$

↑  
Sub-leading loop contribution  $\sim \frac{R_S^2}{b^2}$  - includes absorptive piece

$$\delta_2 \gg \delta_1 \text{ for } R_S \gg l_P$$

# The BFKL equation in Einstein gravity

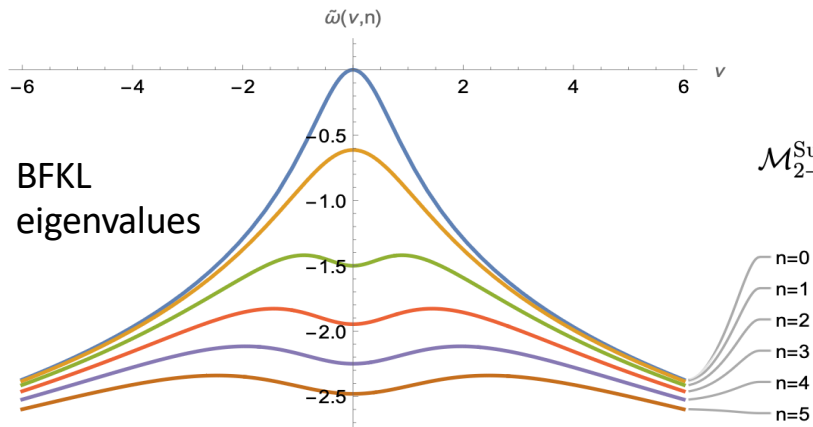
I. Rothstein, M. Saavedra, arXiv:2412.04428  
 H. Raj, RV, arXiv:2507.21252

The BFKL construction follows identically as in QCD...

Integral equation derived by Lipatov for the Mellin amplitude:  $\mathcal{M}_\ell(t) = \frac{t}{16} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2(\mathbf{q} - \mathbf{k})^2} f_\ell(\mathbf{k}, \mathbf{q})$

$$(\ell - \alpha(\mathbf{k}^2) - \alpha((\mathbf{q} - \mathbf{k})^2)) f_\ell(\mathbf{k}, \mathbf{q}) = 1 + \frac{\kappa^2}{4\pi} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \frac{f_\ell(\mathbf{k}', \mathbf{q})}{\mathbf{k}'^2(\mathbf{q} - \mathbf{k}')^2} \mathcal{K}_G(\mathbf{k}, \mathbf{k}')$$

$$C^{\mu_i \nu_i}(k_i, k_{i+1}) C_{\mu_i \nu_i}(q - k_i, q - k_{i+1})$$



$$\mathcal{M}_{2 \rightarrow 2}^{\text{Sudakov}+L^2} \sim \mathcal{M}^{\text{Born}} \left( -\frac{s}{t} \right)^{\kappa^2 t \log(-t/\Lambda_{\text{IR}})/8\pi^2} \frac{1}{3} \left[ 1 + \left( -\frac{s}{t} \right)^{\sqrt{-3\kappa^2 t/8\pi^2}} + \left( -\frac{s}{t} \right)^{-\sqrt{-3\kappa^2 t/8\pi^2}} \right]$$

Growth with energy is slower than the Born amplitude  $\propto s^2$

❖ *Very interestingly, the soft limit of the Lipatov vertex gives the ultrarelativistic limit of Weinberg's radiative amplitude for soft graviton emission*

# Lipatov vertex from shockwave collisions

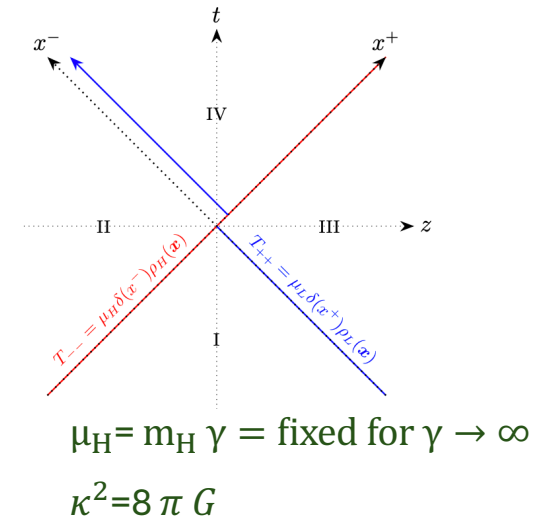
We will now sketch how the Lipatov vertex is recovered in shockwave collisions

Aichelburg-Sexl shockwave metric

$$ds^2 = 2dx^+dx^- - \delta_{ij}dx^i dx^j + f(x^-, \mathbf{x}) (dx^-)^2$$

$$\text{with } f(x^-, \mathbf{x}) = 2\kappa^2 \mu_H \delta(x^-) \frac{\rho_H(\mathbf{x})}{\square_\perp} = \frac{\kappa^2}{\pi} \mu_H \delta(x^-) \int d^2\mathbf{y} \ln \Lambda |\mathbf{x} - \mathbf{y}| \rho_H(\mathbf{y})$$

Soln of Einstein's eqns sourced by the EM tensor  $T_{\mu\nu} = \delta_\mu - \delta_\nu - \mu_H \delta(x^-) \rho_H(\mathbf{x})$



# Shockwave collisions: single shock background

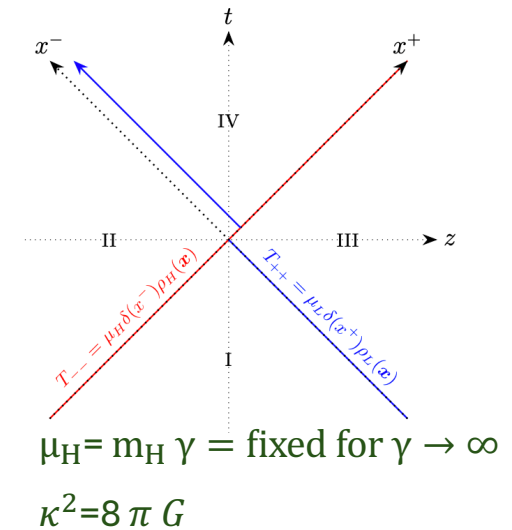
Linearizing around the metric  $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$

fixing light cone gauge  $h_{\mu+}=0$ , find

$$h_{ij}(x^+, x^-, \mathbf{x}) = V(x^-, \mathbf{x}) h_{ij}(x^+, x^- = x_0^-, \mathbf{x})$$

with the gravitational Wilson line  $V(x^-, \mathbf{x}) \equiv \exp\left(\frac{1}{2} \int_{x_0^-}^{x^-} dz^- \bar{g}_{--}(z^-, \mathbf{x}) \partial_+\right)$

Exactly analogous to the QCD case  
with  $A_- \rightarrow g_{--}$  and  $T^a \rightarrow \partial_+$



Melville, Nachulich, Schnitzer, White,  
arXiv:1306.6019

See also Saotome and Akhoury, arXiv:1210.8111

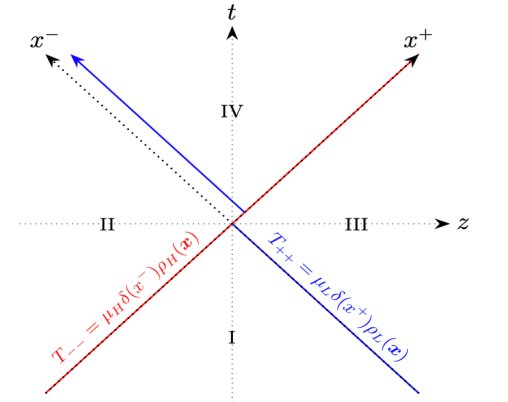
## Shockwave collisions: “dilute-dense” approximation

Now consider the interaction of the “dilute” source  $\rho_L$  with the dense  $\rho_H$  shockwave:

$$T_{\mu\nu} = \delta_{\mu-}\delta_{\nu-}\mu_H\delta(x^-)\rho_H(\mathbf{x}) + \delta_{\mu+}\delta_{\nu+}\mu_L\delta(x^+)\rho_L(\mathbf{x})$$

Solve for metric in region IV – forward lightcone

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \quad \bar{g}_{--} = 2\kappa\mu_H\delta(x^-)\frac{\rho_H(\mathbf{x})}{\square_{\perp}}$$



$$\mu_H = m_H \gamma = \text{fixed for } \gamma \rightarrow \infty$$

$$\kappa^2 = 8\pi G$$

We decompose the perturbation  $h_{\mu\nu}$  into a term linear in  $\rho_L$  and one bi-linear in  $\rho_L\rho_H$  (dilute-dilute limit)

Linearized Einstein’s equations in light-cone gauge ( $h_{+\mu}=0$ ) take the form

$$\bar{g}_{--}\partial_+^2\tilde{h}_{ij} - \square_{\perp}\tilde{h}_{ij} = \kappa^2 \left[ \left(2\partial_i\partial_j - \square_{\perp}\delta_{ij}\right)\frac{1}{\partial_+^2}T_{++} + 2T_{ij} - \delta_{ij}T - \frac{2}{\partial_+} \left(\partial_iT_{+j} + \partial_jT_{+i} - \delta_{ij}\partial_kT_{+k}\right) \right]$$

$$\tilde{h}_{ij} \equiv h_{ij} - \frac{1}{2}\delta_{ij}h \text{ where } h = \delta_{ij}h_{ij}$$

## Shockwave collisions in general relativity: geodesics

Unlike QCD case, sub-eikonal contributions  $T_{+i}, T_{ij}$  are required for consistency of equations of motion

These are not uniquely fixed by energy-momentum conservation, the dynamics of the sources is needed to fix this. In the point particle approximation,

$$T^{\mu\nu}(x) = \frac{\mu_L}{\sqrt{-g}} \int_{-\infty}^{\infty} d\lambda \dot{X}^\mu \dot{X}^\nu \delta^{(4)}(x - X(\lambda))$$

Solution of the corresponding null geodesic equations  $\ddot{X}^\mu + \Gamma_{\nu\rho}^\mu \dot{X}^\nu \dot{X}^\rho = 0$ ,  $g_{\nu\rho} \dot{X}^\nu \dot{X}^\rho = 0$

in shockwave background given by  $X^- = \lambda$ ,  $X^i = b^i - \kappa^2 \mu_H X^- \Theta(X^-) \frac{\partial_i \rho_H(\mathbf{b})}{\square_\perp}$

$$X^+ = -\kappa^2 \mu_H \Theta(X^-) \frac{\rho_H(\mathbf{b})}{\square_\perp} + \frac{\kappa^4 \mu_H^2}{2} X^- \Theta(X^-) \left( \frac{\partial_i \rho_H(\mathbf{b})}{\square_\perp} \right)^2$$

These geodesic solutions allow us to reconstruct the required components of the stress-energy tensor

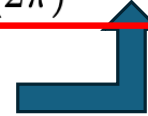
## Shockwave collisions in general relativity: Lipatov vertex

Solving eqns of motion, taking the Fourier transform, and putting the graviton momenta on-shell, one obtains, to  $O(\rho_L \rho_H)$

Gravitational  
radiational field

$$\tilde{h}_{ij}^{(2)}(k) = \frac{2\kappa^3 \mu_H \mu_L}{k^2 + i\epsilon k^-} \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} \Gamma_{ij}(\mathbf{q}_1, \mathbf{q}_2) \frac{\rho_H}{\mathbf{q}_1^2} \frac{\rho_L}{\mathbf{q}_2^2}$$

Gravitational Lipatov vertex

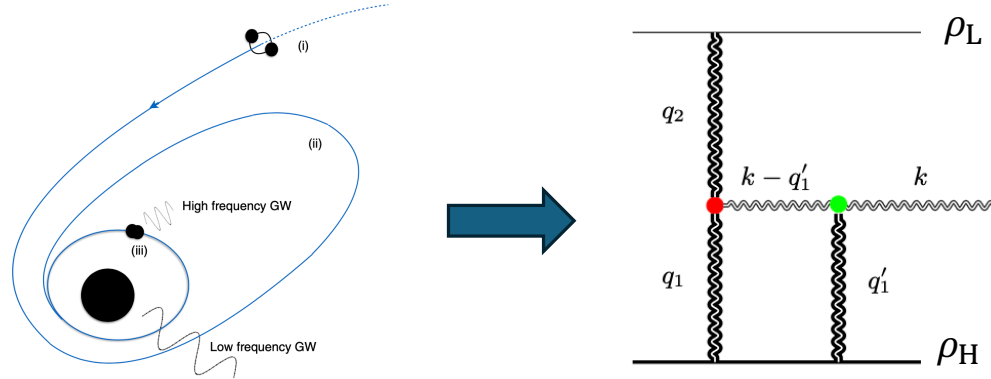


likewise for other components, recovering  $\Gamma_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) \equiv \frac{1}{2} C_\mu(\mathbf{q}_1, \mathbf{q}_2) C_\nu(\mathbf{q}_1, \mathbf{q}_2) - \frac{1}{2} N_\mu(\mathbf{q}_1, \mathbf{q}_2) N_\nu(\mathbf{q}_1, \mathbf{q}_2)$

Compare to gauge theory  
radiation field

$$a_i(k) = \frac{g^3}{k^2 + i\epsilon k^-} \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} C_i(\mathbf{q}_1, \mathbf{q}_2) \frac{\rho_H \cdot T}{\mathbf{q}_1^2} \frac{\rho_L}{\mathbf{q}_2^2}$$

# Shockwave collisions: first dilute-dense “tidal” correction



M. Fite, H. Raj, RV, in preparation

The shockwave computation can be extended straightforwardly to  $O(\rho_L \rho_H^2)$

$$\tilde{h}_{ij}^{(1,2)}(k) = \frac{2\mu_L \mu_H^2 \kappa^6 (-ik^-)}{k^2 + i\epsilon k^-} \int \frac{d^2 \mathbf{q}_1 d^2 \mathbf{q}'_1 d^2 \mathbf{q}_2}{(2\pi)^4} \delta^{(2)}(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}'_1 - \mathbf{q}_2) \frac{\tilde{\rho}_H(\mathbf{q}_1)}{q_1^2} \frac{\tilde{\rho}_H(\mathbf{q}'_1)}{q_1'^2} \frac{\tilde{\rho}_L(\mathbf{q}_2)}{q_2^2} V_{ij}(\mathbf{q}_1, \mathbf{q}'_1, \mathbf{q}_2) .$$

$$\frac{q_2^2}{k^2} \tilde{S}_{ij}(\mathbf{q}_1, \mathbf{q}'_1, \mathbf{q}_2) - \Gamma_{ij}(\mathbf{q}_1, \mathbf{q}_2)$$

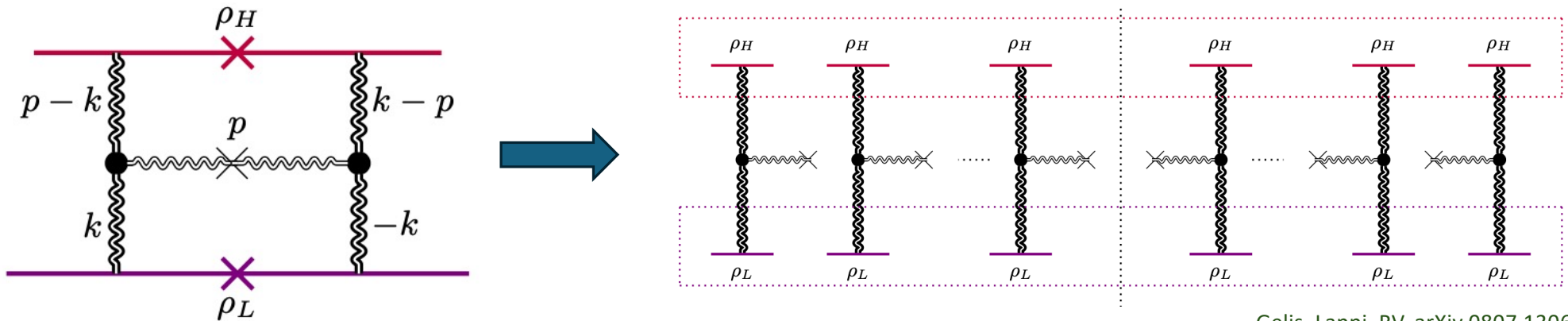
$$\alpha (-2k_i k_j + \mathbf{k}^2 \delta_{ij}) + (q_{1i} q'_{1j} + q'_{1i} q_{1j} - \delta_{ij} \mathbf{q}_1 \cdot \mathbf{q}'_1) + \beta (k_i s_j + k_j s_i - \delta_{ij} \mathbf{k} \cdot \mathbf{s})$$

$$\alpha = \frac{1}{2} + \frac{\mathbf{s} \cdot \mathbf{q}_2 - \mathbf{q}_1 \cdot \mathbf{q}'_1}{k^2} + \frac{4(\mathbf{q}_1 \cdot \mathbf{q}_2)(\mathbf{q}'_1 \cdot \mathbf{q}_2)}{k^4} \quad \beta = 1 + \frac{2 \mathbf{s} \cdot \mathbf{q}_2}{k^2} \quad \mathbf{s} \equiv \mathbf{q}_1 + \mathbf{q}'_1 = \mathbf{k} - \mathbf{q}_2 .$$

## Multi-graviton Lipatov radiation a la CGC EFT in GR?

In QCD, multi-gluon radiation in shockwave scattering (to LLx accuracy) is given by

$$\left\langle \frac{d^n N_n}{d^3 \mathbf{p}_1 \cdots d^3 \mathbf{p}_n} \right\rangle_{\text{LLog}} = \int [D\rho_1] [D\rho_2] W_{Y_1}[\rho_1] W_{Y_2}[\rho_2] \frac{dN}{d^3 \mathbf{p}_1} \Big|_{\text{LO}} \cdots \frac{dN}{d^3 \mathbf{p}_n} \Big|_{\text{LO}}$$



Gelis, Lappi, RV, arXiv 0807.1306

Corresponding “n-particle” distribution is a negative binomial distribution (NBD)  $P_{n;r} = \frac{\Gamma(n+r)}{\Gamma(r)\Gamma(n+1)} \frac{\bar{n}^n r^r}{(\bar{n}+r)^{n+r}}$

Can a similar t-channel fractionation occur in GR in the strong field regime, as  $b \rightarrow R_S$  ?

with  $r = \zeta \frac{(N_c^2 - 1) S_\perp Q_S^2}{2\pi}$

Gelis, Lappi, McLerran, arXiv: 0905.3234

# Multi-graviton radiation: generalized Susskind-Glogower squeezed state?

Stasto, Raj, RV: arXiv 2605.03038

If GR radiation is a NBD, this distribution corresponds to a squeezed state

$$|z; r\rangle = (1 - |z|^2)^{r/2} e^{za^\dagger \sqrt{\hat{N}+r}} |0\rangle \quad \text{where } |z|^2 = p = \frac{\bar{n}}{\bar{n}+r} \quad \text{and } \hat{N} = a^\dagger a$$

which is an eigenstate of the (generalized) Susskind-Glogower operator (gSG)  $\hat{A} = a \sqrt{\frac{1}{\hat{N} + r - 1}}$

$$\hat{A}|z; r\rangle = z|z; r\rangle$$

For NBD parameter  $r=1$ , usual Susskind-Glogower-Barnett-Pegg phase operator  $\hat{A}_{r=1}|\phi\rangle = e^{i\phi}|\phi\rangle$  with  $|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$ .

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For a pure single-mode squeezed state,  $\Delta X^2 \Delta P^2 = \frac{\hbar^2}{4}$

In the gSG case of interest to us,  $(\Delta X)^2 = (\delta + \frac{1}{2}) e^{-2\xi}$ , and  $(\Delta P)^2 = (\delta + \frac{1}{2}) e^{+2\xi}$  with  $\sqrt{(\Delta X)^2(\Delta P)^2} = \frac{1}{2} + \delta$ .  
 $\xi = \frac{1}{4} \ln \frac{(\Delta P)^2}{(\Delta X)^2}$

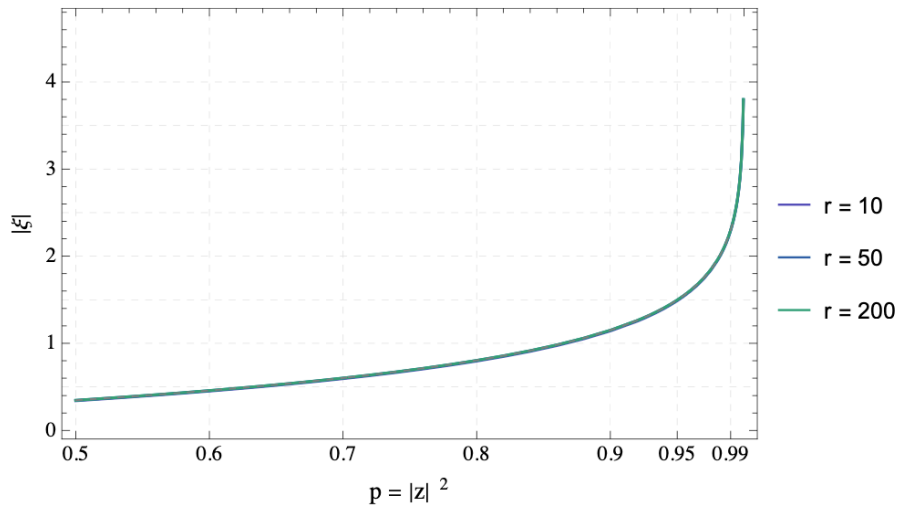
$$(\Delta X)^2 = \frac{1}{2} + \frac{rp}{1-p} + (1-p)^r p A_2(p) - 2(1-p)^{2r} p A_1(p)^2$$

$$(\Delta P)^2 = \frac{1}{2} + \frac{rp}{1-p} - (1-p)^r p A_2(p)$$

$A_1$  and  $A_2$  are infinite sums in powers of  $p$  with binomial prefactors

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Stasto, Raj, RV: arXiv 2605.03038



Wide parameter space for NBD parameter  $r > 1$   
where  $\delta$  is small (close to minimum uncertainty)  
but squeezing parameter  $\xi \gg 1$

For  $r$  large, the squeezing parameter  $\xi = Ln(\bar{n})$

For a gravitational wave measured by LIGO,  $\bar{n} \approx 4 \cdot 10^{36}$ , so very large squeezings are possible

*Specifically, quantum fluctuations on the Planck scale ( $10^{-35}$  m) can be enhanced to detectable levels at current and future gravitational wave observatories*

Caveat: The statistics are “super-Poisson” which means that quantum effects can also be mimicked by classical sources

D. Carney, arXiv:2408.00094

# Geodesic congruence: the geometry of quantum information

The Raychaudhuri equation

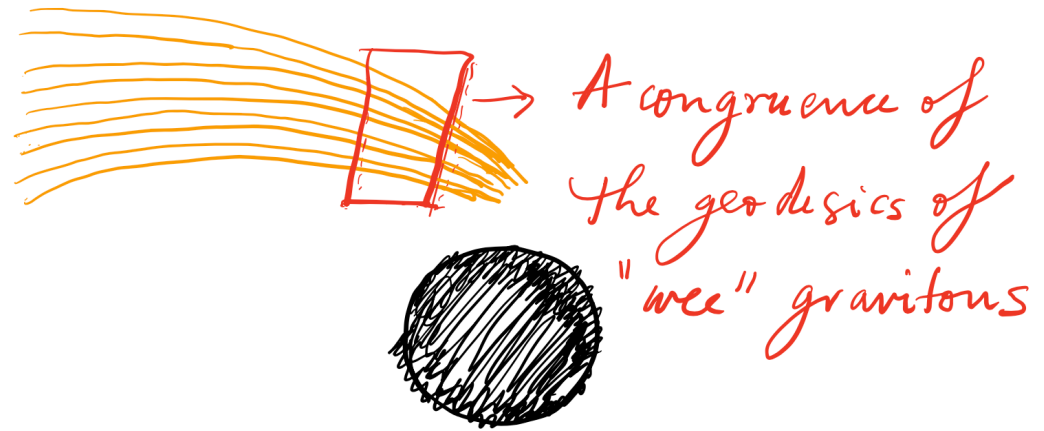
- key in Hawking-Penrose singularity theorems :

Volume change of geodesic convergence

$$\begin{aligned}\dot{\theta} &= -\Omega^i_j \Omega^j_i + K^i_i \\ &= -\frac{1}{3}\theta^2 - \sigma_{ij}\sigma^{ij} + \omega_{ij}\omega^{ij} + K^i_i.\end{aligned}$$

Bulk scalar   Shear tensor   Rotation tensor

Includes Ricci curvature + stochastic graviton noise



H.-T. Cho and B.-L Hu, arxiv:2301.06325  
M. Parikh, F. Wilczek, G. Zaharade, PRL (2021)

