Multi-Loop Amplitudes in the High-Energy Limit in $\mathcal{N} = 4$ SYM

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The Setup

- $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory
- Planar ('t Hooft) limit, $N_C \to \infty$
- Multi-Regge kinematics

Dual conformal symmetry

[Drummond, Henn, Korchemsky, Sokatchev; Drummond, Henn, Smirnov, Sokatchev]

- Fixes all 4&5 point results
- $A_N = A_N^{\text{BDS}} e^{R_N}$

[Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov]

[Drummond, Henn, Korchemsky, Sokatchev]

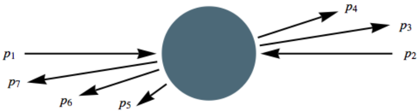
Multi Regge Kinematics (MRK)

Forward scattering:

- $s \gg |t|$
- $\mathcal{A}_4^{(1)} \sim \frac{1}{2} \log^2 \frac{s}{t} + \frac{2\pi^2}{3}$

Multi Regge kinematics:

- Hierarchy in rapidity
- No hierarchy in transverse plane



- $$R_N \sim a^2 \left(\log \tau \, g_{N,\mathrm{LLA}}^{(2)} + g_{N,\mathrm{NLLA}}^{(2)} \right) + a^3 \left(\log^2 \tau \, g_{N,\mathrm{LLA}}^{(3)} + \log \tau \, g_{N,\mathrm{NLLA}}^{(3)} + g_{N,\mathrm{NNLLA}}^{(3)} \right) + \ldots \quad ,$$

  with $\log \tau \gg 1$

Multi Regge Kinematics

Kinematical dependence encoded in $N-2$ transverse momenta $\mathbf{p}_i$.

$\to g_N^{(\ell)}$ are single-valued functions of *transverse dual coordinates* $\mathbf{x}_i$ defined as $\mathbf{p}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$,

$$R_N \sim a^2 \left(\log \tau \, g^{(2)}_{N,\mathrm{LLA}}(\{\mathbf{x}_i\}) + g^{(2)}_{N,\mathrm{NLLA}}(\{\mathbf{x}_i\}) \right) + \dots$$

Or, alternatively of the *cross ratios*

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

Single-valued Polylogarithms

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM in MRK are single-valued iterated integrals

$\rightarrow$ Use *single-valued polylogarithms* $\mathcal{G}$ [Brown]

Fulfil same differential equation as regular Polylogs

$$\partial_z G_{a_1,\ldots,a_n}(z) = \frac{1}{z-a_1} G_{a_2,\ldots,a_n}(z)$$

$$\downarrow$$

$$\partial_z \mathcal{G}_{a_1,\ldots,a_n}(z) = \frac{1}{z-a_1} \mathcal{G}_{a_2,\ldots,a_n}(z)$$

$\mathcal{G}_{a_1,\ldots,a_n}(z)$ has transcendental weight n.

The Goal of This Talk

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Determine the function space of amplitudes in MRK in planar $\mathcal{N} = 4$ SYM to all orders.

[Del Duca, Druc, Drummond, Duhr ,Dulat, RM, Papathanasiou, Verbeek]

The Goal of this Talk

The $\ell-$loop amplitude in MRK in planar $\mathcal{N} = 4$ SYM is a sum of combinations of logarithms $\log\tau$, π, and SVMPLs $\mathcal{G}$ of uniform transcendental weight 2ℓ.

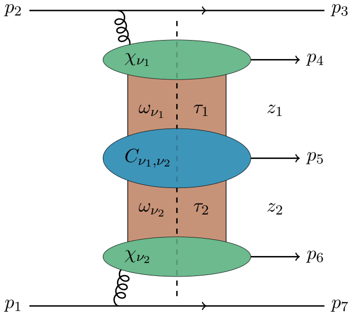
[Del Duca, Druc, Drummond, Duhr ,Dulat, RM, Papathanasiou, Verbeek]

$\rightarrow$ translates into similar (but more involved) statements for the $g_N^{(\ell)}$

The Remainder Function

The Remainder Function

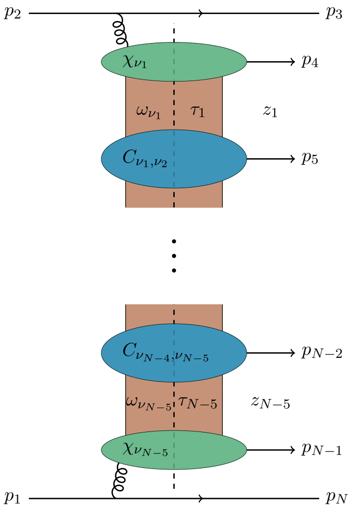
From Claude's talk:



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

$$\sim \mathcal{F}\left[\chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^-\right]$$

The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

The Remainder Function

$$-\omega = aE - \frac{a^2}{4}\left(D^2E - 2VDE + 4\zeta_2 E + 12\zeta_3\right) + \mathcal{O}(a^3)\,,$$

$$\chi^+ = \chi_0^+\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 - NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right]\,,$$

$$\chi^- = \chi_0^-\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 + NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right]\,,$$

$$C_{12}^+ = C_{0,12}^+\Big[1 + a\Big(\tfrac{1}{2}\big[DE_1 - DE_2 + E_1E_2 + \tfrac{1}{4}(N_1+N_2)^2 + V_1V_2 + (V_1 - V_2)\big(M_{12} - E_1 - E_2\big) + 2\zeta_2 + i\pi(V_2 - V_1 - E_1 - E_2)\big] - \tfrac{1}{4}(E_1^2 + E_2^2 + N_1V_1 - N_2V_2) - \tfrac{3}{16}(N_1^2 + N_2^2) - \zeta_2\Big) + \mathcal{O}(a^2)\Big]\,,$$

Define $X_i = X(\nu_i, n_i)$, $X_{ij} = X(\nu_i, n_i, \nu_j, n_j)$.

The Remainder Function

$$-\omega = aE + a^2 \,^{\mathsf{h}}P_3(D,E,V,N,\text{"}\pi\text{"}) + \mathcal{O}(a^3)\,,$$

$$\chi^{\pm} = \chi_0^{\pm}\left[1 + a\,^{\mathsf{h}}P_2(D,E,V,N,\text{"}\pi\text{"}) + \mathcal{O}(a^2)\right]\,,$$

$$C_{12}^{+} = C_{0,12}^{+}\left[1 + a\,^{\mathsf{h}}P_2(D,E,V,N,M,\text{"}\pi\text{"}) + \mathcal{O}(a^2)\right],$$

Only few *elementary building blocks*: D,E,V,N,M (plus $\chi_0^{\pm}, C_{0,12}^{+}$)

The Remainder Function

Define the *vacuum ladder*

$$\varpi_N = \chi^+_{0,1} C^+_{0,12} \cdots C^+_{0(N-6)(N-5)} \chi^-_{0,N-5}$$

All terms that need to be computed have the form

$$\mathcal{F}[\varpi_N P_m(D, E, V, N, M)]$$

We can build the amplitude starting from $\mathcal{F}[\varpi_N]$

Fourier-Mellin Convolutions

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The Fourier-Mellin transform

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2i\nu} F(\nu,n)$$

Products are mapped to convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = \frac{1}{\pi} \int \frac{\mathrm{d}^2 w}{|w|^2} \mathcal{F}[F](w) \mathcal{F}[G] \left(\frac{z}{w}\right)$$

Use this to

- Easily raise loop orders
- Easily flip helicities
- Determine the MRK function space to all orders

Fourier-Mellin Convolutions

From Claude's talk:

$$g_6^{(\ell)} \sim \mathcal{F}[\varpi_6 E^{\ell-1}]$$

$\rightarrow$ Convolute with $\mathcal{F}[E] = -\frac{z+\bar{z}}{2|1-z|^2}$ to raise loop order.

Single-valuedness allows us to solve the convolution integral by computing residues

[Schnetz]

$$\int \frac{\mathrm{d}^2 z}{\pi} f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z) \qquad \partial_{\bar{z}} F(z) = f(z)$$

$\rightarrow$ Increasing loop order as simple as computing residues!

Fourier-Mellin Convolutions

Computed many amplitudes:

MHV:

- All LLA amplitudes through 5 loops
- All NLLA amplitudes through 3 loops

Beyond MHV:

- All LLA amplitudes through 8pt and 4 loops
- All NLLA amplitudes through 8pt and 3 loops

[Del Duca, Druc, Drummond, Duhr ,Dulat, RM, Papathanasiou, Verbeek]

Proof of the All-Order Function Space

Building blocks appear in homog. polynomials ${}^{\mathsf{h}}P_m(D, E, V, N, M, ``\pi")$

$\rightarrow$ good sign for uniform transcendentality

Need to show that each building block raises the weight by 1

$\rightarrow$ Use convolutions!

The Weight of Elementary Building Blocks

Take a function

$$\mathcal{K}(z) = \frac{|z|^2}{(z-a)(\overline{z}-\overline{b})} \qquad (1)$$

and consider

$$\mathcal{G}(a_1,\ldots,a_n;z) * \mathcal{K}(z) = \frac{1}{\pi}\int \mathrm{d}^2 w\, \mathcal{G}(a_1,\ldots,a_n;w)\frac{1}{(w-za)(\overline{w}-\overline{z}\overline{b})}$$

$\rightarrow$ Use residues to get

$$\mathsf{Res}_{\overline{w}=\overline{z}\overline{b}} \frac{\mathcal{G}(za,a_1,\ldots,a_n;w)}{(\overline{w}-\overline{z}\overline{b})} \sim \mathcal{G}(za,a_1,\ldots,a_n;zb)$$

$\rightarrow$ Weight raised by 1!

The Weight of Elementary Building Blocks

$\mathcal{F}[E], \mathcal{F}[N], \mathcal{F}[V]$ have this form!

What about $D = -\mathrm{i}\partial_\nu$?

$$\mathcal{F}[DX] = -\mathrm{i} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2\mathrm{i}\nu} \partial_\nu X$$
$$\overset{\text{IBP}}{=} \mathcal{G}(0, z)\mathcal{F}[X]$$

$\rightarrow$ Raises weight by 1!

The Weight of Elementary Building Blocks

$M \equiv M(\nu_1, n_1, \nu_2, n_2) \rightarrow$ Things get more complicated.

Can write M_{12} as

$$M_{12} = \frac{D_1 C^{+}_{0,12}}{C^{+}_{0,12}} + F_1 = -\frac{D_2 C^{+}_{0,12}}{C^{+}_{0,12}} + F_2 - N_2$$

with a new building-block F and treat derivatives of C using IBP.

$\mathcal{F}[F]$ has the form (1) $\rightarrow$ raises weight by 1.

$\rightarrow$ MHV amplitudes are pure combinations of SVMPLs of maximal transcendentality

The Weight of Elementary Building Blocks

What happens beyond MHV?

Can flip helicities using (see Claude's talk)

$$H = \frac{\chi^-}{\chi^+} = H_0(1 + a^{\mathsf{h}} P_2(D, E, N, V, "\pi") + \mathcal{O}(a^3)),$$

$\rightarrow$ Pure functions of uniform transcendental weight up to LO helicity flips.

$\rightarrow$ Leading singularities R_{bac}

Conclusions & Outlook

- Developed a framework that allows us to compute virtually any amplitude in MRK
- Know the function space in MRK to all orders
- Explicitly checked for
  - All MHV 5-loop amplitudes at LLA
  - 8-point LLA amplitudes for any helicity configuration up to 4 loops
  - All MHV 3-loop amplitudes at NLLA (Not yet published)
  - 7-point NLLA NMHV up to 3 loops
- Can similar things be done outside of $\mathcal{N} = 4$?