Collinear resummations for the non-linear evolution in QCD at high-energy

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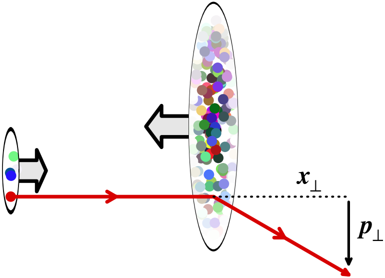
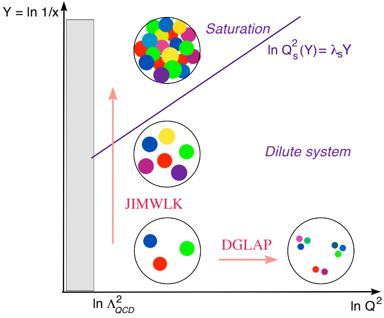
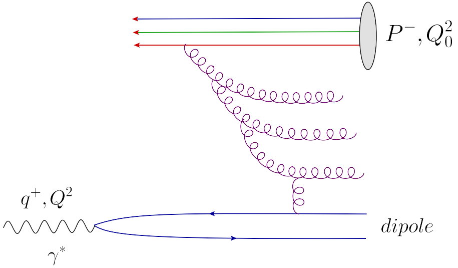


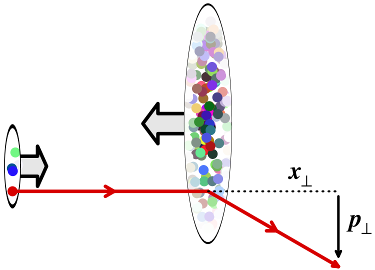


Motivation: Dilute-dense scattering

- DIS (ep or eA) at small Bjorken x, pA (or pp) at forward rapidities ...
  - dilute projectile (γ^*, dipole) & dense target (hadron probed at small x)
  - semi-hard resolution scale: Q^2 in DIS, $p_\perp$ for particle production in pA

$$Q^2,\, p_\perp^2 \;\gg\; Q_0^2 \quad \text{(unitarization scale at low energy)}$$

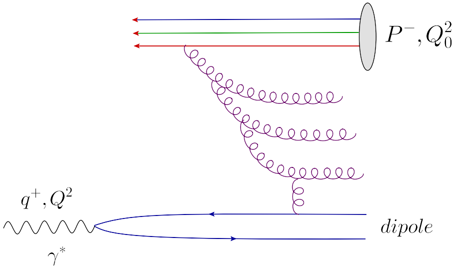


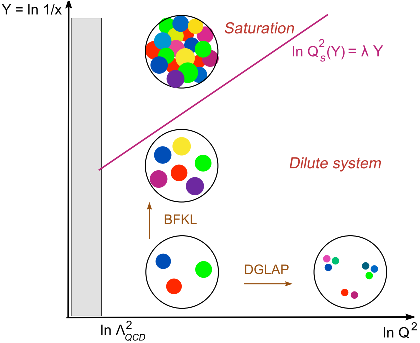


- At high energy, both processes admit a dipole factorization

Motivation: QCD evolution at high energy

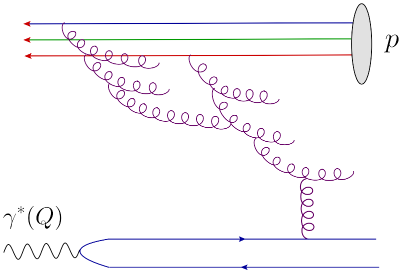
- DIS at high energy/small-x in the hadron IMF (manifest parton picture)
- The gluon distribution in the target is rapidly rising with decreasing x
  - the (linear) BFKL equation in the dilute regime
  - "dilute": gluon occupation number $n \ll \frac{1}{\alpha_s}$

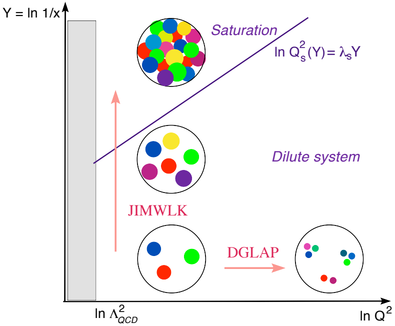




Motivation: QCD evolution at high energy

- DIS at high energy/small-x in the hadron IMF (manifest parton picture)
- The gluon distribution in the target is rapidly rising with decreasing x
  - the (non-linear) BK/JIMWLK equations in the high density regime
  - "non-linear": gluon recombination leading to saturation: $n_{\max} \sim \frac{1}{\alpha_s}$

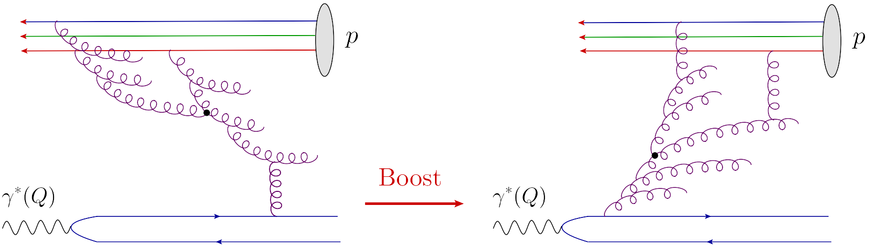




- Saturation line $Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\lambda_s Y}$: strong scattering: $T_{\text{dipole}} \sim 1$

Dipole vs. proton evolution

- The physical picture & the calculation details depend upon frame & gauge
- Via a Lorentz boost, one can transfer the high-energy evolution from the proton to the dipole (dipole frame)



  - recombination ($gg \to g$) gets mapped onto splitting ($g \to gg$)
  - gluon saturation gets mapped onto multiple scattering
- At LO, both pictures for the evolution have been explicitly worked out

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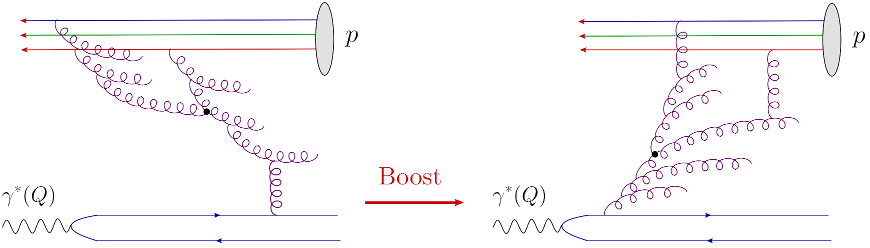

  - dipole evolution: Balitsky hierarchy (96), Balitsky-Kovchegov eq. (99)
  - target evolution: JIMWLK eq. (97-01), CGC effective theory *(Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner)*
- ... with results which are equivalent, as they should !

Dipole vs. proton evolution

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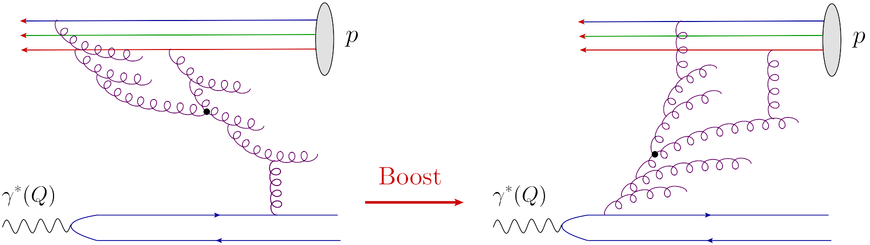


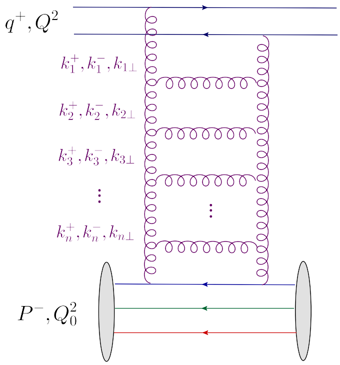
- At NLO, however, only the dipole evolution has been computed

  (Balitsky and Chirilli, 2008-13; Kovner, Lublinsky and Mulian, 2013-16)

- A priori different from the NLO evolution of the target ... and also problematic

Rapidities & Time-ordering

- Strong ordering of successive emissions in longitudinal momenta
  - k^+ for the projectile, k^- for the target
- Typical evolution: $k_\perp$ is increasing from target to projectile: $Q^2 \gg k_\perp^2 \gg Q_0^2$



- Target evolution: strong ordering in k^-

$$P^- \gg k_n^- \gg \cdots \gg k_1^- \gg q^- \equiv \frac{Q^2}{2q^+}$$

$$\eta \equiv \ln \frac{P^-}{q^-} = \ln \frac{s}{Q^2} = \ln \frac{1}{x_{\rm Bj}}$$

- Lifetime of a gluon fluctuation: $\Delta x^- = \frac{2k^-}{k_\perp^2}$
- Typical emissions are also ordered in lifetimes

$$\Delta x_n^- \gg \Delta x_{n-1}^- \gg \cdots \gg \Delta x_1^- \gg \frac{1}{q^+}$$

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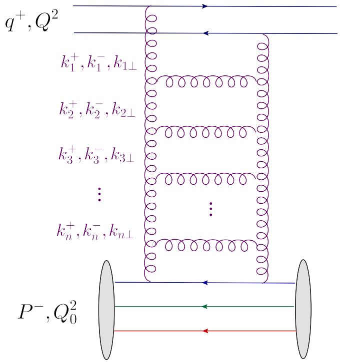

- Projectile evolution: strong ordering in k^+

$$q^+ \gg k_1^+ \gg \cdots \gg k_n^+ \gg P^+ \equiv \frac{Q_0^2}{2P^-}$$

$$Y \equiv \ln \frac{q^+}{P^+} = \ln \frac{s}{Q_0^2} = \eta + \ln \frac{Q^2}{Q_0^2}$$

- Lifetime of a gluon fluctuation: $\Delta x^+ = \frac{2k^+}{k_\perp^2}$
  - the correct time ordering can be violated by the typical evolution at LO

- This is restored via higher-order corrections, which are however large

Rapidities & Time-ordering

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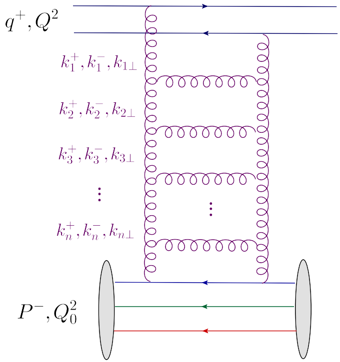

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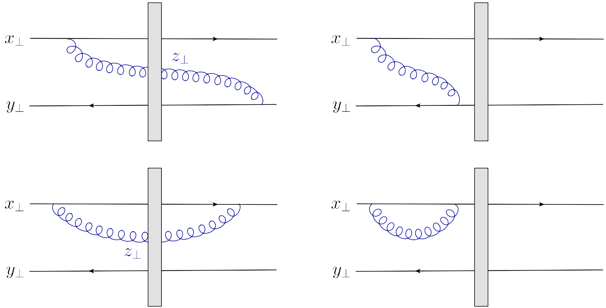
$$\alpha_s \eta \rho = \alpha_s (Y - \rho)\rho = \alpha_s Y \rho - \alpha_s \rho^2 \quad \text{with} \quad \rho \equiv \ln \frac{Q^2}{Q_0^2} > 0$$

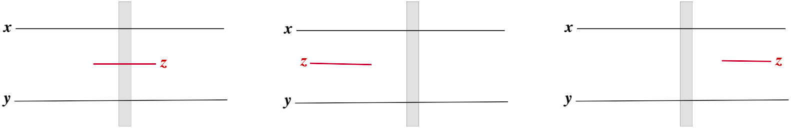
LO evolution in Y

- $x_{\rm Bj} \ll 1 \;\Rightarrow\; \frac{2q^+}{Q^2} \gg \frac{1}{P^-}$: the dipole scatters off a shockwave
- Multiple scattering can be resummed in the eikonal approximation
  - transverse coordinates are "good quantum numbers"
- Soft gluon emissions, which occur well before, or well after the shockwave
- To construct the evolution equation, it is enough to look at the first emission



BK equation at LO

- Large N_c : the original dipole splits into two new dipoles



$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right]$$

  - dipole kernel: probability for the dipole to emit a soft gluon at $\boldsymbol{z}$

- Non-linear completion of the BFKL equation for $T = 1 - S$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

- Unitarity: the "black disk limit" $T = 1$ is a fixed point

BK equation at LO

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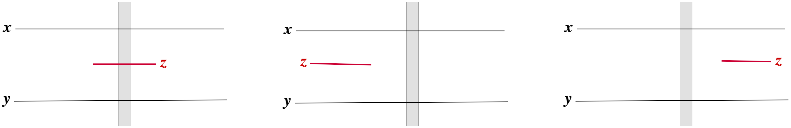

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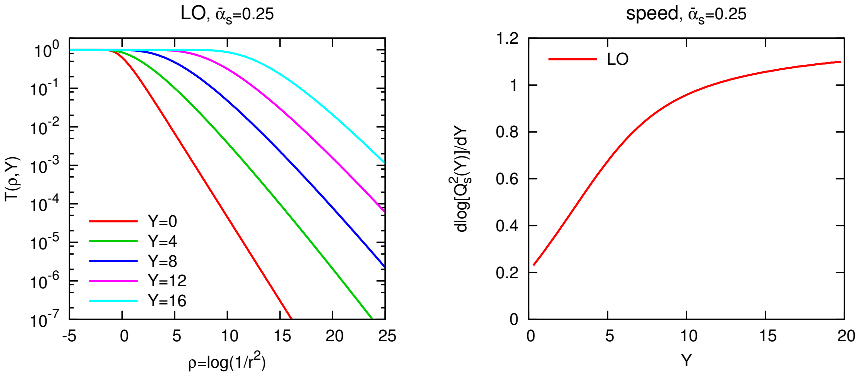
- Saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 1/Q_s(Y)$

The saturation front

- $T(Y,r)$ as a function of $\rho \equiv \ln \frac{1}{r^2 Q_0^2}$ with increasing Y

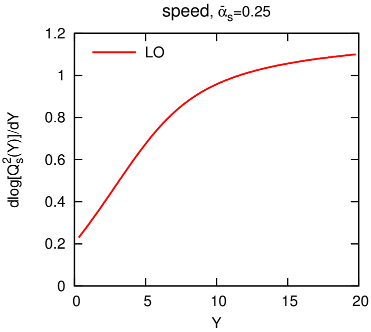


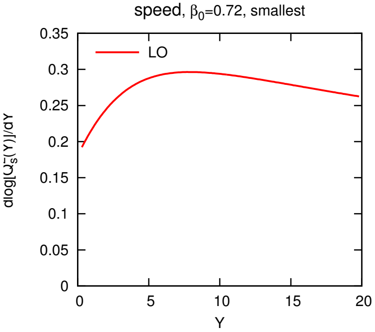
$$T(Y,r) \simeq \begin{cases} 1 & \text{for } rQ_s(Y) \gtrsim 1 \\ \left(r^2 Q_s^2(Y)\right)^{\gamma_s} & \text{for } rQ_s(Y) \ll 1 \end{cases}, \qquad Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\lambda_s Y}$$

- Saturation exponent $\lambda_s \simeq 4.88\bar{\alpha}_s$, anomalous dimension $1-\gamma_s \simeq 0.37$

Adding running coupling: rcBK

- Saturation exponent: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$ for $Y \gtrsim 5$: much too large
  - phenomenology requires a much smaller valuer $\lambda_s \simeq 0.2 \div 0.3$
- Including running coupling dramatically slows down the evolution

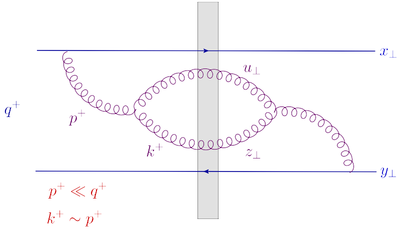


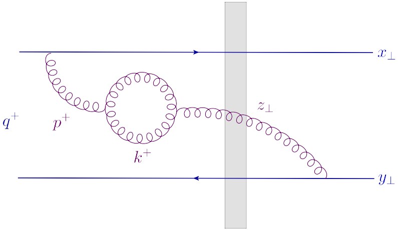


- Rather successful phenomenology based on rcBK
- ... but what about the other NLO corrections ?

Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}_s^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha}_s)$ correction to the r.h.s. of BK eq.





- The prototype: two successive, soft, emissions, with similar longitudinal momentum fractions: $p^+ \sim k^+ \ll q^+$

- Exact kinematics (full QCD vertices, as opposed to eikonal)

- New color structures, up to 3 dipoles at large N_c

- NLO BFKL: *Fadin, Lipatov, Camici, Ciafaloni ... 95-98*

BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar\alpha_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}}-S_{\boldsymbol{xy}}\right)\Bigg\{1+$$

$$+\,\bar\alpha_s\Bigg[\bar b\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2-\bar b\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\Bigg]\Bigg\}$$

$$+\,\frac{\bar\alpha_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\,\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}}-S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

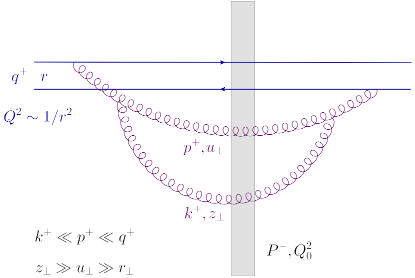
$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- green : leading-order (LO) terms
- violet : running coupling corrections
- blue : single collinear logarithm (DGLAP)
- red : double anti-collinear logarithm : troublesome !

The double anti-collinear logarithm

- Important in the "hard-to-soft" evolution: very large daughter dipoles

$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\simeq -\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2} \quad \text{if} \quad |\boldsymbol{z}-\boldsymbol{x}|\simeq|\boldsymbol{z}-\boldsymbol{y}|\gg r$$



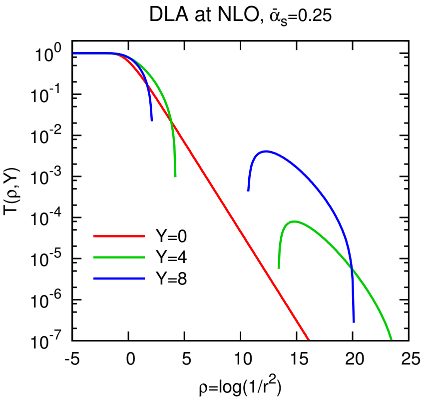
- NLO correction restoring time ordering

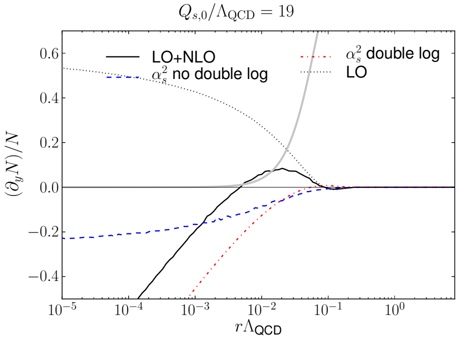
$$\frac{2q^+}{Q^2}\gg\frac{2p^+}{p_\perp^2}\gg\frac{2k^+}{k_\perp^2}\gg\frac{1}{P^-}$$

- In coordinate space: $p^+u_\perp^2 \gg k^+z_\perp^2$
- Integrate out the harder gluon $(p^+, u_\perp)$

$$\bar{\alpha}_s\int_{r^2}^{z^2}\frac{\mathrm{d}u^2}{u^2}\int_{k^+}^{q^+}\frac{\mathrm{d}p^+}{p^+}\,\Theta(p^+u^2-k^+z^2) = \bar{\alpha}_s Y\ln\frac{z^2}{r^2}-\frac{\bar{\alpha}_s}{2}\ln^2\frac{z^2}{r^2}$$

Unstable numerical solution





- Left: LO BK + the double collinear logarithm
- Right: full NLO BK *(Lappi, Mäntysaari, arXiv:1502.02400)*
- The main source of instability: the double collinear logarithm

Collinear resummations in Y (1)

- Different pieces generated by time-ordering are included in different orders
  - an infinite series of terms $\propto (\alpha\rho^2)^n$, with $n \geq 1$ and alternating signs
- This whole series can be resummed by enforcing TO within LO BK eq.
- Two "collinearly improved" versions of BK equation: local & non-local (in Y)
- Same kernel as at LO, but non-local in rapidity *(G. Beuf, 2014)*

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[S_{\boldsymbol{xz}}(Y-\Delta_{\boldsymbol{xz}})S_{\boldsymbol{zy}}(Y-\Delta_{\boldsymbol{zy}}) - S_{\boldsymbol{xy}}(Y)\right]$$

  - rapidity shift important only if daughter dipole larger than the parent:

$$\Delta_{\boldsymbol{xz}} \equiv \Theta\left((\boldsymbol{x}-\boldsymbol{z})^2 - r^2\right) \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}, \quad r = |\boldsymbol{x}-\boldsymbol{y}|$$

  - this shift is not unique beyond the double-log accuracy
  - a boundary value problem: $Y = \eta + \rho \geq \rho \equiv \ln \frac{1}{r^2 Q_0^2}$

Collinear resummations in Y (2)

- Local equation, but with all-order resummed kernel & initial condition

  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)

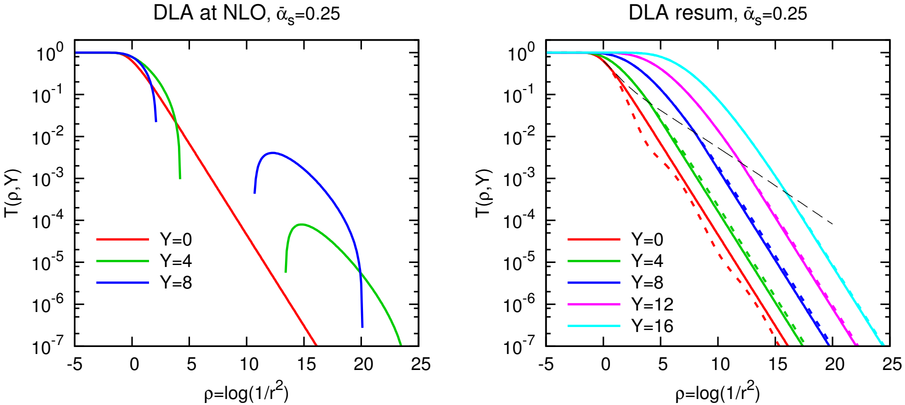
$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \mathcal{K}_{\rm DLA}(\rho_{\boldsymbol{xyz}}) \left[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)\right]$$

$$\mathcal{K}_{\rm DLA}(\rho) \equiv \frac{\mathrm{J}_1\left(2\sqrt{\bar{\alpha}_s\rho^2}\right)}{\sqrt{\bar{\alpha}_s\rho^2}} = 1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{(\bar{\alpha}_s\rho^2)^2}{12} + \cdots,$$

- Initial value problem: $S_{\boldsymbol{xy}}(Y = Y_0) = S_0^{\rm unphys}(\boldsymbol{x}, \boldsymbol{y})$
  - unphysical “initial condition” chosen such that $S(Y = \rho) = S_0^{\rm phys}(\rho)$
  - the physical S-matrix is reproduced for $Y \geq \rho$ (where it should)
  - $S_0^{\rm unphys}$ is explicitly known only to double-log accuracy
- N.B.: this is the strategy implicitly followed in standard pQCD for BFKL/BK

Numerical solutions: saturation front

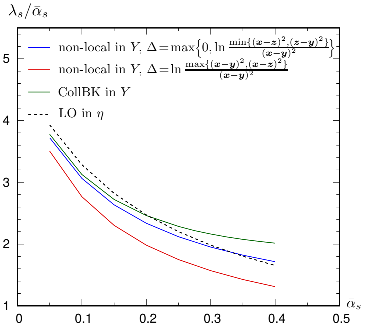
- The resummation stabilizes and slows down the evolution

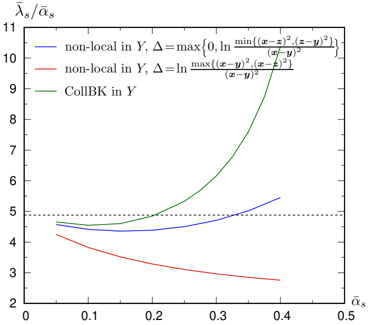


- left: the NLO double-log alone
- right: double collinear logs resummed to all orders

Resummations in Y: saturation exponent

- Both strategies suffer from similar drawbacks:
  - ambiguities beyond double log accuracy (powers of $\alpha_s \rho^2$)
  - difficulties with formulating the initial condition
- Saturation exponent in Y (left) and in $\eta \equiv Y - \rho$ (right)





  - For the physical evolution in $\eta = \ln \frac{1}{x_{\rm Bj}}$: strong scheme dependence
  - The resummed evolutions are stable but lack predictive power

What about evolving in η ?

- Enforcing time-ordering in the evolution with Y appears to be problematic
- Why not work directly in η ? (TO would be automatic !)
- A similar strategy proposed for NLO BFKL *(Salam; Ciafaloni, Colferai, 98)*
  - "the choice of the energy scale" (in the definition of the rapidity)
  - for DIS the proper scale is $Q^2 \Rightarrow \eta = \ln \frac{s}{Q^2}$
- This choice is only the first step in the "collinear-improvement" program
  - collinear resummations are still need (milder instabillites)
  - w/o resummations: complex saddle point, oscillating amplitude
- For the non-linear evolution, even this first step seems difficult to achieve
  - evolution in $\eta \Leftrightarrow$ evolution of the target wavefunction
  - complicated in the presence of gluon saturation

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- This choice is only the first step in the "collinear-improvement" program
  - collinear resummations are still need (milder instabillites)
  - w/o resummations: complex saddle point, oscillating amplitude
- For the non-linear evolution, even this first step seems difficult to achieve
  - evolution in $\eta \Leftrightarrow$ evolution of the target wavefunction
  - complicated in the presence of gluon saturation
- Not fully right ! In pQCD, NLO corrections in η can be obtained from those in Y via a simple change of variables !

Projectile evolution in η at NLO (1)

- Recall: NLO BK equation in Y *(Balitsky and Chirilli, 2008)*

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$-\frac{\bar{\alpha}_s^2}{4\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$+\ \bar{\alpha}_s^2 \times \text{“regular”}.$$

- Change of variable: $\eta \equiv Y-\rho$, $S_{\boldsymbol{xy}}(Y) \equiv \bar{S}_{\boldsymbol{xy}}(\eta)$, with $\rho \equiv \ln\frac{1}{(\boldsymbol{x}-\boldsymbol{y})^2 Q_0^2}$
  - the rapidity shift $Y-\rho$ is formally of $\mathcal{O}(\bar{\alpha}_s)$:

$$S_{\boldsymbol{xz}}(\eta+\rho) = \bar{S}_{\boldsymbol{xz}}\left(\eta + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right) \simeq \bar{S}_{\boldsymbol{xz}}(\eta) + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\frac{\partial \bar{S}_{\boldsymbol{xz}}(\eta)}{\partial\eta}$$

  - use the LO BK equation to evaluate $\frac{\partial \bar{S}_{\boldsymbol{xz}}(\eta)}{\partial\eta}$
  - replace $S(Y) \to \bar{S}(\eta)$ in the NLO terms

Projectile evolution in η at NLO (2)

- NLO BK equation in η *(Ducloué et al, arXiv:1902.06637)*

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$- \frac{\bar{\alpha}_s^2}{4\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

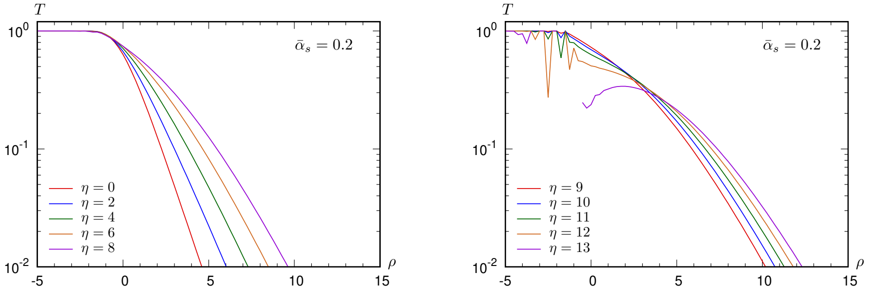
$$+ \frac{\bar{\alpha}_s^2}{2\pi^2} \int_{\boldsymbol{z},\boldsymbol{u}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \bar{S}_{\boldsymbol{xu}}(\eta) \left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{uy}}(\eta)\right]$$

$$+ \ \bar{\alpha}_s^2 \times \text{“regular”}.$$

- The 3rd term, coming from the change of variables, cancels the double anti-collinear log for large daughter dipoles: $|\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$
- But it generates new, collinear, double logs when one of the daughter dipoles is small : $|\boldsymbol{z}-\boldsymbol{x}| \ll r$ or $|\boldsymbol{z}-\boldsymbol{y}| \ll r$
- Such atypical configurations are allowed by BFKL diffusion

NLO BK evolution in η

- Numerical solutions to "NLO BK in η" (LO BK + the double collinear log)



- Although disfavoured by the typical "hard-to-soft" evolution, the collinear double-logs do still entail a (mild) instability
  - the instability develops only for sufficiently large η
  - it first appears for relatively large dipole sizes, close to $1/Q_s$

- Fluctuations leading to large dipoles which then fragment into smaller ones

Collinear resummation in η

(Ducloué, E.I., Mueller, Soyez, Triantafyllopoulos, arXiv:1902.06637)

- The whole series of double collinear logs can be resummed via an appropriate rapidity shift $\Rightarrow$ non-local evolution in η

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{xz}})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zy}}) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

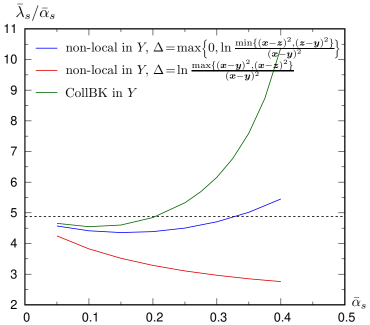
  - rapidity shift if one daughter dipole is much smaller than its parent:

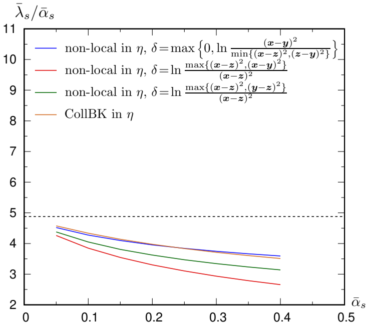
$$\delta_{\boldsymbol{xz}} \equiv \Theta\left(r^2-(\boldsymbol{x}-\boldsymbol{z})^2\right)\,\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}$$

- A genuine initial value problem: $\bar{S}(\eta_0, r) = S_0(r)$
- Linear level (NLO BFKL): equivalent to the "ω-shift" prescription by Salam
- As before, this prescription is not unique beyond double-log accuracy
- Extension to full NLO accuracy possible (for a given prescription)

Resummed BK evolution in η: fixed coupling

- $\bar{\lambda}_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}\eta}$: the speed of the saturation front in η (for a fixed coupling)
  - recall: LO result $\bar{\lambda}_s \simeq 4.88\bar{\alpha}_s \sim \mathcal{O}(1)$ (way too large)





- Left: resummations in Y: strong scheme dependence, no clear pattern
- Right: resummations in η: weak scheme dependence $\sim \mathcal{O}(\alpha_s^2)$
  - it would be even weaker after completing the equation to NLO accuracy

Resummed BK evolution in η: fixed coupling

- $\bar{\lambda}_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}\eta}$: the speed of the saturation front in η (for a fixed coupling)
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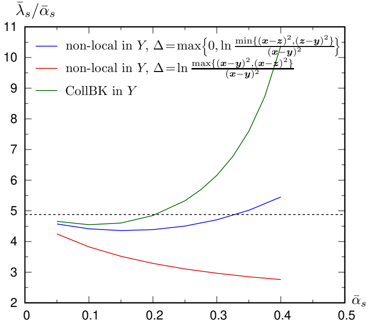
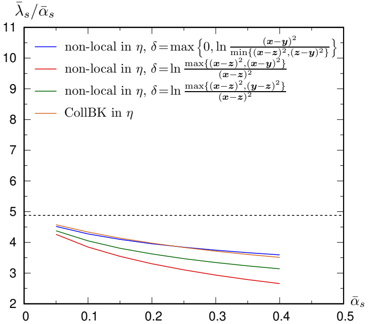
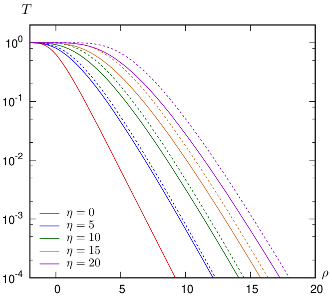


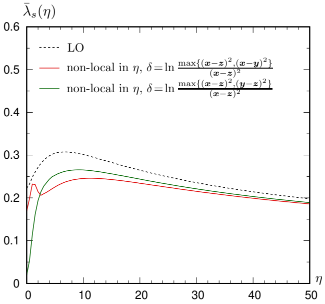


- Left: resummations in Y: strong scheme dependence, no clear pattern
- Right: resummations in η: systematic reduction by $30 \div 40$ w.r.t. LO
  - but this is not yet sufficient for the phenomenology !

Resummed BK evolution in η: running coupling

- Recall: phenomenology requires $\bar{\lambda}_s \simeq 0.20 \div 0.25$
- The main reduction comes from the use of a running coupling
  - below: $\bar{\alpha}_s(r_{\min})$ where $r_{\min} = \min\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{y}-\boldsymbol{z}|\}$

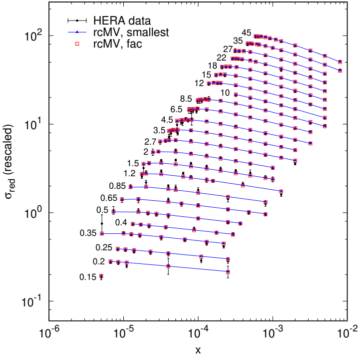


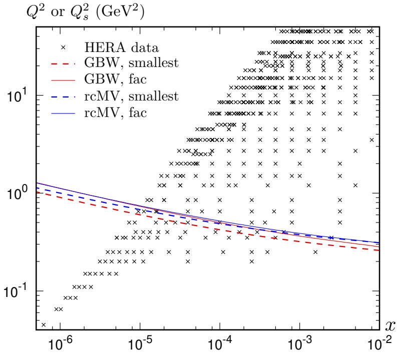


- Left: saturation fronts in η: collBK (full lines) vs. LO BK (dashed)
- Right: saturation exponent: $\bar{\lambda}_s \simeq 0.2$ at large η ☺

A fit to DIS at HERA

- Excellent fit to the HERA data at small x: $x_{\rm Bj} \leq 0.01$, $Q^2 \leq 50$ GeV^2
  - 4 free parameters, all encoded in the initial condition $T(\eta_0, r)$
  - 2 prescriptions for the running of the coupling (equivalent to 1-loop)
  - also resummation of the NLO single logs (DGLAP): essential





- Right: the saturation scale given by the fit on top of the data points

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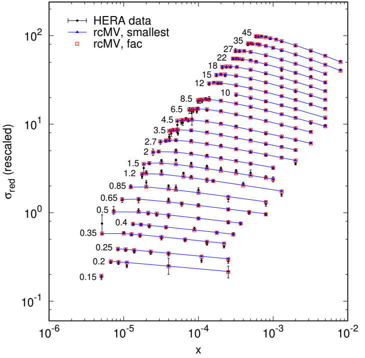

init cdt.	RC schm	double logs	single logs	χ^2/npts for $Q^2_{\rm max}$			
				50	100	200	400
GBW	small	yes	no	2.05	2.17	2.27	2.24
GBW	small	no	yes	1.26	1.26	1.35	1.46
GBW	small	yes	yes	1.18	1.21	1.31	1.39
GBW	fac	yes	no	1.65	1.75	1.94	2.01
GBW	fac	no	yes	1.19	1.23	1.37	1.51
GBW	fac	yes	yes	1.14	1.17	1.25	1.32
rcMV	small	yes	no	1.72	1.86	1.93	1.92
rcMV	small	no	yes	1.07	1.08	1.04	1.03
rcMV	small	yes	yes	1.03	1.04	1.01	1.00
rcMV	fac	yes	no	1.31	1.34	1.35	1.33
rcMV	fac	no	yes	0.98	0.98	0.95	0.95
rcMV	fac	yes	yes	1.01	1.03	1.01	1.00

Table 2: Evolution of the fit quality when increasing $Q^2_{\rm max}$ (in GeV^2).

- Right: the quality of the best fit remains constant up to $Q^2_{\rm max} = 400$ GeV^2

Conclusions

- High-energy evolution in the presence of saturation is most conveniently computed in terms of the rapidity Y of the dilute projectile
  - double anticollinear logs associated with violations of time ordering
  - unstable, no predictive power
- Resummations in Y suffer from a series of problems
  - not a simple initial condition problem
  - strong scheme dependence
- Perturbative series for the evolution in η (the target rapidity) can be deduced via a change of variables
  - double collinear logs which are large only for atypical evolution
- Resummations in η are much better under control
  - reasonably weak scheme dependence
  - initial value problem
  - can be promoted to full NLO accuracy