Parton Shower and Colour Evolution

Zoltán Nagy
DESY-HH

In collaboration with Dave Soper

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Perturbative Cross Section

The main focus of this workshop is to calculate the pQCD cross sections as precise as possible, thus we have a pretty integral

$$\sigma[O_J] = \sum_m \frac{1}{m!} \sum_{\{a,b,f_1,\ldots,f_m\}} \int_0^1 d\eta_\mathrm{a} \overbrace{\int_{\eta_\mathrm{a}}^1 \frac{dz}{z} \varGamma_{aa'}^{-1}(z,\mu^2)\, f_{a'/A}(\eta_\mathrm{a}/z,\mu^2)}^{\textit{Bare PDF}}$$

$$\times \int_0^1 d\eta_\mathrm{b} \int_{\eta_\mathrm{b}}^1 \frac{d\bar{z}}{\bar{z}} \varGamma_{bb'}^{-1}(\bar{z},\mu^2)\, f_{b'/A}(\eta_\mathrm{b}/\bar{z},\mu^2)$$

$$\times \int d\phi(\eta_\mathrm{a}\eta_\mathrm{b}s,\{p,f\}_m) \left\langle M(\{p,f\}_m)\right| \underbrace{O_J(\{p,f\}_m)}_{\textit{IR safe measurement operator}} \overbrace{\left|M(\{p,f\}_m)\right\rangle}^{\textit{Partonic matrix element}}$$

$$+\, \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{\mu_J^2}\right)$$

Error of the factorization
(Cannot be beaten by calculating higher and higher order.)

and here the MSbar parton in parton renormalised PDF is

$$\varGamma_{aa'}(z,\mu^2) = \delta(1-z)\delta_{aa'} - \frac{\alpha_\mathrm{s}(\mu^2)}{2\pi} \frac{1}{\epsilon} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{aa'}(z) + \cdots \quad .$$

Motivation

For a generic IR safe observable we can do either fixed order or parton shower calculations

Fixed order calculations

✓ Systematically improvable by working to higher order.

- The procedure is well defined and can be carried out order by order. The definition of cross section tells us what to do.
- The subtraction procedure regularizes the α_S series and turns the d=4-2ε dimensional expression to a d=4 dimensional one.
- Counter-terms are defined order by order
- The result is independent of the ambiguities of the counter-terms order by order.

✗ Only few partons represent a jet.

✗ Suffers from large logarithms

Parton shower algorithms

✗ But what about parton showers?

- Are they just QCD inspired or fit into a scheme that can be systematically improved by working to higher order?
- Is the (*all order*) shower cross section equal to the pQCD (*all order*) cross section?
- Is there a shower way to regularize α_S series?

✓ A jet consists of many partons

✓ Sums up logarithms (only for some observable).

What is the relation between fixed order and parton shower?

Motivation

Fixed order NLO PDF is a well defined and systematically improvable approximation of the usual LO PDF:

$$f(\eta,\mu^2) = f_{1\mathrm{GeV}}(\eta) + \int_{1\mathrm{GeV}^2}^{\mu^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\frac{\alpha_s(\tilde{\mu}^2)}{2\pi}\int_\eta^1 \frac{dx}{x}P^{(1)}(x) f_{1\mathrm{GeV}}(\eta/x) + \cdots$$

But we never use this and we prefer the **fully exponentiated** solution of the DGLAP equation

$$f(\eta,\mu^2) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} dN\,\eta^{-N}\exp\left(\int_{1\mathrm{GeV}^2}^{\mu^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\frac{\alpha_s(\tilde{\mu}^2)}{2\pi}\int_0^1 dx\,x^{N-1}P^{(1)}(x)\right) \times \int_0^1 dx\,x^{N-1} f_{1\mathrm{GeV}}(x)$$

A statement: The fixed order NLO, NNLO and N^{k}LO calculations are just approximations to the fully exponentiated LO, NLO and N^{k-1}LO calculations.

An aim: The parton shower can provide the fully exponentiated LO, NLO and N^{k-1}LO calculations.

Statistical Space

Introducing the statistical space we can represent the QCD density operator as a vector

$$\sigma[O_J] = \underbrace{(1|}_{\text{All the initial and final state sums and integrals}} \mathcal{O}_J \overbrace{\left[\mathcal{F}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right]}^{\text{Bare PDFs for both incoming hadrons}} \underbrace{|\rho(\mu^2))}_{|M\rangle\langle M|}$$

QCD density operator
Describes the fully exclusive partonic final states.

The physical cross section is RG invariant as well as the QCD density operator and the bare PDF.

$$\mu^2 \frac{d}{d\mu^2} |\rho(\mu^2)) = \mu^2 \frac{d}{d\mu^2} \left[\mathcal{F}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right] = 0 + \mathcal{O}(\alpha_s^{k+1})$$

Perturbative expansion of the density operator

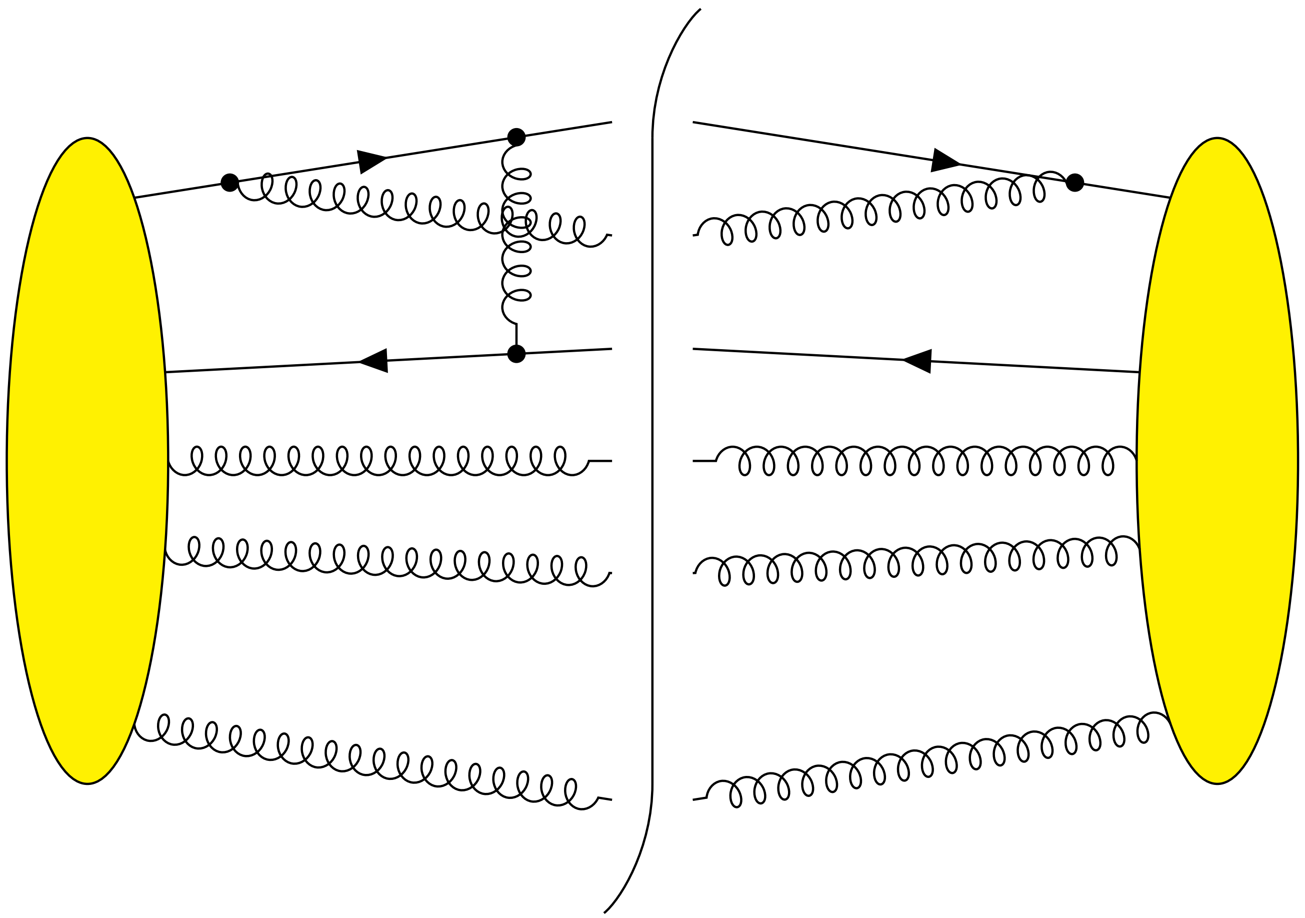
$$|\rho(\mu^2)) = \sum_{n=0}^{k} \left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^n \sum_{\substack{n_R=0 \\ n_R+n_V=n}}^{n} \sum_{n_V=0}^{n} |\rho^{(n_R, n_V)}(\mu^2))$$

Number of real radiations (n_R)

Number of loops (n_V)

Infrared Sensitive Operator

Amplitudes have soft or collinear singularities and they have divergences 1/ε from the loops

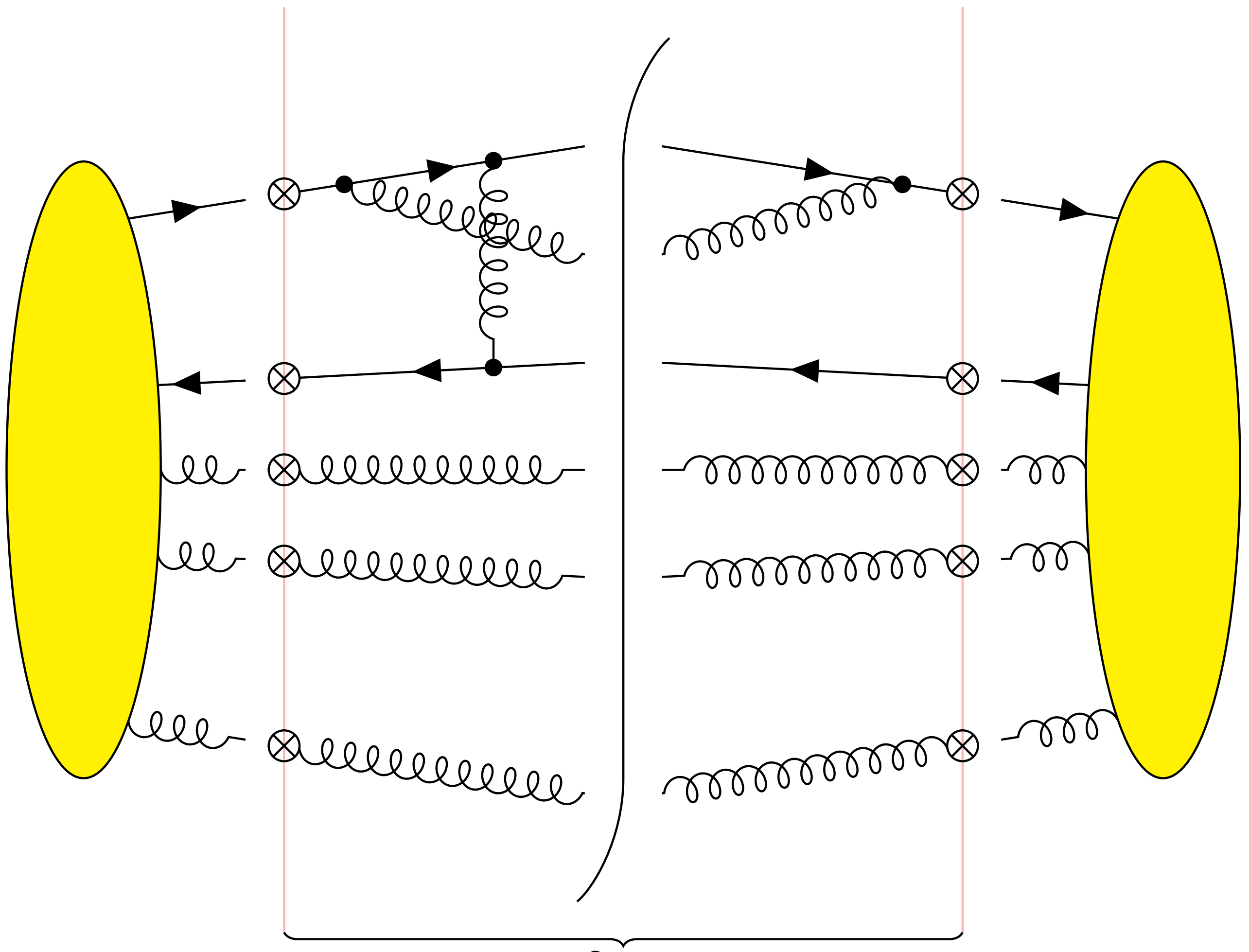


➠ We want to describe the singularity structure in **process independent way**.

➠ Everything in the yellow blobs is considered hard.

Infrared Sensitive Operator

Consider the momenta coming from the hard part as fixed and on shell.

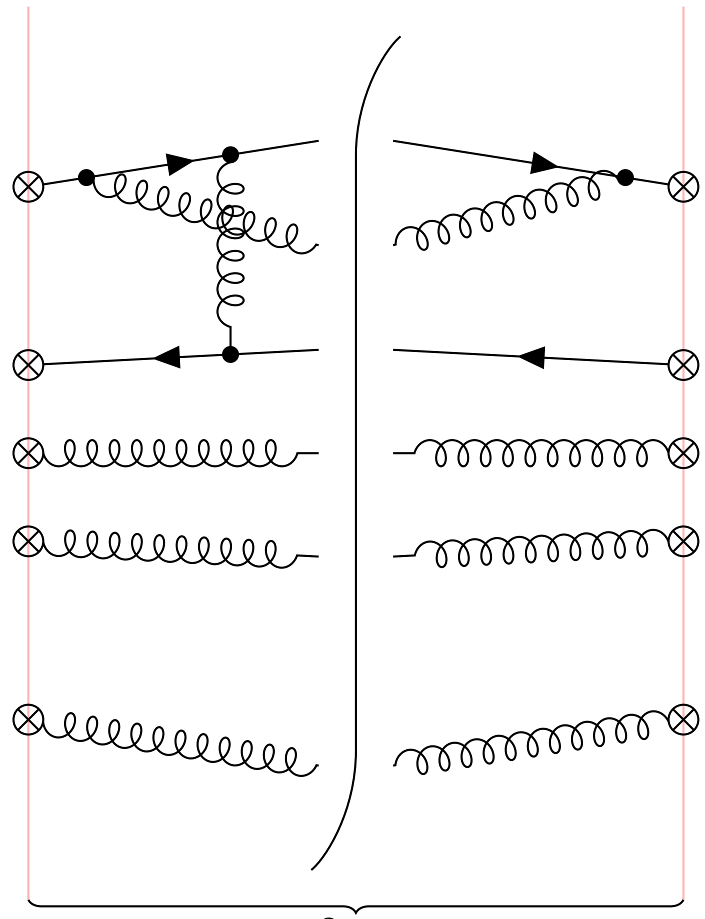


This gives us an operator as

$$
\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{s},\hat{s}',\hat{c},\hat{c}'\}_{m+n_R}\big|\rho(\mu^2)\big) \\
&\quad \sim \frac{1}{m!}\int[d\{p\}_m]\sum_{\{f\}_m}\sum_{\{s,s',c,c'\}_m} \\
&\qquad \times \big(\{\hat{p},\hat{f},\hat{s},\hat{s}',\hat{c},\hat{c}'\}_{m+n_R}\big|\mathcal{D}(\mu^2)\big|\{p,f,s,s',c,c'\}_m\big) \\
&\qquad \times \big(\{p,f,s,s',c,c'\}_m\big|\rho_{\rm hard}(\mu^2)\big)
\end{aligned}
$$

Infrared Sensitive Operator

We can consider a more constructive approach to build the full infrared sensitive operator. This operator basically represents the QCD density operator of a $m \to X$ *(anything)* process.



$$\mathcal{D}(\mu^2) = 1 + \sum_{n=1}^{k} \left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^n \sum_{\substack{n_R=0 \\ n_R+n_V=n}}^{n} \sum_{n_V=0}^{n} \mathcal{D}^{(n_R,n_V)}(\mu^2)$$

$$\left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^2 \mathcal{D}^{(1,1)}(\mu^2)$$

The structure is rather straightforward:

$$\big(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+n_R}\big|\mathcal{D}^{(n_R,n_V)}(\mu^2,\mu_S^2)\big|\{p,f,s',c',s,c\}_m\big)$$
$$= \sum_{G\in\mathrm{Graphs}} \int d^d\{\ell\}_{n_V}\ {}_D\big\langle\{\hat{s},\hat{c}\}_{m+n_R}\big|\boldsymbol{V}_L(G;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V},\mu^2)\big|\{s,c\}_m\big\rangle$$
$$\times \big\langle\{s,c\}_m\big|\boldsymbol{V}_R^\dagger(G;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V},\mu^2)\big|\{\hat{s},\hat{c}\}_{m+n_R}\big\rangle_D$$
$$\times \sum_{I\in\mathrm{Regions(G)}} \big(\{\hat{p},\hat{f}\}_{m+n_R}\big|\mathcal{P}_G(I)\big|\{p,f\}_m\big)\underbrace{\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V};\mu_S^2)}$$

Constrains the off-shellness of the hard partons

Infrared Sensitive Operator

➠ We have to introduce an **ultraviolet cutoff to capture only the IR part** of the amplitudes. At first order level in the real graphs it is just a cut on an infrared sensitive variable of the splitting:

$$\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_{\rm R}},\{\ell\}_{n_{\rm V}};\mu_{\rm S}^2)\sim\theta(k_\perp^2<\mu_{\rm S}^2)$$

➠ The D operator depends on two scales (renormalization scale μ and the **shower scale** $\mu_{\rm S}$) but we always set them equal.

$$\mu_{\rm S}^2=\mu^2$$

➠ We don't do eikonal approximation in the soft gluon exchange between two external lines because that messes up the **Glauber region**.

➠ We also need a **momentum mapping**. This can be tricky at higher order level and not necessary the simpler is the better. We prefer "global" momentum mapping.

$\mathrm{N^kLO}$ calculations

Subtractions

Singularities cancel each other here

$$\sigma[O_J] = \overbrace{\big(1\big|\big[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\big]\,\mathcal{D}(\mu^2)}^{\text{Singularities cancel each other here}}\;\underbrace{\mathcal{D}^{-1}(\mu^2)\,\mathcal{O}_J\,\big|\rho(\mu^2)\big)}_{=|\rho_{\mathrm{H}}(\mu^2))}$$

$$+\,\mathcal{O}(\alpha_{\mathrm{s}}^{k+1}L^{2k+2})$$

$$+\,\mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)$$

Hard part, finite in d=4 dimension

Normally $\mathcal{D}^{-1}(\mu^2)$ is constructed by hand and $\mathcal{D}(\mu^2)$ is its inverse.

$$\begin{aligned}\mathcal{D}^{-1}(\mu^2)|\rho(\mu^2)) &= |\rho^{(0)}(\mu^2)) + \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\Big[|\rho^{(1)}(\mu^2)) - \mathcal{D}^{(1)}|\rho^{(0)}(\mu^2))\Big] \\ &+ \left[\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right]^2\Big\{|\rho^{(2)}(\mu^2)) - \mathcal{D}^{(1)}|\rho^{(1)}(\mu^2)) - \big[\mathcal{D}^{(2)}(\mu^2) - \mathcal{D}^{(1)}(\mu^2)\mathcal{D}^{(1)}(\mu^2)\big]|\rho^{(0)}(\mu^2))\Big\} \\ &+ \mathcal{O}(\alpha_{\mathrm{s}}^3)\end{aligned}$$

$\mathrm{N^kLO}$ calculations

Collecting all the singularities in an operator,

$$\mathcal{X}(\mu^2) = \left[\mathcal{F}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right] \mathcal{D}(\mu^2)\, \mathcal{F}^{-1}(\mu^2)$$

Then we have found that $\left(1\middle|\mathcal{X}(\mu^2)\middle|\{p,f,c',c,s',s\}_m\right) = \mathrm{finite}$. Now **define a finite operator** that **leaves the momenta and flavors unchanged** in such a way that

$$\big(1\big|\underbrace{\mathcal{V}(\mu^2)}_{\text{IR finite operator}} = \big(1\big|\mathcal{X}(\mu^2)$$

With this we have the usual fixed order cross section structure:

$$\sigma[O_J] = \big(1\big|\mathcal{X}(\mu^2)\,\mathcal{F}(\mu^2)\,\mathcal{D}^{-1}(\mu^2)\,\mathcal{O}_J\,\big|\rho(\mu^2)\big) + \mathcal{O}(\alpha_s^{k+1}L^{2k+2}) + \mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)$$

$$= \boxed{\big(1\big|\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\mathcal{D}^{-1}(\mu^2)\,\mathcal{O}_J\,\big|\rho(\mu^2)\big) + \mathcal{O}(\alpha_s^{k+1}L^{2k+2}) + \mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)}$$

Shower Cross Section

At this point we have everything to **derive** the shower cross section. Let us do it!

Start with the fixed order (all order) cross section

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\mathcal{X}(\mu^2)\,\mathcal{F}(\mu^2)\,\underbrace{\mathcal{D}^{-1}(\mu^2)\,\big|\rho(\mu^2)\big)}_{=|\rho_{\rm H}(\mu^2))}$$

Insert an unit operator

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\,\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_{\rm H}(\mu^2)\big)$$

and restructure the expression

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\big[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\big]\,\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_{\rm H}(\mu^2)\big)$$

Shower Cross Section

Insert another unit operator

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J \left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]\left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]^{-1}\left[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\right] \mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

and restructure the expression

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J \left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]\left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]^{-1}\left[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\right] \mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

Shower Cross Section

Let us play this game one more time!

Insert another unit operator

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\big[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\big]\,\big[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\big]^{-1}\big[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\big]$$
$$\mathcal{V}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\,\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

and restructure the expression

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\big[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\big]\,\big[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\big]^{-1}\big[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\big]$$
$$\mathcal{V}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\,\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

Shower Cross Section

We can simplify this further introducing the evolution operators. Thus we have

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J \left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right] \overbrace{\left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]^{-1}\left[\mathcal{X}(\mu^2)\,\mathcal{V}^{-1}(\mu^2)\right]}^{\mathcal{U}(\mu_\mathrm{f}^2,\mu^2)}$$

$$\mathcal{V}(\mu_\mathrm{f}^2)\,\underbrace{\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\,\mathcal{V}(\mu^2)}_{\mathcal{U}_\mathcal{V}(\mu_\mathrm{f}^2,\mu^2)}\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

and with this notation the cross section is

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J \left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right] \mathcal{U}(\mu_\mathrm{f}^2,\mu^2)\,\mathcal{V}(\mu_\mathrm{f}^2)\,\mathcal{U}_\mathcal{V}(\mu_\mathrm{f}^2,\mu^2)\,\mathcal{F}(\mu^2)\,\big|\rho_\mathrm{H}(\mu^2)\big)$$

Shower Cross Section

We have to deal with the singular part.

$$\sigma[O_J] = \left(1\middle|\mathcal{O}_J\left[\mathcal{X}(\mu_f^2)\,\mathcal{V}^{-1}(\mu_f^2)\right]\mathcal{U}(\mu_f^2,\mu^2)\,\mathcal{V}(\mu_f^2)\,\mathcal{U}_{\mathcal{V}}(\mu_f^2,\mu^2)\,\mathcal{F}(\mu^2)\middle|\rho_H(\mu^2)\right)$$

When $\mu_f^2 \sim \Lambda_{QCD}^2$

$$\left(1\middle|\mathcal{O}_J\left[\mathcal{X}(\mu_f^2)\,\mathcal{V}^{-1}(\mu_f^2)\right]\middle|\{p,f,...\}_m\right) = \left(1\middle|\mathcal{O}_J\middle|\{p,f,...\}_m\right) + \mathcal{O}\left(\frac{\mu_f^2}{\mu_J^2}\right)$$

and

$$\mathcal{V}(\mu_f^2) \approx 1$$

since this operator is finite.

$$\sigma[O_J] = \left(1\middle|\mathcal{O}_J\,\overbrace{\mathcal{U}(\mu_f^2,\mu^2)}^{\text{Unitary shower}}\,\underbrace{\mathcal{U}_{\mathcal{V}}(\mu_f^2,\mu^2)}_{\text{Resummation of threshold effects}}\,\mathcal{F}(\mu^2)\middle|\rho_H(\mu^2)\right) + \mathcal{O}(\overbrace{\alpha_s^{k+1}L^n}^{\text{Hopefully n << 2k+1}}) + \mathcal{O}(\mu_f^2/\mu_J^2)$$

Threshold Logs

The threshold operator is defined by

$$\mathcal{U}_{\mathcal{V}}(\mu_{\mathrm{f}}^2, \mu_{\mathrm{H}}^2) = \mathcal{V}^{-1}(\mu_{\mathrm{f}}^2)\,\mathcal{V}(\mu_{\mathrm{H}}^2) = \mathbb{T}\exp\left(\int_{\mu_{\mathrm{f}}^2}^{\mu_{\mathrm{H}}^2} \frac{d\mu^2}{\mu^2}\,\mathcal{S}_{\mathcal{V}}(\mu^2)\right)$$

where the generators are

$$\frac{1}{\mu^2}\,\mathcal{S}_{\mathcal{V}}(\mu^2) = \mathcal{V}^{-1}(\mu^2)\,\frac{d\mathcal{V}(\mu^2)}{d\mu^2}$$

$$= \mathcal{V}^{-1}(\mu^2)\,\frac{\partial}{\partial\mu_{\mathrm{S}}^2}\mathcal{V}(\mu^2, \mu_{\mathrm{S}}^2)\Big|_{\mu_{\mathrm{S}}^2=\mu^2} - \underbrace{\frac{d\mathcal{F}(\mu^2)}{d\mu^2}\mathcal{F}^{-1}(\mu^2)}_{\text{pure DGLAP evolution}}$$

- Doesn't create new partons.
- Provides perturbative corrections to the hard part.
- **Sums up threshold logarithms**

Unitary Shower

The unitary shower operator is

$$\mathcal{U}(\mu_\mathrm{f}^2, \mu_\mathrm{H}^2) = \left[\mathcal{X}(\mu_\mathrm{f}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{f}^2)\right]^{-1} \mathcal{X}(\mu_\mathrm{H}^2)\,\mathcal{V}^{-1}(\mu_\mathrm{H}^2) = \mathbb{T}\exp\left(\int_{\mu_\mathrm{f}^2}^{\mu_\mathrm{H}^2} \frac{d\mu^2}{\mu^2}\, \mathcal{S}(\mu^2)\right)$$

where the generators are

$$\frac{1}{\mu^2}\mathcal{S}(\mu^2) = \mathcal{V}(\mu^2)\mathcal{F}(\mu^2)\mathcal{D}^{-1}(\mu^2)\frac{d}{d\mu^2}\left[\mathcal{D}(\mu^2)\mathcal{F}^{-1}(\mu^2)\mathcal{V}^{-1}(\mu^2)\right]$$
$$= \mathcal{V}(\mu^2)\mathcal{F}(\mu^2)\mathcal{D}^{-1}(\mu^2)\ \frac{\partial\mathcal{D}(\mu^2,\mu_\mathrm{S}^2)}{\partial\mu_\mathrm{S}^2}\left[\mathcal{V}(\mu^2)\mathcal{F}(\mu^2)\right]^{-1}\Big|_{\mu_\mathrm{S}^2=\mu^2}$$
$$-\ \frac{\partial\mathcal{V}(\mu^2,\mu_\mathrm{S}^2)}{\partial\mu_\mathrm{S}^2}\mathcal{V}^{-1}(\mu^2)\Big|_{\mu_\mathrm{S}^2=\mu^2}$$

- Creates new partons.
- Preserves probabilities: $\big(1\big|\mathcal{U}(\mu_\mathrm{f}^2, \mu_\mathrm{H}^2) = \big(1\big|$
- **Sums up "*visible*" logarithms** (accuracy can depend on the observable)

Shower Kernel

The generators of the unitary shower can be expanded in the coupling:

$$S(\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} S^{(1)}(\mu^2) + \left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^2 S^{(2)}(\mu^2) + \cdots$$

and the first order term is rather simple

$$\frac{1}{\mu_s^2} S^{(1)}(\mu^2) = \frac{\partial}{\partial \mu_s^2}\left[\mathcal{F}(\mu^2)\mathcal{D}^{(1,0)}(\mu^2,\mu_s^2)\mathcal{F}^{-1}(\mu^2) + \mathcal{D}^{(0,1)}(\mu^2,\mu_s^2)\right]_{\mu_s^2=\mu^2} - \left.\frac{\partial \mathcal{V}^{(1)}(\mu^2,\mu_s^2)}{\partial \mu_s^2}\right|_{\mu_s^2=\mu^2}$$

$$= \Bigg[\underbrace{\mathcal{F}(\mu^2)\frac{\partial \mathcal{D}^{(1,0)}(\mu^2,\mu_s^2)}{\partial \mu_s^2}\mathcal{F}^{-1}(\mu^2)} - \underbrace{\frac{\partial \mathcal{F}(\mu^2)\circ \overline{\mathcal{D}}^{(1,0)}(\mu^2,\mu_s^2)}{\partial \mu_s^2}\mathcal{F}^{-1}(\mu^2)} + \underbrace{\mathrm{Im}\frac{\partial \mathcal{D}^{(0,1)}(\mu^2,\mu_s^2)}{\partial \mu_s^2}}\Bigg]_{\mu_s^2=\mu^2}$$

Real operator
all the quantum numbers of the emitted parton is resolved

Integrated real operator
- all the quantum numbers of the emitted parton is integrated out
- it is not the contribution of the virtual graphs

Glauber gluon
imaginary part of the virtual graphs
$\sim i\pi$

Note, the first order kernel is **independent of** the real part of the virtual graphs.

Shower Kernel

At second order level we are not that lucky. The shower kernel is much more complicated:

$$\frac{1}{\mu_S^2} S^{(2)}(\mu^2) = \mathcal{F}(\mu^2) \left(\frac{\partial \mathcal{D}^{(2)}(\mu^2, \mu_S^2)}{\partial \mu_S^2} - \mathcal{D}^{(1)}(\mu^2) \frac{\partial \mathcal{D}^{(1)}(\mu^2, \mu_S^2)}{\partial \mu_S^2} \right)_{\mu_S^2 = \mu^2} \mathcal{F}^{-1}(\mu^2)$$
$$- \left(\frac{\partial \mathcal{V}^{(2)}(\mu^2, \mu_S^2)}{\partial \mu_S^2} - \mathcal{V}^{(1)}(\mu^2) \frac{\partial \mathcal{V}^{(1)}(\mu^2, \mu_S^2)}{\partial \mu_S^2} \right)_{\mu_S^2 = \mu^2}$$
$$+ \left[\mathcal{V}^{(1)}(\mu^2), \frac{1}{\mu^2} \mathcal{S}^{(1)}(\mu^2) \right]$$

This is highly non-trivial operator and cancelation of all the singularities in the first term is rather delicate.

$$\mathcal{D}^{(2)}(\mu^2, \mu_S^2) = \overbrace{\mathcal{D}^{(2,0)}(\mu^2, \mu_S^2)}^{\text{Double real}} + \underbrace{\mathcal{D}^{(1,1)}(\mu^2, \mu_S^2)}_{\text{Real-virtual}} + \overbrace{\mathcal{D}^{(0,2)}(\mu^2, \mu_S^2)}^{\text{Double virtual}}$$

and

$$\mathcal{D}^{(1)}(\mu^2, \mu_S^2) = \overbrace{\mathcal{D}^{(1,0)}(\mu^2, \mu_S^2)}^{\text{Single real}} + \underbrace{\mathcal{D}^{(0,1)}(\mu^2, \mu_S^2)}_{\text{Single virtual}}$$

Summary

- **Fixed order** calculations

$$\sigma[O_J] = \big(1\big|\mathcal{V}(\mu^2)\,\mathcal{F}(\mu^2)\,\mathcal{D}^{-1}(\mu^2)\,\mathcal{O}_J\,\big|\rho(\mu^2)\big) + \mathcal{O}(\alpha_s^{k+1}L^{2k+2}) + \mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)$$

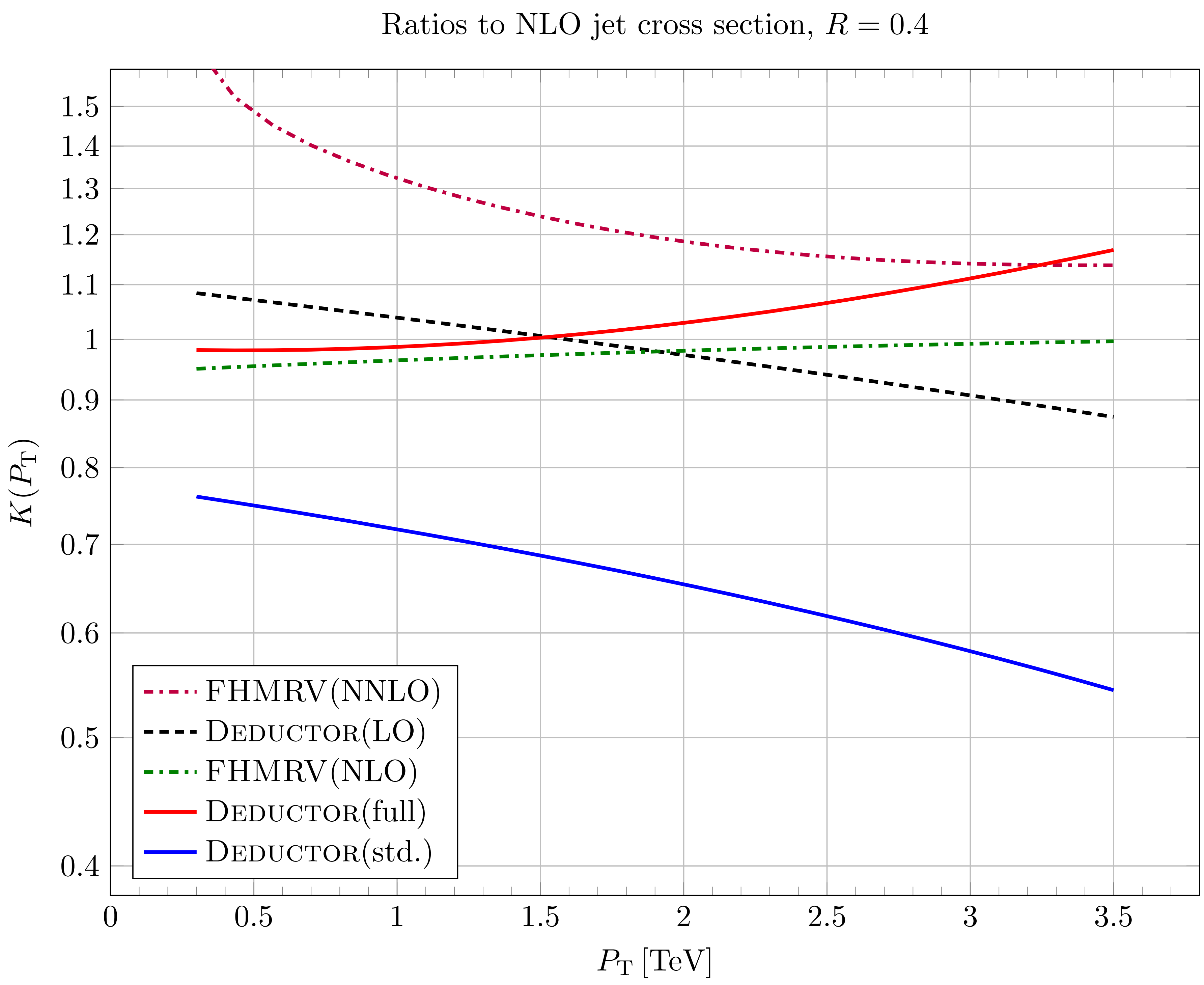
can be systematically improved by working to higher order.

- **Parton shower** calculations

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\mathcal{U}(\mu_f^2,\mu^2)\,\mathcal{U}_{\mathcal{V}}(\mu_f^2,\mu^2)\,\mathcal{F}(\mu^2)\,\mathcal{D}^{-1}(\mu^2)\big|\rho(\mu^2)\big) + \mathcal{O}(\alpha_s^{k+1}L^n) + \mathcal{O}(\mu_f^2/\mu_J^2)$$

can be systematically improved by working to higher order.

Threshold Effect in Jet Production



Dealing with Colour Perturbatively

Unitary Shower Operator

Here we focus on the the unitary shower

$$\mathcal{U}(\mu_{\rm f}^2, \mu_{\rm H}^2) = \mathbb{T} \exp\left(\int_{\mu_{\rm f}^2}^{\mu_{\rm H}^2} \frac{d\mu^2}{\mu^2} \frac{\alpha_{\rm s}(\mu^2)}{2\pi} \left[\mathcal{S}^{(1)}(\mu^2) + \frac{\alpha_{\rm s}(\mu^2)}{2\pi} \mathcal{S}^{(2)}(\mu^2) + \cdots \right] \right)$$

Here we are interested only at first order level:

$$\frac{\alpha_{\rm s}(\mu^2)}{2\pi} \mathcal{S}^{(1)}(\mu^2) = \overbrace{\mathcal{H}(\mu^2)}^{\text{Real emissions}} - \underbrace{\mathcal{V}_{\rm Re}(\mu^2)}_{\text{Inclusive splitting operator}} - \overbrace{{\rm i}\pi \mathcal{V}_{{\rm i}\pi}(\mu^2)}^{\text{Imaginary part of the 1-loop contributions}}$$

Unitary condition tells us:

$$\big(1\big|\mathcal{S}^{(1)}(\mu^2) = \big(1\big|\big[\mathcal{H}(\mu^2) - \mathcal{V}_{\rm Re}(\mu^2)\big] = \big(1\big|\mathcal{V}_{{\rm i}\pi}(\mu^2) = 0$$

Evolution Equation

The shower operator obeys the following integral equation:

$$\mathcal{U}(\mu_2^2, \mu_1^2) = \mathcal{N}(\mu_2^2, \mu_1^2) + \int_{\mu_2^2}^{\mu_1^2} \frac{d\mu^2}{\mu^2} \mathcal{U}(\mu_2^2, \mu^2)\mathcal{H}(\mu^2)\mathcal{N}(\mu^2, \mu_1^2)$$

where the no-splitting (**Sudakov**) operator is

$$\mathcal{N}(\mu_2^2, \mu_1^2) = \mathbb{T}\exp\left\{-\int_{\mu_2^2}^{\mu_1^2} \frac{d\mu^2}{\mu^2} \underbrace{\left[\mathcal{V}_{\mathrm{Re}}(\mu^2) + \mathrm{i}\pi\mathcal{V}_{\mathrm{i}\pi}(\mu^2)\right]}\right\}$$

This is not a diagonal operator and it is impossible to diagonalise when we have ~*O(10)* partons.

Let's define a systematic approximation!

LC+ Approximation

ZN, D. Soper, **JHEP06** (2012) 044

The **real splittings** are described by

$$\mathcal{H}(\mu^2)\big|\{p,f,c',c\}_m\big) \propto \sum_{l,k} H_{lk}(\mu^2)\big|\{p,f\}_m\big) \left\{ t_l^\dagger \big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big| t_k + t_k^\dagger \big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big| t_l \right\}$$

The index l always represents the emitter parton and the emitted parton can be collinear only with l.

The **inclusive splitting operator** is

$$\mathcal{V}_{\rm Re}(\mu^2)\big|\{p,f,c',c\}_m\big) \propto \sum_{l,k} V_{lk}(\mu^2)\big|\{p,f\}_m\big) \left\{ \big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|[t_k\cdot t_l^\dagger] + [t_l\cdot t_k^\dagger]\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big| \right\}$$

We need an approximation (only in the color space) that

- **can handle color interference** contributions
- is as minimal approximation as possible
- is **exact in the collinear and soft-collinear** regions
- makes some **harm only in the wide angle soft** region
- preserves unitary

LC+ Approximation

We insert a projection **only on the spectator side**

$$t_k^\dagger \big|\{c\}_m\big\rangle \longrightarrow C(l, m+1) t_k^\dagger \big|\{c\}_m\big\rangle$$
$$\big\langle\{c'\}_m\big| t_k \longrightarrow \big\langle\{c'\}_m\big| t_k C(l, m+1)^\dagger$$

The **operator** ***C(l, m+1)*** is defined by it action on the basis states:

$$C(l, m+1)\big|\{\hat{c}\}_{m+1}\big\rangle = \begin{cases} \big|\{\hat{c}\}_{m+1}\big\rangle & \text{if } l \text{ and } m+1 \text{ are color connected in } \{\hat{c}\}_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

(In string basis l and m+1 are color connected when they are next to each other along the fermion line.)

In the inclusive splitting operator, the color simplifies a lot:

$$[t_l \cdot t_k^\dagger]\big|\{c\}_m\big\rangle \longrightarrow [t_l \cdot C(l, m+1) t_k^\dagger]\big|\{c\}_m\big\rangle = \big|\{c\}_m\big\rangle \frac{t_l^2}{1+\delta_{\mathrm{g}f_l}}$$
$$\big\langle\{c'\}_m\big|[t_k \cdot t_l^\dagger] \longrightarrow \big\langle\{c'\}_m\big|[t_k C(l, m+1)^\dagger \cdot t_l^\dagger] = \frac{t_l^2}{1+\delta_{\mathrm{g}f_l}}\big\langle\{c'\}_m\big|$$

LC+ Approximation

In LC+ approximation every basis state is eigenstate of the inclusive splitting operator

$$\mathcal{V}^{\rm LC+}(\mu^2)\big|\{p,f,c',c\}_m\big) = \lambda(\{p,f,c',c\}_m,\mu^2)\big|\{p,f,c',c\}_m\big)$$

and we have **Sudakov factor** instead of Sudakov operator

$$\mathcal{N}^{\rm LC+}(\mu_2^2,\mu_1^2)\big|\{p,f,c',c\}_m\big) = \exp\left\{-\int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\lambda(\{p,f,c',c\}_m,\mu^2)\right\}\big|\{p,f,c',c\}_m\big)$$

Based on this we can define the **LC+ parton shower** and its evolution equation is

$$\mathcal{U}^{\rm LC+}(\mu_2^2,\mu_1^2) = \mathcal{N}^{\rm LC+}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\mathcal{U}^{\rm LC+}(\mu_2^2,\mu^2)\mathcal{H}^{\rm LC+}(\mu^2)\mathcal{N}^{\rm LC+}(\mu^2,\mu_1^2)$$

Beyond LC+

ZN, D. Soper, **Phys.Rev. D99** (2019) no.5, 054009

Now we can define the operators of the soft wide angle emissions

$$\mathcal{H}(\mu^2) = \mathcal{H}^{\rm LC+}(\mu^2) + \Delta\mathcal{H}(\mu^2)$$
$$\mathcal{V}(\mu^2) = \mathcal{V}_{\rm Re}(\mu^2) + {\rm i}\pi\mathcal{V}_{{\rm i}\pi}(\mu^2) = \mathcal{V}^{\rm LC+}(\mu^2) + \Delta\mathcal{V}(\mu^2)$$

With these the full shower is

$$\mathcal{U}(\mu_2^2,\mu_1^2) = \mathcal{U}^{\rm LC+}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{U}(\mu_2^2,\mu^2)\left[\Delta\mathcal{H}(\mu^2) - \Delta\mathcal{V}(\mu^2)\right]\mathcal{U}^{\rm LC+}(\mu^2,\mu_1^2)$$

One can expand this in terms of the soft wide angle operators at a given order. In principle this is what we want, but this form is not efficient for implementation.
Let's **try something else**:

$$\mathcal{N}(\mu_2^2,\mu_1^2) = \underbrace{\mathcal{X}(\mu_2^2,\mu_1^2)}\,\mathcal{N}^{\rm LC+}(\mu_2^2,\mu_1^2)$$

Hopefully it is **simple enough** to deal with it perturbatively.

Beyond LC+

The evolution equation is

$$\mathcal{U}(\mu_2^2,\mu_1^2) = \mathcal{X}(\mu_2^2,\mu_1^2)\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\mathcal{U}(\mu_2^2,\mu^2)\left[\mathcal{H}^{\text{LC+}}(\mu^2) + \Delta\mathcal{H}(\mu^2)\right]\mathcal{X}(\mu^2,\mu_1^2)\mathcal{N}^{\text{LC+}}(\mu^2,\mu_1^2)$$

When we iterate this equation we can **control the number of ΔH** operator **insertions**.

The X operator obeys its evolution equation:

$$\mathcal{X}(\mu_2^2,\mu_1^2) = 1 - \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{X}(\mu_2^2,\mu^2)\,\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu^2)\Delta\mathcal{V}(\mu^2)\,\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu^2)^{-1}$$

It is not immediately obvious but this operator depends **only pure soft** contributions

$$\mathcal{X}(\mu_2^2,\mu_1^2) = 1 - \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{X}(\mu_2^2,\mu^2)\,\mathcal{N}_{\text{soft}}^{\text{LC+}}(\mu_2^2,\mu^2)\Delta\mathcal{V}(\mu^2)\,\mathcal{N}_{\text{soft}}^{\text{LC+}}(\mu_2^2,\mu^2)^{-1}$$

and the **Sudakov factor** is

$$\mathcal{N}_{\text{soft}}^{\text{LC+}}(\mu_2^2,\mu_1^2)\big|\{p,f,c',c\}_m\big) = \exp\left\{-\int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\underbrace{\lambda_{\text{soft}}(\{p,f,c',c\}_m,\mu^2)}\right\}\big|\{p,f,c',c\}_m\big)$$

It is rather **simple** and can be computed "***quasi-analytically***".

Beyond LC+

Expanding the shower operator in terms of ΔH and ΔV operators, we can write

$$\mathcal{U}(\mu_2^2, \mu_1^2) = \mathcal{U}^{\mathrm{LC+}}(\mu_2^2, \mu_1^2) + \underbrace{\mathcal{U}^{(1)}(\mu_2^2, \mu_1^2)}_{\sim\mathcal{O}([\Delta\mathcal{H}(\mu^2) - \Delta\mathcal{V}(\mu^2)])} + \overbrace{\mathcal{U}^{(2)}(\mu_2^2, \mu_1^2)}^{\sim\mathcal{O}([\Delta\mathcal{H}(\mu^2) - \Delta\mathcal{V}(\mu^2)]^2)} + \dots$$

This expansion is systematic and the **unitary condition** is satisfied term by term,

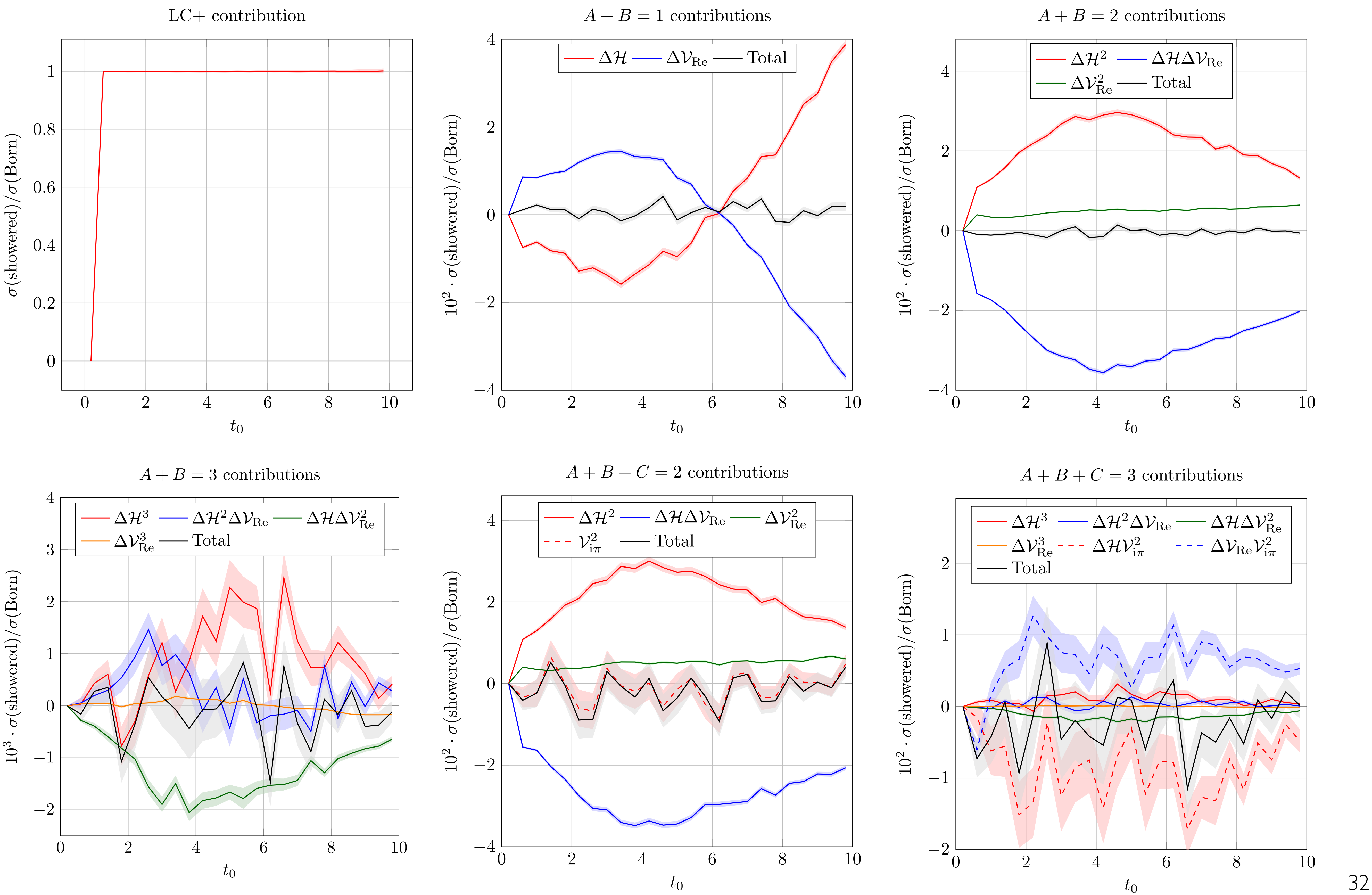
$$\big(1\big|\mathcal{U}^{\mathrm{LC+}}(\mu_2^2, \mu_1^2) = \big(1\big|$$

and for the corrections

$$\big(1\big|\mathcal{U}^{(k)}(\mu_2^2, \mu_1^2) = 0 \qquad \text{for } k = 1, 2, 3, \dots$$

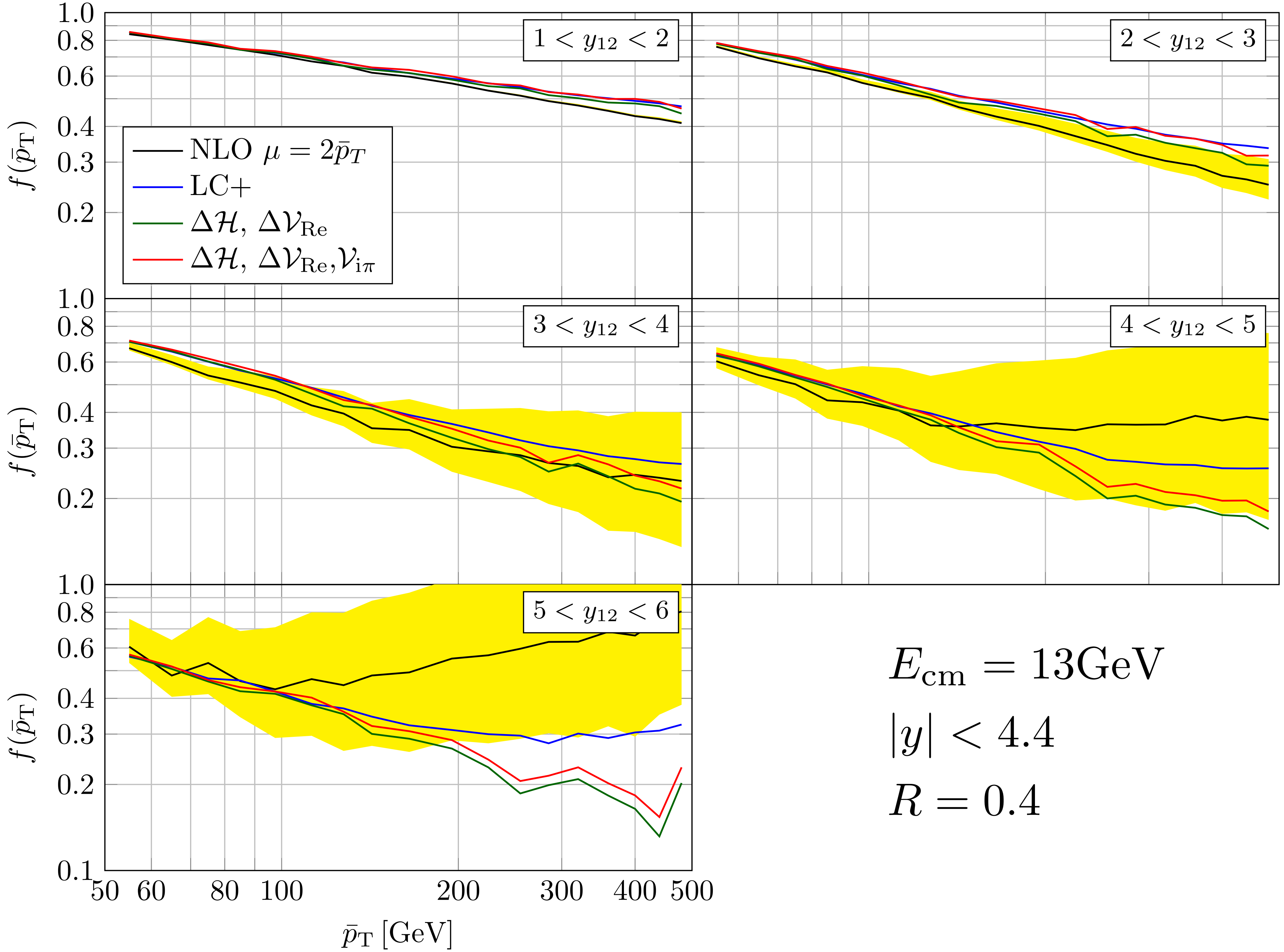
We can test this numerically!

Unitary Test



Rapidity Gap Survival

ZN, D. Soper, **arXiv:1905.07176**

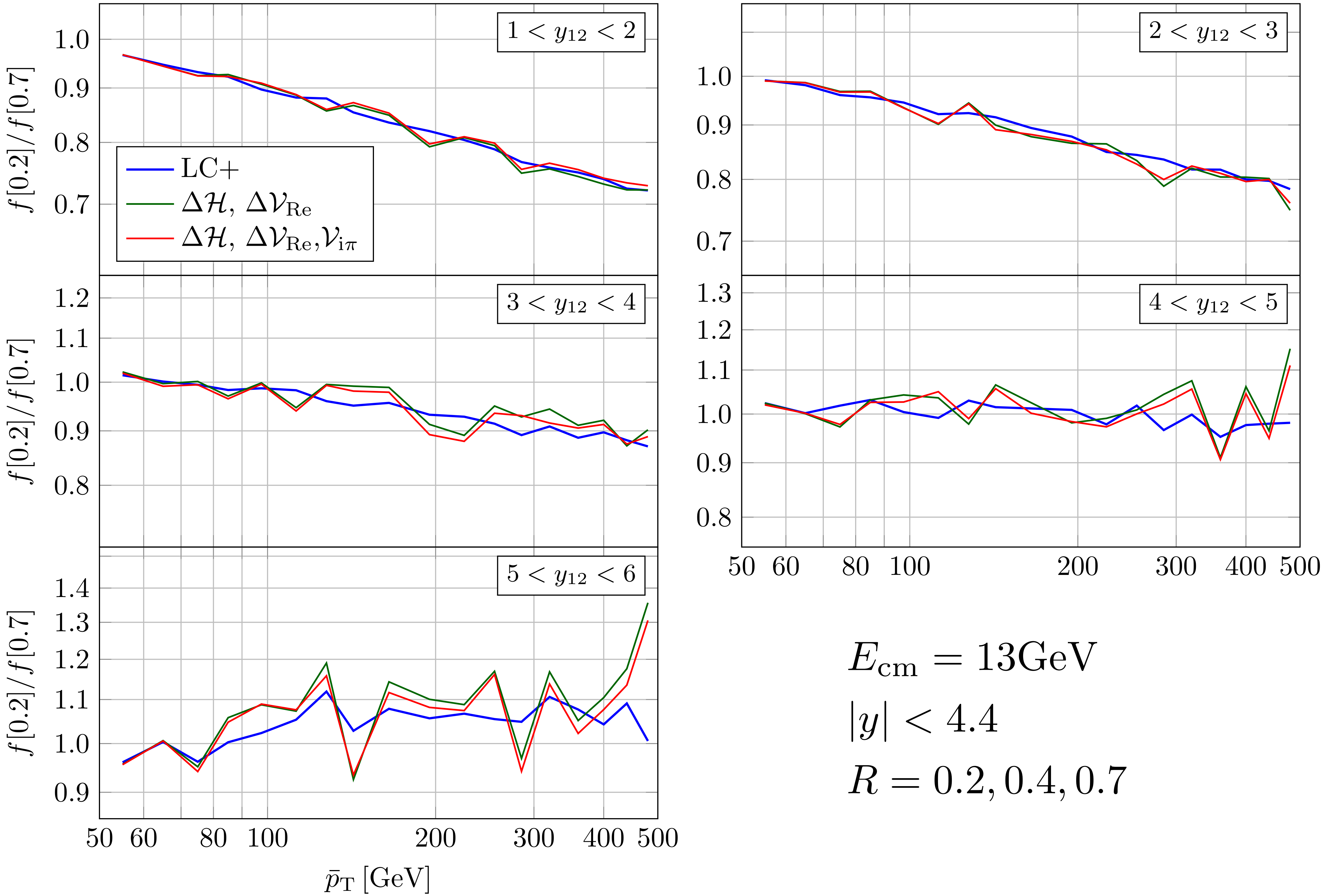


$E_{\rm cm} = 13{\rm GeV}$

$|y| < 4.4$

$R = 0.4$

Rapidity Gap Survival



$E_{\mathrm{cm}} = 13\mathrm{GeV}$

$|y| < 4.4$

$R = 0.2, 0.4, 0.7$

Exponentiating the Phase Terms

Evolution Equation

The evolution equation is

$$\mathcal{U}(\mu_2^2,\mu_1^2) = \mathcal{X}(\mu_2^2,\mu_1^2)\mathcal{N}^{\rm LC+}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\mathcal{U}(\mu_2^2,\mu^2)\left[\mathcal{H}^{\rm LC+}(\mu^2)+\Delta\mathcal{H}(\mu^2)\right]\mathcal{X}(\mu^2,\mu_1^2)\mathcal{N}^{\rm LC+}(\mu^2,\mu_1^2)$$

The X operator obeys its evolution equation:

$$\mathcal{X}(\mu_2^2,\mu_1^2) = 1 - \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{X}(\mu_2^2,\mu^2)\,\mathcal{N}^{\rm LC+}(\mu_2^2,\mu^2)[\Delta\mathcal{V}_{\rm Re}(\mu^2) + \underbrace{{\rm i}\pi\mathcal{V}_{{\rm i}\pi}(\mu^2)]}\,\mathcal{N}^{\rm LC+}(\mu_2^2,\mu^2)^{-1}$$

Can we exponentiate this term?

Here

$${\rm i}\pi\mathcal{V}_{{\rm i}\pi}(t) = -4{\rm i}\pi\frac{\alpha_{\rm s}}{2\pi}\big([(T_{\rm a}\cdot T_{\rm b})\otimes 1] - [1\otimes(T_{\rm a}\cdot T_{\rm b})]\big)$$

and

$$\mathcal{N}^{\rm LC+}(t_2,t_1) = \exp\left[-\int_{t_1}^{t_2}d\tau\;\mathcal{V}^{\rm LC+}(\tau)\right]$$

$$\Rightarrow\quad \mathcal{N}^{\rm E}(t_2,t_1) = \exp\left[-\int_{t_1}^{t_2}d\tau\;(\mathcal{V}^{\rm LC+}(\tau) + {\rm i}\pi\mathcal{V}_{{\rm i}\pi}(\tau)\right]$$

No-splitting Operator

Statistical space → **Quantum space**

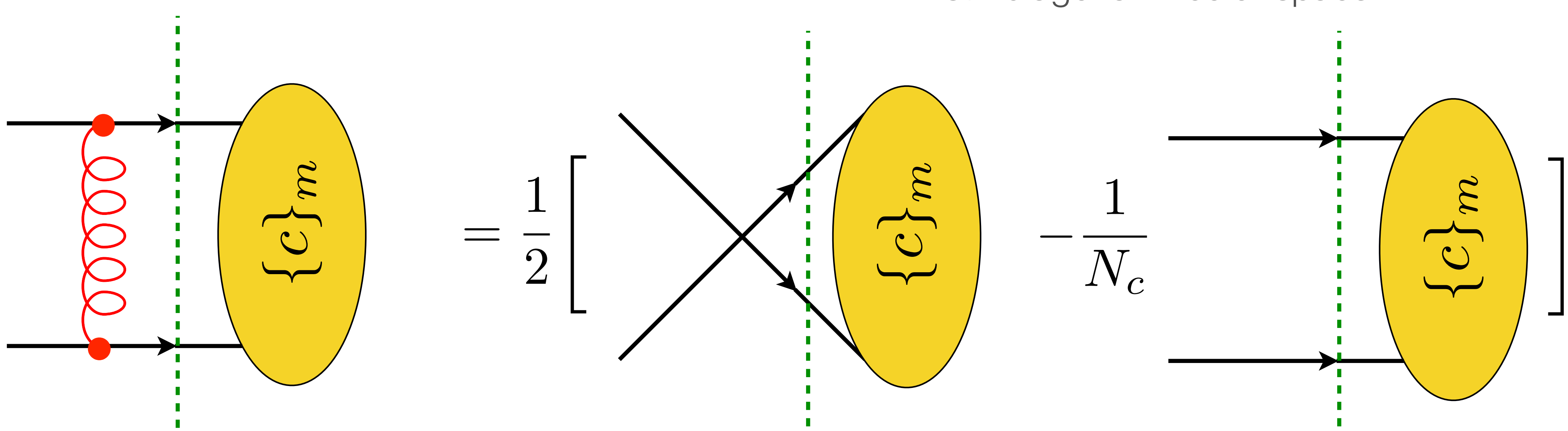
$$\mathcal{N}^{\mathrm{E}}(t_2,t_1)\big|\{p,f,c'c\}_m\big) = n(t_2,t_1,\{p,f\}_m)\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|n(t_2,t_1,\{p,f\}_m)^{\dagger}$$

We have to compute the the no-splitting operator in the quantum space and that is

How messy is this?

$$n(t_2,t_1,\{p,f\}_m) = \mathbb{T}\exp\left[-\int_{t_1}^{t_2} dt\,\frac{\alpha_{\mathrm{s}}(t)}{2\pi}\left(\underbrace{a^{\mathrm{LC+}}(t,\{p,f\}_m)}_{\text{LC+ part}} - 4\mathrm{i}\pi\,\overbrace{T_{\mathrm{a}}\cdot T_{\mathrm{b}}}\right)\right]$$

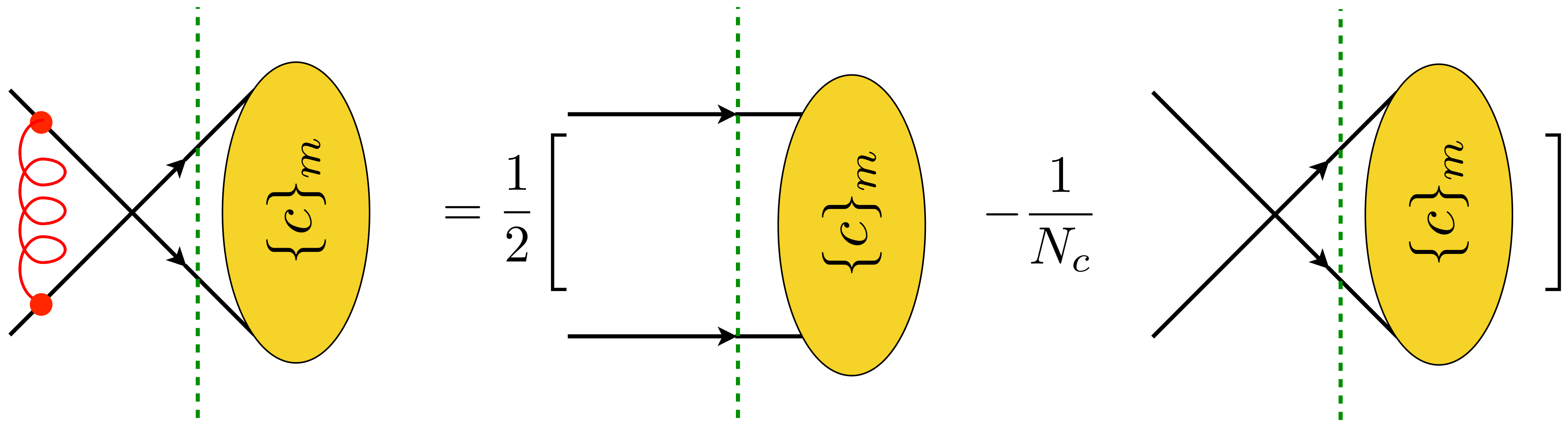
LC+ part
still diagonal in color space



A new state: $\big|\{c_1\}_m\big\rangle$ **Original state:** $\big|\{c\}_m\big\rangle$

No-splitting Operator

Now, what happens when phase operator acts on the new state?



$$\color{red}{T_{\rm a} \cdot T_{\rm b}}\big|\{c_1\}_m\big\rangle \qquad \big|\{c_1\}_m\big\rangle \qquad \big|\{c\}_m\big\rangle$$

This is good! The phase operator rotates **only** in the **2 dimensional** subspace.

$$\color{red}{T_{\rm a} \cdot T_{\rm b}}\big|\{c_n\}_m\big\rangle = \sum_{n'} M_{n'n}\big|\{c_{n'}\}_m\big\rangle$$

$$M = -\frac{1}{2}\begin{bmatrix} 1/N_{\rm c} & -1 \\ -1 & 1/N_{\rm c} \end{bmatrix}$$

Exponentiating a 2x2 matrix is easy!

No-splitting Operator

How about gluon incomings? For **quark-gluon** incomming partons we have a 4D subspace and the corresponding 4x4 matrix is

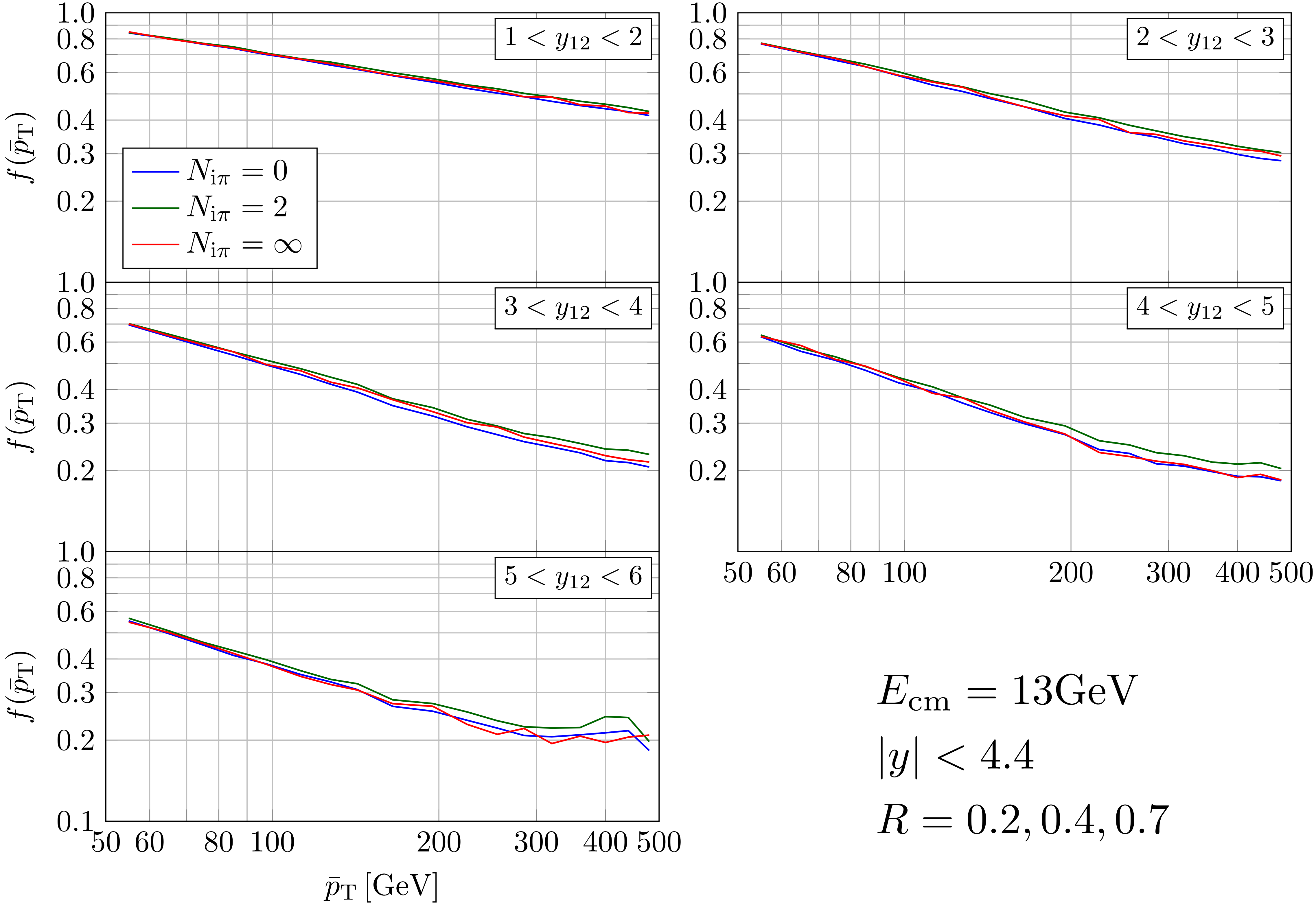
$$M = -\frac{1}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & N_c & 0 \\ 1 & 0 & 0 & N_c \end{bmatrix}$$

In the most complicated case, gluon-gluon initiated process, we have to deal with a 14D subspace and then we have

$$M = -\frac{1}{2}\begin{bmatrix}
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & N_c & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_c & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2N_c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2N_c
\end{bmatrix}$$

It is still easy to exponentiate numerically.

Rapidity Gap Survival



$E_{cm} = 13\text{GeV}$

$|y| < 4.4$

$R = 0.2, 0.4, 0.7$

Implementation

- **Deductor** is designed to do a better job with color, spin and resummation of large logarithms compared to other shower generators.
    - Lambda, kT and angular ordering
    - LC+ color treatment. It allows us to do color evolution at amplitude level
    - Threshold log resummation
    - Spin correlations are not yet computed
    - Fully exponentiated Glauber (Coulomb) gluon effects
    - Wide angle soft gluon effects perturbatively.
- Next version is available soon…
    - NLO matching at density operator level
- It is available from

http://www.desy.de/~znagy/deductor
http://pages.uoregon.edu/soper/deductor

Summary, Outlook

- **We have theory of parton shower algorithm.**
  - Defined at all order level and derived from fixed order perturbation theory (with the help of renormalization group).
  - It is implemented at LO level.
  - All the classical shower algorithm fit into this formalism.
- **But how good it is at summing up large logarithms?**
  - … stay tuned!