QCD AT SMALL x

A PERSONAL PERSPECTIVE

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TOWARDS ACCURACY AT SMALL x EDINBURGH, SEPTEMBER 10, 2019

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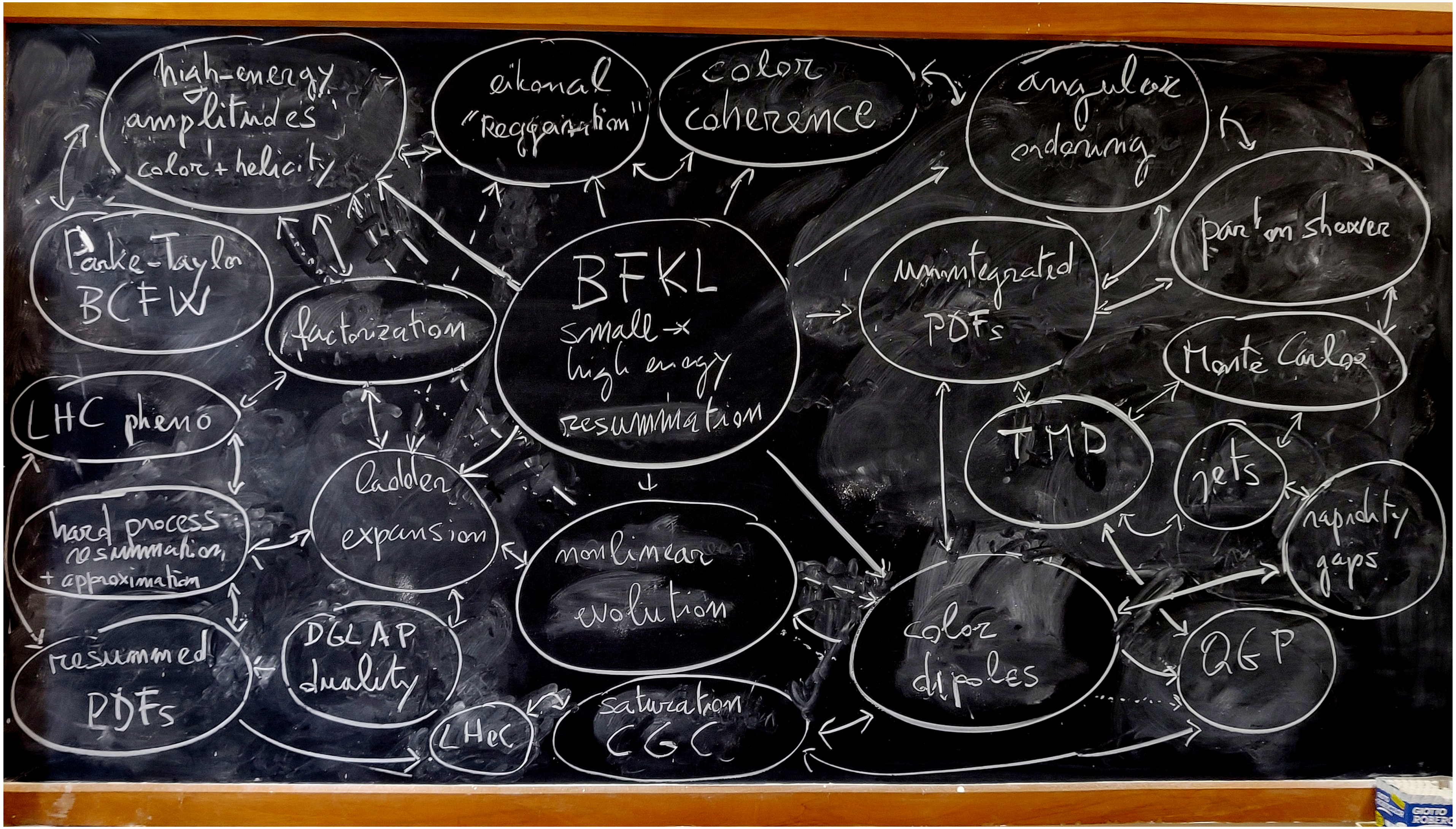
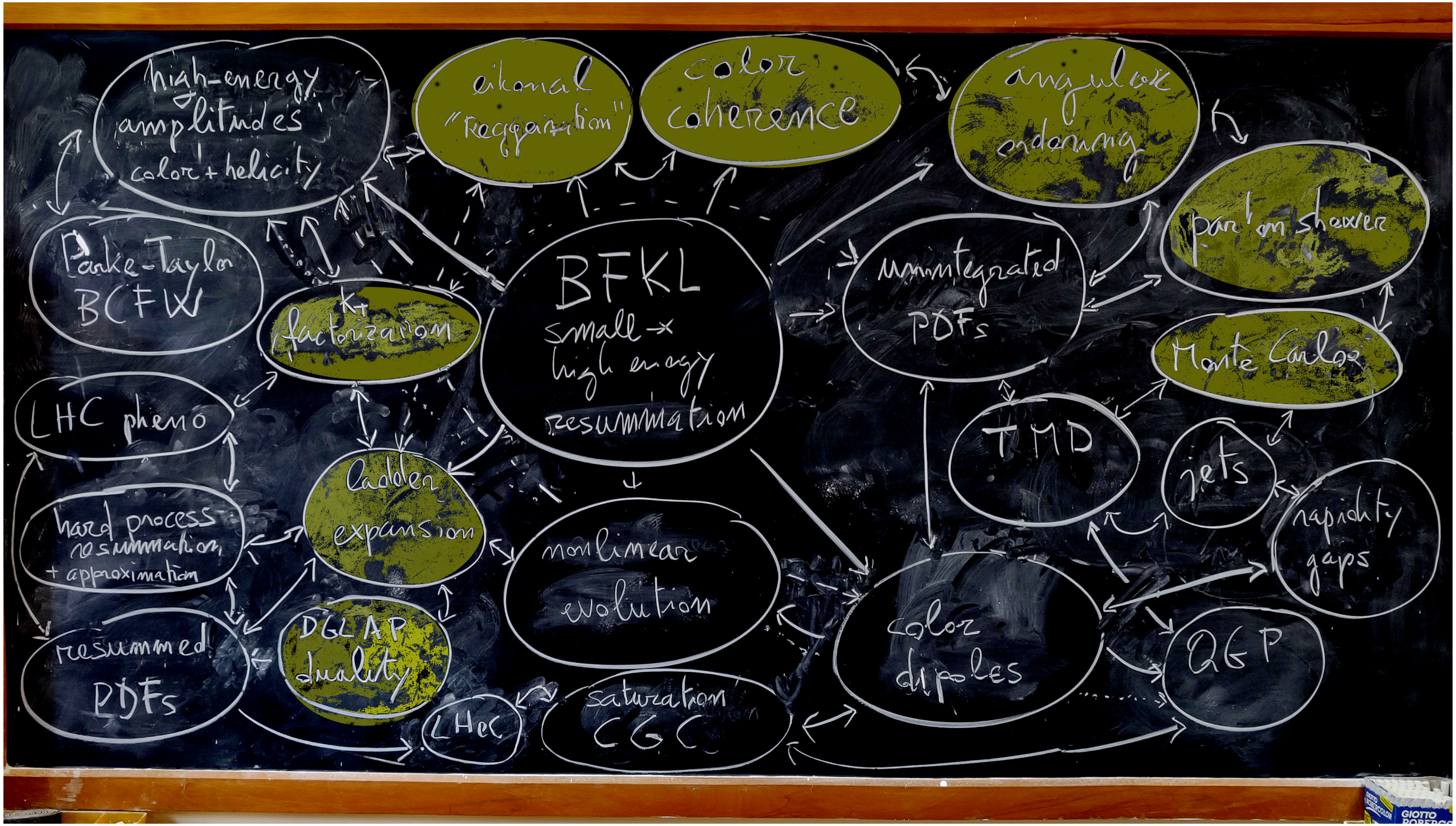
QCD AT SMALL x



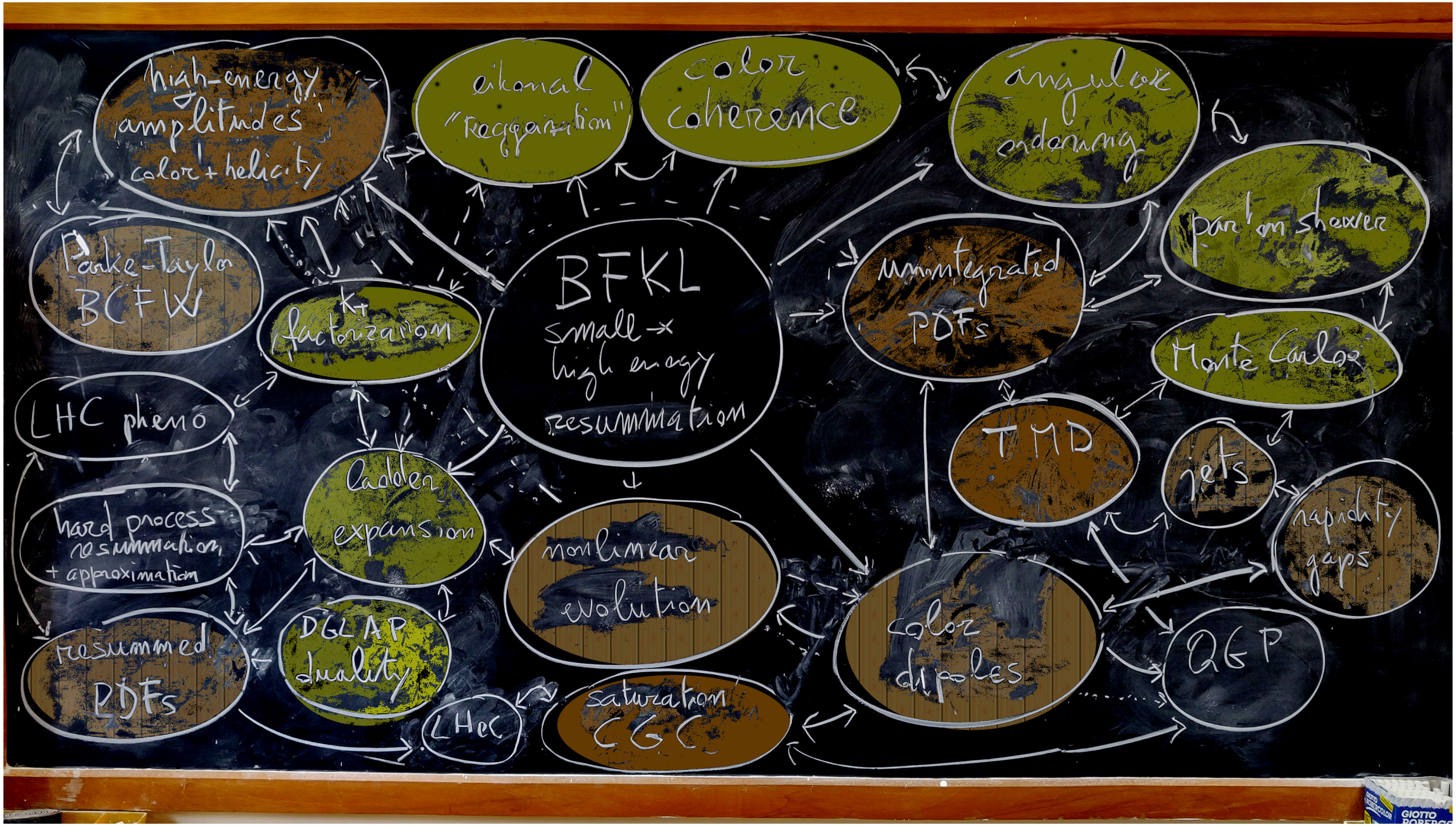
A TIMELINE

L. N. Lipatov, " Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories" Yad. Fiz. 23 (1976) 642
E.A. Kuraev, L.N. Lipatov, V. S. Fadin "Multiregge Processes in Yang-Mills Theory" ZETF 71 (1976) 840
E.A. Kuraev, L.N. Lipatov, V. S. Fadin "The Pomeranchuk Singularity in Nonabelian Gauge Theories" ZETF 72 (1977) 377
I.I. Balitsky, L.N. Lipatov, "The Pomeranchuk Singularity in Quantum Chromodynamics", Yad.Fiz. 28 (1978) 1597

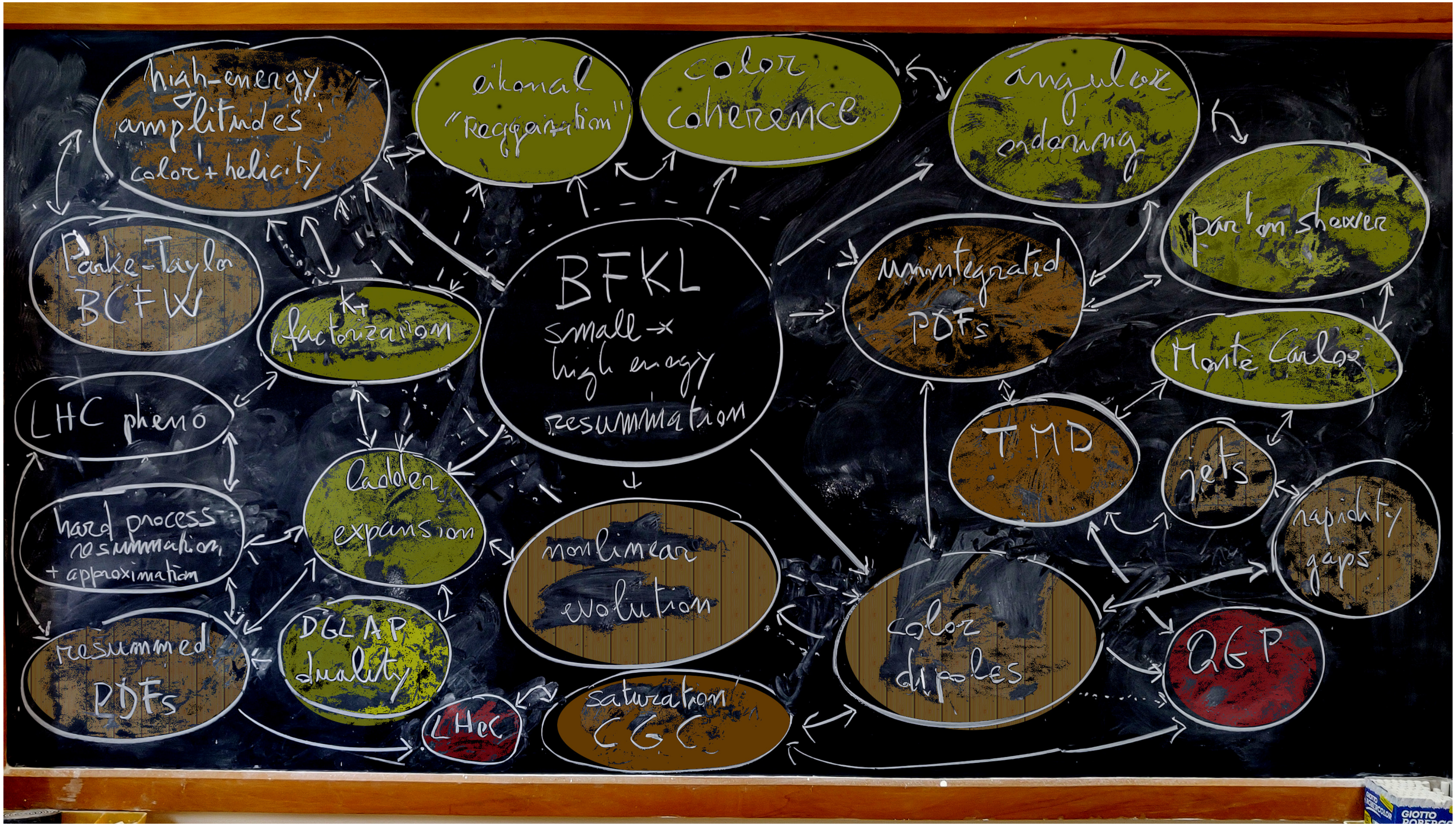
- before 1980: 9 CITES
- before 1990: 62 CITES
  - Ciafaloni (1979-1988)+ Bassetto, Marchesini: COLOR COHERENCE
  - Ciafaloni, Bassetto, Marchesini (1983): LADDER EXPANSION, k_t FACTORIZATION
  - Jaroszewicz (1982): LO DGLAP DUALITY
  - Marchesini, Webber (1988): MONTE CARLO WITH ANGULAR ORDERING
  - Catani, Fiorani, Marchesini (1990): ANGULAR ORDERED EVOLUTION EQUATION


high-energy amplitudes color+helicity
eikonal "Reggeization"
color coherence
angular ordering
Parke-Taylor BCFW
k_T factorization
BFKL
small-x
high energy
resummation
unintegrated PDFs
parton shower
LHC pheno
Monte Carlo
TMD
jets
ladder expansion
hard process resummation + approximation
nonlinear evolution
rapidity gaps
DGLAP duality
resummed PDFs
color dipoles
QGP
LHeC
saturation CGC

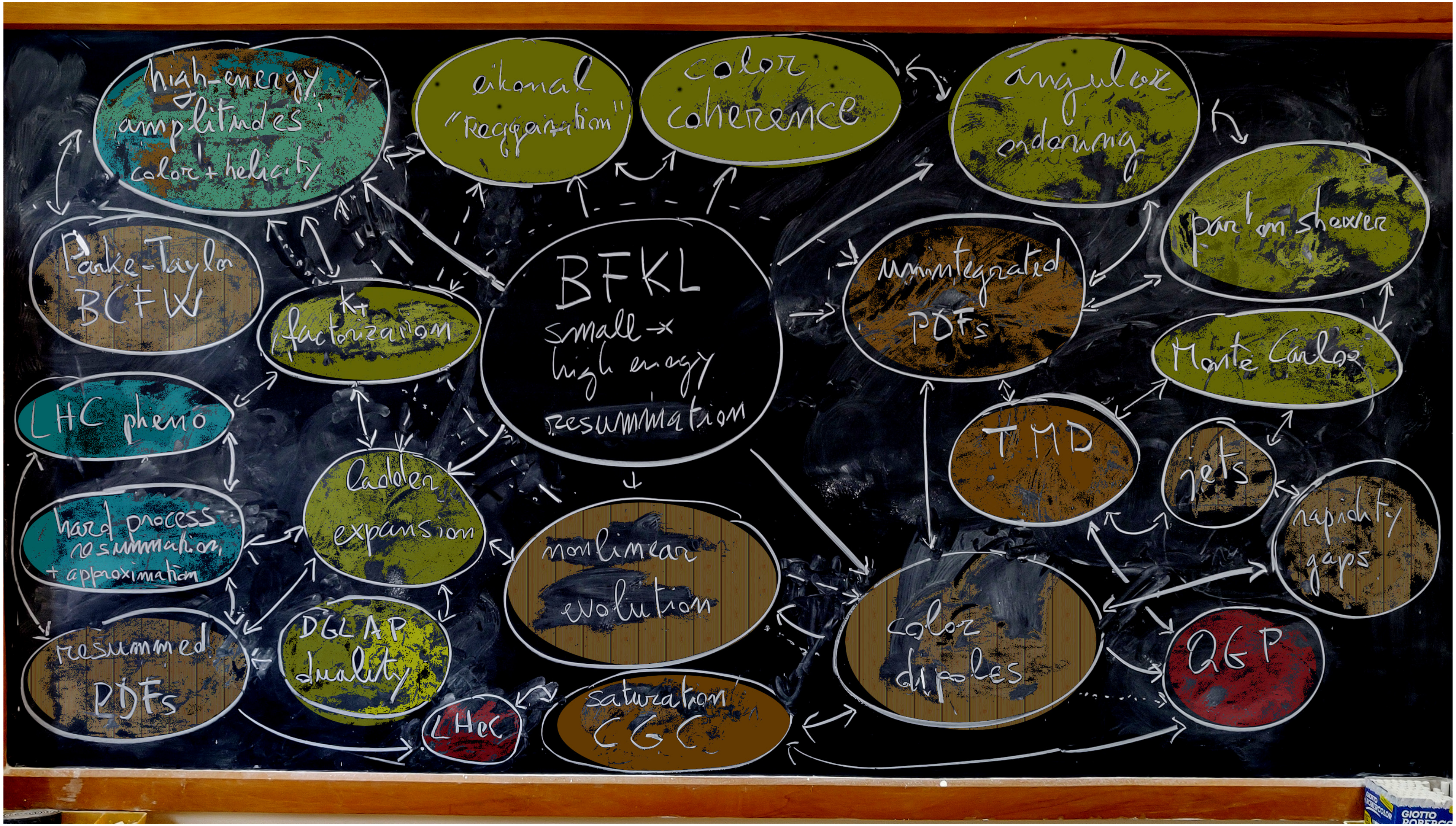
- before 2000: 948 CITES



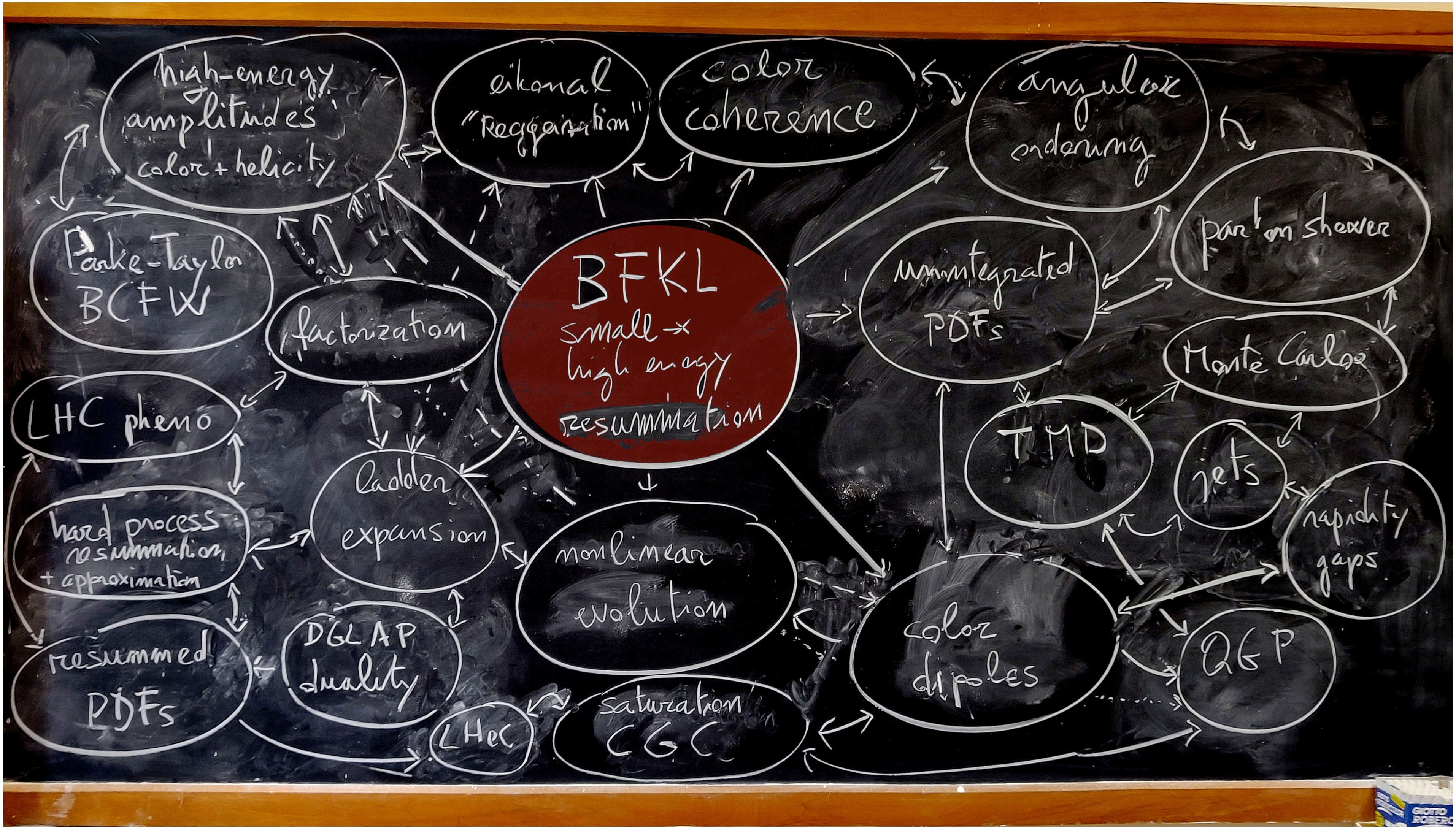
- before 2010: 2121 CITES



- now: 3159 CITES

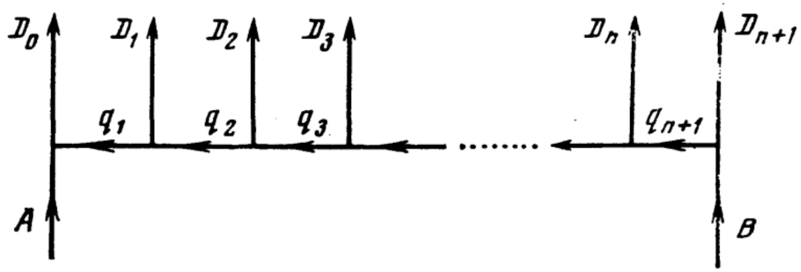


E.A. Kuraev, L.N. Lipatov, V. S. Fadin “The Pomeranchuk Singularity in Nonabelian Gauge Theories” ZETF 72 (1977) 377



MULTIGLUON AMPLITUDES

THE GOAL: DETERMINE THE t-CHANNEL PARTIAL WAVE AMPLITUDES



$2 \to 2+n$ SCATTERING IN MULTIREGGE KINEMATICS

$$s_k=(p_{D_{k-1}}+p_{D_k})^2\gg m^2, \quad -t_i=-q_i^2\sim m^2, \quad \prod_{i=1}^{n+1} s_i = s\prod_{i=1}^{n}(m_i^2+\mathbf{p}_{D_i}^{\perp 2}),$$

$\Rightarrow$ LARGE RAPIDITY GAP BETWEEN SUBSEQUENT EMISSIONS

AMPLITUDE

$$A_{2\to2+n} = s\Gamma_{AD_0}^{i_1} \frac{(s_1/m^2)^{\alpha(t_1)}}{t_1-m^2}\gamma_{i_1 i_2}^{D_1}(q_1,q_2)\frac{(s_2/m^2)^{\alpha(t_2)}}{t_2-m^2} \ldots \gamma_{i_n i_{n+1}}^{D_n}(q_n,q_{n+1})\frac{(s_{n+1}/m^2)^{\alpha(t_{n+1})}}{t_{n+1}-m^2}\Gamma_{BD_{n+1}}^{i_{n+1}},$$

LIPATOV VERTEX

$$\gamma_{ij}^D(q_1,q_2)=ige_{dij}\left[-(q_1+q_2)^\mu - p_A^\mu\left(\frac{2p_Bp_D}{p_Ap_B}-\frac{m^2-q_1^2}{p_Ap_D}\right)+p_B^\mu\left(\frac{2p_Ap_D}{p_Ap_B}-\frac{m^2-q_2^2}{p_Bp_D}\right)\right]e_\mu(\lambda_D)$$

REGGE TRAJECTORY

$$j=1+\alpha(t)=1+\frac{g^2}{(2\pi)^3}(t-m^2)\int\frac{d^2k}{[(k-q)^2-m^2][k^2-m^2]},$$
$$(t=q^2,\quad q=q_\perp,\quad k=k_\perp,\quad k^2=-\mathbf{k}_\perp^2)$$

HELICITY PROJ.

Γ

THE BFKL EQUATION

UNITARITY $\Rightarrow$ AMPLITUDE (Sommerfeld-Watson form)

$$A^{(T)}(s,q)=\frac{s}{4i}\int_{\delta-i\infty}^{\delta+i\infty} d\omega \left(\frac{s}{m^2}\right)^{\omega} \frac{e^{-i\pi\omega}-(-1)^T}{\sin\pi\omega} F_\omega^{(T)}(q^2),$$

ON-SHELL $\Leftarrow$ OFF-SHELL AMPLITUDE COLOR T ANGULAR MOMENTUM ω

$$F_\omega^{(T)}(q^2)=F_\omega^{(T)}(k,q-k)\,|_{k^2=(q-k)^2=m^2},$$

BETHE-SALPETER-LIKE EQUATION

$$[\omega-\alpha(k^2)-\alpha((q-k)^2)]F_\omega^{(T)}(k,q-k)$$

$$=\frac{\omega}{A_T}+\frac{g^2}{(2\pi)^3}\int d^2k' \frac{K^{(T)}(k,k')}{(k'^2-m^2)[(q-k')^2-m^2]}F_\omega^{(T)}(k',q-k').$$

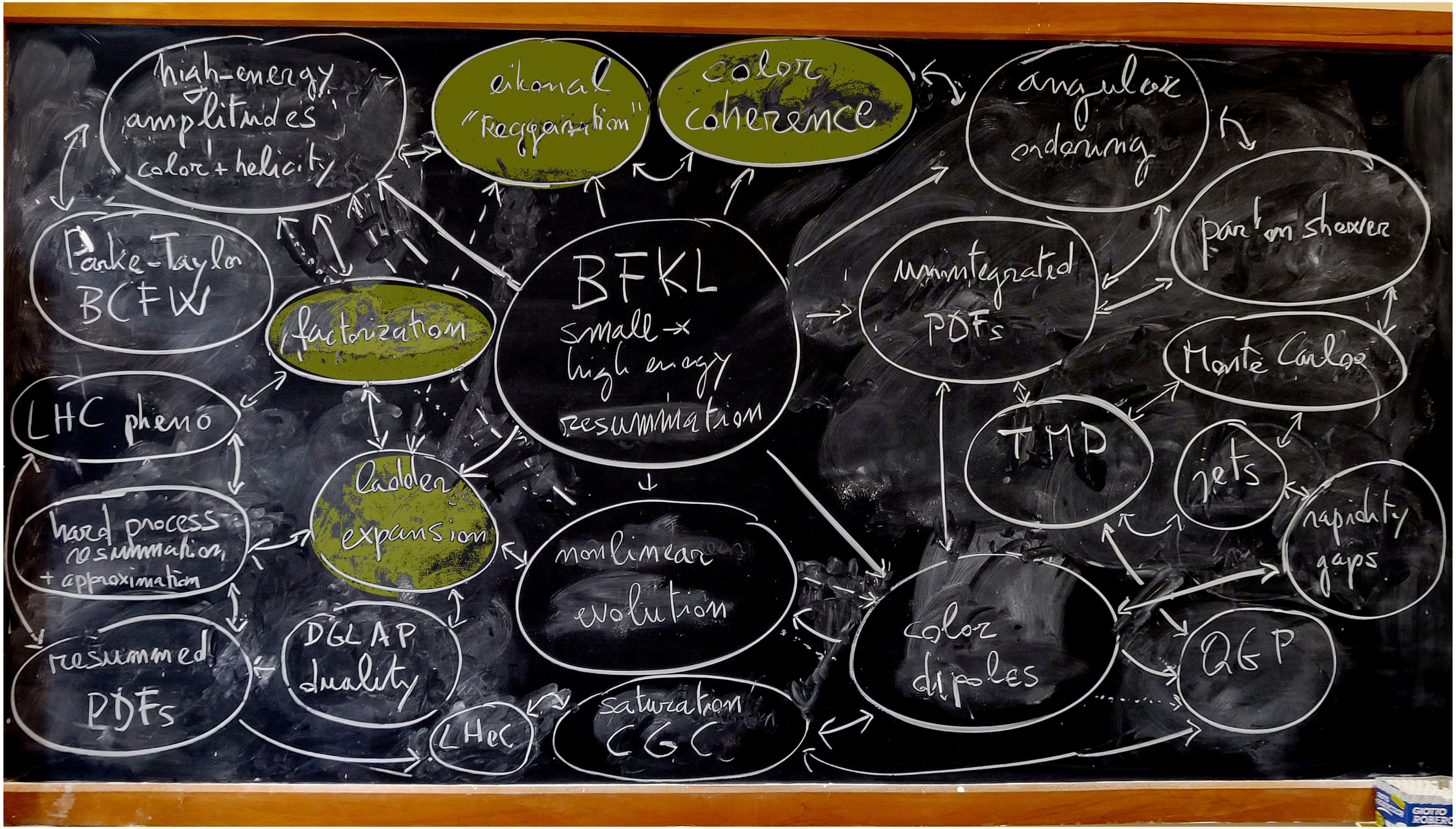
KERNEL

$$K^{(T)}(q_1,q_2)=A_T-C_T\frac{(q_1^2-m^2)[(q-q_2)^2-m^2]+(q_2^2-m^2)[(q-q_1)^2-m^2]}{(q_1-q_2)^2-m^2}.$$

- ANGULAR MOMENTUM POLES $\Rightarrow$ CONJUGATE LOGS
- ITERATIVE STRUCTURE $\Rightarrow$ EXPONENTIATION

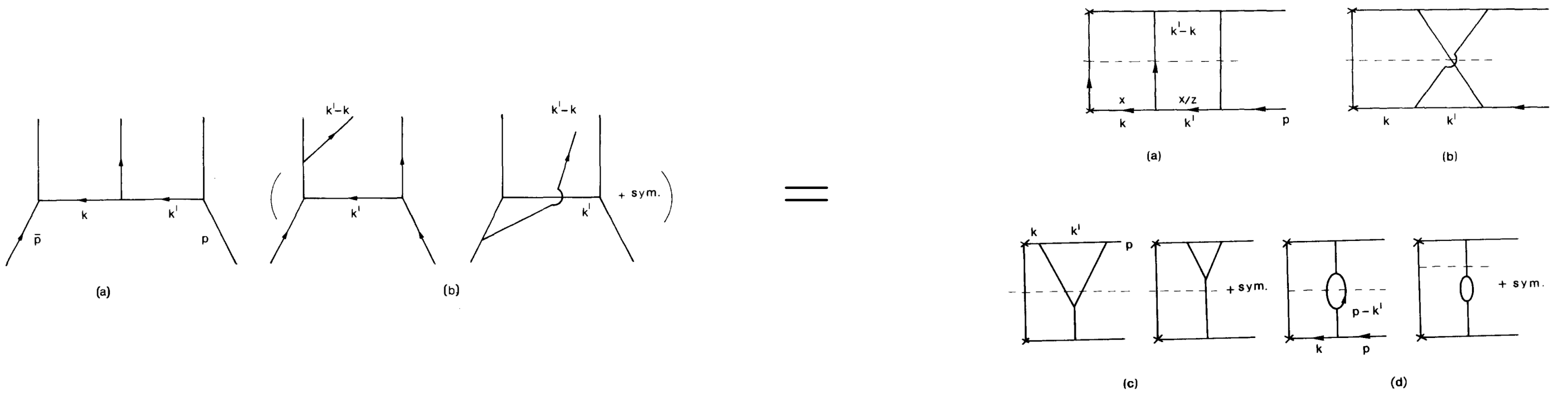
☞ **CHIRILLI'S TALK**

A. Bassetto, M. Ciafaloni, G. Marchesini "Jet Structure and Infrared Sensitive Quantities in Perturbative QCD", Phys.Rept. 100 (1983) 201



DIS AT SMALL x

COMPUTE STRUCTURE FUNCTION WITH OFF-SHELL $(k_\perp \neq 0)$ GLUONS:



STRONG ORDERING $k_\perp^2 > {k'_\perp}^2 > {k''_\perp}^2 \ldots,\ x < x' < x'' \ldots$

REAL EMISSION

$$F_R^{(2)} = \frac{\bar{\alpha}_s^2}{x} \int \frac{dz}{z} \frac{dk_\perp^2\, dk_\perp'^2}{|k_\perp^2 - k_\perp'^2|\, k_\perp'^2}\, \Theta\left(\frac{k_\perp^2}{z} - k_\perp'^2\right),$$

VIRTUAL CORRNS

$$\left(\frac{1}{k_\perp^2 - k_\perp'^2}\right)_{\rm Reg} = \frac{1}{|k_\perp^2 - k_\perp'^2|} - \delta(k_\perp^2 - k_\perp'^2)\, 2 \int_0^{(k_\perp^2 - 0)} \frac{dq_\perp^2}{|k_\perp^2 - q_\perp^2|}.$$

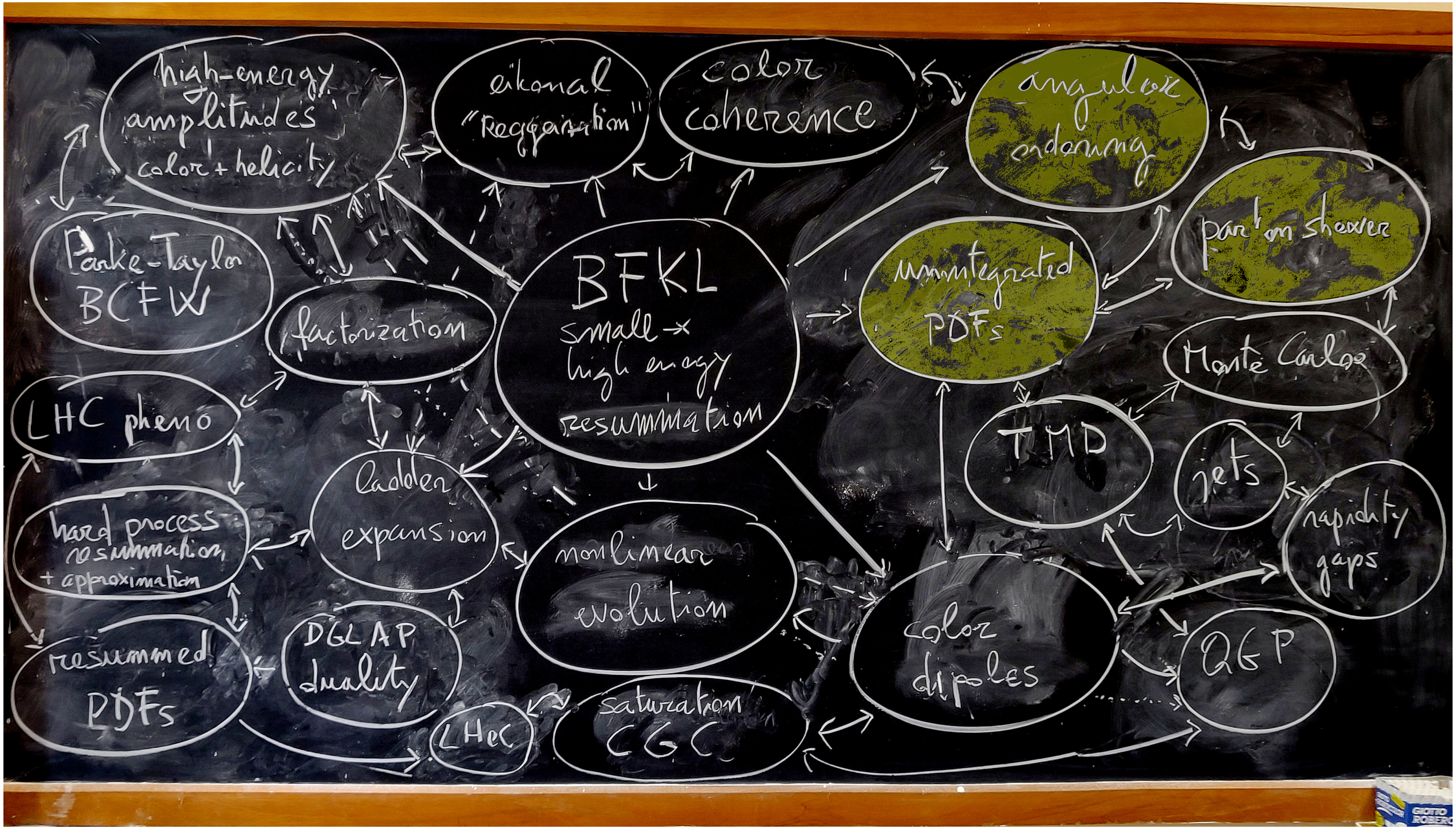
ITERATION

$$\mathscr{F}(k_\perp^2, x) = \delta(x-1) + \bar{\alpha}_s \int \frac{dz}{z} \left(\frac{dk_\perp'^2}{k_\perp'^2 - k_\perp^2}\right)_{\rm Reg} \Theta\left(\frac{k_\perp^2}{z} - k_\perp'^2\right) \mathscr{F}\left(k_\perp'^2, \frac{x}{z}\right),$$

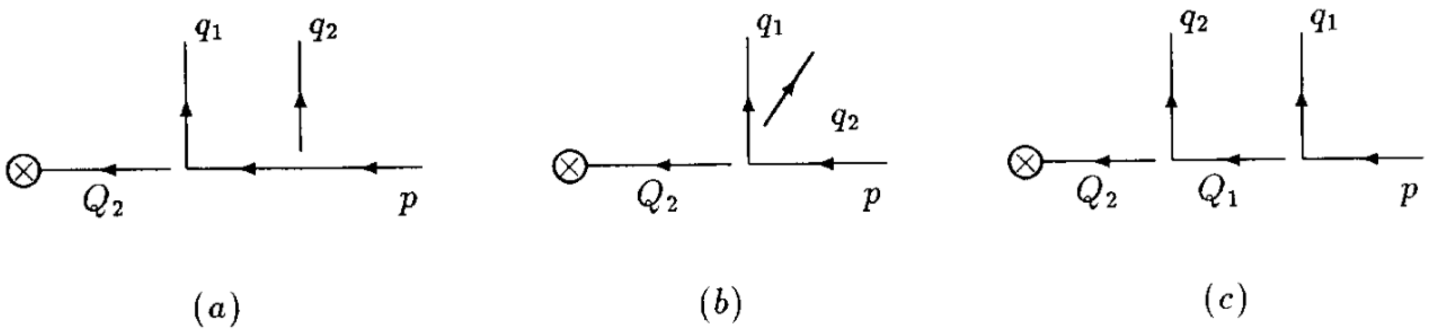
- COLOR COHERENCE $\Leftrightarrow$ EIKONAL EMISSION (SYMMETRY OF 3-GLUON VERTEX)
- k_T FACTORIZATION & ORDERING: UNINTEGRATED PDFS?

☞ **FORSHAW'S TALK**
NAGY'S TALK

S. Catani, F. Fiorani, G. Marchesini, “Small x Behavior of Initial State Radiation in Perturbative QCD” Nucl.Phys. B336 (1990) 18



ANGULAR ORDERING



- PROOF OF FACTORIZATION AND EXPONENTIATION FOR DIS AT SMALL x
- INSERTION OF EIKONAL+NON-EIKONAL CURRENTS:

$$\boldsymbol{J}_{\rm tot}^{(n-1)}(q_n) = \boldsymbol{J}_{\rm eik}^{(n-1)}(q_n) + \boldsymbol{J}_{\rm ne}(Q_n, q_n),$$

$$\boldsymbol{J}_{\rm eik}^{(n-1)}(q) = -\boldsymbol{T}_p \frac{p}{pq} + \boldsymbol{T}_{p'} \frac{p'}{p'q} + \sum_{l=1}^{n-1} \boldsymbol{T}_l \frac{q_l}{q_l q},$$

$$\boldsymbol{J}_{\rm ne}(Q_n, q_n) = \frac{2(Q_{n-1} - x_{n-1} p) \cdot \epsilon(q_n)}{Q_{n-1}^2} \boldsymbol{T}_{p'},$$

UNINTEGRATED PDF

$$F(Q^2, x) = \frac{1}{x} \int d^2 Q_t \, \tilde{F}(\boldsymbol{Q}_t, x)$$

$$= \delta(1-x) + \frac{1}{x} \int d^2 Q_t \int_{\mu^2}^{Q^2} \frac{d^2 q_t}{\pi q_t^2} \int_x^1 \frac{dz}{z} \mathscr{F}(x, \boldsymbol{Q}_t, z, q_t).$$

THE CCFM EQUATION

- RECURSION RELATION $\Rightarrow$ EVOLUTION EQUATION

$$\mathscr{F}(x, \boldsymbol{Q}_t, z, q_t) = \Theta(Q_t^2 - z q_t^2)\Big[\bar{\alpha}_s \delta^2(\boldsymbol{Q}_t + \boldsymbol{q}_t)\, \delta\Big(1 - \frac{x}{z}\Big)\, \Delta_{ne}(x, Q_t, q_t)$$

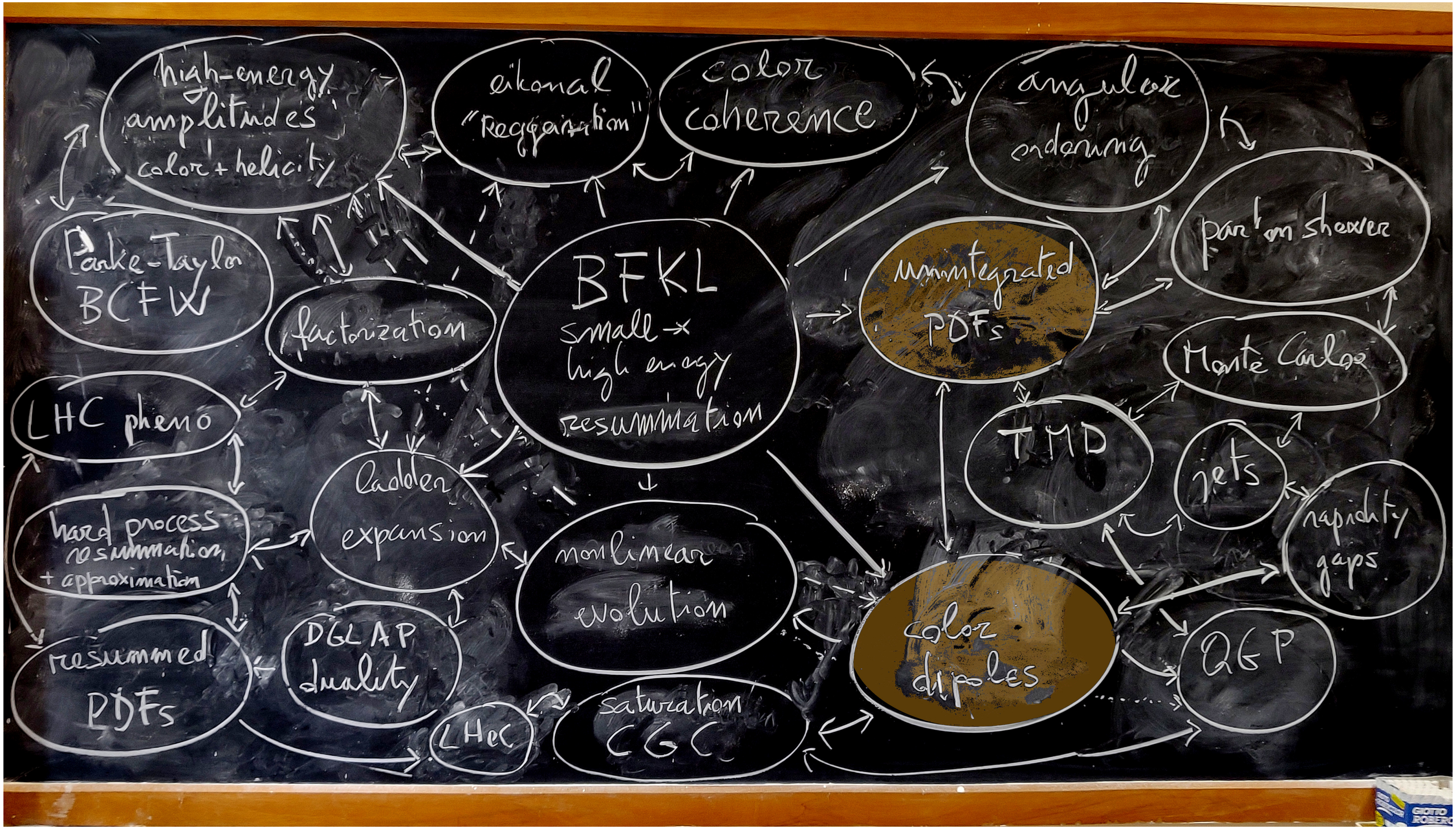
$$+ \bar{\alpha}_s \int_{x/z}^{1} \frac{dz'}{z'} \int_{\mu^2}^{Q^2} \frac{d^2 q_t'}{\pi q_t'^2} \Theta(q_t - z' q_t')\, \Delta_{ne}(z, Q_t, q_t)\, \mathscr{F}\Big(\frac{x}{z}, \boldsymbol{Q}_t + \boldsymbol{q}_t, z', q_t'\Big)\Big],$$

WITH FORM FACTOR (KERNEL)

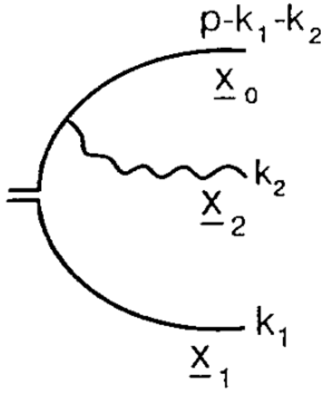
$$\Delta_{ne}(z_r, Q_{rt}, q_{rt}) = \exp\left[-\bar{\alpha}_s \int_{z_r}^{1} \frac{dz}{z} \int_{(z q_{rt})^2}^{Q_{rt}^2} \frac{dq_t^2}{q_t^2}\right].$$

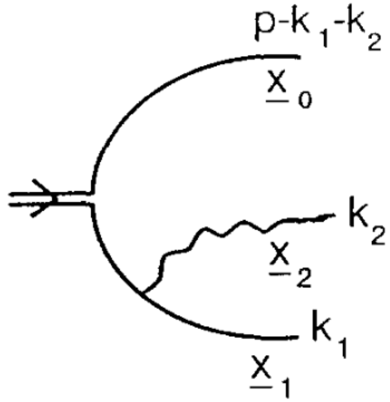
☞ **HAUTMANN'S TALK**
ANDERSEN'S TALK

A. H. Mueller, "Soft gluons in the infinite momentum wave function and the BFKL pomeron", Nucl.Phys. B415 (1994) 373



THE ONIUM WAVE FUNCTION





ONE EXTRA SOFT GLUON: WAVE FUNCTION...

$$\psi^{(1)a}_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2; z_1, z_2) = -\frac{igT^a}{\pi}\psi^{(0)}_{\alpha\beta}(\mathbf{x}_1, z_1)\left(\frac{\mathbf{x}_{20}}{x^2_{20}} - \frac{\mathbf{x}_{21}}{x^2_{21}}\right)\cdot \boldsymbol{\epsilon}^{\lambda}_2,$$

...AND PROBABILITY: ONE EXTRA GLUON

$$\Phi^{(1)}(\mathbf{x}_1, z_1) = \int \frac{\mathrm{d}^2\mathbf{x}_2}{2\pi}\int_{z_0}^{z_1}\frac{\mathrm{d}z_2}{2z_2}\sum_{\substack{a\lambda\\ \alpha\beta}}|\psi^{(1)a}_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2; z_1, z_2)|^2$$

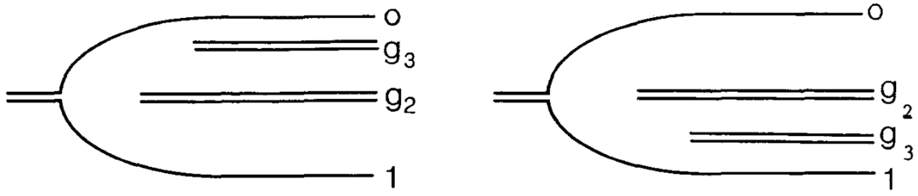
$$\Phi^{(1)}(\mathbf{x}_1, z_1) = \int \mathrm{d}^2\mathbf{x}_2\int_{z_0}^{z_1}\frac{\mathrm{d}z_2}{z_2}\frac{\alpha C_{\mathrm{F}}}{\pi^2}\frac{x^2_{10}}{x^2_{20}x^2_{21}}\Phi^{(0)}(\mathbf{x}_1, z_1).$$

N GLUONS: THE GENERATING FUNCTIONAL:

$$\frac{1}{n!}\frac{\delta}{\delta u(\mathbf{x}_2, z_2)}\frac{\delta}{\delta u(\mathbf{x}_3, z_3)}\cdots\frac{\delta}{\delta u(\mathbf{x}_{n+1}, z_{n+1})}\Phi(\mathbf{x}_1, z_1, u(\mathbf{x}, z))|_{u=0}$$

$$= \Phi^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{n+1}; z_1, z_2 \ldots z_{n+1}),$$

THE INCLUSIVE GLUON DISTRIBUTION
ITERATION AND RECURSION:



$$\Phi(\mathbf{x}_1, z_1, u) = \Phi^{(0)}(\mathbf{x}_1, z_1) Z(\mathbf{x}_1, \mathbf{x}_0, z_1, u),$$

$$Z(\mathbf{x}_1, \mathbf{x}_0, z_1, u) = \exp\left\{-\frac{4\alpha C_F}{\pi}\ln\left(\frac{x_{10}}{\rho}\right)\ln(z_1/z_0)\right\} + \frac{\alpha C_F}{\pi^2}\int_{z_0}^{z_1}\frac{dz_2}{z_2}\int_{R(x_{20},\, x_{10})}$$
$$\times\exp\left\{-\frac{4\alpha C_F}{\pi}\ln\left(\frac{x_{10}}{\rho}\right)\ln(z_1/z_2)\right\}$$
$$\times\frac{d^2\mathbf{x}_2 x_{10}^2}{x_{20}^2 x_{21}^2} u(\mathbf{x}_2, z_2) Z(\mathbf{x}_2, \mathbf{x}_1, z_2, u) Z(\mathbf{x}_2, \mathbf{x}_0, z_2, u).$$

THE UNINTEGRATED PDF:

$$zF(z, Q^2) = \int d^2\mathbf{x}_1 \int_0^1 dz_1 \Phi^{(0)}(\mathbf{x}_1, z_1) T(x_{10}, z_1; Q, z),$$

$$T(x_{10}, z_1; Q, z) = \frac{2\alpha C_F}{\pi} v(Qx_{10}) \exp\left\{-\frac{4\alpha C_F}{\pi}\ln\left(\frac{x_{10}}{\rho}\right)\ln(z_1/z)\right\}$$
$$+\frac{4\alpha C_F}{\pi}\int_z^{z_1}\frac{dz_2}{z_2}\int_{R(x_{20},\, x_{10})}\exp\left\{-\frac{4\alpha C_F}{\pi}\ln\left(\frac{x_{10}}{\rho}\right)\ln(z_1/z_2)\right\}$$
$$\times\tilde{K}(x_{10}, x_{12})\, dx_{12} T(x_{12}, z_2; Q, z)$$

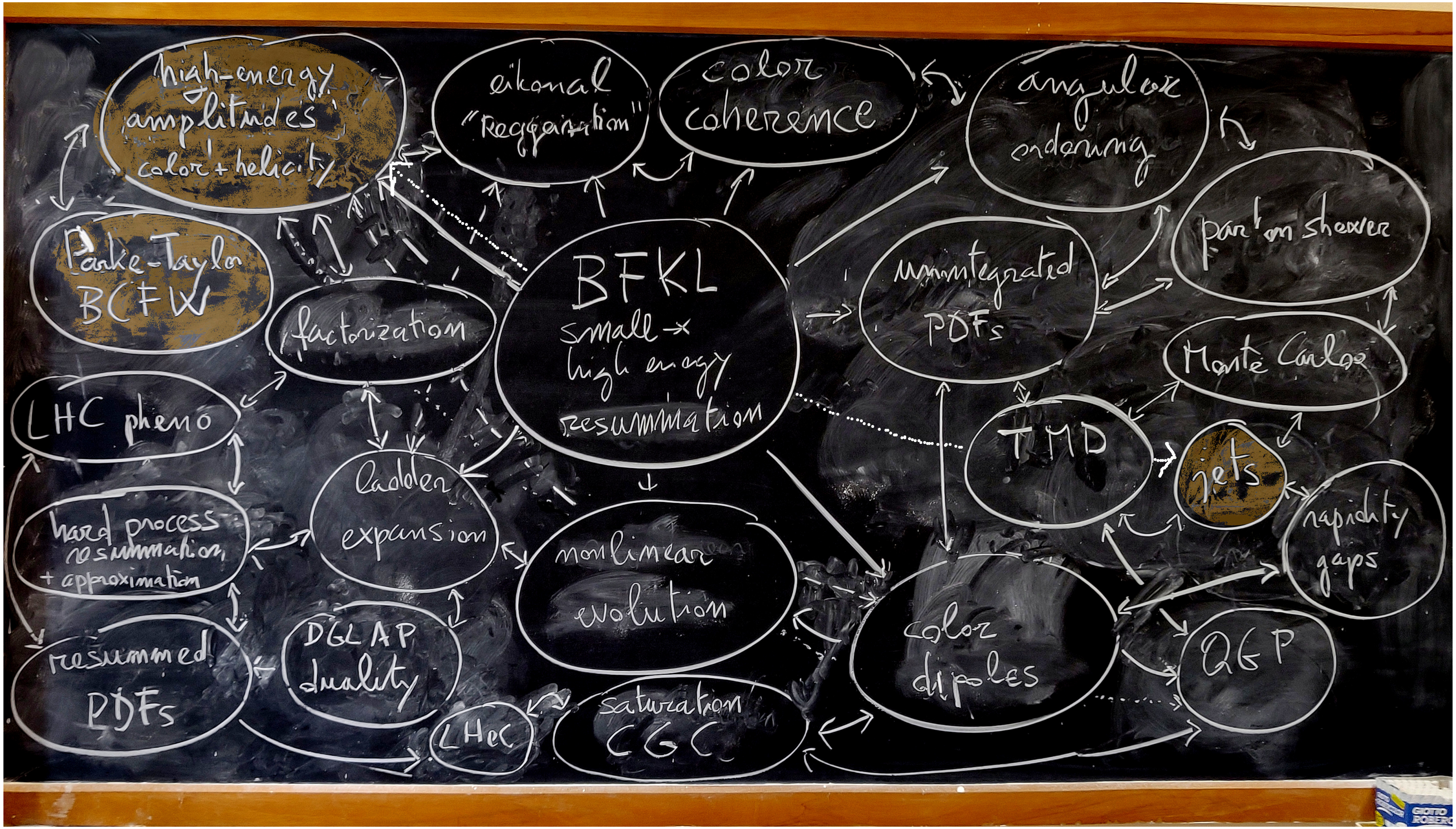
THE BFKL KERNEL

$$\tilde{K}(x_{10}, x_{12}) = \frac{1}{2\pi}\int_R \frac{x_{10}^2}{x_{12}^2 x_{20}^2} J(x_{21}, x_{20})\, dx_{20}.$$

$$J(x_{21}, x_{20}) = \frac{4x_{21}x_{20}}{\sqrt{\left[(x_{21}+x_{20})^2 - x_{10}^2\right]\left[x_{10}^2 - (x_{21}-x_{20})^2\right]}},$$

☞ **WALLON'S TALK**

V. Del Duca, “Equivalence of the Parke-Taylor and the Fadin-Kuraev-Lipatov amplitudes in the high-energy limit”, Phys.Rev. D52 (1995) 1527



THE PARKE-TAYLOR AMPLITUDES

- EXACT $2 \to n+2$ GLUON HELICITY MHV AMPLITUDES
  (ALL GLUONS BUT TWO WITH +)

- MULTIREGGE KINEMATICS

$$y_0 \gg y_1 \gg \cdots \gg y_{n+1}, \qquad |p_{i\perp}| \simeq |p_\perp| .$$

- KINEMATIC SIMPLIFICATION:

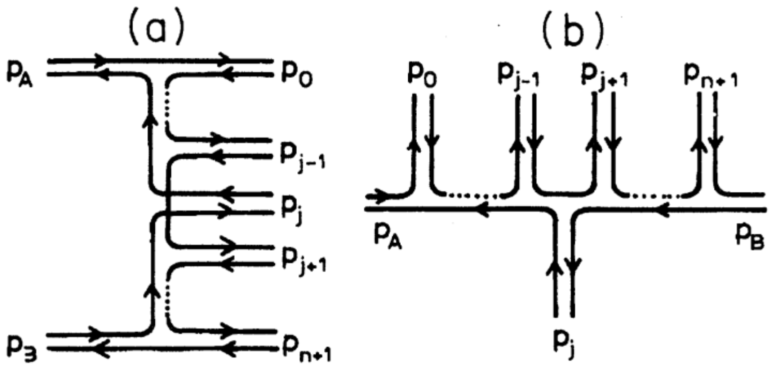
$$\hat{s} \simeq |p_{0\perp}||p_{n+1\perp}|e^{y_0 - y_{n+1}} ,$$
$$\hat{s}_{Ai} \simeq -|p_{0\perp}||p_{i\perp}|e^{y_0 - y_i} ,$$
$$\hat{s}_{Bi} \simeq -|p_{i\perp}||p_{n+1\perp}|e^{y_i - y_{n+1}} ,$$
$$\hat{s}_{ij} \simeq |p_{i\perp}||p_{j\perp}|e^{|y_i - y_j|} .$$

- SURVIVING HELICITIES: (TWO GLUONS WITH -:)

$$(A,B), \quad (A,n+1), \quad (B,0), \quad (0,n+1) .$$

COMPARISON TO BFKL

- DETERMINE LEADING COLOR CONFIGURATIONS: COLOR ORDERING $\leftrightarrow$ HELICITY ORDERING



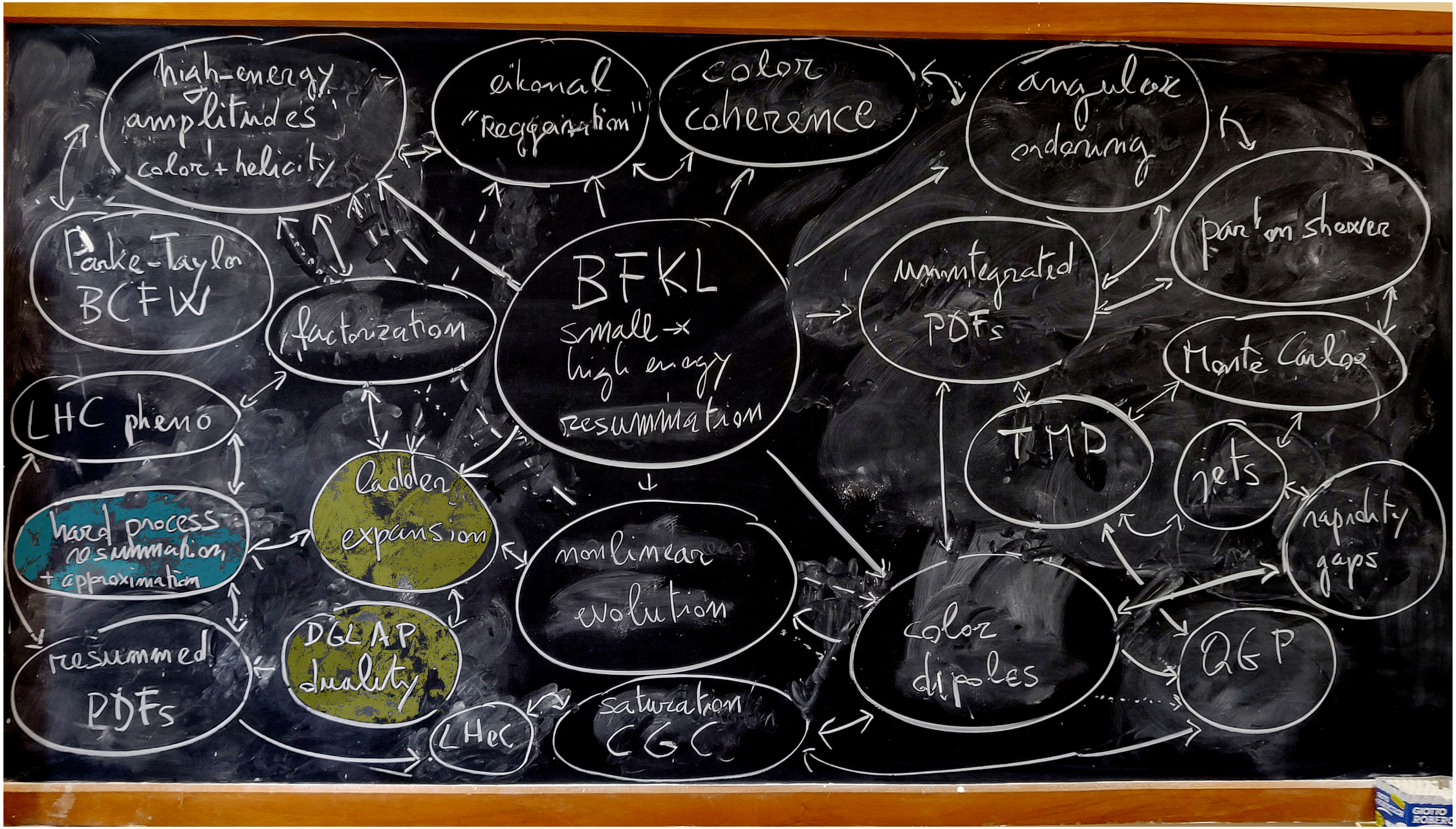
THE AMPLITUDE

$$iM(-p_A,-;p_0,+;...;p_{n+1},+;-p_B,-) \simeq i\,(-1)^{n+1}\,2^{2+n/2}\,g^{n+2}\,\hat{s}\,\frac{1}{\prod_{i=0}^{n+1} p_{i\perp}}\,\mathrm{tr}\left(\lambda^a\,\left[\lambda^{d_0},\left[\lambda^{d_1},...,\left[\lambda^{d_{n+1}},\lambda^b\right]\right]\right]\right)\,.$$

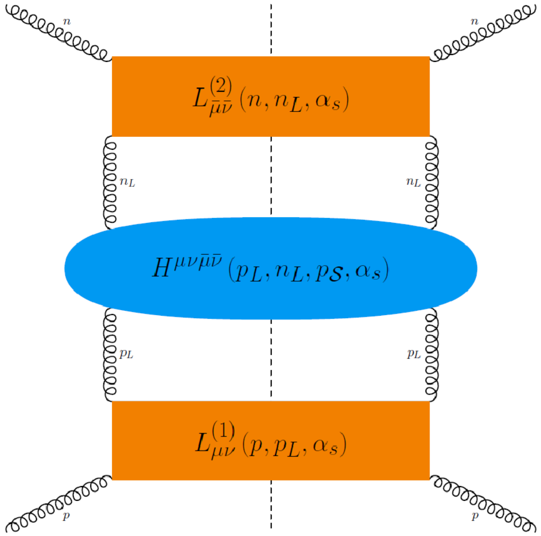
- COMMON HELICITY CONFIGURATIONS, PARKE-TAYLOR AND BFKL COINCIDE FOR REAL EMISSION
- REGGEIZATION $\Rightarrow$ SAME COLOR AND HELICITY
- LOOP-LEVEL AMPLITUDES MULTIGLUON AT HIGH-ENERGY SAME AS PARKE-TAYLOR

DUHR'S TALK
VERNAZZA'S TALK

S. Catani, M. Ciafaloni, F. Hautmann "High-energy factorization and small x heavy flavor production", Nucl.Phys. B366 (1991) 135; ABF 1998-2008; SF+Caola, Marzani et al 2011-2016



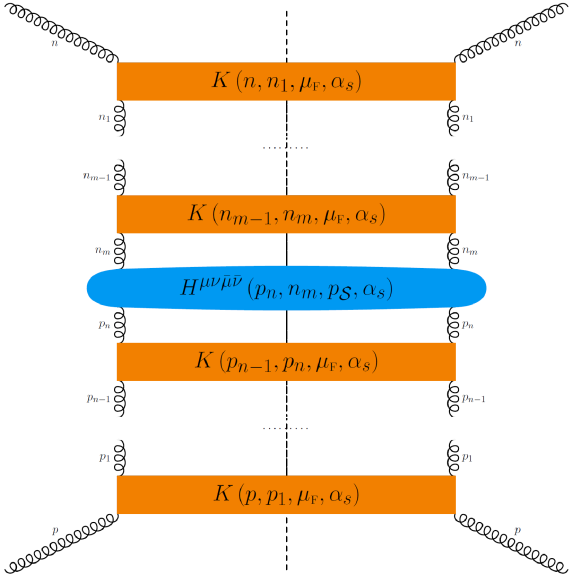
HIGH-ENERGY RESUMMATION: FACTORIZATION



$$\sigma\left(\frac{Q^2}{s}, \frac{\mu_f^2}{Q^2}, \frac{\mu_r^2}{Q^2}\right) = \int \frac{Q^2}{2s} H^{\mu\nu\bar\mu\bar\nu}\left(n_L, p_L, \Omega_{\mathcal{S}}, \mu_r^2, \mu_f^2, \alpha_s\right) L_{\mu\nu}\left(p_L, p, \mu_r^2, \mu_f^2, \alpha_s\right) L_{\bar\mu\bar\nu}\left(n_L, n, \mu_r^2, \mu_f^2, \alpha_s\right)[dp_L][dn_L]$$

- SUDAKOV PARM. FOR p_L, n_l; HIGH ENERGY LIMIT: $z \ll 1$, $\frac{k_t^2}{s} \ll 1$
- 2GI HARD + 2GR LADDER CONTRIBUTES IN HIGH ENERGY LIMIT, UP TO POWER SUPPRESSION
- LORENTZ DECOMPOSITION OF LADDERS & HARD:
  ONLY LONGITUDINAL
  CONTRIBUTES IN HIGH ENERGY LIMIT

THE LADDER EXPANSION



$$
\sigma^{n,m}\left(N, \frac{\mu_f^2}{Q^2}, \alpha_s; \epsilon\right) = \int_0^\infty \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,n}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,n}^2}{{k_{t\,n}^2}^{1+\epsilon}} \times \int_0^\infty \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,m}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,m}^2}{{\bar{k}_{t\,n}^2}^{1+\epsilon}}
$$

$$
\times\, C\left(N, {k_t}^2, {\bar{k}_t}^2, \alpha_s; \epsilon\right)
$$

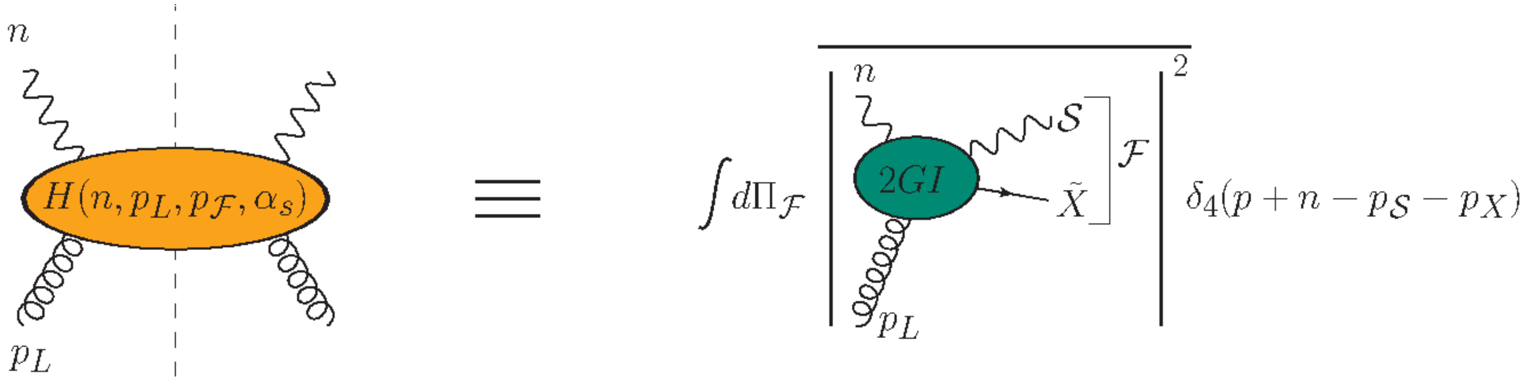
$$
\times \int_0^{k_{t\,n}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,n-1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,n-1}^2}{{k_{t\,n-1}^2}^{1+\epsilon}} \times \cdots \times \int_0^{k_{t\,2}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,1}^2}{{k_{t\,1}^2}^{1+\epsilon}}
$$

$$
\times \int_0^{\bar{k}_{t\,m}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{\bar{k}_{t\,m-1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,m-1}^2}{{\bar{k}_{t\,m-1}^2}^{1+\epsilon}} \times \cdots \times \int_0^{\bar{k}_{t\,2}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,1}^2}{{\bar{k}_{t\,1}^2}^{1+\epsilon}}.
$$

- LADDER OBTAINED BY ITERATION OF A 2GI KERNEL
- THE INTEGRATED KERNEL IS A LLx ANOMALOUS DIMENSION:
  $$K\left(N, \left(\frac{\mu_f^2}{{k_t}^2}\right)^\epsilon, \alpha_s; \epsilon\right) = \gamma\left(N, \left(\frac{\mu_f^2}{{k_t}^2}\right)^\epsilon, \alpha_s; \epsilon\right)$$

RESUMMATION AND THE OFF-SHELL CROSS-SECTION

THE TOTAL CROSS-SECTION



$$\sigma_{\rm res}\left(N, \alpha_s\right) = h\left(N, \gamma\left(\tfrac{\alpha_s}{N}\right), \gamma\left(\tfrac{\alpha_s}{N}\right), \alpha_s\right)$$

$$h\left(N, M_1, M_2, \alpha_s\right) = M_1 M_2\, R\left(M_1\right) R\left(M_2\right) \int_0^\infty dk_t{}^2\, {k_t{}^2}^{M_1-1} \int_0^\infty d\bar{k_t}{}^2\, {\bar{k_t}{}^2}^{M_2-1}\, C\left(N, k_t{}^2, \bar{k_t}{}^2, \alpha_s\right)$$

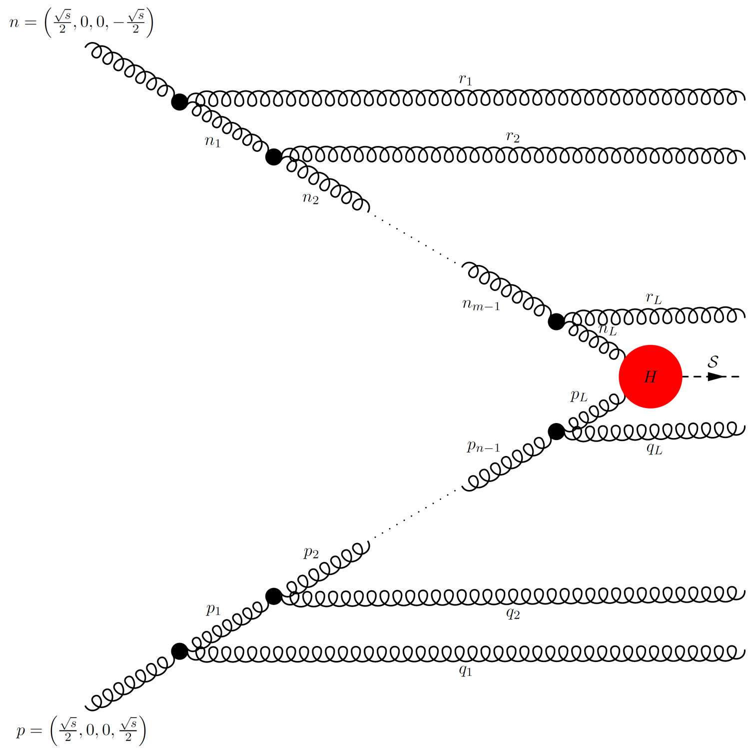
- THE ITERATED KERNEL (ANOMALOUS DIMENSION) EXPONENTIATES
- THE CONVOLUTIONS LOOK LIKE k_t-SPACE MELLIN-TRANSFORMS (k_t^2 GLUON OFF-SHELLNESS)
- RESUMMATION $\Leftrightarrow$ OFF-SHELL CROSS-SECTION WITH $M = \gamma$ (DUALITY)

☞ VAN HAMEREN'S TALK

RESUMMATION OF DIFFERENTIAL DISTRIBUTIONS

$$
\begin{aligned}
p_1 &= z_1 p - \mathbf{k}_1 \\
q_1 &= (1 - z_1)\, p + \mathbf{k}_1 \\
p_2 &= z_2 z_1 p - \mathbf{k}_2 \\
q_2 &= (1 - z_1 z_2)\, z_1 p + \mathbf{k}_2 - \mathbf{k}_1 \\
&\cdots\cdots\cdots \\
p_L &= z p - \mathbf{k} \\
q_L &= (1 - z)\, p + \mathbf{k} - \mathbf{k_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
n_1 &= \bar{z}_1 p - \bar{\mathbf{k}}_1 \\
r_1 &= (1 - \bar{z}_1)\, p + \bar{\mathbf{k}}_1 \\
n_2 &= \bar{z}_2 \bar{z}_1 p - \bar{\mathbf{k}}_2 \\
r_2 &= (1 - \bar{z}_1 \bar{z}_2)\, \bar{z}_1 p + \bar{\mathbf{k}}_2 - \bar{\mathbf{k}}_1 \\
&\cdots\cdots\cdots \\
n_L &= \bar{z} n - \bar{\mathbf{k}} \\
r_L &= (1 - \bar{z})\, n + \bar{\mathbf{k}} - \bar{\mathbf{k}}_{\mathbf{m-1}}
\end{aligned}
$$



- TRANSVERSE MOMENTUM INTEGRATIONS ARE INDEPENDENT FROM EACH OTHER
- TRANSVERSE MOMENTUM DEPENDENCE LINKS INCOMING AND OUTGOING MOMENTA IN THE HARD PART
- LONGITUDINAL MOMENTUM RESCALING $\Rightarrow$ RAPIDITY BOOST
- RESUMMATION $\Leftrightarrow$ OFF-SHELL DIFFERENTIAL CROSS-SECTION WITH $M = \gamma \pm ib$ (DUALITY)

$$
\frac{d\hat{\sigma}^{\rm res}}{dy\, d\xi_p}\left(N, \xi_p, b, \alpha_s\right) = h_{p_t, y}\left(N, \gamma\left(\frac{\alpha_s}{N - ib/2}\right), \gamma\left(\frac{\alpha_s}{N + ib/2}\right), \xi_p,\ b, \alpha_s\right)
$$

$$
h_{p_t, y}\left(N, M_1, M_2, \xi_p, b, \alpha_s\right) = M_1 M_2 R\left(M_1\right) R\left(M_2\right) \int_0^\infty d\xi\, \xi^{M_1 - 1} \int_0^\infty d\bar{\xi} \bar{\xi}^{M_2 - 1}\, C_{p_t, y}\left(N, \xi, \bar{\xi}, \xi_p, b, \alpha_s\right)
$$

RESUMMED EVOLUTION

DUALITY OF THE ANOMALOUS DIMENSIONS

THE ALTARELLI-PARISI EQN IS AN INTEGRO–DIFFERENTIAL EQUATION $\Rightarrow$ IT CAN BE EQUIVALENTLY VIEWED AS Q^2–EVOLUTION EQUATION FOR x–MOMENTS (usual RG eqn.), OR x–EVOLUTION EQUATION FOR Q^2–MOMENTS(BFKL eqn.)

EVOLUTION IN $t = \ln Q^2$

$$\frac{d}{dt} G(N,t) = \gamma(N,\alpha_s)\; G(N,t)$$

MELLIN x–MOMENTS

$$G(N,t) = \int_0^\infty d\xi\, e^{-N\xi}\; G(\xi,t)$$

EVOLUTION IN $\xi = \ln 1/x$

$$\frac{d}{d\xi} G(\xi,M) = \chi(M,\alpha_s)\; G(\xi,M)$$

MELLIN Q^2–MOMENTS

$$G(\xi,M) = \int_{-\infty}^\infty dt\, e^{-Mt}\; G(\xi,t)$$

THE TWO EQUATIONS HAVE THE SAME SOLUTIONS PROVIDED THE EVOLUTION KERNELS ARE RELATED BY

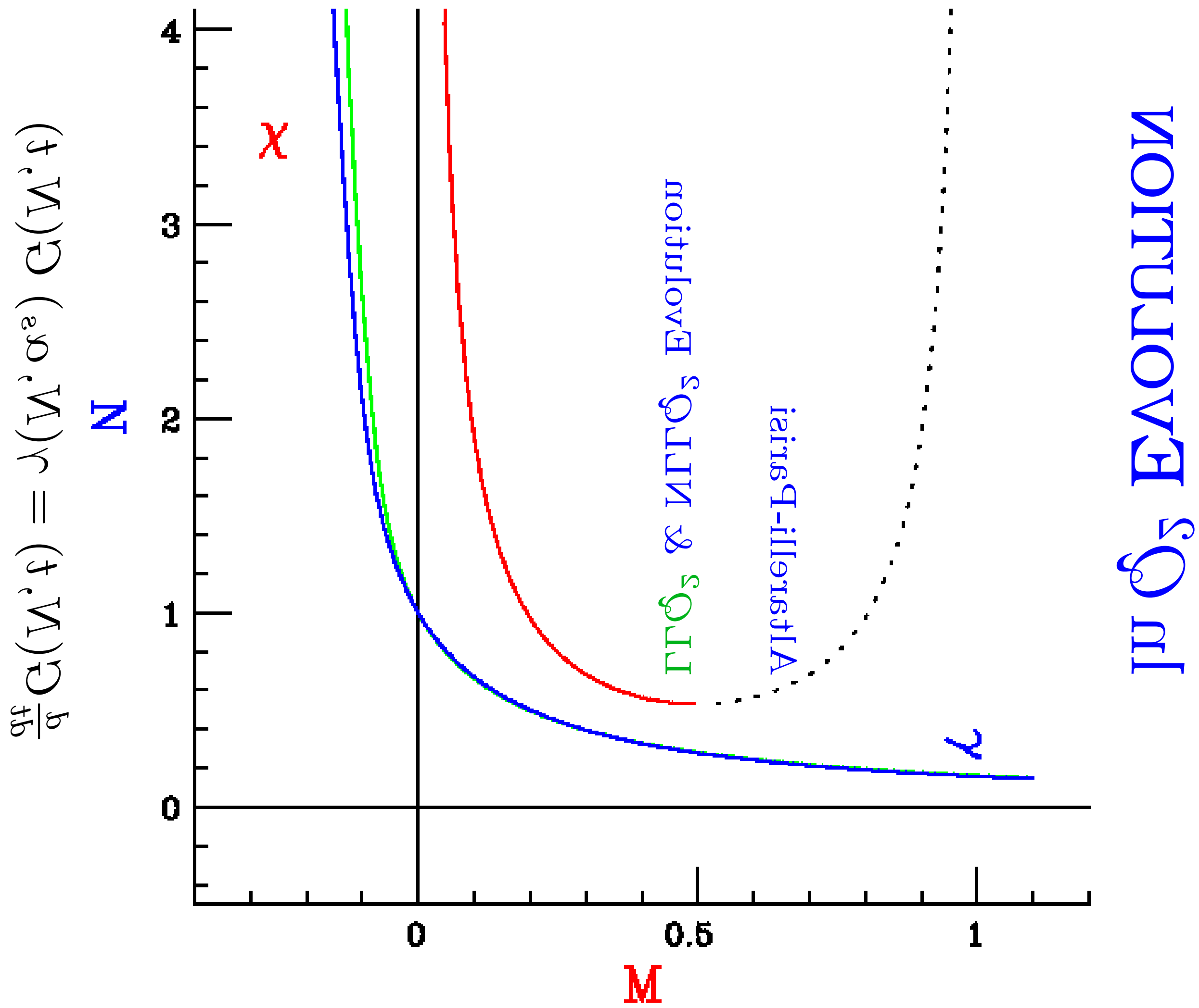
$$\chi(\gamma(N,\alpha_s),\alpha_s) = N$$
$$\gamma(\chi(M,\alpha_s),\alpha_s) = M$$

& BOUNDARY CONDITIONS RELATED BY

$$H_0[M] \to G_0(N) = H_0[\gamma(N,\alpha_s)]/\chi'(\gamma(N,\alpha_s))$$

. . . can switch from LLQ^2 to LL$1/x$ choosing the Evolution Kernel

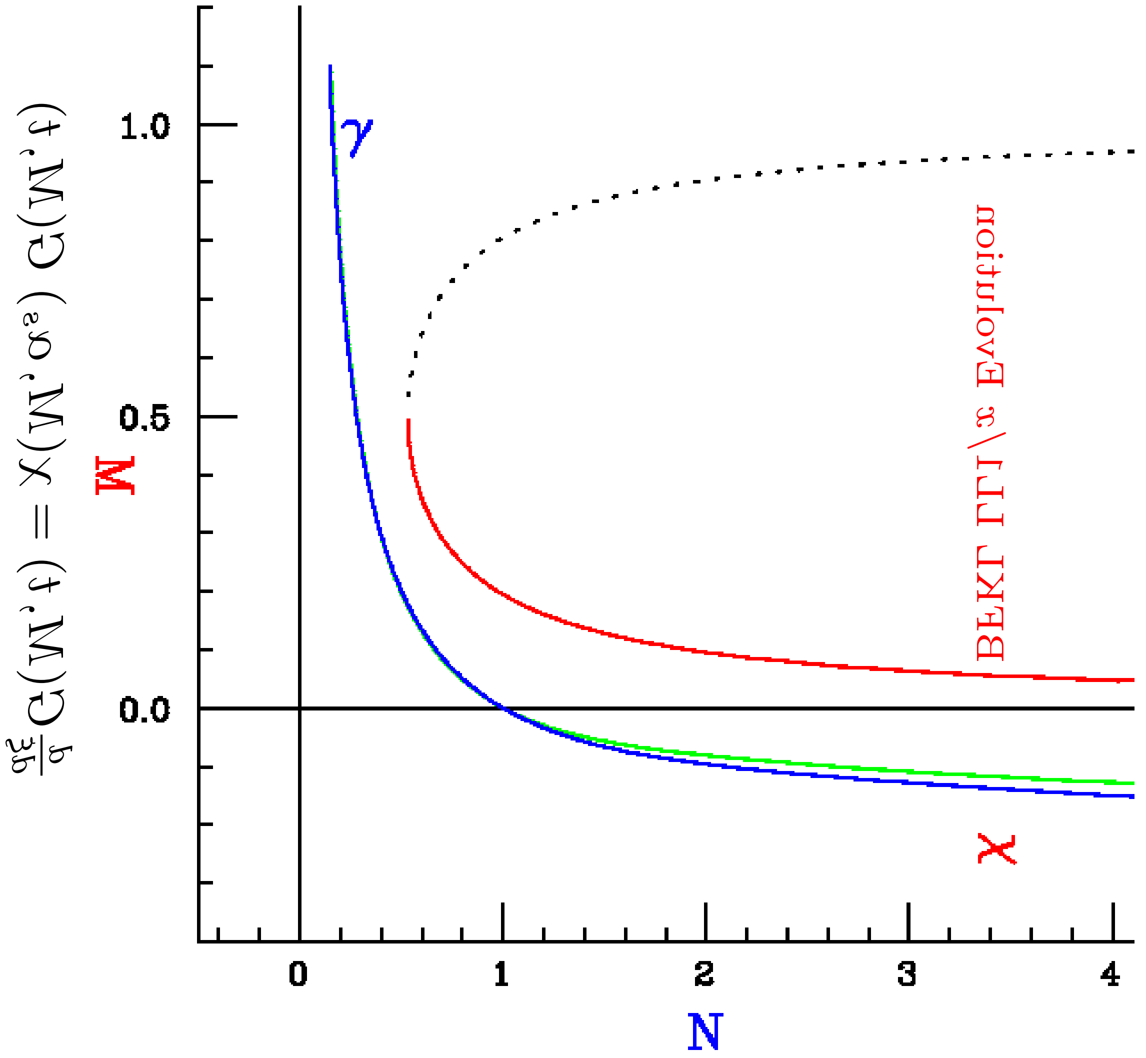
$\ln 1/x$ Evolution



$$\frac{d}{d\xi} G(M,t) = \chi(M, \alpha_s)\ G(M,t)$$

. . . IN EITHER EQUATION!

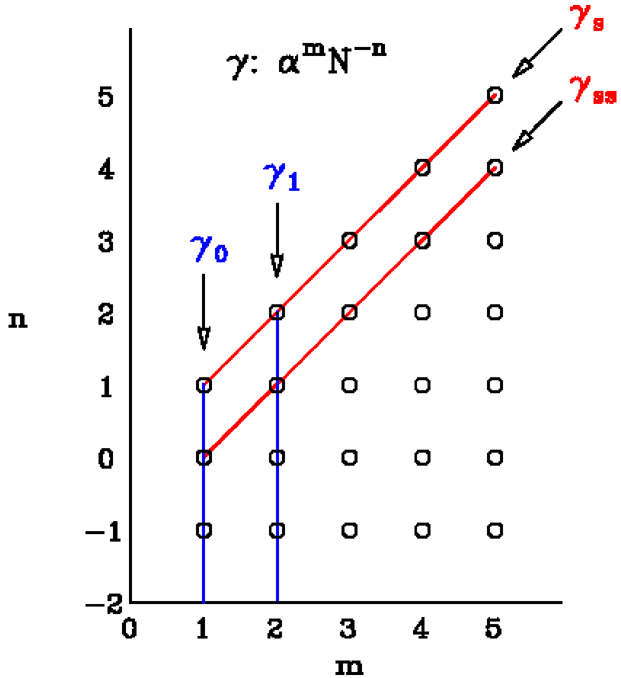
$\ln Q^2$ EVOLUTION



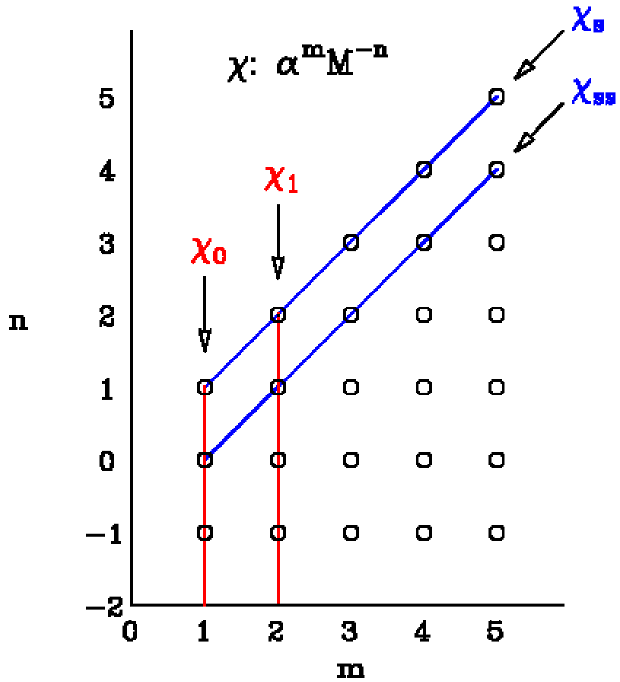
ln 1/x EVOLUTION

$$\frac{d}{dt}G(N,t) = \gamma(N,\alpha_s)\ G(N,t)$$

DUAL PERTURBATIVE EXPANSIONS



$\ln Q^2$ EVOLUTION



$\ln 1/x$ EVOLUTION

$$\gamma(N) = \alpha\left(\frac{c_{-1}^{(1)}}{N} + c_0^{(1)} + \dots\right) + \alpha^2\left(\frac{c_{-2}^{(2)}}{N^2} + \frac{c_{-1}^{(2)}}{N} + \dots\right)$$

$$\chi(M) = \alpha\left(\frac{\tilde{c}_{-1}^{(1)}}{M} + \tilde{c}_0^{(1)} + \dots\right) + \alpha^2\left(\frac{\tilde{c}_{-2}^{(2)}}{Q^2} + \frac{\tilde{c}_{-1}^{(2)}}{M} + \dots\right)$$

$$\gamma_s(N) = c_{-1}^{(1)}\frac{\alpha}{N} + c_{-2}^{(2)}\frac{\alpha^2}{N^2} + \dots$$

$$\chi_s(M) = \tilde{c}_{-1}^{(1)}\frac{\alpha}{M} + \tilde{c}_{-2}^{(2)}\frac{\alpha^2}{Q^2} + \dots$$

$1/N$ POLES $\Leftrightarrow \ln 1/x$

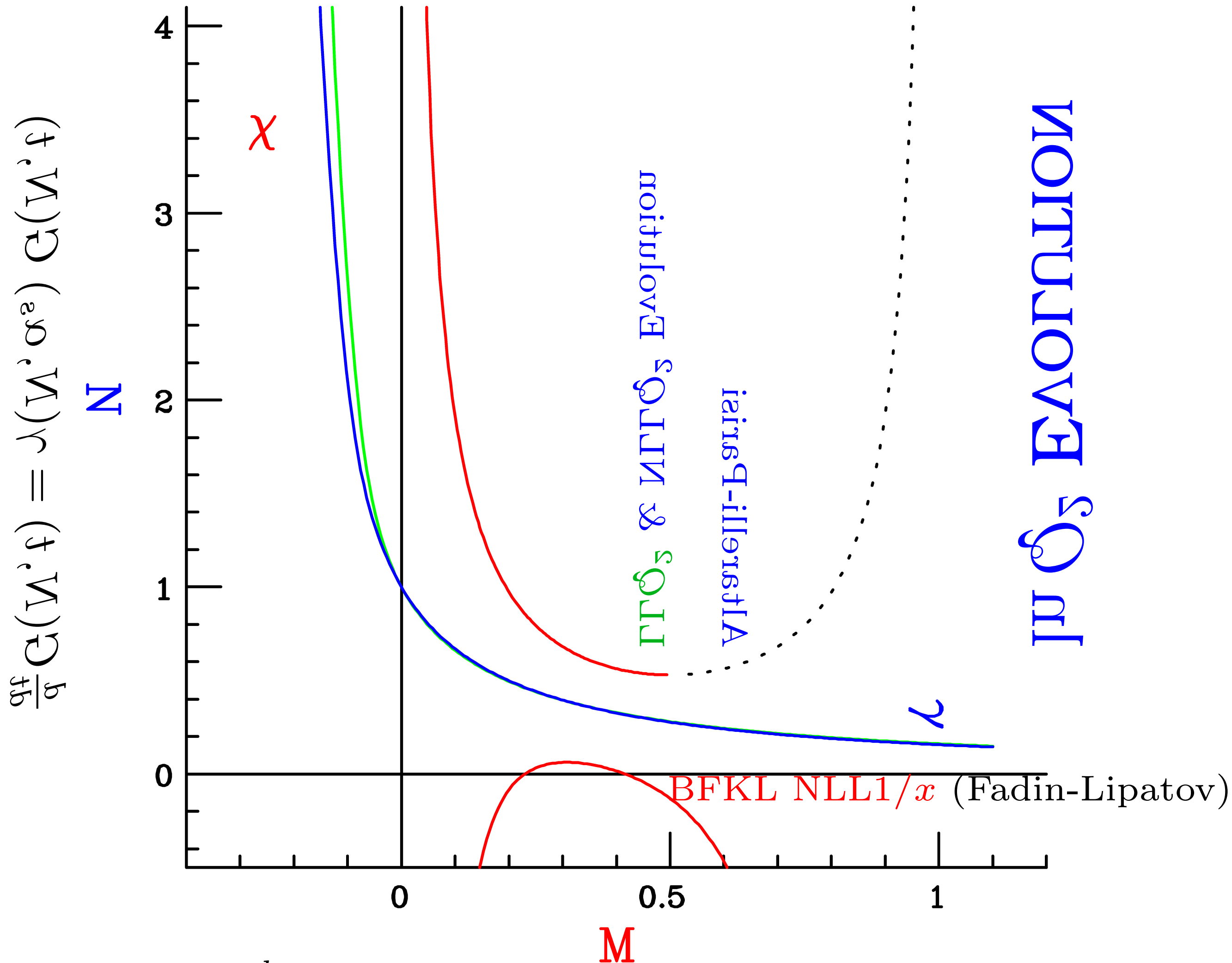
$1/M$ POLES $\Leftrightarrow \ln Q^2$

$$\gamma_0(N) \Longleftrightarrow \chi_s(\alpha_s/M)$$

$$\gamma_s(\alpha_s/N) \Longleftrightarrow \chi_0(M)$$

STABILIZING THE LL$1/x$ EVOLUTION
NL CORRECTIONS!

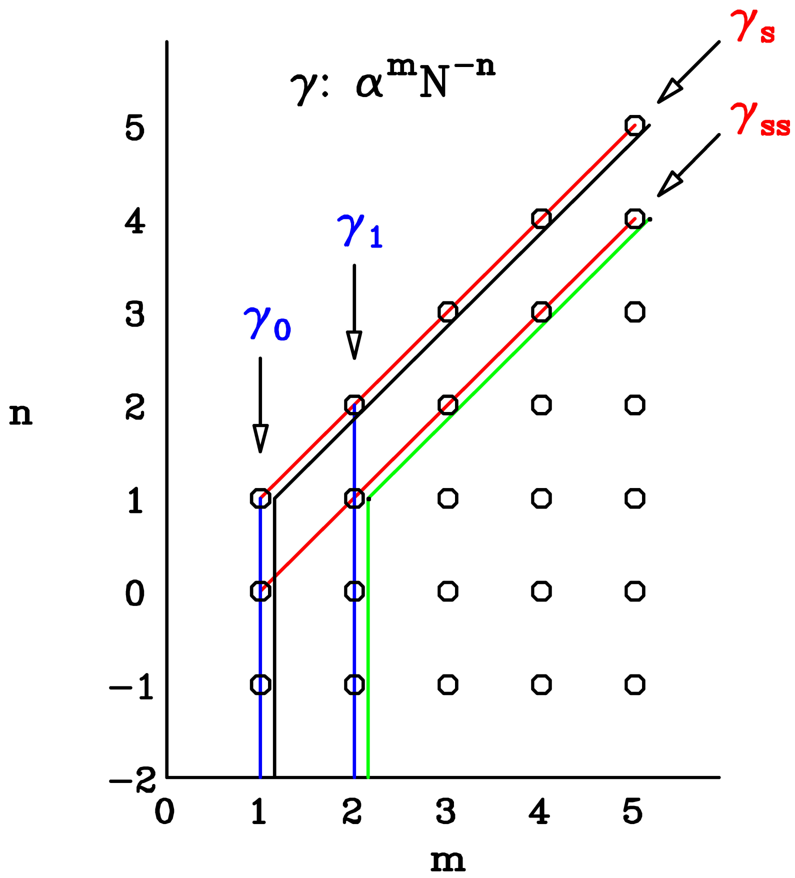
$\ln 1/x$ EVOLUTION

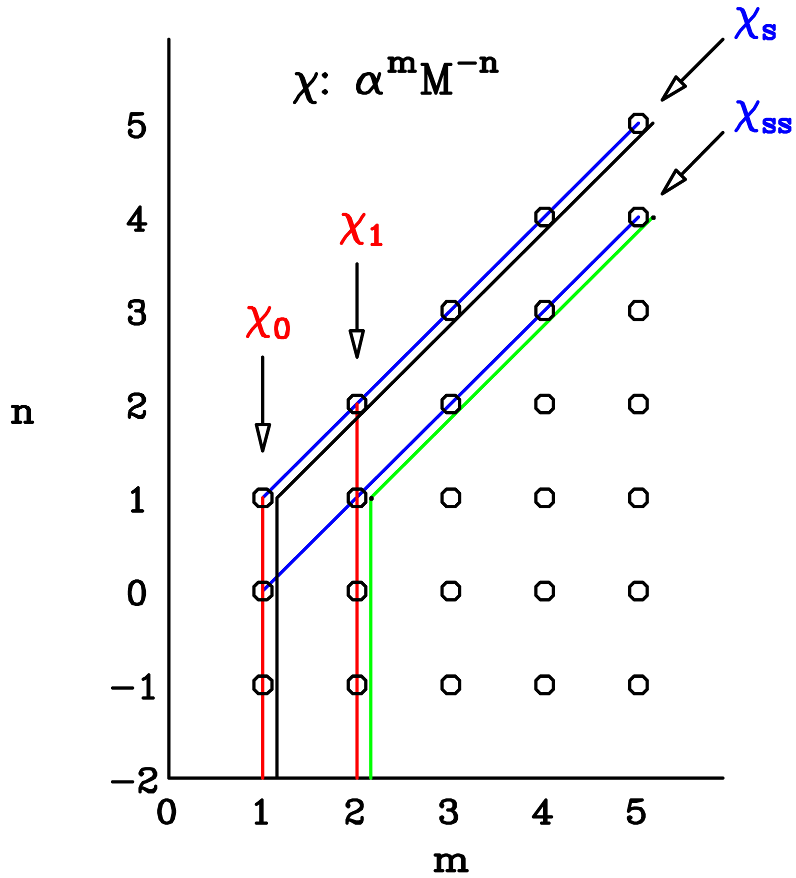


$$\frac{d}{d\xi} G(M,t) = \chi(M,\alpha_s)\; G(M,t)$$

- THE LLQ^2 AND LL$1/x$ KERNELS GREATLY DIFFER FROM EACH OTHER
- THE EXPANSION OF THE LL$1/x$ KERNEL LOOKS VERY UNSTABLE

THE DOUBLE–LEADING EXPANSION





$$\gamma(N,\alpha_s) = \left[\alpha_s\gamma_0(N) + \gamma_s\left(\frac{\alpha_s}{N}\right) - \frac{n_c\alpha_s}{\pi N}\right] \Leftarrow \Rightarrow \chi(M,\alpha_s) = \left[\alpha_s\chi_0(M) + \chi_s\left(\frac{\alpha_s}{M}\right) - \frac{n_c\alpha_s}{\pi M}\right]$$

$$+\alpha_s\left[\alpha_s\gamma_1(N) + \gamma_{ss}\left(\frac{\alpha_s}{N}\right) - \alpha_s\left(\frac{e_2}{N^2} + \frac{e_1}{N}\right) - e_0\right] \quad +\alpha_s\left[\alpha_s\chi_1(M) + \chi_{ss}\left(\frac{\alpha_s}{M}\right) - \alpha_s\left(\frac{f_2}{M^2} + \frac{f_1}{M}\right) - f_0\right]$$

$$+\cdots \qquad +\cdots$$

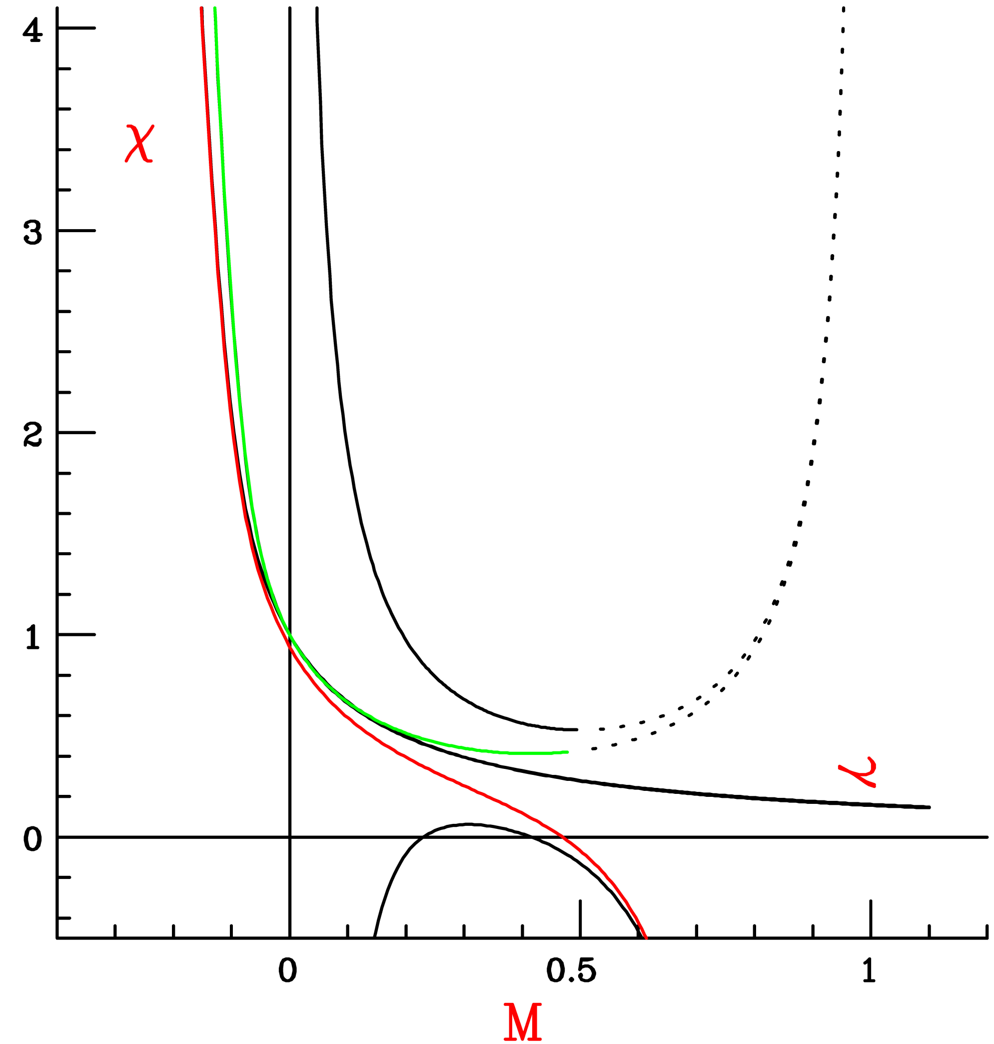
DUALITY HOLDS ORDER–BY–ORDER IN THE DOUBLE–LEADING EXPANSION:

the dual of $\chi_{\rm DL}^{\rm LO}$ is $\gamma_{\rm DL}^{\rm LO}$ up to terms of order $\gamma_{\rm DL}^{\rm NLO}$, and conversely

Double–Leading Evolution

$\chi_s \underset{M\to 0}{\simeq} \frac{\alpha_s+M}{\alpha_s} = \frac{M}{\alpha_s} - \frac{M^2}{\alpha_s^2} + \frac{M^3}{\alpha_s^3} + \dots$

$\chi(\gamma) = M \to \chi(0) = 1$



Momentum Conservation

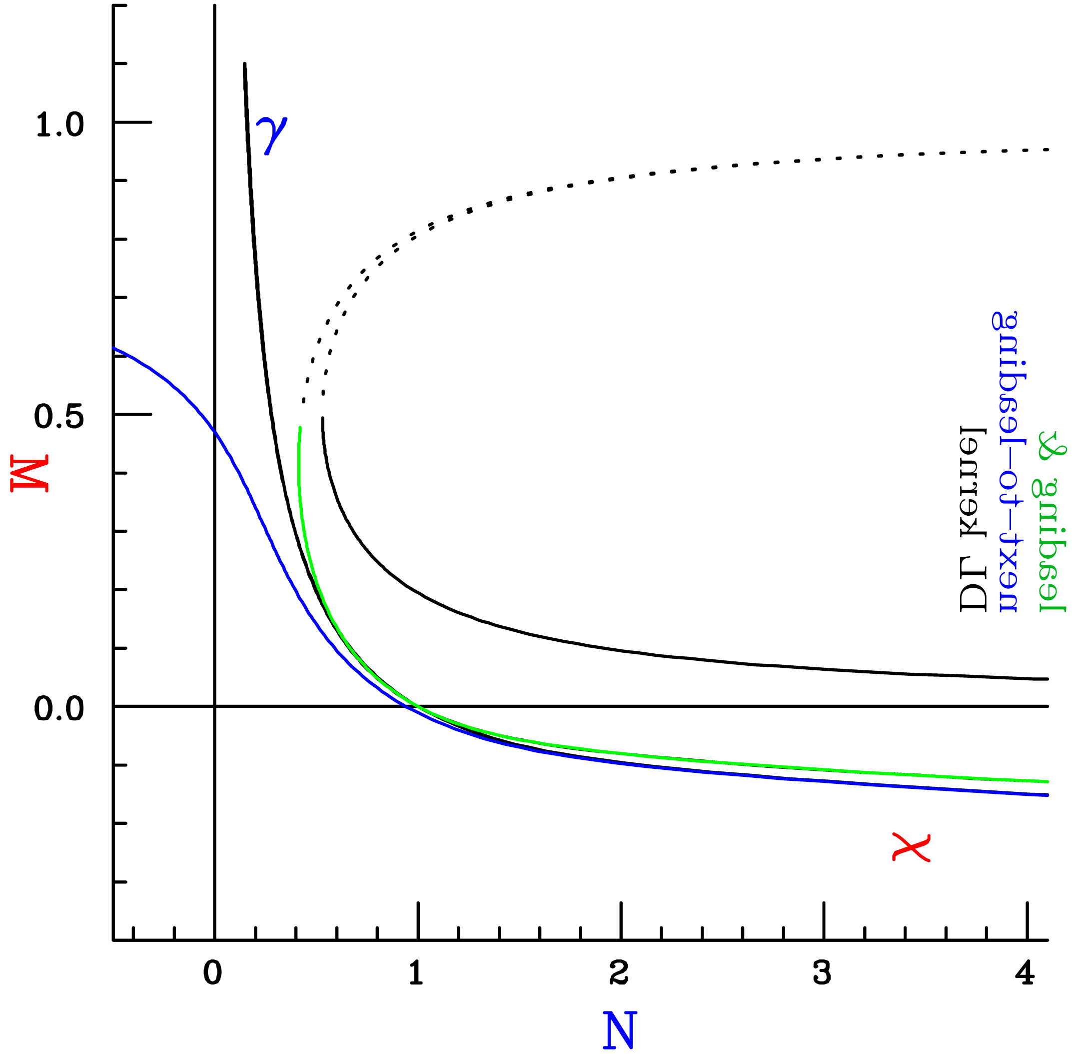
$\gamma(1) = 0$

- perturbative expansion stabilized in $M \sim 0$ region

Double–Leading Evolution

Momentum Conservation!

$$\gamma(1) = 0$$



$$\chi(\gamma) = N \to \chi(0) = 1$$

$$\chi_s(M) \underset{M \to 0}{\sim} \frac{\alpha}{\alpha + M} = \frac{\alpha}{M} - \frac{\alpha^2}{M^2} + \frac{\alpha^3}{M^3} + \dots$$

STABILIZING THE ANTICOLLINEAR REGION: EXCHANGE SYMMETRY

DIAGRAMS FOR $\ln 1/x$ EVOLUTION KERNEL

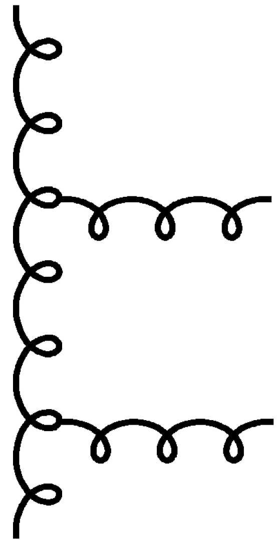
$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha_s)\; G(\xi, M)$$

$$\chi(\xi, M) = \int_{-\infty}^{\infty} \frac{dQ^2}{Q^2} \left(\frac{Q^2}{k^2}\right)^{-M} \chi(\xi, \tfrac{Q^2}{k^2})$$

SYMMETRIC UPON INTERCHANGE
OF INITIAL AND FINAL PARTON VIRTUALITIES

$k^2 \Leftrightarrow Q^2$

$Q^2 \Leftrightarrow k^2$



$$Q^2 \leftrightarrow k^2 \Leftrightarrow M \leftrightarrow 1 - M$$

COLLINEAR RES. OF $\frac{1}{M}$ POLES $\leftrightarrow$ ANTICOLLINEAR RES. OF $\frac{1}{1-M}$ POLES

SYMMETRY BREAKING

- DIS KINEMATIC VARIABLES $s = \frac{Q^2}{x}$ (small x)
- RUNNING OF THE COUPLING $\alpha_s(Q^2)$

BOTH CAN BE DETERMINED EXACTLY

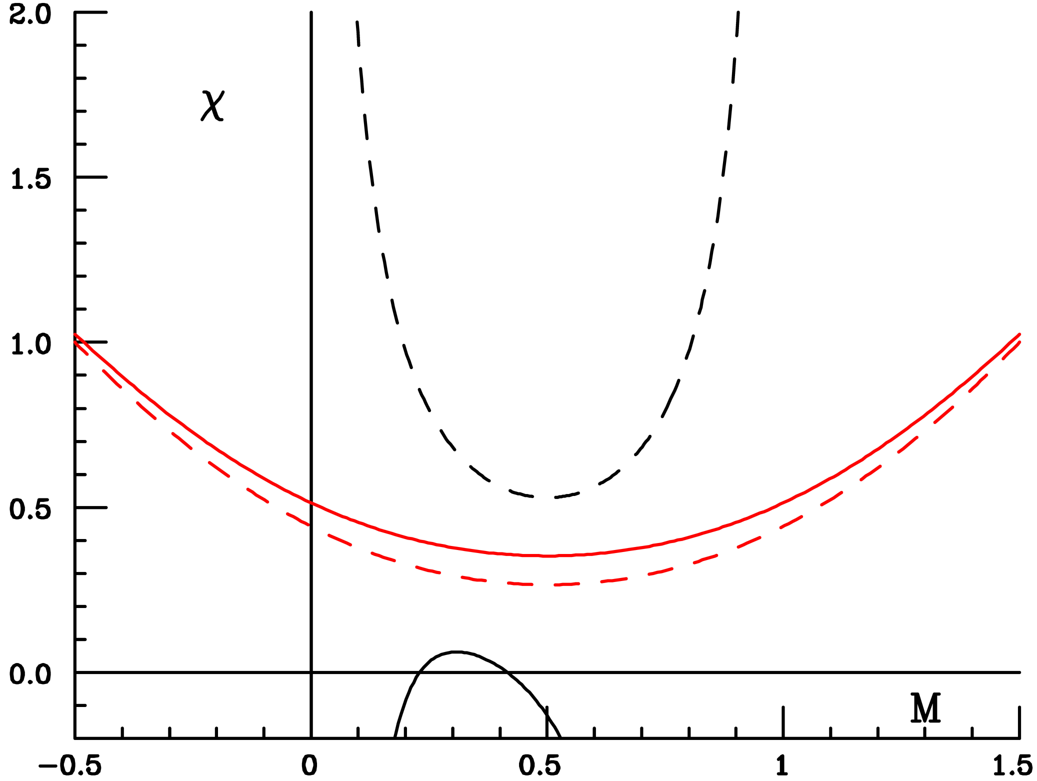
SYMMETRIZED EXPANSION

THE χ KERNEL

MOMENTUM CONSERVATION + SYMMETRY$\Rightarrow\chi$ ALWAYS HAS A MINIMUM

SYMMETRIC VARIABLES

- LO, NLO SYMMETRIC RESUMMED CLOSE TO EACH OTHER
- χ IS AN ENTIRE FCTN (QUADRATIC APPROX. IS EXCELLENT!)
- RESUMMED NLO HIGHER THAN LO



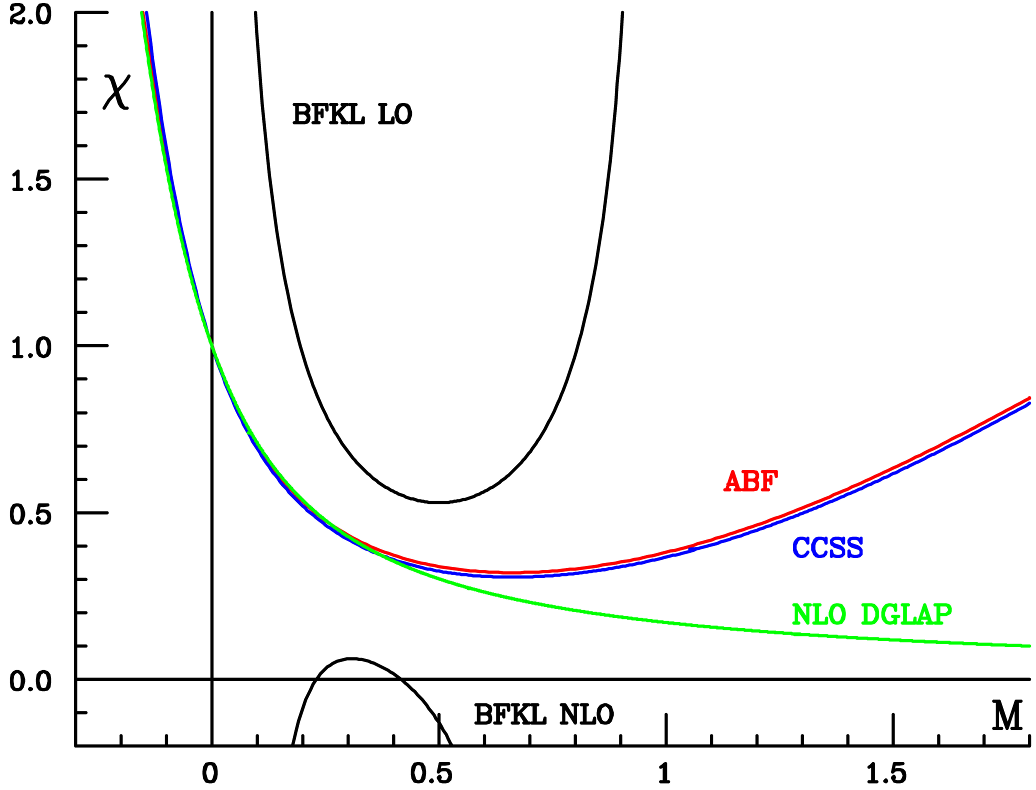
SYMMETRIZED EXPANSION

THE χ KERNEL

MOMENTUM CONSERVATION + SYMMETRY$\Rightarrow\chi$ ALWAYS HAS A MINIMUM

ASYMMETRIC VARIABLES

- LO, NLO SYM. CLOSE TO EACH OTHER
- LO, NLO SYM. CLOSE TO AP
- CURVATURE & INTERCEPT SAME IN SYM. & ASYM. VARIABLES



- RESULT DETERMINED BY MOM. CONS. + SYM.
- AMBIGUITIES MINIMAL, (CFR. ABF VS. CCSS) BUT MATCHING TO GLAP CRUCIAL

STABILIZING RUNNING COUPLING CORRECTIONS

- THE RUNNING OF THE COUPLING $\alpha(t) = \alpha_\mu[1 - \beta_0 \alpha_\mu t + \dots]$
  ($t \equiv \ln \frac{Q^2}{\mu^2}$) IS LEADING LOG Q^2, BUT NEXT–TO–LEADING LOG $\frac{1}{x}$
- UPON M-MELLIN TRANSFORMATION ($\ln x$ EVOLUTION)
  $\alpha_s(t)$ BECOMES AN OPERATOR:

$$\alpha_s(M) = \alpha_{\mu^2}[1 + \beta_0 \alpha_{\mu^2} \frac{d}{dM} + \dots]$$

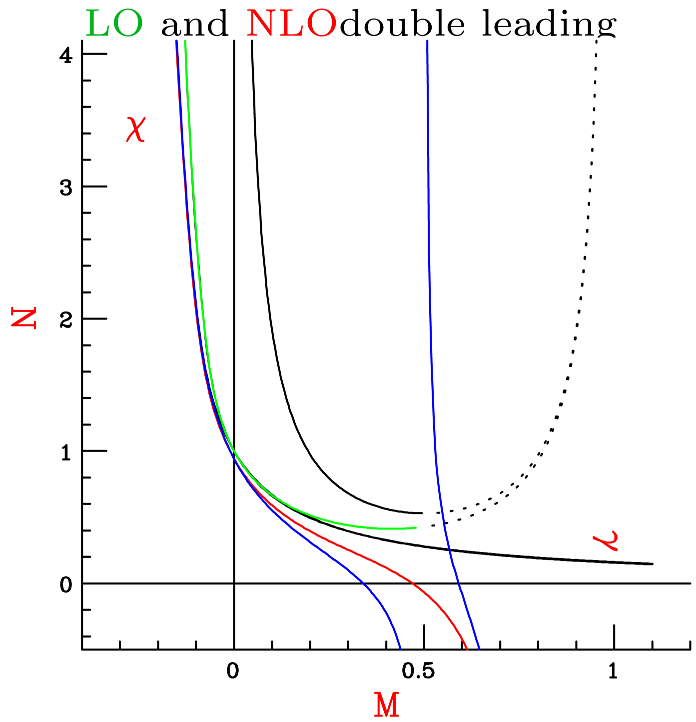
$\Rightarrow$ EVOLUTION EQUATION for $G(N, M)$ with b.c. $H_0(M)$

$$\left(1 - \frac{\alpha_\mu}{N}\right) \chi(M) G(N, M) - H_0(M) = \beta_0 \alpha_\mu \frac{d}{dM} G(N, M)$$

- BAD NEWS: PERTURBATIVE INSTABILITY

NLO R.C. CORRECTION
NOT UNIFORMLY SMALL AS $x \to 0$:

$$\frac{\Delta P_{ss}(\alpha_s, \xi)}{P_s(\alpha_s, \xi)} \underset{\xi \to \infty}{\sim} (\alpha_s \xi)^2$$



RUNNING COUPLING DUALITY
THE OPERATOR APPROACH:

DUAL KERNEL INVERSION

$$\chi(\hat{\alpha}_s, \gamma(\hat{\alpha}_s, N)) = N$$
$$\gamma(\hat{\alpha}_s, \chi(\hat{\alpha}_s, M)) = M$$

ACTING ON $G(N, M)$

DUALITY STILL HOLDS TO ALL ORDERS!:

- CAN DETERMINE γ (χ) AS A FUNCTIONAL OF FIXED-COUPLING DUAL γ_s (χ_s):
at LO, using $\chi_0 = N\hat{\alpha}^{-1}$, $\gamma \neq \gamma_s$ because $[N\hat{\alpha}^{-1}, \chi_0(M)] \neq 0$, so

$$\gamma\left(N\hat{\alpha}^{-1}\right) = \gamma_s\left(N\hat{\alpha}^{-1}\right) - \frac{1}{2}N\beta_0\gamma_s''\left(N\hat{\alpha}^{-1}\right)/\gamma_s'\left(N\hat{\alpha}^{-1}\right) + \ldots$$

RESULTS UP TO NLO FOR χ, NNNLO FOR γ (including β_1 terms)

- PERTURBATIVE SERIES OF UNSTABLE R.C. CORRNS. CAN BE RESUMMED

EXACT ASYMPTOTIC SOLUTION

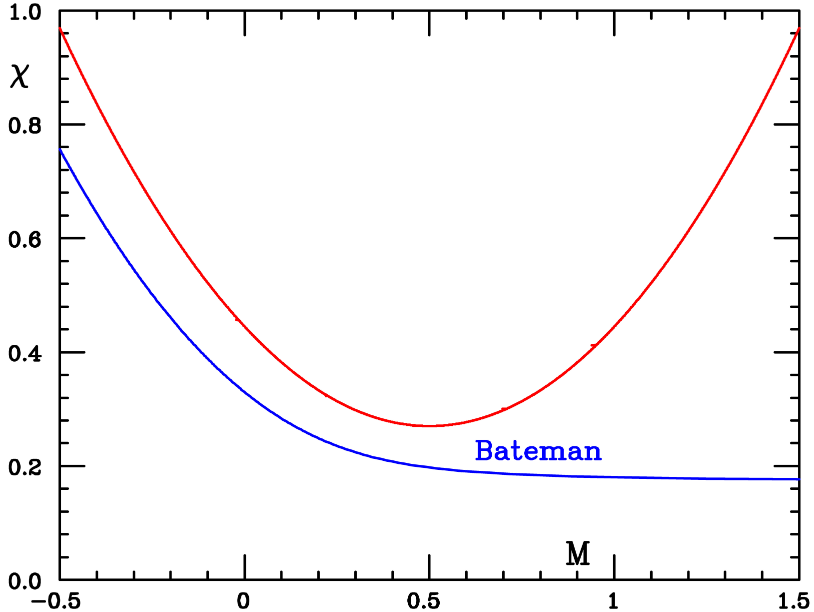
ASYMPTOTIC BEHAVIOUR CONTROLLED BY

MINIMUM OF $\chi(M) \Leftrightarrow$ RIGHTMOST SING. OF $\gamma(N)$

QUADRATIC KERNEL $\chi_q(\hat{\alpha}_s, M) = [c(\hat{\alpha}_s) + \frac{1}{2}\kappa(\hat{\alpha}_s)(M - M_s)^2]$

CAN SOLVE EXACTLY WITH LINEARIZED $c(\hat{\alpha}_s)$, $\kappa(\hat{\alpha}_s)$
IN TERMS OF BATEMAN FUNCTION $K_\nu(x)$:

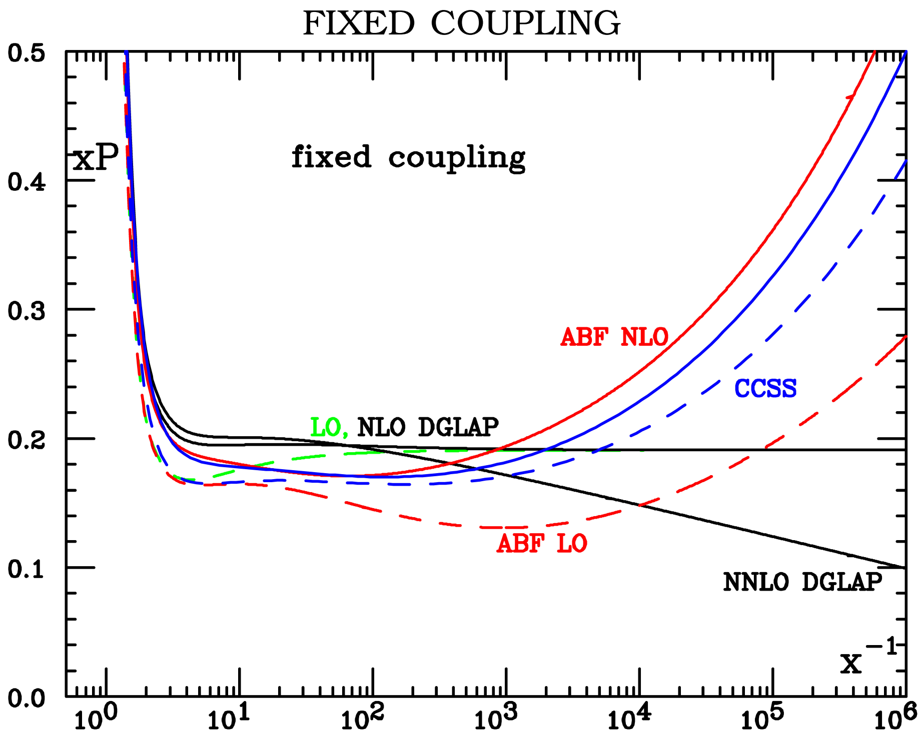
THE EFFECTIVE RESUMMED KERNEL

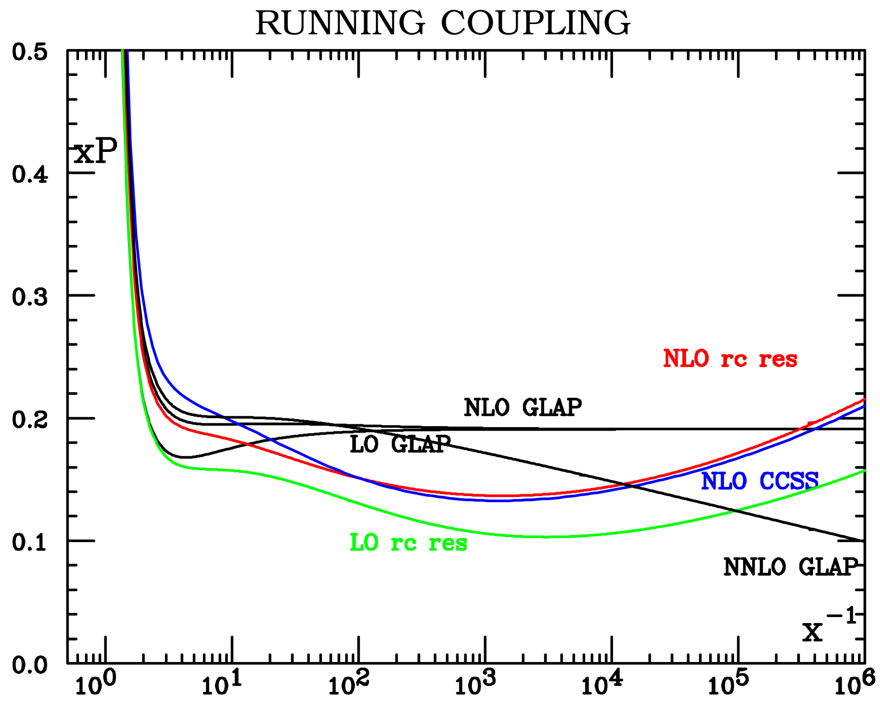


- $G(N,t) \propto K_{2B(\alpha_s,N)} \left[\frac{1}{\beta_0 \bar{\alpha_s}(t) A(\alpha_s,N)}\right]$
  A, B DEPEND ON α_s, N THROUGH c, κ
- ASYMPTOTIC LEADING LOG SMALL x SERIES RESUMMED
- BRANCH CUT IN γ REPLACED BY SIMPLE POLE

FINAL RESUMMED EVOLUTION: GENERAL FEATURES

THE SPLITTING FUNCTION $(n_f = 0)$





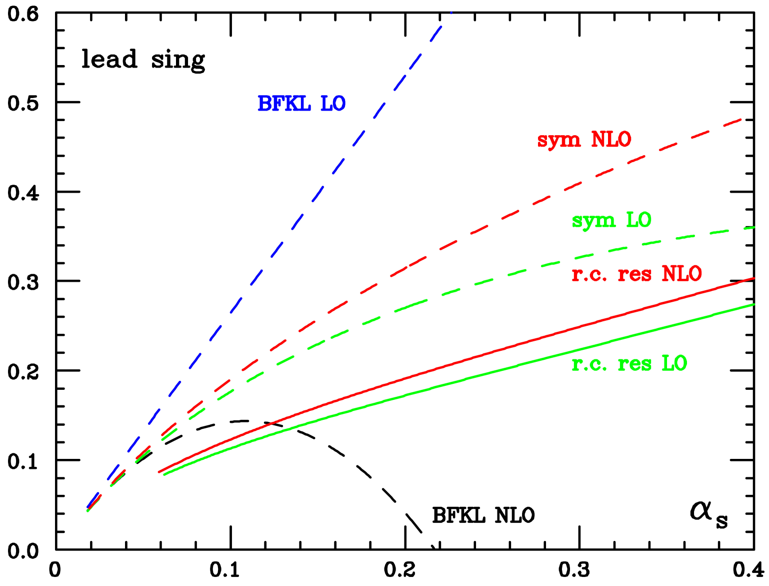
- RESUMMED EXPANSION CONVERGES RAPIDLY
  ESPECIALLY WITH RUNNING COUPLING
- BEHAVIOUR FOR $x < 10^{-2}$ VERY STABLE
- CAREFUL MATCHING OF SMALL x RUNNING COUPLING TERMS REQUIRED
  compare with CCSS $x \sim 0.2$
- DOMINANT QUALITATIVE BEHAVIOUR: DIP
  (SUPPRESSION) OF SPLITTING FUNCTION AT MEDIUM–SMALL x

RESUMMATION: GENERAL FEATURES

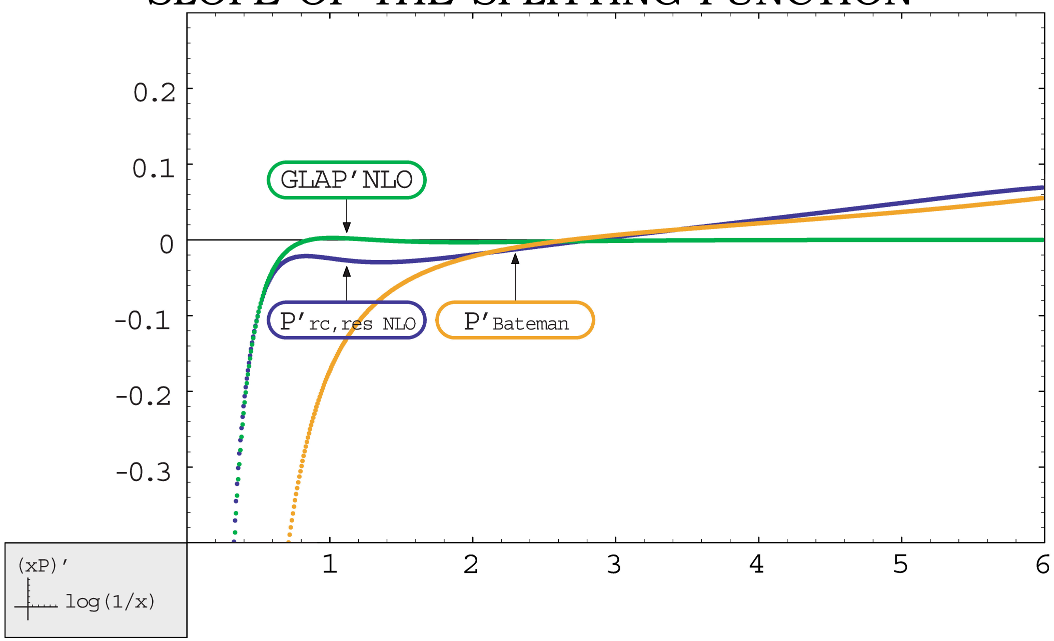
SMALL x BEHAVIOUR

SINGULARITY IN ANOM. DIM. AT $N = \alpha \Rightarrow$ ASYMPT. SMALL-x POWER $G \sim x^{-\alpha}$

SMALL-x POWER VS. α_s



SLOPE OF THE SPLITTING FUNCTION



- SUBLEADING TERMS (SYM. + R.C.) MANDATORY FOR STABLE PERTURBATIVE EXPANSION
- AT LARGE x ($x \gtrsim 0.2$) SPLITTING FUNCTION COINCIDES WITH NLO GLAP
- SMALL x INTERCEPT & CURVATURE DETERMINE RESUMMED BEHAVIOUR

☞ **BONVINI'S TALK**