Scattering Amplitudes in multi-Regge kinematics in planar N=4 SYM

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based on work in collaboration with

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Towards accuracy in small-x

Higgs Center for Theoretical Physics

Edinburgh, 12 September 2019

N=4 Super Yang-Mills

- Supersymmetric cousin of $SU(N_c)$ Yang-Mills theory.
- Spectrum:
  - ➡ Gluon (spin 1, 2 pol.)
  - ➡ Gluino (spin 1/2, 2 pol., 4 kinds)
  - ➡ Scalar (spin 0, 6 kinds)

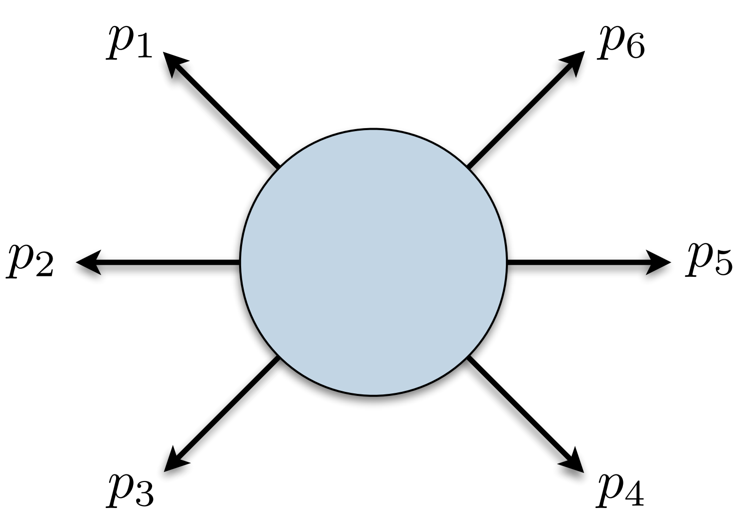
8 bosonic and
8 fermionic d.o.f.

- Conformal at the quantum level.
- Expected to be dual to string theory on $AdS_5 \times S^5$ via AdS/CFT correspondence.
  - ➡ Allows to explore strongly coupled regime.
- Could be looking at the first exactly solvable gauge theory in 4D.
  - ➡ N=4 SYM is the 'hydrogen atom of the 21st centruy'.

Dual conformal symmetry

- In the planar limit $N_c \to \infty$ scattering amplitudes in N=4 SYM have additional symmetries.
  - ➡ Result of a duality between amplitudes and Wilson loops.

[Alday, Maldacena; Branshuber, Heslop, Spence, Travaglini; Drummond, Henn, Korchemsky, Sokatchev]



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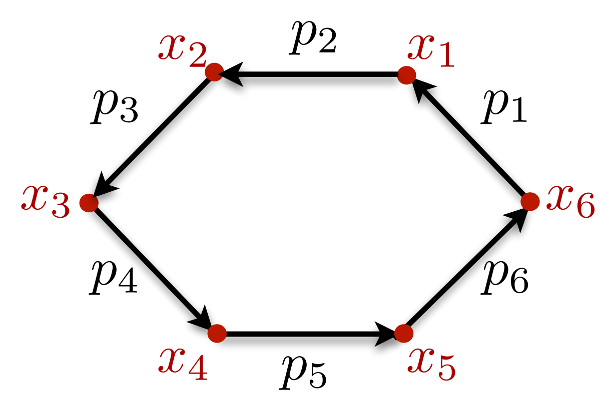


$$p_i = x_i - x_{i-1}$$

  - ➡ Dual conformal symmetry = conformal symmetry in the x_i.
  - ➡ Closes with ordinary conformal symmetry into an infinite-dimensional Yangian symmetry. [Drummond, Henn, Plefka]
  - ➡ Sign of integrability!?

Amplitudes in N=4 SYM

- Symmetry fixes 4 & 5-point amplitudes completely.
- From 6 points: amplitude determined up to a function of conformally invariant cross ratios ('remainder function').

$$u_{ijkl} = \frac{x_{ik}^2 \, x_{jl}^2}{x_{ij}^2 \, x_{kl}^2} \qquad\qquad x_{ij} = x_i - x_j$$

- 6-point remainder function:
  - ➡ MHV (--++++) known through 7 loops.
  - ➡ NMHV (---+++) known through 5 loops.

[Caron-Huot, Del Duca, Dixon, CD, Dulat, Drummond, Goncharov, Henn, von Hippel, McLeod, Smirnov, Spradlin, Pennington, Vergu, Volovich, …]

- 7-point remainder function:
  - ➡ MHV (---++++) known through 2 loops (+3&4-loop symbol).

[Golden, Spradlin; Drummond, Papathansiou, Spradlin]

- Some results at strong and finite coupling.

[Alday, Maldacena; Alday, Gaiotto, Madacena, Sever, Vieira; Basso, Sever, Vieira]

N=4 Super Yang-Mills

- Mysterious property: 'Maximal transcendentality'
  - ➡ An L loop amplitude only contains polylogarithms of 'transcendentality'/weight $2L$.

$$\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3} \qquad G(\underbrace{a_1,\ldots,a_n}_{\text{weight } n};z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

$$G(0;z) = \log z \qquad \log(-1) = i\pi$$

  - ➡ MHV (--++...) amplitudes are 'pure': coefficients in front of polylogarithms are rational numbers (not functions!)
  - ➡ Currently there is no explanation or proof.
  - ➡ It is known to hold for very large classes of amplitudes, correlation functions, form factors, anomalous dimension, ...

Multi-Regge kinematics

- Definition of MRK:

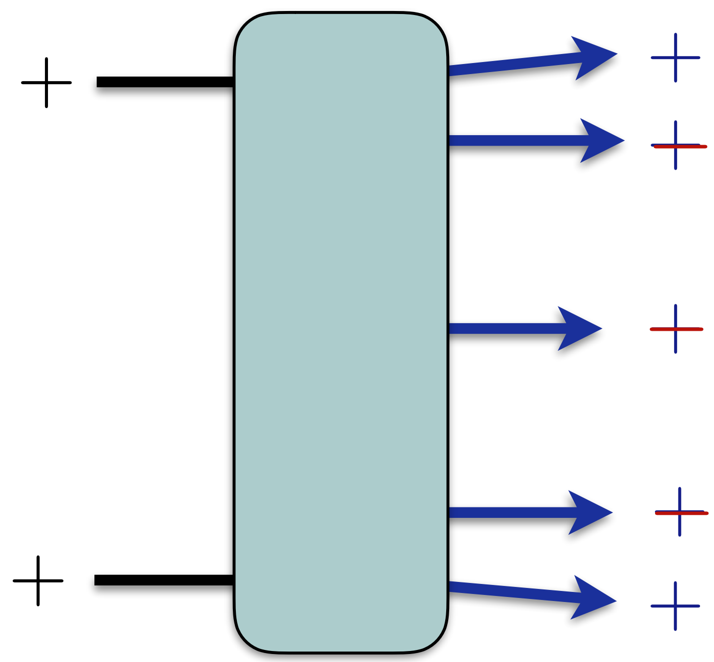
$$p_3^+ \gg p_4^+ \gg \ldots p_{N-1}^+ \gg p_N^+ \,, \qquad |\mathbf{p}_3| \simeq \ldots \simeq |\mathbf{p}_N| \qquad \mathbf{p}_k = p_k^x + i p_k^y$$

- We know all remainder functions in the Euclidean region in MRK:
  - ➡ For all multiplicities and helicity configurations.
  - ➡ For all values of the coupling.

0

Multi-Regge kinematics

- This is no longer true if we go to other Riemann sheets!



$\neq 0$ in MRK.

- Origin: Taking the multi-Regge limit does not commute with analytic continuation.

[Bartels, Lipatov, Sabio-Vera]

➡ First $u \to 1$, then $1-u \to e^{2\pi i}(1-u)$:

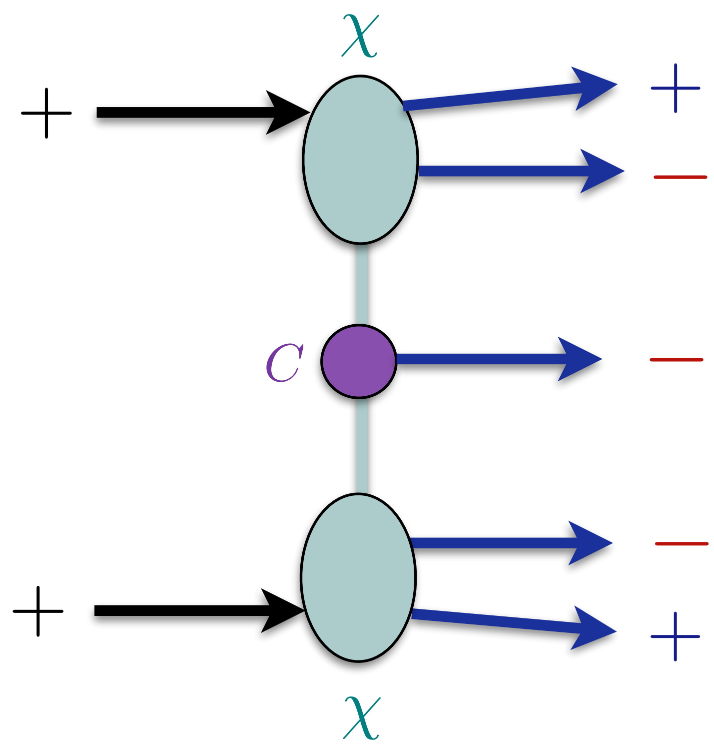
$$\mathrm{Li}_2(1-u) \to 0 \to 0$$

➡ First $1-u \to e^{2\pi i}(1-u)$, then $u \to 1$:

$$\mathrm{Li}_2(1-u) \to \mathrm{Li}_2(1-u) + 2\pi i \log(1-u) \to 2\pi i \log 0$$

Multi-Regge kinematics

- The remainder is described by a BFKL-type equation.



[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygaryn, Schnitzer; Bartels, Lipatov, Kormilitzin, Prygaryn; Del Duca, Druc, Drummond, CD, Dulat, Marzucca, Papathanasiou, Verbeek]

BFKL eigenvalue (octet)

$$\sim \prod_{j=1,2} \sum_{n_j} \left(\frac{z_j}{\bar{z}_j}\right)^{n_j/2} \int \frac{d\nu_j}{2\pi} |z_j|^{2i\nu_j} \chi_1^{h_3} \tau_1^{\omega_1} C_{12}^{h_4} \tau_2^{\omega_2} \chi_2^{-h_5}$$

Impact factor   Central emission vertex

$$\tau_i \sim \text{‘}\frac{s_i}{t_i}\text{’} \qquad z_i \sim \text{transverse d.o.f.}$$

➡ Obvious generalisation to higher points.

Outline

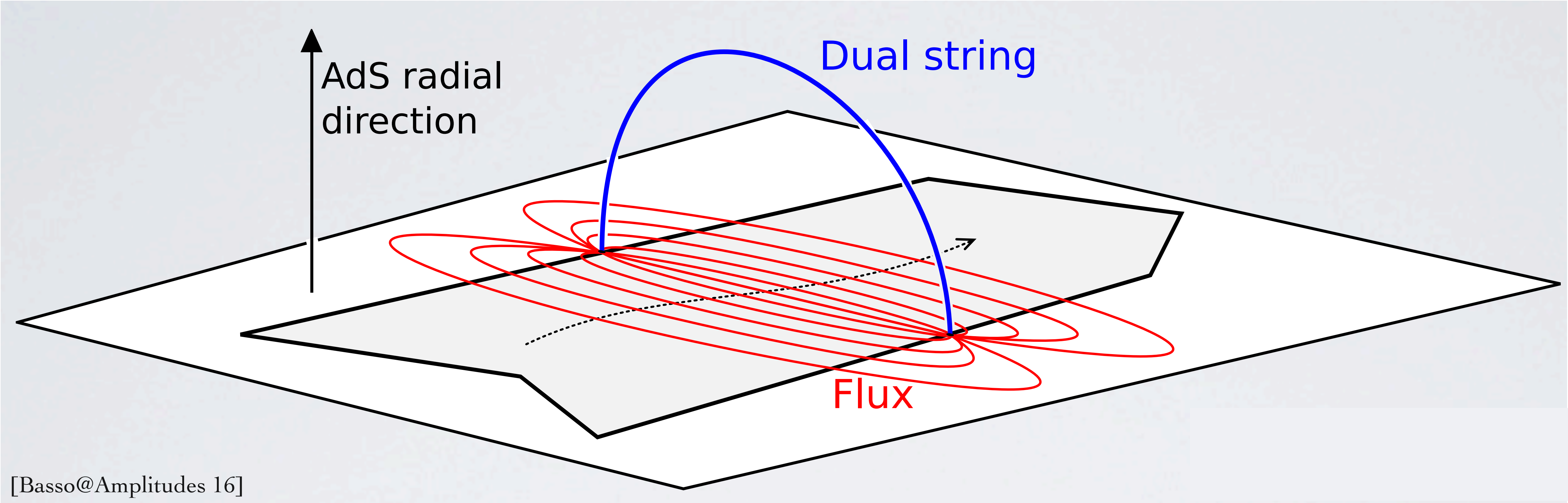
- The Fourier-Mellin space story:
  - ➡ Integrability and all order results.
- The momentum-space story:
  - ➡ The geometry of multi-Regge kinematics.
- The perturbative story:
  - ➡ A complete picture of MRK to all orders.
  - ➡ See also Robin's talk!

The Fourier-Mellin space story

Integrability and all order results

Flux tube picture

- The sides of the polygon source a flux tube.



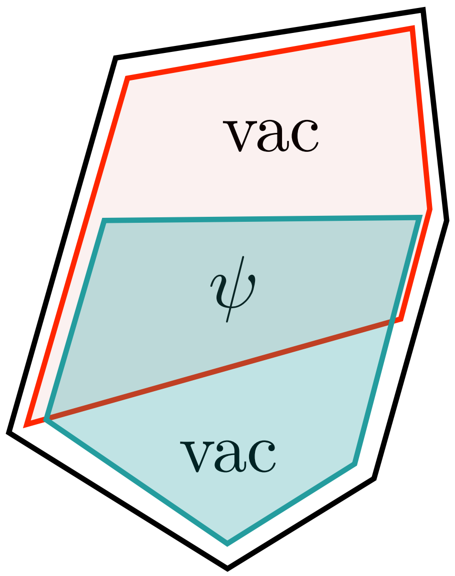
[Alday, Gaiotto, Madacena, Sever, Vieira]

- Can describe the Wilson loop/amplitude via the excitations of the flux tube.
  ➡ The spectrum of excitations is known from integrability.

[Basso]

Pentagon transitions

- In principle: Fully non-perturbative description of amplitudes.



$$= \sum_{\psi} \int d\mu \, P(0|\psi) \, e^{-E\tau + ip + im\phi_1} \, P(\psi|0)$$

[Basso, Sever, Vieira]

➡ Transition probability $P(\psi_1|\psi_2)$ known from integrability.

- In practise: Hard to make it concrete.

➡ So far only used for low numbers of points to obtain a series expansion around the collinear limit.

➡ But first results on 6-point amplitude at finite coupling!

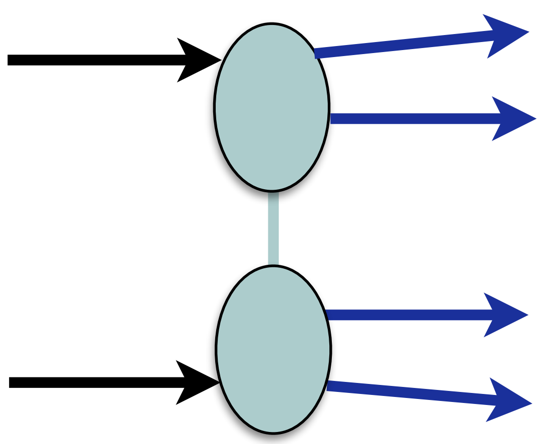
[Basso, Sever, Vieira]

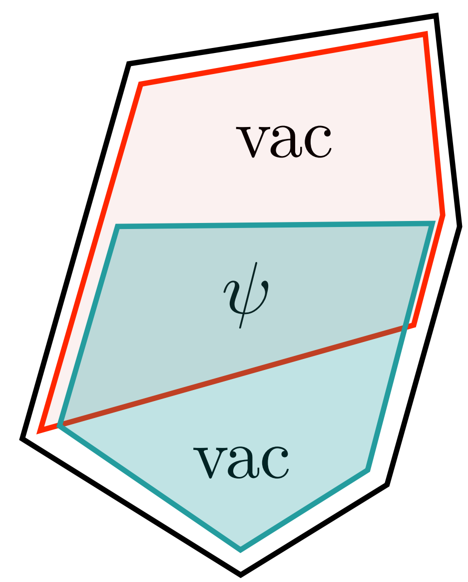
MRK vs Flux Tube

- BFKL-type equation very reminiscent of flux tube formula!

BFKL eigenvalue ⟷ Spectrum of excitations

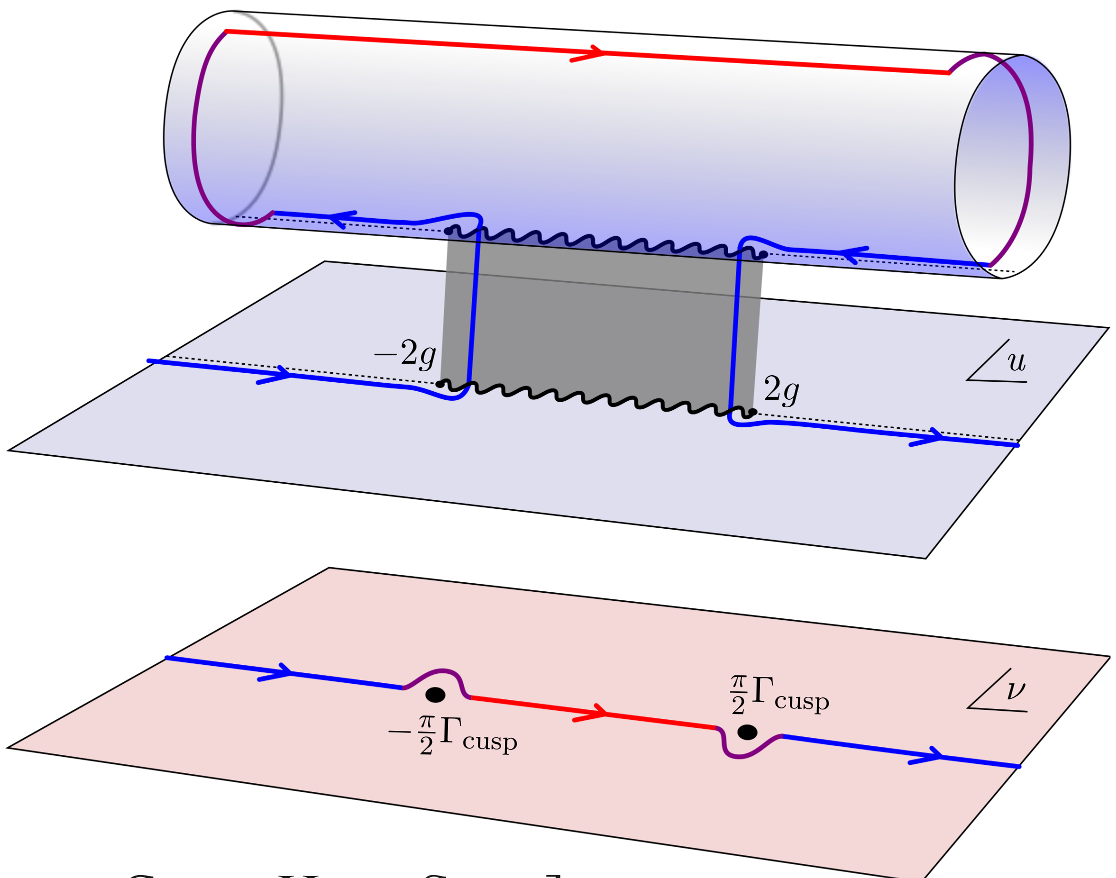
Impact factor & central emission block ⟷ Transition probability $P(\psi_1|\psi_2)$





$$\sum_n \left(\frac{z}{\bar{z}}\right)^{n/2} \int \frac{d\nu}{2\pi} |z|^{2i\nu} \chi_1^{h_3} \tau_1^{\omega_1} \chi_1^{-h_4} \qquad \sum_\psi \int d\mu \, P(0|\psi)\, e^{-E\tau+ip+im\phi_1}\, P(\psi|0)$$

MRK vs Flux Tube



[Basse, Caron-Huot, Sever]

- Flux-tube building blocks obtained from integrability for all values of the coupling.
- BFKL building blocks obtained by analytically continuing the flux-tube building blocks.

- In a landmark paper Basso, Caron-Huot and Sever have determined the octet BFKL eigenvalue and impact factors for all values of the coupling!

  ➡ Sufficient to compute 6-point amplitude to all orders in MRK.

Central emission vertex

- We have recently determined the central emission vertex to all orders in the coupling.

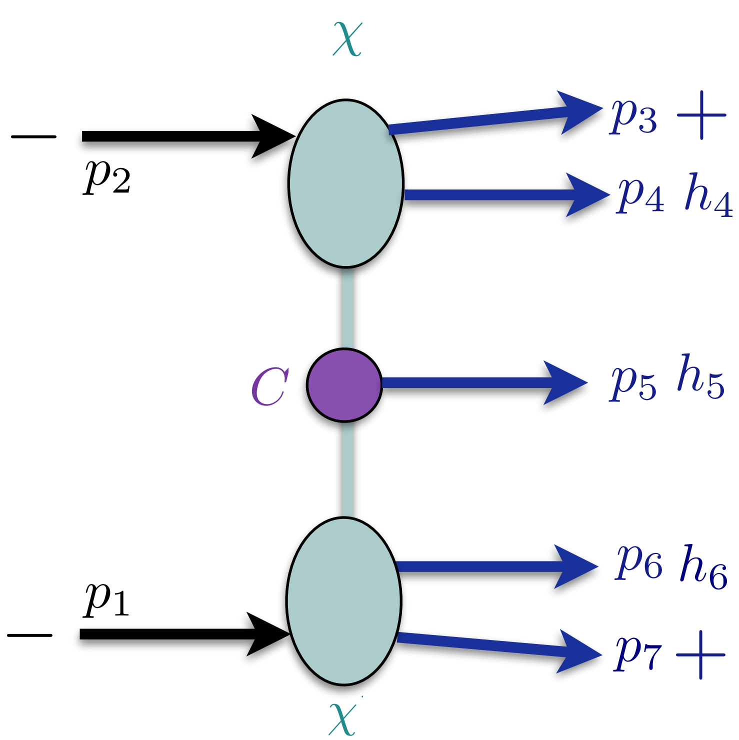
[Del Duca, Druc, Drummond, CD, Dulat, Marzucca, Papathanasiou, Verbeek, to appear]

- Basic idea:
  - ➡ Write all-orders ansatz inspired by all order formulas for eigenvalue and impact factor.
  - ➡ Match ansatz to available perturbative data through 3 loops.
  - ➡ Extrapolate beyond 3-loops.

- This conjecturally provides the last missing building block for all order formula for amplitudes in MRK.

Convolutions

- Next step: what happens in momentum space?



$$\sim \prod_{j=1,2} \sum_{n_j} \left(\frac{z_j}{\bar{z}_j}\right)^{n_j/2} \int \frac{d\nu_j}{2\pi} |z_j|^{2i\nu_j} \chi^{h_3} \tau_1^{aE_{\nu_1 n_1}} C^{h_4} \tau_2^{aE_{\nu_2 n_2}} \chi^{-h_5}$$

$$\sim \sum_{i_1,i_2} \frac{a^{i_1+i_2}}{i_1! i_2!} \log^{i_1}\tau_1 \log^{i_2}\tau_2\, g^{(i_1,i_2)}_{h_3 h_4 h_5}(z_1, z_2)$$

[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygaryn, Schnitzer; Bartels, Lipatov, Kormilitzin, Prygaryn]

- Fourier-Mellin transform: $\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} F(\nu, n)$

- Which $F(\nu, n)$ can appear?

FM building blocks

- Integrability: In perturbation theory, integrand is a polynomial in multiple zeta values and

$$E(\nu,n) = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)$$

$$F(\nu,n) = -2\psi(1) + \psi\left(1+i\nu-\frac{n}{2}\right) + \psi\left(1-i\nu-\frac{n}{2}\right),$$

$$V(\nu,n) = \frac{i\nu}{\nu^2+\frac{n^2}{4}}, \qquad N(\nu,n) = \frac{n}{\nu^2+\frac{n^2}{4}}, \qquad D_\nu^n \equiv (-i)^n \partial_\nu^n,$$

$$M(\nu_k,n_k,\nu_l,n_l) = \psi\left(i(\nu_k-\nu_l) - \frac{n_k-n_l}{2}\right) + \psi\left(1-i(\nu_k-\nu_l) - \frac{n_k-n_l}{2}\right),$$

- Example: NLO BFKL eigenvalue

$$E_{\nu,n}^{(1)} = -\frac{1}{4}\, D_\nu^2 E_{\nu,n} + \frac{1}{2}\, V\, D_\nu E_{\nu,n} - \zeta_2\, E_{\nu,n} - 3\,\zeta_3$$

$$E_{\nu,n}^{(2)} = \frac{1}{8}\Big\{\frac{1}{6}\, D_\nu^4 E_{\nu,n} - V\, D_\nu^3 E_{\nu,n} + (V^2+2\zeta_2)\, D_\nu^2 E_{\nu,n} - V\,(N^2+8\zeta_2) D_\nu E_{\nu,n}$$
$$+ \zeta_3(4\,V^2+N^2) + 44\zeta_4\, E_{\nu,n} + 16\zeta_2\zeta_3 + 80\zeta_5\Big\},$$

FM building blocks

- Conclusion: We only need to understand how to compute FM transforms that involve products of these building blocks.

- FM transform maps products into convolutions:

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g \ = \frac{1}{\pi} \int \frac{d^2 w}{|w|^2}\, f(w)\ g\left(\frac{z}{w}\right)$$

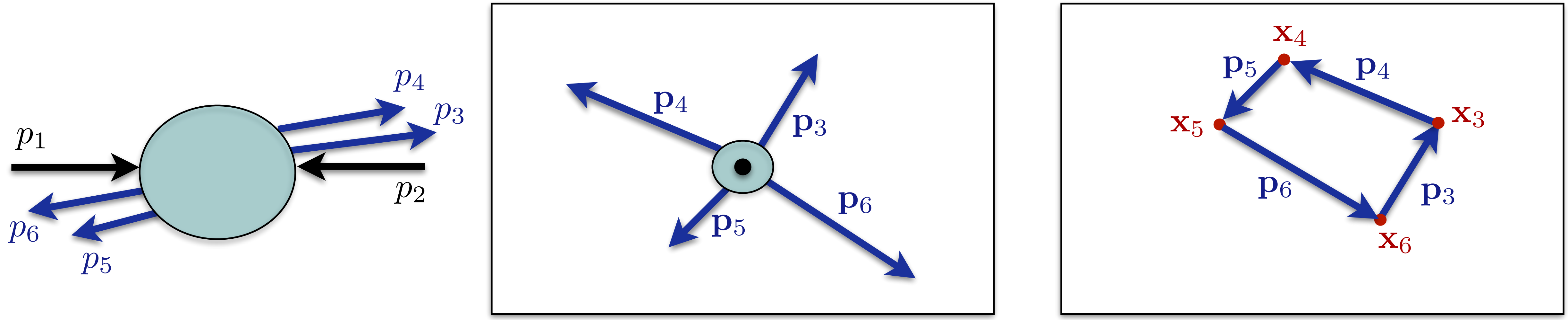
- What can we say about these integrals…?

The momentum-space story

The geometry of multi-Regge kinematics

Multi-Regge kinematics

- Non-trivial kinematical dependence in transverse plane.



  ➡ Kinematics encoded into $N-2$ points in transverse plane.

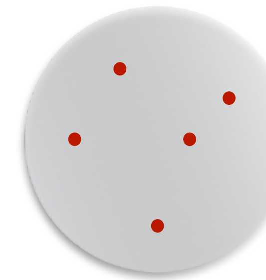
- Dual conformal invariance in transverse plane:

  ➡ Functional dependence only on $N-5$ cross ratios in transverse plane:

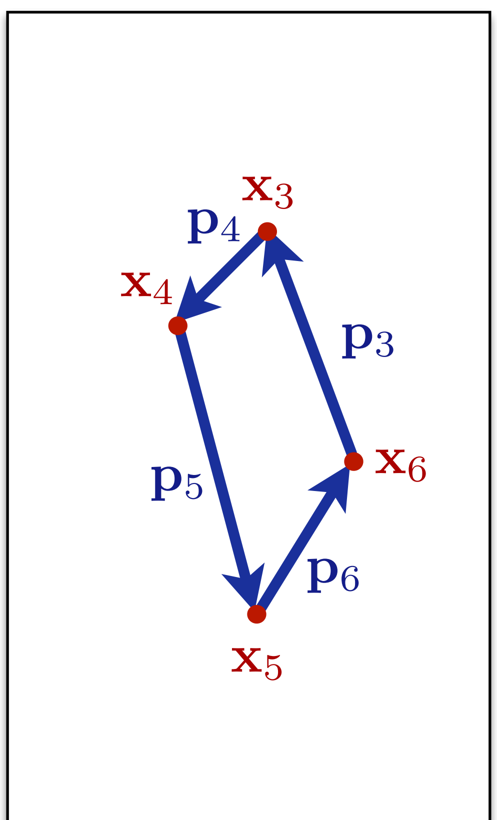
$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

The moduli space $\mathfrak{M}_{0,n}$

- $\mathfrak{M}_{0,n}$ = moduli space space of Riemann spheres with n marked points.
  = space of configurations of n points on the Riemann sphere.



- For $n = N - 2 : \mathfrak{M}_{0,N-2}$ is 'phase space' of MRK.



➡ Fix 3 points to $0, 1, \infty$.

➡ $\dim_{\mathbb{C}} \mathfrak{M}_{0,n} = n - 3$

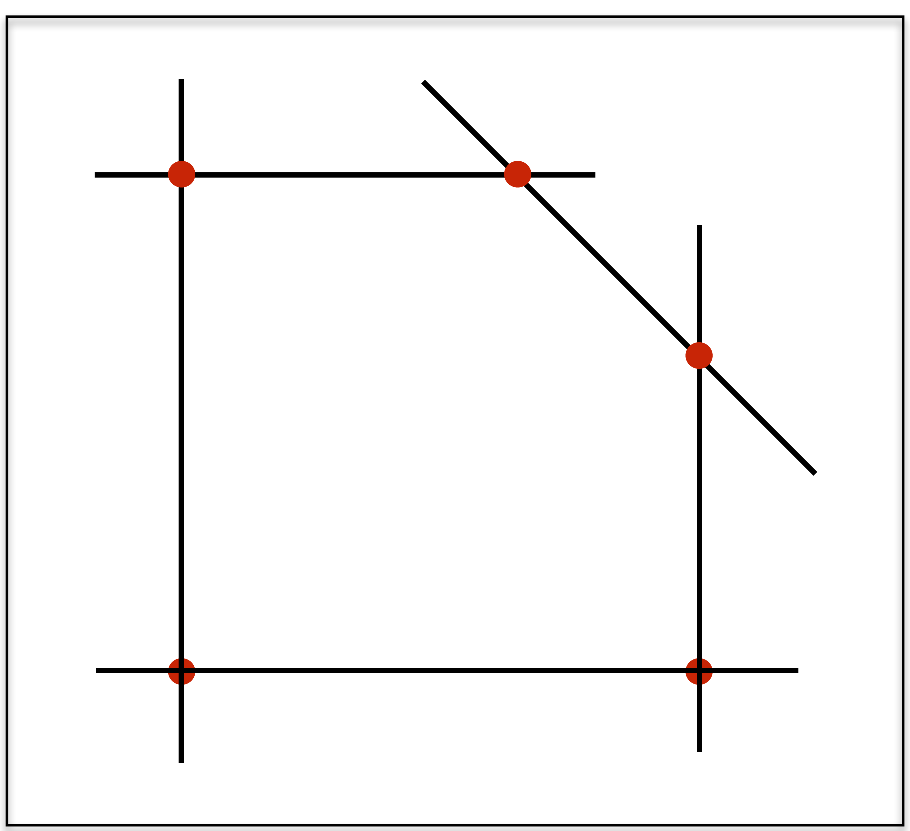
➡ Coordinates are collection of $n - 3 = N - 5$ cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

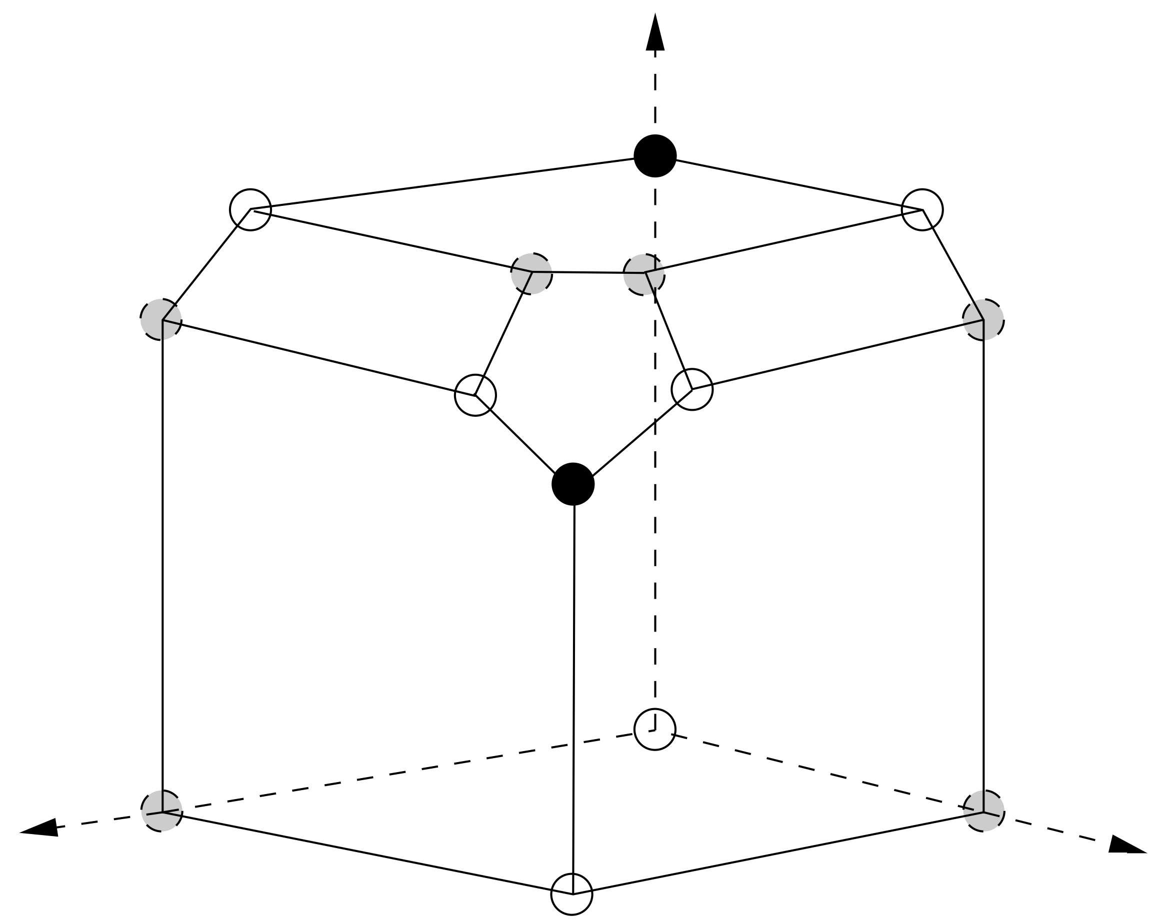
The moduli space $\mathfrak{M}_{0,n}$

- Fix three points to $0, 1, \infty$.
- $\mathfrak{M}_{0,4}$ = complex plane with the points $0, 1, \infty$ removed.

$\mathfrak{M}_{0,5}$



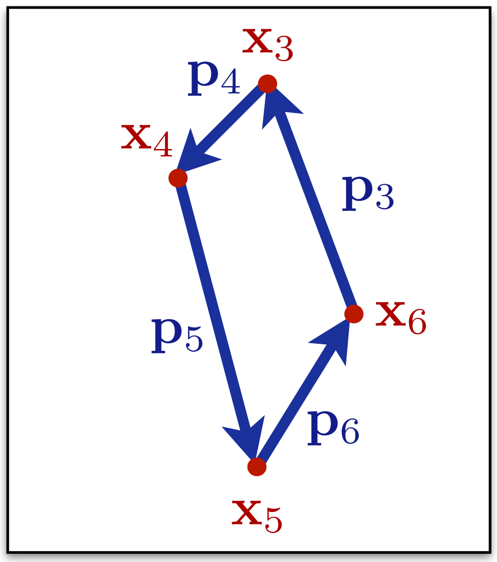
$\mathfrak{M}_{0,6}$



[Figure: F. Brown]

Iterated integrals on $\mathfrak{M}_{0,n}$

- Singularities: 'Degenerate' configurations of points.



= 2 points become equal.

➡ Physically: momentum is soft.

- What are 'natural integrals' on this space?

➡ Should have singularities at most when $\mathbf{x}_i = \mathbf{x}_j$.

- All iterated integrals on $\mathfrak{M}_{0,n}$ can be written in terms of polylogarithms. [Brown]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0,1;z) = -\mathrm{Li}_2(z)$$

- Consequence: Amplitudes in MRK can be written in terms of polylogarithms.

➡ Must have branch cuts dictated by unitarity!

Hopf algebras

- The geometry of $\mathfrak{M}_{0,n}$ and polylogarithms are well studied in modern mathematics.
- Polylogarithms form a Hopf algebra. [Goncharov; Brown]
  - ➡ Algebra: Vector space with an operation that allows one to 'fuse' two elements into one (multiplication).
  - ➡ Coalgebra: Vector space with an operation that allows one to break one element apart (coproduct Δ).
- A Hopf algebra is
  - ➡ at the same time an algebra & a coalgebra
  - ➡ such that the product and coproduct are compatible

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$

  - ➡ together with an antipode, a linear map S such that

$$m(S \otimes \mathrm{id})\Delta(x) = 0\,, \text{ if } x \notin \mathbb{Q}$$

Polylogarithms

- Polylogarithms form a Hopf algebra. [Goncharov; Brown]

$$\Delta(G(1;z)) = G(1;z)\otimes 1 + 1\otimes G(1;z) \qquad G(1;z) = \log(1-z)$$

$$\Delta(G(0,1;z)) = G(0,1;z)\otimes 1 + G(1;z)\otimes G(0;z) + 1\otimes G(0,1;z)$$

- Meaning of the two entries: discontinuities and derivatives.

$$\Delta \mathrm{Disc} = (\mathrm{Disc}\otimes \mathrm{id})\Delta \qquad \Delta\partial_z = (\mathrm{id}\otimes\partial_z)\Delta$$

- Example: $G(0,1;z) = -\mathrm{Li}_2(z)$

  ➡ $G(0,1;z)$ has a branch cut starting at $z=1$ ...

  ➡ ... but not at $z=0$.

Polylogarithms & Amplitudes

- Branch cuts are dictated by unitarity:

  ➡ Coproduct of amplitudes must 'know' about unitarity.

- Optical theorem: branch cuts can only start at $x_{ij}^2 = (x_i - x_j)^2 = 0$.
- Consequence: Logarithms that appear in first factor of coproduct are highly constrained!

$$\Delta(\mathcal{A}) \sim \log x_{ij}^2 \otimes \ldots \qquad\qquad \Delta \text{Disc} = (\text{Disc} \otimes \text{id})\Delta$$

- Consequence for MRK:

$$\Delta(\mathcal{A}^{MRK}) \sim \log \underbrace{|\mathbf{x}_i - \mathbf{x}_j|^2}_{>0,\ \text{if } i \neq j} \otimes \ldots$$ ➡ Single-valued function!

- Conclusion:

> N-point scattering amplitudes in planar N=4 SYM in MRK are single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$.

Single-valued functions

- Single-valued polylogarithms = combinations of polylogarithms and their complex conjugates such that all branch cuts cancel.
- One way to construct them: A map $\mathbf{s}$ that assigns to each polylogarithm its single-valued version:

$$\mathbf{s} = \mu(\tilde{S} \otimes \mathrm{id})\Delta$$

[cf. Brown for MZV case]

$\mu =$ multiplication

$\Delta =$ coproduct

$\tilde{S} =$ Complex conjugate of the antipode (up to a sign)

- Examples:

$$\mathcal{G}_a(z) = \mathbf{s}(G(a;z)) = G(a;z) + G(\bar{a};\bar{z}) = \log\left|1 - \frac{z}{a}\right|^2$$

$$\mathcal{G}_{a,b}(z) = \mathbf{s}(G(a,b;z)) = G(a,b;z) + G(\bar{b},\bar{a};\bar{z}) + G(b;a)\,G(\bar{a};\bar{z}) + G(\bar{b};\bar{a})\,G(\bar{a};\bar{z}) - G(a;b)\,G(\bar{b};\bar{z}) + G(a;z)\,G(\bar{b};\bar{z}) - G(\bar{a};\bar{b})\,G(\bar{b};\bar{z})$$

Single-valued functions

- Preserves multiplication: $\mathbf{s}(a\cdot b)=\mathbf{s}(a)\cdot\mathbf{s}(b)$
- Preserves relations among polylogarithms.
- Commutes with holomorphic differentiation: $\partial_z\,\mathbf{s}=\mathbf{s}\,\partial_z$
- Antipode corresponds to complex conjugation: $\bar{\mathbf{s}}=\mathbf{s}\,\tilde{S}$
  - ➡ Example:

$$\mathcal{G}(\bar a,\bar b;\bar z)=\bar{\mathbf{s}}(G(a,b;z))\;=\mathcal{G}(b,a;z)+\mathcal{G}(b;a)\,\mathcal{G}(a;z)-\mathcal{G}(a;b)\,\mathcal{G}(b;z)$$

- Does not commute with anti-holomorphic differentiation:
  - ➡ Example:

$$\bar\partial_z\mathcal{G}(a,b;z)=\frac{1}{\bar z-\bar a}\,\mathcal{G}(b;a)+\frac{1}{\bar z-\bar b}(\mathcal{G}(a;z)-\mathcal{G}(a;b))$$

The perturbative story

A complete picture of MRK to all orders

Single-valued functions

- The theory of single-valued polylogarithms almost trivialises the computation of the convolution integrals.
- Main idea:
  - ➡ Recursive structure in the loop order (for fixed log accuracy).

$$\mathcal{F}[\chi^+_{\nu n} E^{\ell+1}_{\nu n} \chi^-_{\nu n}] = \mathcal{F}[\chi^+_{\nu n} E^{\ell}_{\nu n} \chi^-_{\nu n}] * \mathcal{F}[E_{\nu n}]$$

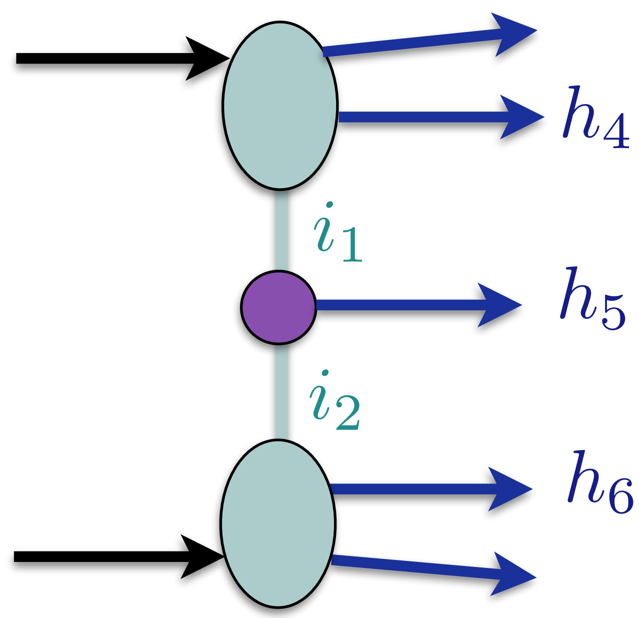
$$\mathcal{F}[E_{\nu n}] = -\frac{z+\bar{z}}{2|1-z|^2}$$

  - ➡ Convolution integral is a simple residue computation.
  - ➡ See Robin's talk!
- Here: Focus on general consequences.

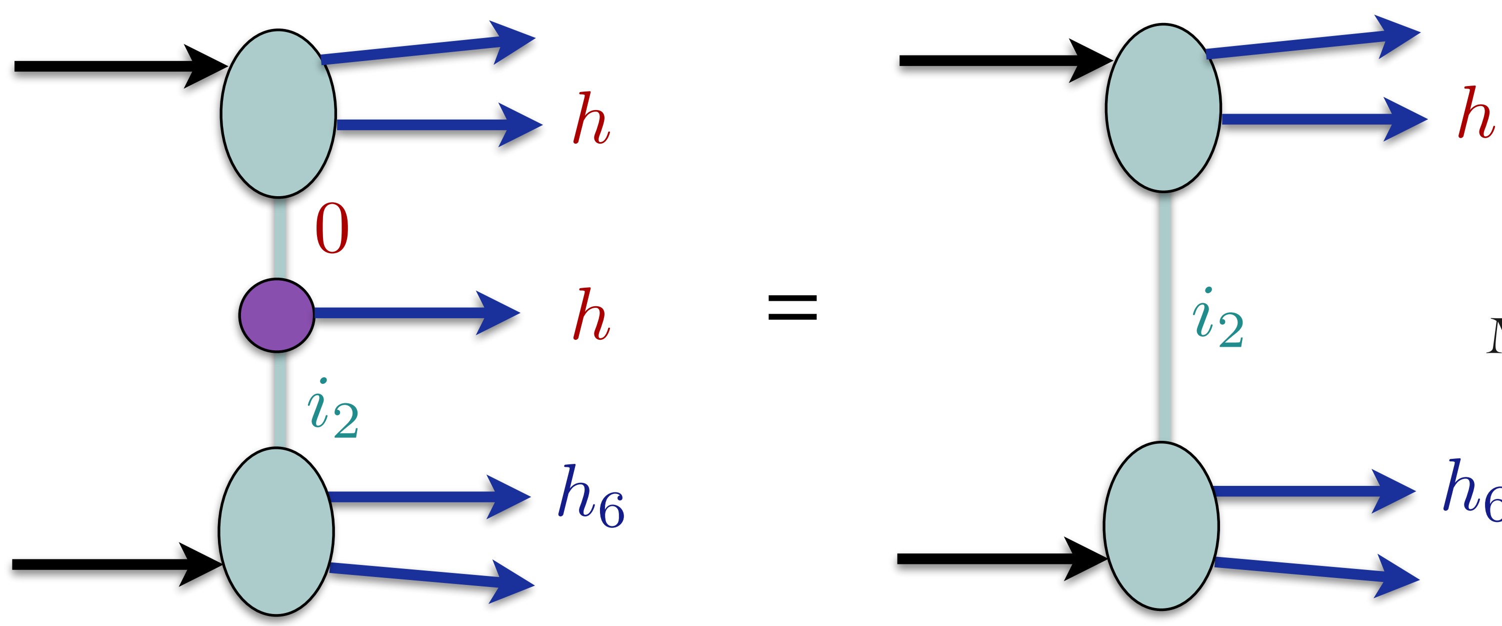
Factorisation

- Consequence 1: Convolutions imply a factorisation theorem!

$$g^{(i_1,i_2)}_{h_4 h_5 h_6} \sim$$



- Theorem:



[Del Duca, Druc, Drummond, CD ,Dulat, Marzucca, Papathanasiou, Verbeek; Bargheer]

➡ Implies relations between amplitudes with different numbers of legs.

Factorisation for MHV

- **Consequence:** At L loops an MHV amplitudes in MRK is determined by amplitudes with at most $L+4$ external legs.

- **Two loops, LLA:** Reduces to known factorisation:

$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\le i\le N-5} \log\tau_i \, g^{(1)}_{++}(\rho_i)$$

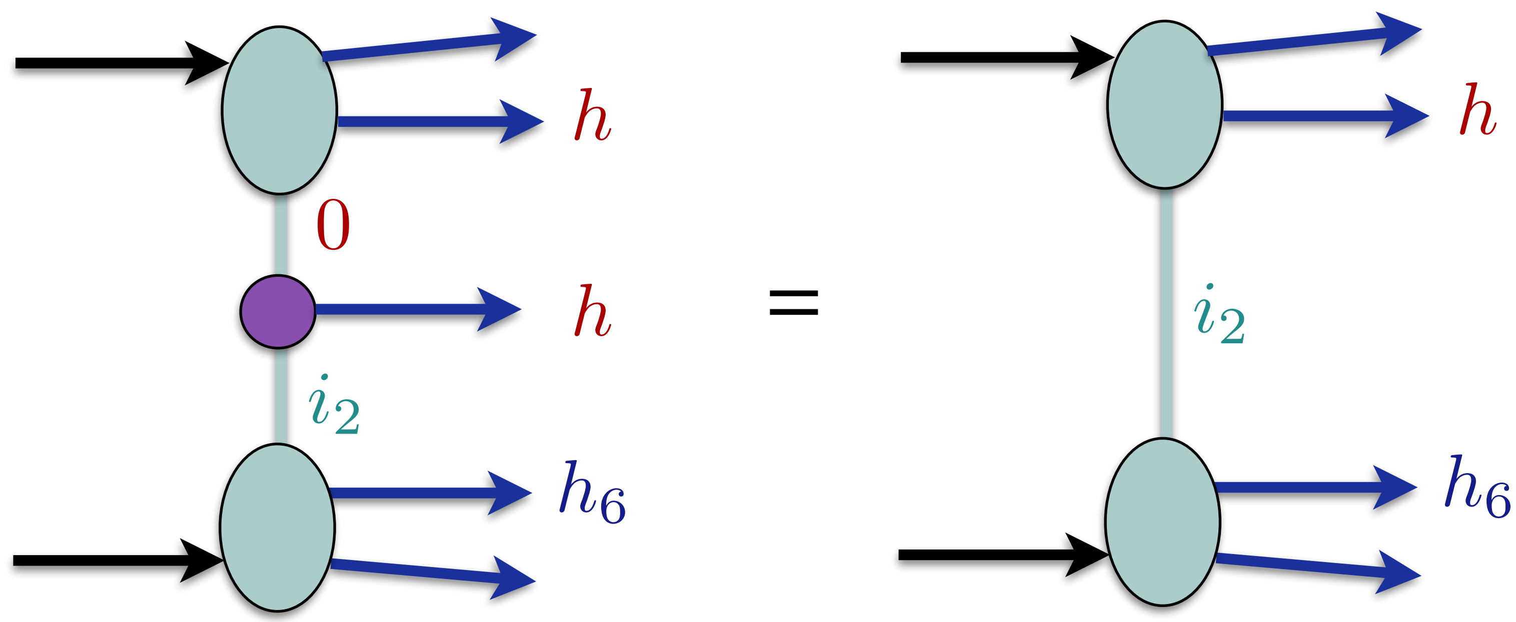
[Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]

- **Three loops, LLA:**

$$\mathcal{R}^{(3)}_{+\ldots+} = \frac{1}{2}\sum_{1\le i\le N-5} \log^2\tau_i \, g^{(2)}_{++}(\rho_i) + \sum_{1\le i<j\le N-5} \log\tau_i \, \log\tau_j \, g^{(1,1)}_{+++}(\rho_i,\rho_j)\,.$$

Factorisation for non-MHV

- Factorisation theorem still holds for non-MHV amplitudes.



➡ Unlike MHV: infinite number building blocks per loop.

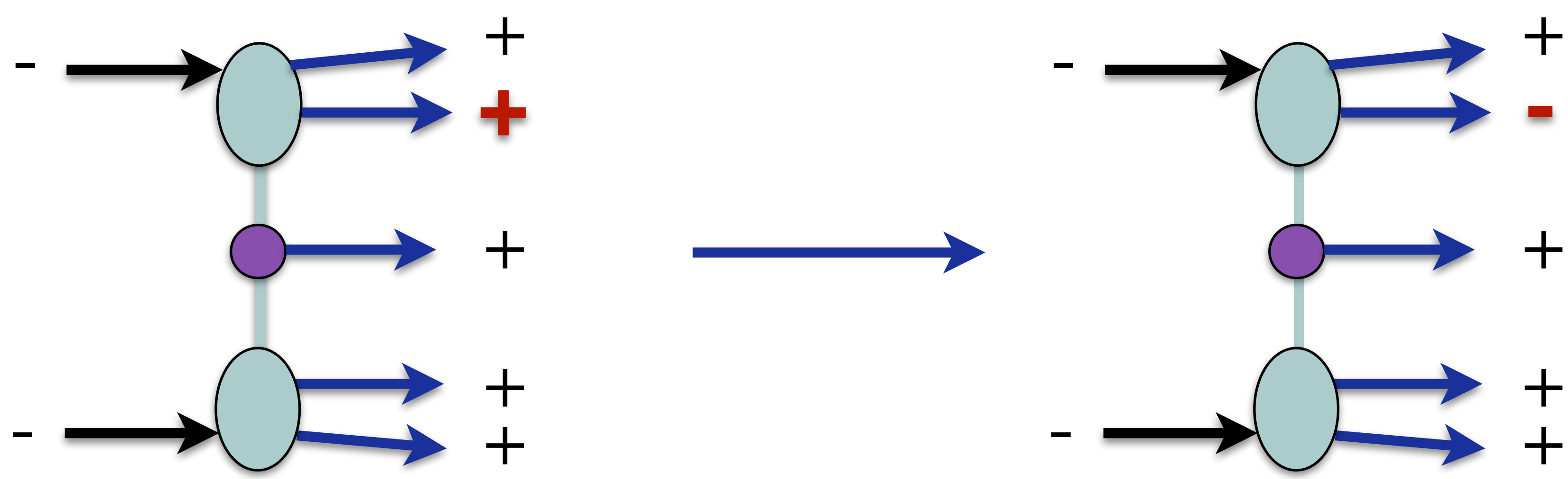
- Example:

$$\mathcal{R}^{(2)}_{-+\dots} = \log\tau_1\, g^{(1)}_{-+}(\rho_1) + \sum_{j=2}^{N-5} \log\tau_j\, g^{(0,1)}_{-++}(\rho_1,\rho_j)$$

$$\mathcal{R}^{(2)}_{+-+\dots} = \log\tau_1\, g^{(1,0)}_{+-+}(\rho_1,\rho_2) + \log\tau_2\, g^{(0,1)}_{+-+}(\rho_1,\rho_2) + \sum_{j=3}^{N-5} \log\tau_j\, g^{(0,0,1)}_{+-++}(\rho_1,\rho_2,\rho_j)$$

Helicity flips

- Consequence 2: Non-MHV amplitudes from MHV ones.



$$\mathcal{F}\left[\chi^{+}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

$$\mathcal{F}\left[\chi^{-}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

$$\sim \mathcal{F}\left[\chi^{-}/\chi^{+}\right] * \mathcal{F}\left[\chi^{+}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

- Helicity flip kernel: $\mathcal{F}\left[\chi^{-}/\chi^{+}\right] = -\dfrac{z}{(1-z)^2}$

- Helicity flips on central emission block are similar.

Transcendentality

- **Consequence 3:** Complete characterisation of the function space.

- **Integrability:** In perturbation theory, integrand is a polynomial in multiple zeta values and
  of uniform weight

$$E_{\nu n} \quad N(\nu, n) \quad V(\nu, n) \quad M(\nu_1, n_1, \nu_2, n_2) \quad F(\nu, n) \quad D_\nu$$ weight 1

- **Example:** NLO BFKL eigenvalue

$$E^{(1)}_{\nu,n} = -\frac{1}{4} D_\nu^2 E_{\nu,n} + \frac{1}{2} V\, D_\nu E_{\nu,n} - \zeta_2\, E_{\nu,n} - 3\,\zeta_3$$ weight 3

$$E^{(2)}_{\nu,n} = \frac{1}{8}\left\{\frac{1}{6} D_\nu^4 E_{\nu,n} - V\, D_\nu^3 E_{\nu,n} + (V^2 + 2\zeta_2)\, D_\nu^2 E_{\nu,n} - V\,(N^2 + 8\zeta_2) D_\nu E_{\nu,n} + \zeta_3 (4\,V^2 + N^2) + 44\zeta_4\, E_{\nu,n} + 16\zeta_2\zeta_3 + 80\zeta_5\right\},$$ weight 5

Transcendentality

- Theorem: If $\mathcal{A}(z)$ is a pure combination of SVMPLs of uniform weight n, then $\mathcal{A}(z) * \mathcal{F}[X]$, with $X \in \{E, V, N, M, F, D\}$, is a pure combination of SVMPLs of uniform weight $n+1$.

- All two-loop MHV amplitudes in MRK are known, e.g., at LLA:

$$\mathcal{R}^{(2)}_{+\dots+} = \sum_{1\le i\le N-5} \log \tau_i \, g^{(1)}_{++}(\rho_i)$$

[Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]

[NLLA: Bargheer, Papathanasiou, Schomerus; Del Duca, Druc, Drummond, CD, Dulat, Marzucca, Papathanasiou, Verbeek; Del Duca, CD, Dulat, Penante]

➡ They are all pure functions of uniform weight!

- We can recursively characterise the function space to all orders.

Transcendentality

Theorem: All amplitudes in MRK in planar N=4 SYM are combinations of uniform weight of SVMPLs, (single-valued) multi zeta values and powers of $2\pi i$.

In addition:

- MHV amplitudes are pure functions (no rational prefactors).
- Non-MHV amplitudes are not pure.

[Del Duca, Druc, Drummond, CD, Dulat, Marzucca, Papathanasiou, Verbeek, to appear]

- Why are MHV amplitudes not pure?

$$\mathcal{F}\left[\chi^{-}/\chi^{+}\right] * \mathcal{F}\left[\chi^{+}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right] \qquad \mathcal{F}\left[\chi^{-}/\chi^{+}\right] = -\frac{z}{(1-z)^2}$$

Requires residues at double pole ~ derivative

Conclusion & Outlook

- We have a solution of scattering amplitudes of planar N=4 SYM in MRK in this Mandelstam region to all orders in the coupling.
  - ➡ (Conjectural) all-order formula for BFKL eigenvalue, impact factor & central emission vertex.
  - ➡ Geometric picture of the kinematics.
  - ➡ Algorithmic way of doing all Fourier-Melling integrals
- Applications: (see Robin's talk)
  - ➡ Explicit results.
  - ➡ Complete description of MRK function space.