MULTILOOP CORRECTIONS TO TWO-PARTON SCATTERING AMPLITUDES IN THE REGGE LIMIT FROM BFKL EVOLUTION

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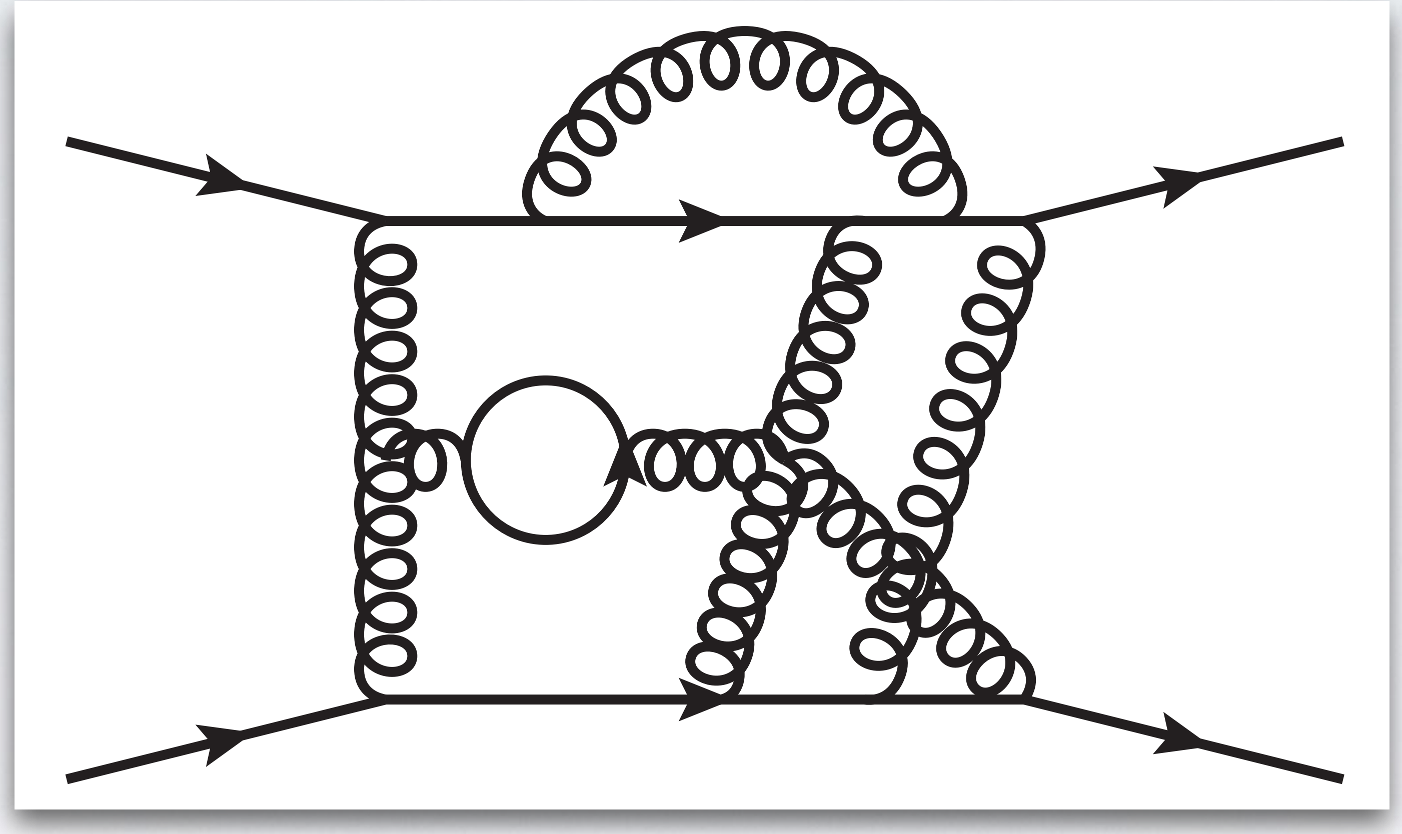
Towards accuracy at small x, Edinburgh, 12/09/2019

OUTLINE

- **Factorisation of amplitudes in the high-energy limit**
- **The two-Reggeon cut: scattering amplitudes by iterated solution of the BFKL equation**
- **Infrared singularities to all orders**
- **The finite wavefunction and amplitude**
  - ***In collaboration with***
    Simon Caron-Huot, Einan Gardi and Joscha Reichel
  - ***Based on***
    arXiv:1701.05241 (JHEP 1706 (2017) 016)
    arXiv:1711.04850 (JHEP 1803 (2018) 098),
    and in preparation

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY



HIGH-ENERGY LIMIT

- Very interesting theoretical problem:
  - retain rich dynamic in the 2d transverse plane,
  - toy model for full amplitude,
  - non-trivial function spaces,
  - predict amplitudes and other observables in overlapping limits:

    → soft limit, infrared divergences.

MRK in N=4 SYM: Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2016, …

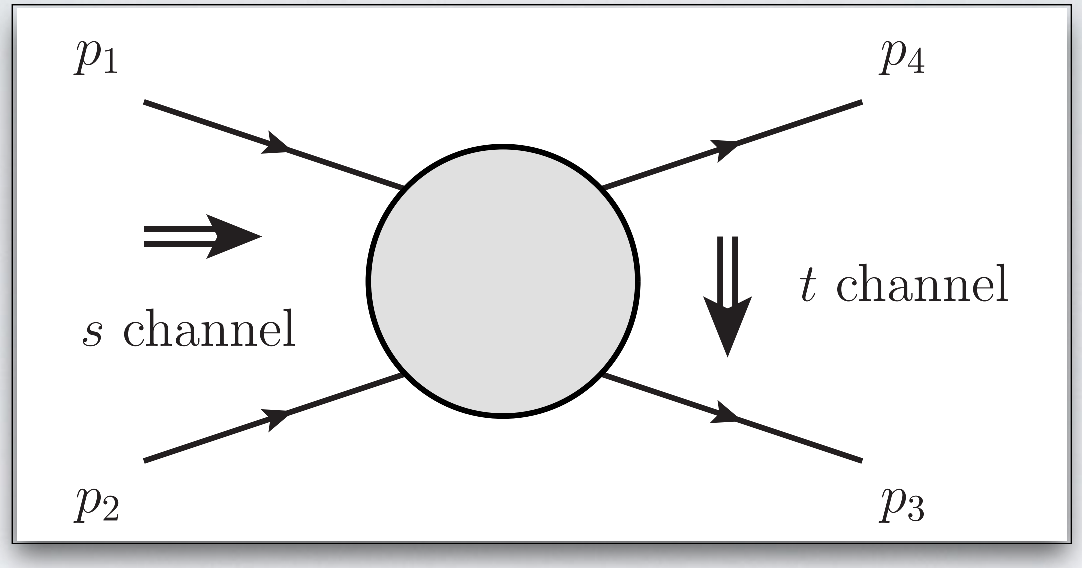
- Relevant for phenomenology at the LHC and future colliders:
  - perturbative phenomenology of forward scattering, e.g.

    → Deep inelastic scattering/saturation (small x = Regge, large Q^2 = perturbative),

    → Mueller-Navelet: $pp \to X + 2$jets, forward and backward.

See talks by **Andersen, Bonvini, Chirilli, Forshaw, Forte, Giani, Hautmann, Iancu, Mulian, Nagy, Newman, Wallon**

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



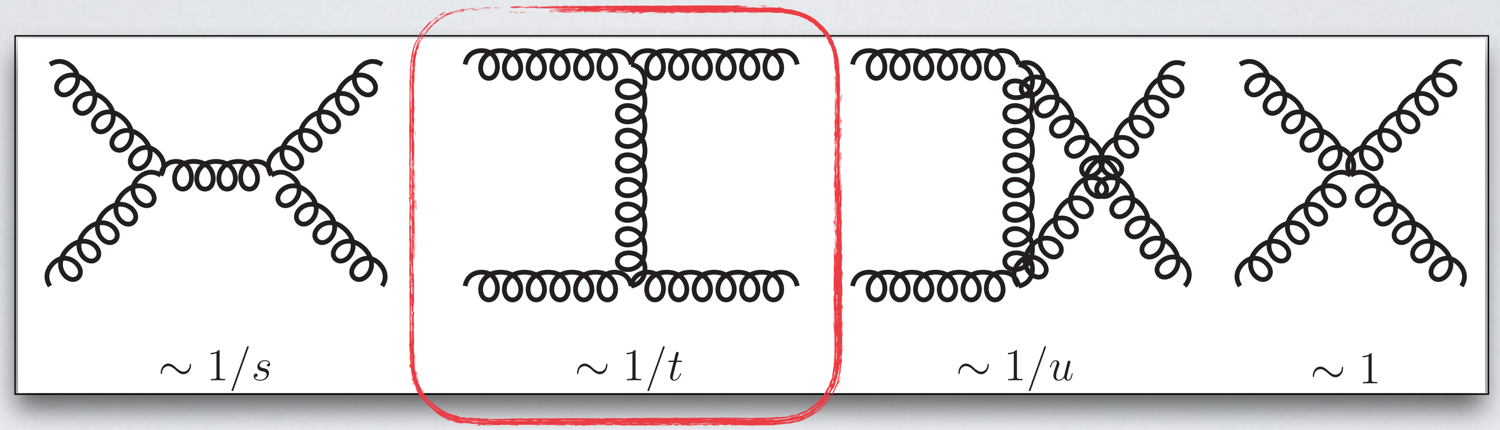
- Consider 2 → 2 scattering amplitudes in the high-energy limit:

$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0.$$

- The amplitude is expanded in the small ratio $|t/s|$; we consider here the leading power term:

$$\mathcal{M}_{ij\to ij}(s,t,\mu^2) = \frac{s}{t}\mathcal{M}^{[-1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \mathcal{M}^{[0]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \frac{t}{s}\mathcal{M}^{[1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \ldots$$

HIGH ENERGY LIMIT AT LL: REGGE POLES



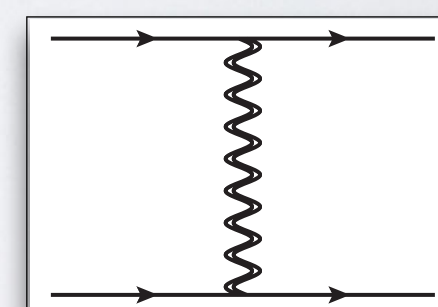
- At leading power, gluon exchanges in the t-channel:

$$\mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t}\,(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}\,.$$

- The amplitude contains logarithms of the ratio $|s/t|$.
  → Characterised in terms of Regge poles at LL:

Regge, Gribov

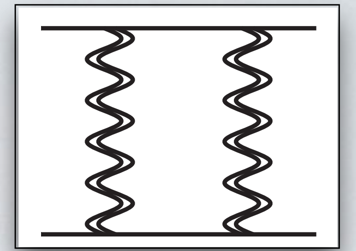
$$\mathcal{M}_{ij\to ij}|_{\rm LL} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}\,C_A\,\alpha_g^{(1)}(t)}\;4\pi\alpha_s\,\mathcal{M}^{(0)}_{ij\to ij},$$



- The function $\alpha_g(t)$ is known as the Regge trajectory:

$$\alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \stackrel{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon}, \qquad r_\Gamma = e^{\epsilon\gamma_{\rm E}}\,\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1-\frac{1}{2}\zeta_2\,\epsilon^2-\frac{7}{3}\zeta_3\,\epsilon^3+\ldots$$

BEYOND LL: REGGE CUTS



- Crossing symmetry $s \leftrightarrow u$:

  → project onto eigenstates of signature:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

  → Express amplitudes in terms of the signature-even combination of logs:

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

  → $M^{(+)}$ and $M^{(-)}$ are respectively imaginary and real.

- Color: beyond tree level

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\,\mathcal{M}^{[i]}(s,t).$$

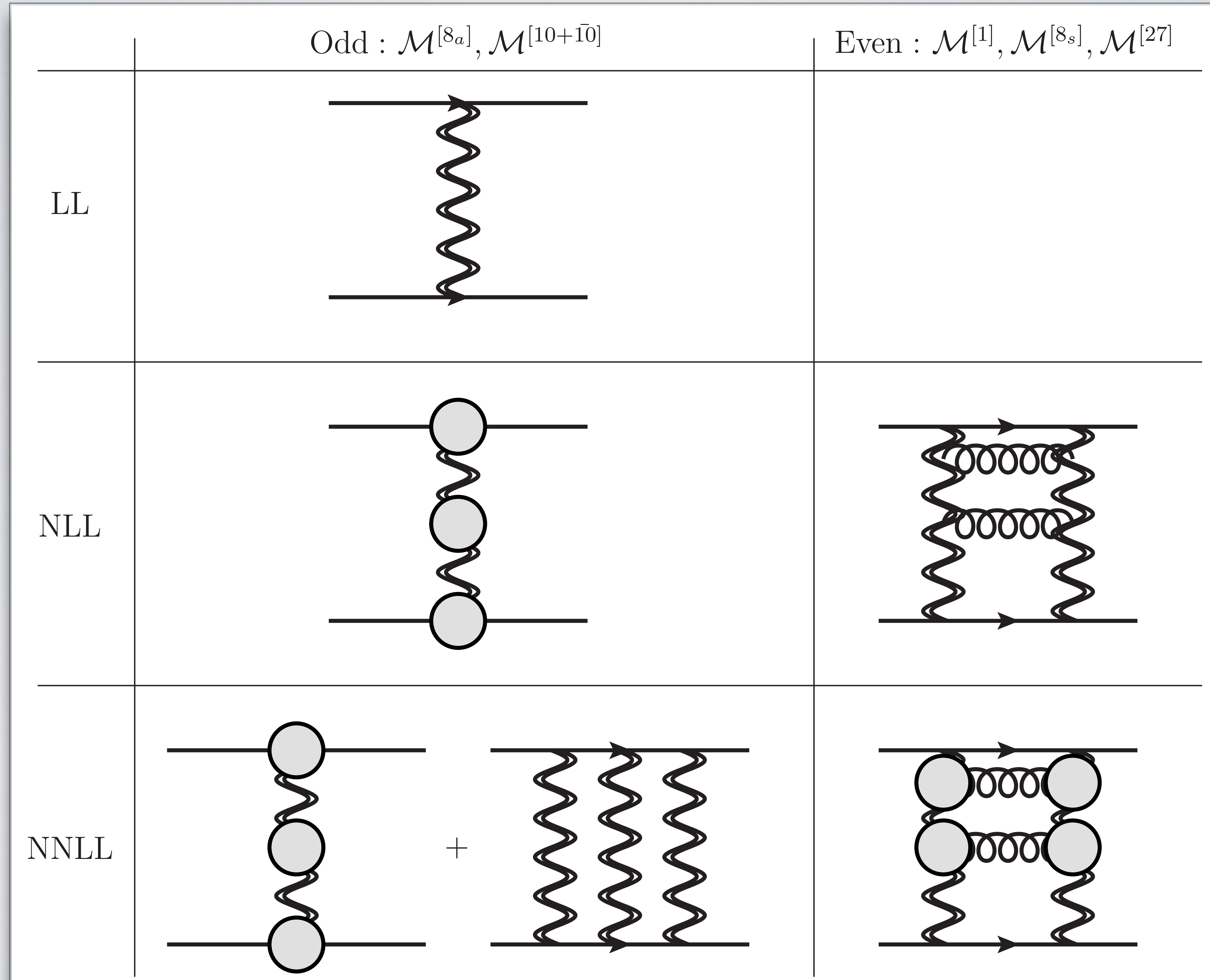
  → Decompose the amplitude in a color orthonormal basis in the t-channel

$$8 \otimes 8 = 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27$$

  → Invoking Bose symmetry we deduce (gg scattering)

$$\text{odd: } \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even: } \mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}.$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



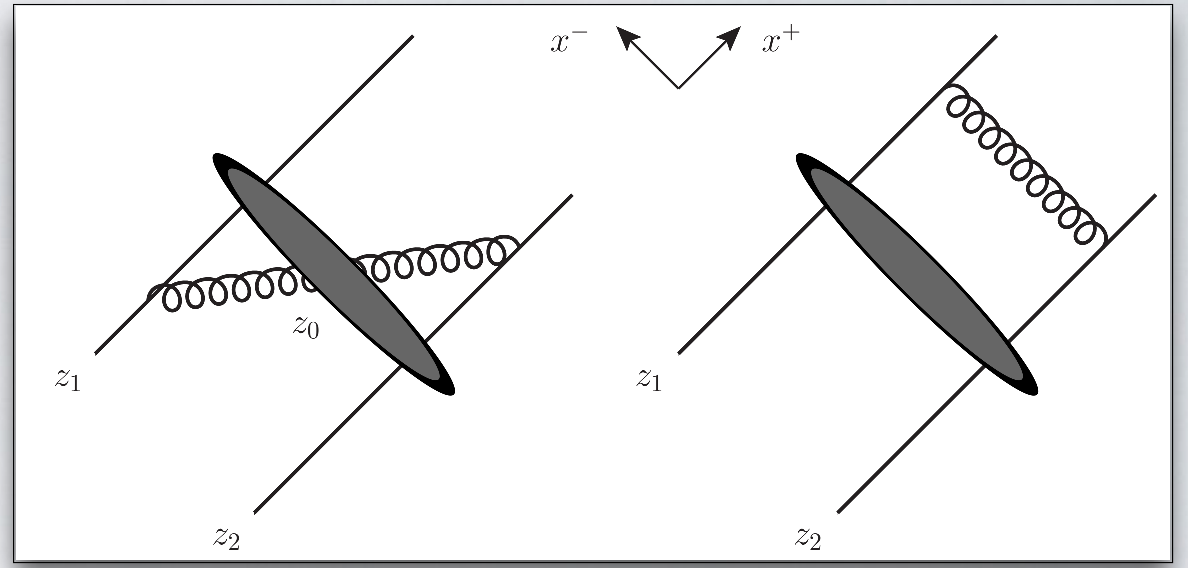
Caron-Huot, 2013; Del Duca, Falcioni, Magnea, LV, 2014, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- High-energy limit = forward scattering: to leading power, the fast projectile and target described in terms of Wilson lines:

$$U(z_\perp) = \mathcal{P}\exp\left[ig_s \int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+ T^a\right].$$

Korchemskaya, Korchemsky, 1994, 1996; Babansky, Balitsky, 2002, Caron-Huot, 2013



- The Wilson line stretches from $-\infty$ to $+\infty$ and thus develops rapidity divergencies. The regularised Wilson lines obeys the (non linear!) Balitsky-JIMWLK evolution equation:

$$-\frac{d}{d\eta}\Big[U(z_1)\ldots U(z_n)\Big] = \sum_{i,j=1}^{n} H_{ij}\cdot\Big[U(z_1)\ldots U(z_n)\Big],$$

with

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int [dz_i][dz_j][dz_0]K_{ij;0}\Big[T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\Big] + \mathcal{O}(\alpha_s^2).$$

- Evolution in rapidity resums the high-energy log:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

FROM BALITSKY-JIMWLK TO AMPLITUDES

- In perturbation theory the unitary matrices $U(z) \sim \mathbb{1}$: parametrize in terms of a field W

$$U(z) = e^{i g_s \, T^a W^a(z)} \, .$$

Kovner Lublinsky, 2005; Caron-Huot, 2013

- The color-adjoint field W sources a BFKL Reggeised gluon: a generic projectile is expanded at weak coupling as

$$|\psi_i\rangle \equiv g_s \, D_{i,1}(t) \, |W\rangle + g_s^2 \, D_{i,2}(t) \, |WW\rangle + g_s^3 \, D_{i,3}(t) \, |WWW\rangle + \ldots$$

- Focus on the Regge-cut contributions: define a "reduced" amplitude by removing the Reggeized gluon and collinear divergences

$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\, \mathbf{T}_t^2 \, \alpha_g(t) \, L} \, \mathcal{M}_{ij\to ij} \, ,$$

- Scattering amplitude: expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s} \hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s} \left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij} \right) \equiv \langle \psi_j^{(+)} | e^{-\hat{H} L} | \psi_i^{(+)} \rangle + \langle \psi_j^{(-)} | e^{-\hat{H} L} | \psi_i^{(-)} \rangle .$$

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- An $m \to m+k$ transition from the leading-order Balitsky-JIMWLK equation is proportional to g_s^{2l+k}. Thus for $k \geq 0$, all the interactions can be extracted from the leading-order equation.

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

$$H \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \cdots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \cdots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \cdots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \cdots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \cdots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix} .$$

LO BFKL kernel

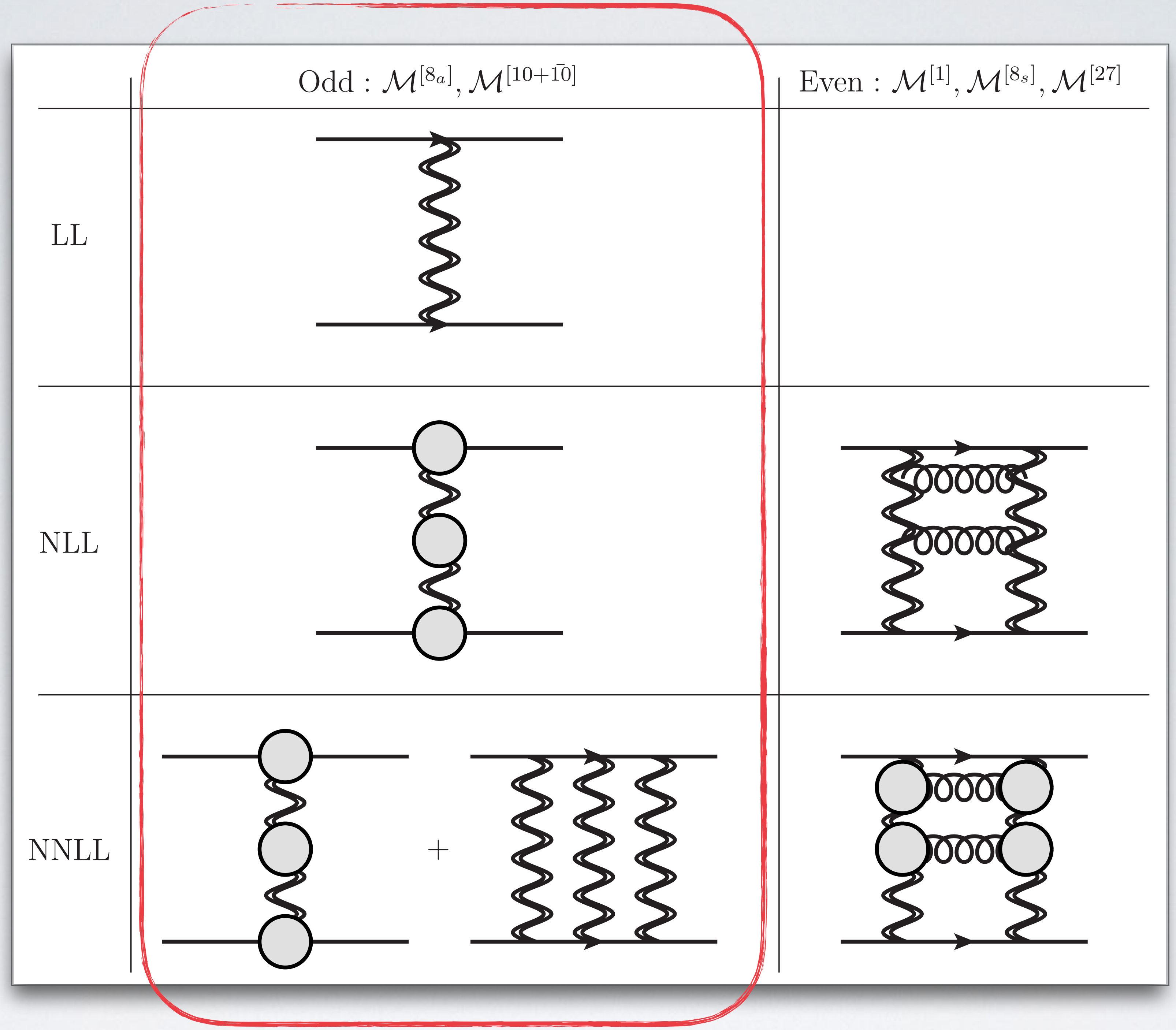
From LO B-JIMWLK

Terms in NNLO B-JIMWLK - predicted by symmetry $H = H^T$

- Interactions with $k < 0$ are suppressed by at least $g_s^{2l+|k|}$, which means that they can first appear in the $(|k|+l)$-loop Balitsky-JIMWLK Hamiltonian.
- At NLL we need $m \to m$ transition only $\to$ the LO BFKL kernel.

See Rothstein, Stewart 2016 for a SCET approach

THE ODD AMPLITUDE UP TO THREE LOOPS



Del Duca, Glover, 2001;

Del Duca, Falcioni, Magnea, LV, 2013, 2014,

Caron-Huot, Gardi, LV, 2017

THE ODD AMPLITUDE UP TO THREE LOOPS

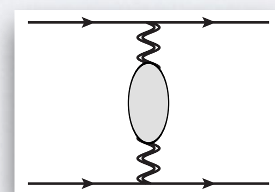
- Odd and even sectors are orthogonal and closed under the action of $\hat{H}$ (signature symmetry):

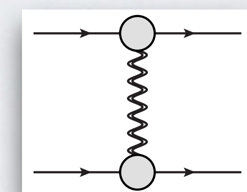
$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij}\right) \equiv \langle\psi_j^{(+)}|e^{-\hat{H}L}|\psi_i^{(+)}\rangle + \langle\psi_j^{(-)}|e^{-\hat{H}L}|\psi_i^{(-)}\rangle.$$

- The signature odd amplitude becomes to three loops:

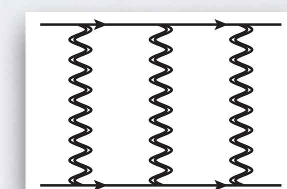
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{tree}}_{ij\to ij} = \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{LO})},$$

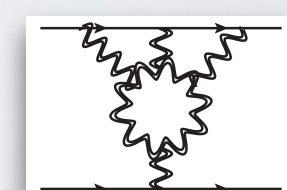
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = -L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NLO})},$$



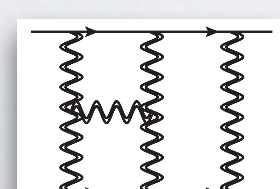


$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} = +\frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{LO})} - L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NLO})} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NNLO})},$$





$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{3-loops}}_{ij\to ij} = -\frac{1}{6}L^3\langle\psi_{j,1}|(\hat{H}_{1\to1})^3|\psi_{i,1}\rangle^{(\text{LO})} + \frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{NLO})} - L\Big\{\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NNLO})} + \Big[\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle + \langle\psi_{j,3}|\hat{H}_{1\to3}|\psi_{i,1}\rangle + \langle\psi_{j,1}|\hat{H}_{3\to1}|\psi_{i,3}\rangle\Big]^{(\text{LO})}\Big\} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{NLO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}.$$



THE ODD AMPLITUDE UP TO THREE LOOPS

- Needs *1→3* transitions:

$$H\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \cdots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \cdots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \cdots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix}$$

$$H_{k\to k+2} = \frac{\alpha_s^2}{3\pi}\int [dz_i][dz_0]\, K_{ii;0}\,(W_i-W_0)^x W_0^y (W_i-W_0)^z\, \mathrm{Tr}\big[F^x F^y F^z F^a\big]\frac{\delta}{\delta W_i^a}$$
$$+\frac{\alpha_s^2}{6\pi}\int [dz_i][dz_j][dz_0]\, K_{ij;0}\,(F^x F^y F^z F^t)^{ab}\Big[(W_i-W_0)^x W_0^y W_0^z (W_j-W_0)^t$$
$$- W_i^x (W_i-W_0)^y W_0^z (W_j-W_0)^t - (W_i-W_0)^x W_0^y (W_j-W_0)^z W_j^t\Big]\frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

Caron-Huot, Gardi, LV, 2017

THE ODD AMPLITUDE AT NNLL TO THREE LOOPS

- Up to three loops the amplitude reads

$$\hat{\mathcal{M}}^{(-,1)}_{ij\to ij} = \left(D_i^{(1)} + D_j^{(1)}\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij}\,,$$

$$\hat{\mathcal{M}}^{(-,2)}_{ij\to ij} = \left[D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)} + \pi^2 R^{(2)}\left((\mathbf{T}^2_{s-u})^2 - \tfrac{1}{12}(C_A)^2\right)\right]\hat{\mathcal{M}}^{(0)}_{ij\to ij}\,,$$

$$\hat{\mathcal{M}}^{(-,3)}_{ij\to ij} = \Big[D_i^{(3)} + D_j^{(3)} + D_i^{(2)}D_j^{(1)} + D_i^{(1)}D_j^{(2)}$$
$$+\,\pi^2\left(R_A^{(3)}\,\mathbf{T}^2_{s-u}[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}] + R_B^{(3)}\,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R_C^{(3)}\,(C_A)^3\right)\Big]\hat{\mathcal{M}}^{(0)}_{ij\to ij}\,,$$

with loop functions *R* (Regge cut-contribution)

$$R^{(2)} = (r_\Gamma)^2\left(-\tfrac{1}{8\epsilon^2} + \tfrac{3}{4}\epsilon\zeta_3 + \tfrac{9}{8}\epsilon^2\zeta_4 + \ldots\right)\,, \quad R_A^{(3)} = (r_\Gamma)^3\left(\tfrac{1}{48\epsilon^3} + \tfrac{37}{24}\zeta_3 + \ldots\right)\,,$$

$$R_B^{(3)} = (r_\Gamma)^3\left(\tfrac{1}{24\epsilon^3} + \tfrac{1}{12}\zeta_3 + \ldots\right)\,, \quad R_C^{(3)} = (r_\Gamma)^3\left(\tfrac{1}{864\epsilon^3} - \tfrac{35}{432}\zeta_3 + \ldots\right)\,.$$

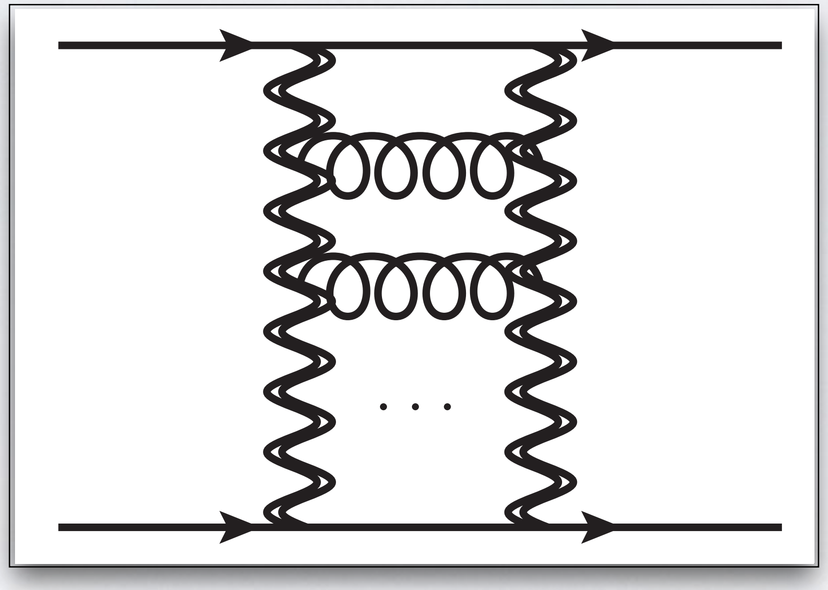
✓ Agreement with the quadrupole correction to the soft anomalous dimension

Almelid, Duhr, Gardi 2015, 2016;

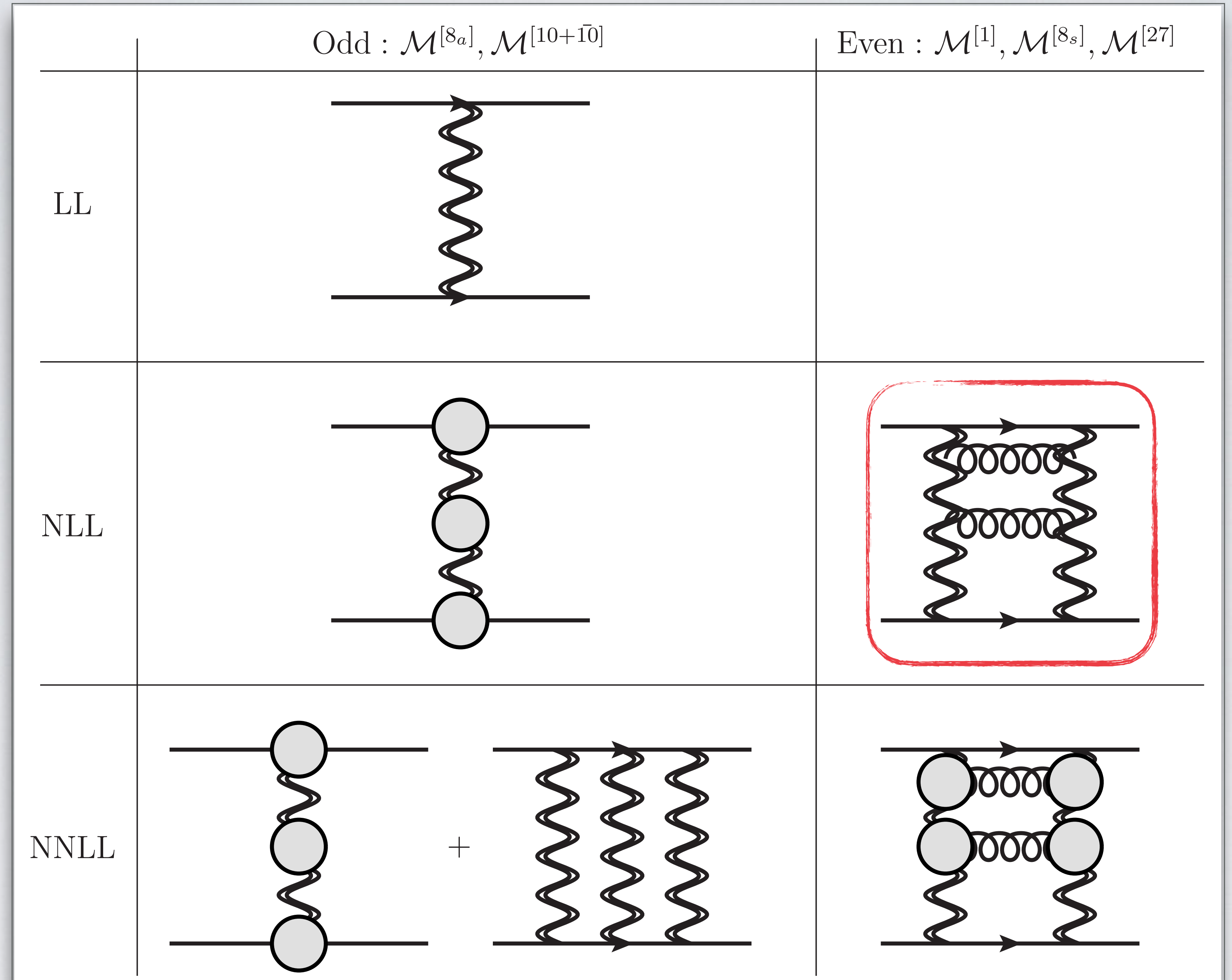
✓ Agreement with three-loop computation of gluon-gluon amplitude in N=4 SYM

Henn, Mistlberger, 2016

THE TWO REGGEON CUT



THE TWO-REGGEON CUT



$$\frac{i}{2s}\hat{\mathcal{M}}^{(+),\mathrm{NLL}}_{ij\to ij} \equiv \langle \psi^{(+)}_{j,2} | e^{-\hat{H}L} | \psi^{(+)}_{i,2} \rangle .$$

THE TWO-REGGEON CUT

- The BFKL equation for the even amplitude takes the form

$$\frac{d}{dL}\Omega(p,k) = \frac{\alpha_s B_0(\epsilon)}{\pi}\hat{H}\,\Omega(p,k)\,, \qquad B_0 = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- The BFKL kernel is non-trivial, and we do not know how to diagonalise it. However, we can derive an iterative (perturbative) solution: define

$$\Omega(p,k) = \sum_{\ell=1}^{\infty}\left(\frac{\alpha_s}{\pi}B_0\right)^{\ell}\frac{L^{\ell-1}}{(\ell-1)!}\,\Omega^{(\ell-1)}(p,k),$$

the "wavefunction" is defined order by order by iterating the BFKL kernel

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k),$$

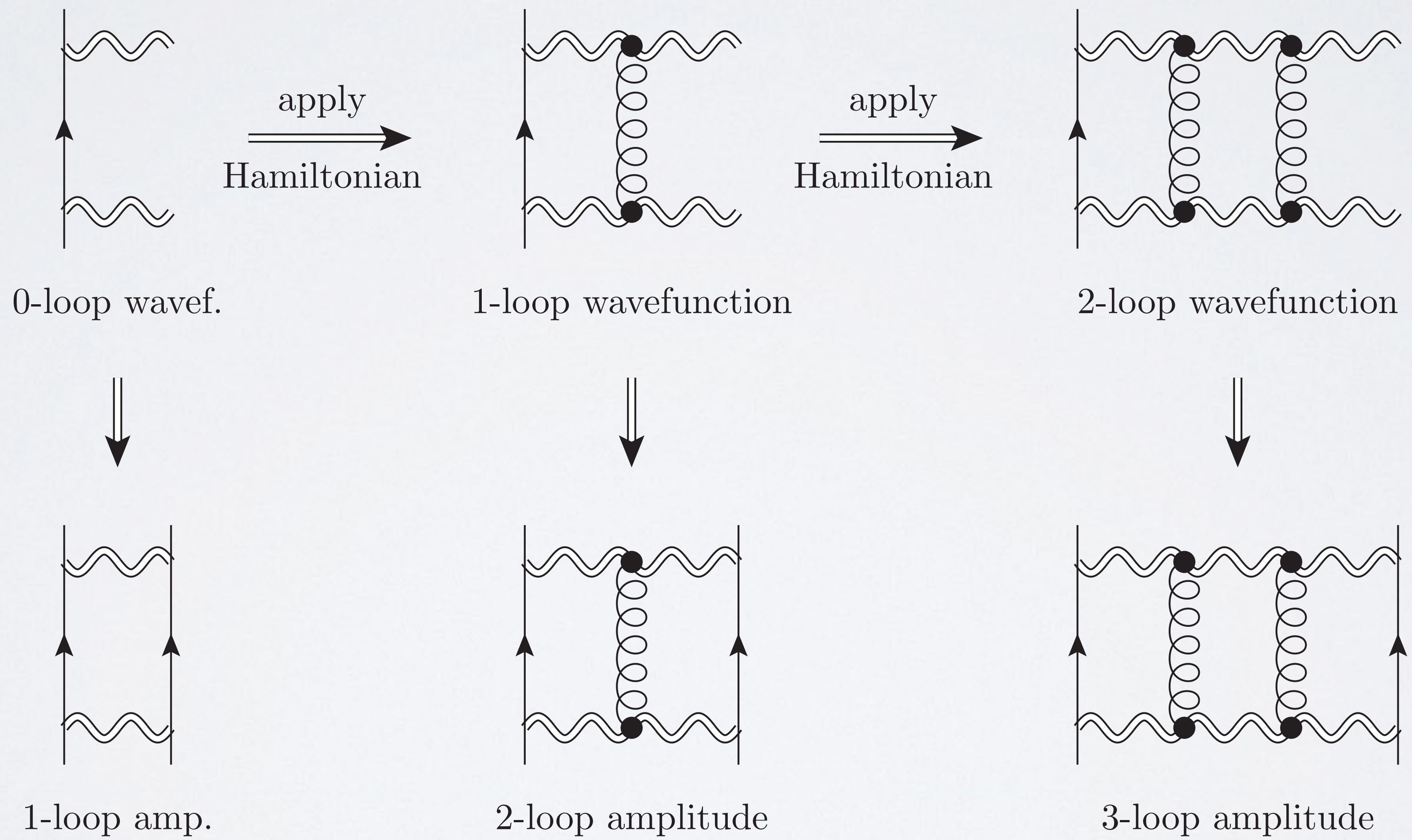
with initial condition

$$\Omega^{(0)}(p,k) = 1.$$

THE TWO-REGGEON CUT

- At each order the amplitude is:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [Dk]\, \frac{p^2}{k^2(k-p)^2}\, \Omega^{(\ell-1)}(p,k)\, \mathbf{T}^2_{s-u}\, \mathcal{M}^{(0)}, \qquad [Dk] = \frac{\pi}{B_0}\left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon} \frac{d^{2-2\epsilon}}{(2\pi)^{2-2\epsilon}}.$$



THE TWO-REGGEON CUT

- At each order the amplitude is:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [Dk]\, \frac{p^2}{k^2(k-p)^2}\, \Omega^{(\ell-1)}(p,k)\, \mathbf{T}^2_{s-u}\, \mathcal{M}^{(0)}.$$

- One "rung" = apply once the BFKL kernel on the "target averaged wave function":

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\, \Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\, \hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2)\, \hat{H}_{\rm m}$$

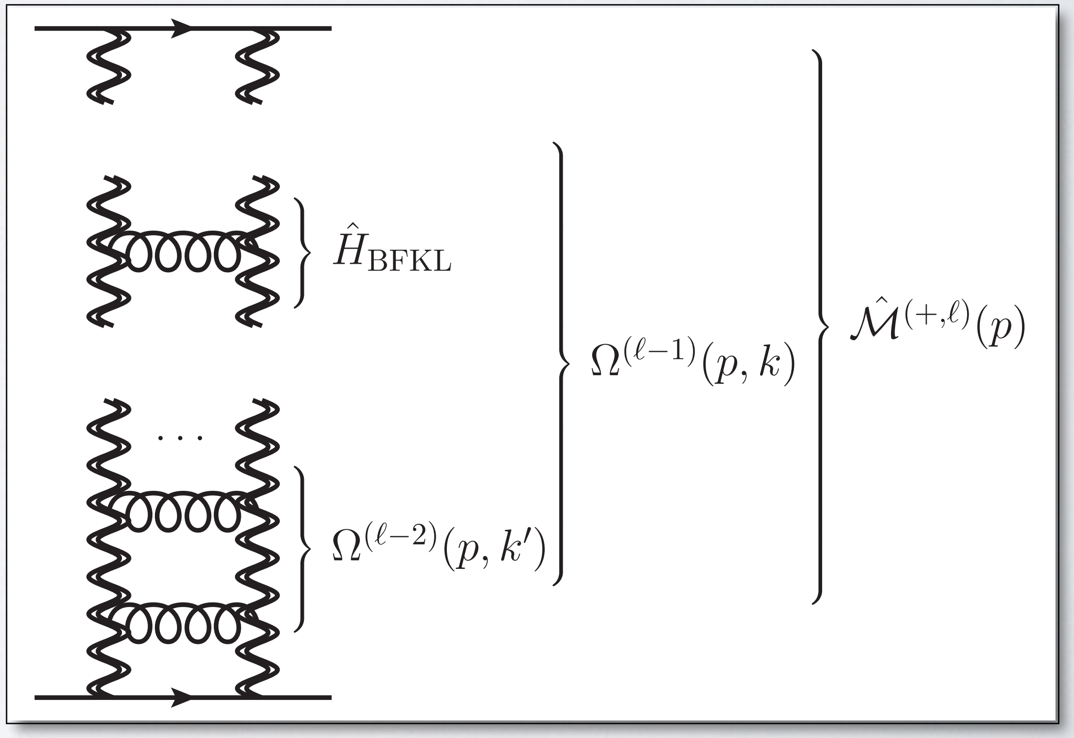
- "Integration" part:

$$\hat{H}_{\rm i}\, \Psi(p,k) = \int [{\rm D}k']\, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2}.$$

- "Multiplication" part:

$$\hat{H}_{\rm m}\Psi(p,k) = \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^\epsilon - \left(\frac{p^2}{(p-k)^2}\right)^\epsilon\right]\Psi(p,k).$$



THE TWO-REGGEON CUT

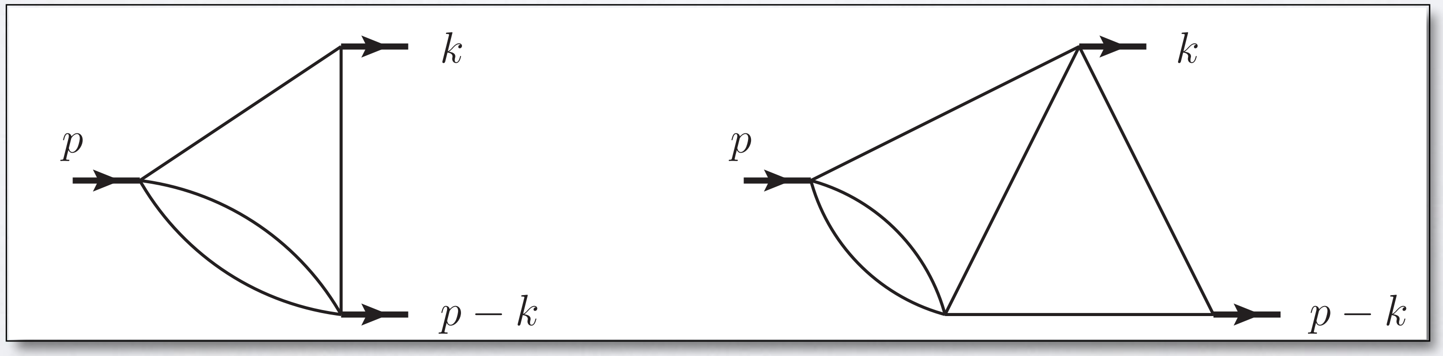
$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm m}$$

- First few orders:

$$\Omega^{(0)} = 1\,,$$
$$\Omega^{(1)} = (C_A - \mathbf{T}_t^2)\,J(p,k)\,,$$
$$\Omega^{(2)} = (C_A - \mathbf{T}_t^2)^2\,J^2(p,k) + (2C_A - \mathbf{T}_t^2)(C_A - \mathbf{T}_t^2)\int [Dk]\,f(p,k,k')\,\big[J(p,k') - J(p,k)\big]\,,$$

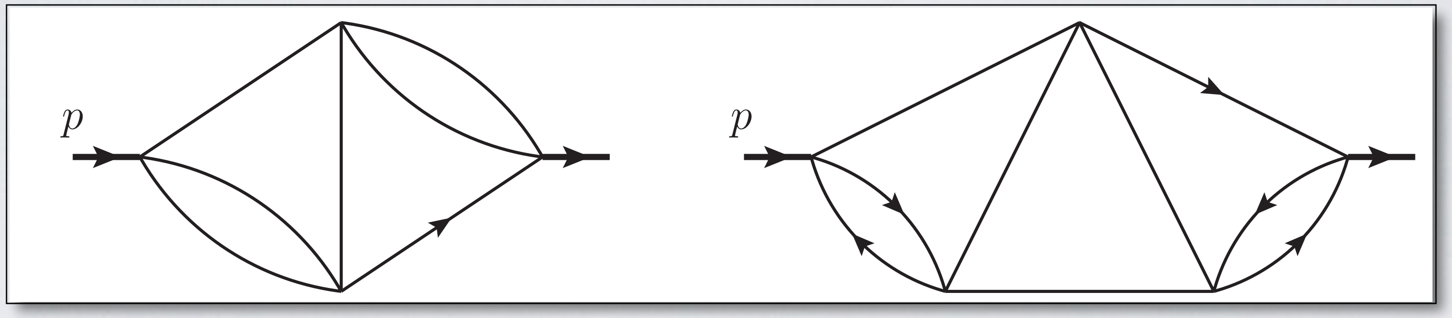
- At higher orders one gets increasingly difficult integrals.

$$\Omega^{(2)} \ni \int [Dk']\,\frac{(p-k)^2}{(p-k')^2(k-k')^2}\left(\frac{p^2}{(k')^2}\right)^{\epsilon},$$
$$\Omega^{(3)} \ni \int [Dk'][Dk'']\,\frac{k^2(p-k'')^2}{(k'')^2(p-k')^2(k-k'')^2(k'-k'')^2}\left(\frac{p^2}{(k')^2}\right)^{\epsilon}.$$



THE TWO-REGGEON CUT

- At the level of the amplitude these integrals corresponds to

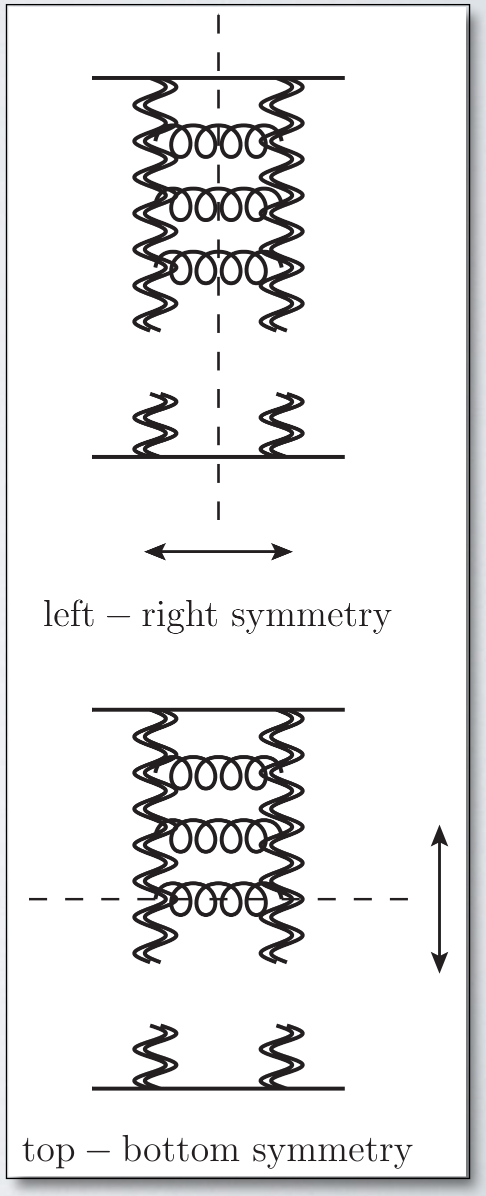


- We obtain the amplitude analytically to 4 loops, numerically (pySecDec) to 5 loops:

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \frac{B_0^4}{4!}\Big\{(C_A - \mathbf{T}_t^2)^3 \left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) + C_A(C_A - \mathbf{T}_t^2)^2 \left(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2)\right)\Big\}\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

Caron-Huot, 2013

$$\hat{\mathcal{M}}^{(+,5)}_{\rm NLL} = i\pi \frac{B_0^5}{5!}\Big\{(C_A - \mathbf{T}_t^2)^4 \left(\frac{1}{32\epsilon^5} - \frac{53\zeta_5}{2}\right) + C_A(C_A - \mathbf{T}_t^2)^3 \left(-\frac{\zeta_3}{16\epsilon^2} - \frac{3\zeta_4}{32\epsilon} + \frac{253\zeta_5}{16}\right) - \frac{5}{2}C_A^2(C_A - \mathbf{T}_t^2)^2\zeta_5\Big\}\mathbf{T}^2_{s-u}\mathcal{M}^{(0)}.$$



- General features:
  - → "top-bottom" and "left-right" ladder symmetry;
  - → outermost rungs are always easy (multiplication);
  - → first non-trivial integration at 4-loops.

TWO-REGGEON CUT: SOFT APPROXIMATION

- Observations:

  1) The wavefunction Ω(n)(p,k) is finite as $\epsilon \to 0$:

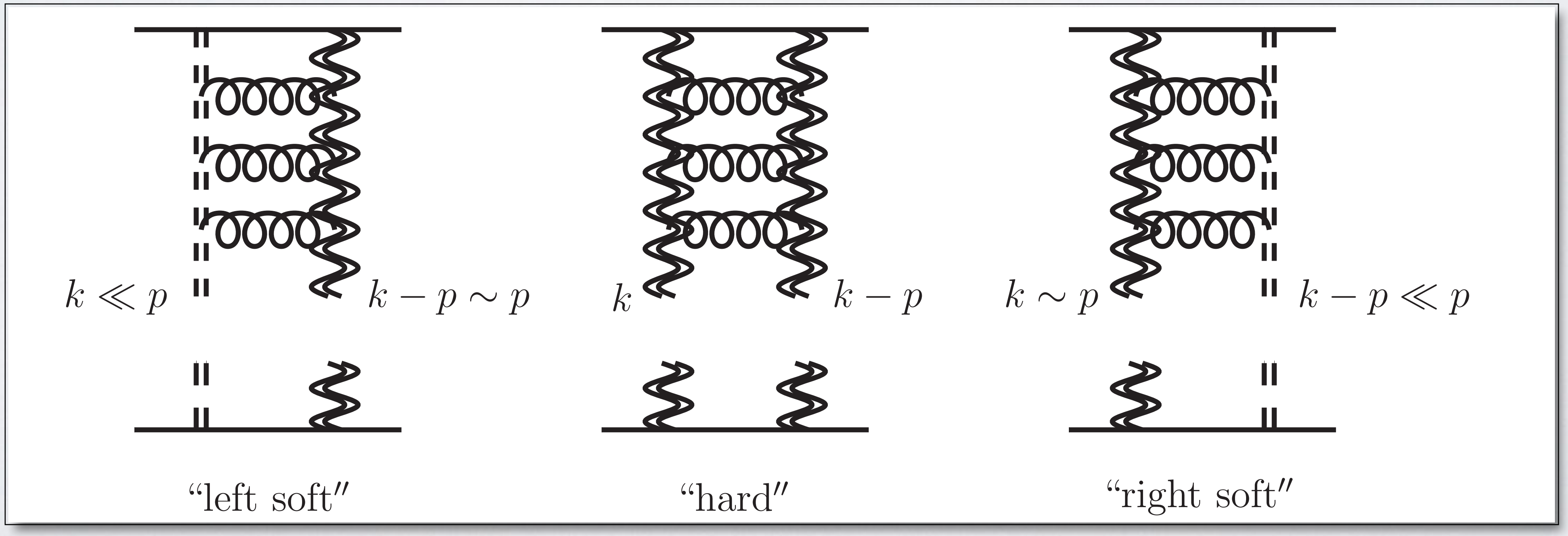
     → poles can only appear from final integration.

  2) Evolution closes in the soft limit:

$$\int_{k\to 0} \Omega^{(\ell)}(p,k).$$

  → IR divergences occur only when a full rail goes soft!

  → compute evolution in the (left) soft region and multiply by two.



TWO-REGGEON CUT: SOFT APPROXIMATION

- The soft wavefunction is polynomial in $(p^2/k^2)^\epsilon$:

Γ functions

$$\hat{H}_{\rm i}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon},$$

$$\hat{H}_{\rm m}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = \frac{1}{2\epsilon}\left[\left(\frac{p^2}{k^2}\right)^{n\epsilon} - \left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}\right].$$

- This allows to obtain the wavefunction at order *(ℓ-1):*

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p}{k}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 - \hat{B}_m(\epsilon)\frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)}\right\},$$

with

$$\hat{B}_n(\epsilon) \equiv 1 - \frac{B_n(\epsilon)}{B_0(\epsilon)} = 2n(2+n)\zeta_3\epsilon^3 + 3n(2+n)\zeta_4\epsilon^4 + \ldots.$$

Caron-Huot, Gardi, Reichel, LV, 2017

2-REGGEON CUT: SOFT APPROXIMATION

- It is easy to compute the amplitude at order ℓ, accurate to $O(\epsilon^{-l})$:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} (1+\hat{B}_{-1}) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n}$$

$$\times \prod_{m=0}^{n-2} \left[1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2} \right] \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

- The result is highly constrained, the wavefunction has to be finite!

  → The amplitude reduces to a geometric series, to $O(\epsilon^{-l})$:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL, s_{simpl.}} = \frac{i\pi}{(2\epsilon)^\ell} \frac{(B_0(\epsilon))^\ell}{\ell!} (C_A - \mathbf{T}_t^2)^{\ell-1} \frac{B_{-1}(\epsilon)}{B_0(\epsilon)} \left(1 - \hat{B}_{-1}(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)} \right)^{-1} \mathbf{T}_{s-u}^2 M^{(0)}.$$

In this form, the amplitude can be summed into a closed form expression:

$$\hat{\mathcal{M}}_{\rm NLL, s_{simpl.}} = \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \left\{ \left(e^{\frac{B_0}{2\epsilon}(C_A - \mathbf{T}_t^2)x} - 1 \right) \frac{B_{-1}(\epsilon)}{B_0(\epsilon)} \left(1 - \hat{B}_{-1}(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)} \right)^{-1} \right\} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)},$$

where $x = L\, \alpha_s/\pi$.

Caron-Huot, Gardi, Reichel, LV, 2017

TWO REGGEON CUT: SOFT APPROXIMATION

- A few orders:

Caron-Huot, Gardi, Reichel, LV, 2017

$$\bar{\mathcal{M}}^{(+,1)}_{\rm NLL}|_s = i\pi \left[\frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,2)}_{\rm NLL}|_s = i\pi \frac{C_A - \mathbf{T}_t^2}{2!} \left[\frac{1}{(2\epsilon)^2} + \mathcal{O}(\epsilon^0)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,3)}_{\rm NLL}|_s = i\pi \frac{(C_A - \mathbf{T}_t^2)^2}{3!} \left[\frac{1}{(2\epsilon)^3} + \mathcal{O}(\epsilon^0)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

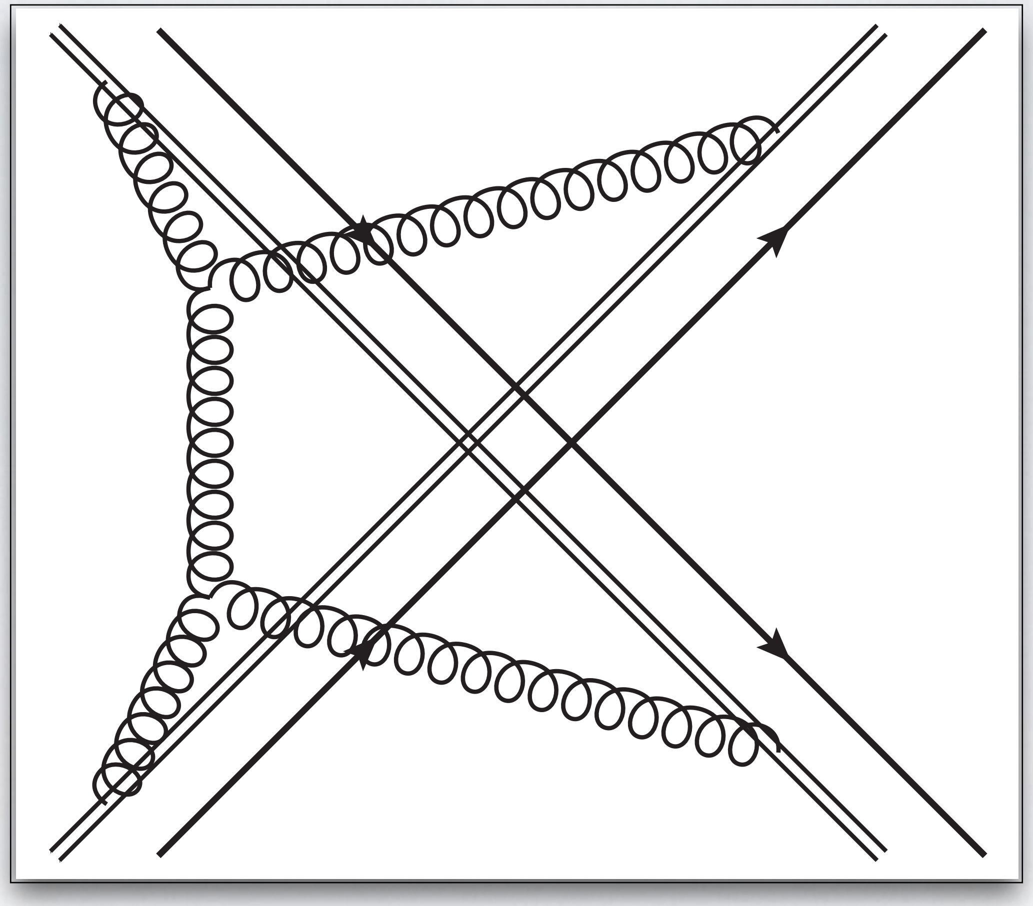
$$\bar{\mathcal{M}}^{(+,4)}_{\rm NLL}|_s = i\pi \frac{(C_A - \mathbf{T}_t^2)^3}{4!} \left[\frac{1}{(2\epsilon)^4} - \frac{1}{2\epsilon}\frac{\zeta_3 C_A}{4(C_A - \mathbf{T}_t^2)} + \mathcal{O}(\epsilon^0)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,5)}_{\rm NLL}|_s = i\pi \frac{(C_A - \mathbf{T}_t^2)^4}{5!} \left[\frac{1}{(2\epsilon)^5} - \frac{1}{(2\epsilon)^2} - \frac{1}{2\epsilon}\frac{3\zeta_4 C_A}{16(C_A - \mathbf{T}_t^2)} + \mathcal{O}(\epsilon^0)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(0)}.$$

iteration of lower loops

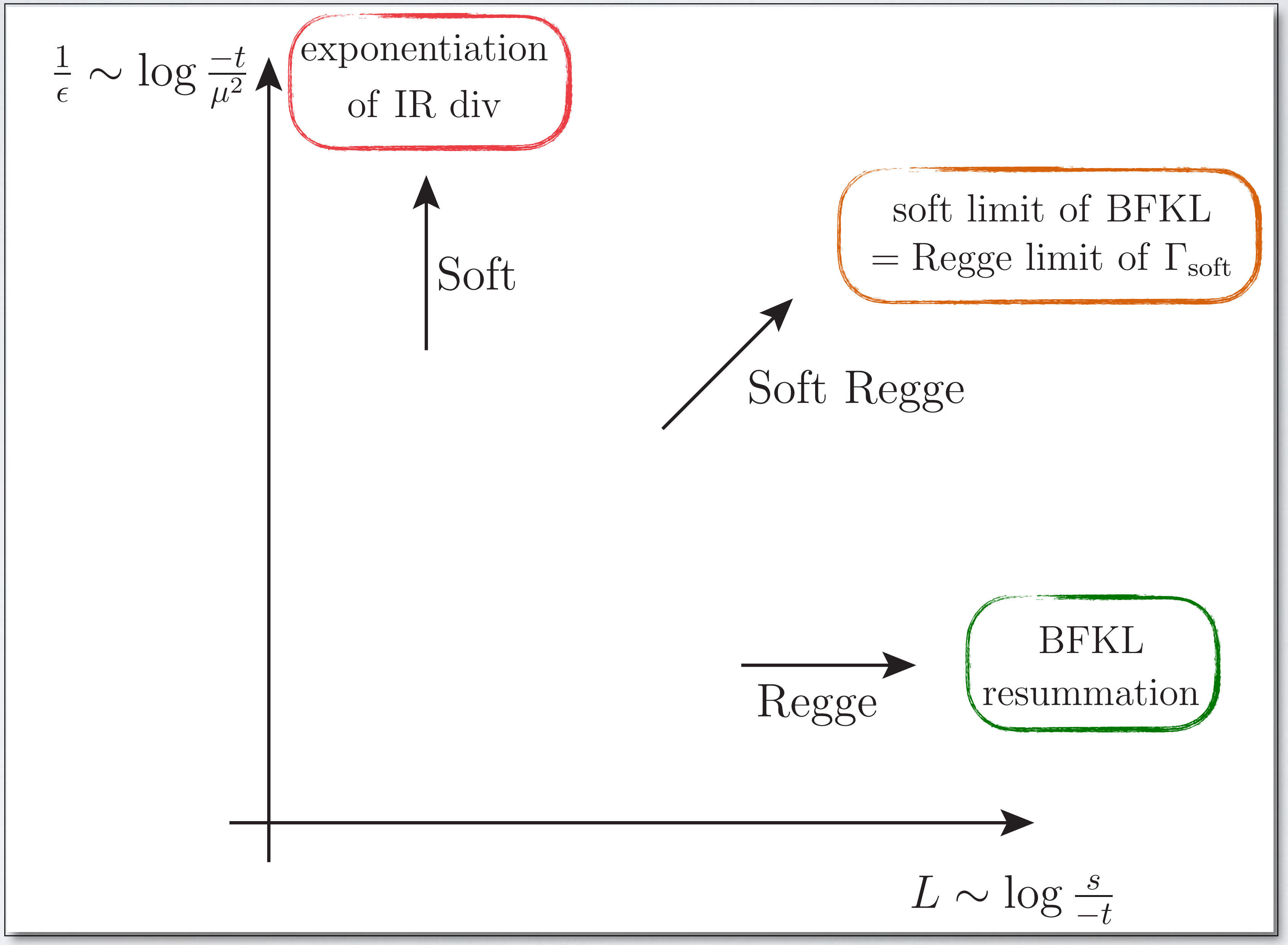
determines the soft anomalous dimension

2-REGGEON CUT: INFRARED SINGULARITIES



REGGE VS INFRARED FACTORISATION

- $2 \rightarrow 2$ kinematic limits:



- Application: test (and predict) the analytic structure of infrared divergences.

REGGE VS INFRARED FACTORISATION

- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\mathbf{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

Becher, Neubert, 2009; Gardi, Magnea, 2009

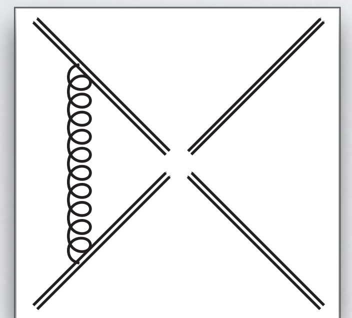
$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

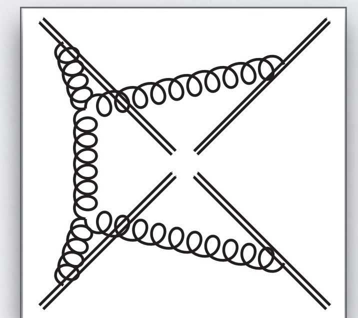
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\rm dip.}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

REGGE VS INFRARED FACTORISATION

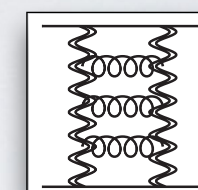
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$



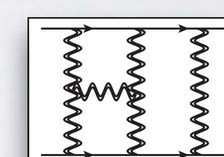
"dipole formula"



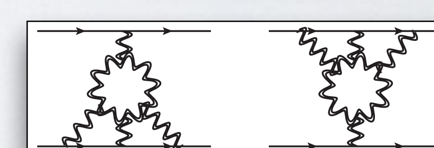
"quadrupole correction"



- Early studies of constraints from soft-collinear factorisation, collinear limits, and the high-energy limit in Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012;
- First evidence of "beyond dipole" contribution at four loops in Caron-Huot, 2013;



- Calculated a three loops in Almelid, Duhr, Gardi, 2015, 2016;
- Confirmed, in 2 ➔ 2 scattering in N=4 SYM in Henn, Mistlberger, 2016;
- Confirmed, in the high energy limit, in Caron-Huot, Gardi, LV, 2017;



- Re-derived based on a bootstrap approach in Almelid, Duhr, Gardi, McLeod, White, 2017.

2-REGGEON CUT: INFRARED SINGULARITIES

- Expand the soft anomalous dimension in the high-energy logarithm:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s(\lambda), L\right) + \ldots .$$

- At LL gluon Reggeization fixes $\mathbf{\Gamma}_{\rm LL}$ from gluon trajectory:

$$\mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda)\right) = \frac{\alpha_s(\lambda)}{\pi}\frac{\gamma_K^{(1)}}{2} L\,\mathbf{T}_t^2 = \frac{\alpha_s(\lambda)}{\pi} L\,\mathbf{T}_t^2 .$$

- At NLL

$$\mathbf{\Gamma}_{\rm NLL} = \mathbf{\Gamma}^{(+)}_{\rm NLL} + \mathbf{\Gamma}^{(-)}_{\rm NLL},$$

- with

$$\mathbf{\Gamma}^{(+)}_{\rm NLL} = \frac{\alpha_s(\lambda)}{\pi}\sum_{i=1}^{2}\left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right) + \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\,\mathbf{T}_t^2,$$

$$\mathbf{\Gamma}^{(-)}_{\rm NLL} = i\pi\frac{\alpha_s(\lambda)}{\pi}\,\mathbf{T}^2_{s-u} + O(\alpha_s^4 L^3) .$$

Del Duca, Duhr, Gardi, Magnea, White, 2011

2-REGGEON CUT: INFRARED SINGULARITIES

- Derive an Infrared-factorised representation of the reduced amplitude: start from

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \exp\left\{ -\frac{\alpha_s(\mu)}{\pi}\frac{B_0(\epsilon)}{2\epsilon}\, L\mathbf{T}_t^2\right\} \Big[\mathbf{Z}^{(-)}_{\rm NLL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(-)}_{\rm LL}\left(\{p_i\},\mu,\alpha_s(\mu)\right)$$

$$+\, \mathbf{Z}^{(+)}_{\rm LL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(+)}_{\rm NLL}\left(\{p_i\},\mu,\alpha_s(\mu)\right)\Big],$$

No poles

- we obtain

$$\exp\left\{\frac{1-B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t^2)\right\}\hat{\mathcal{M}}_{\rm NLL}$$

$$= -\int_0^p \frac{d\lambda}{\lambda}\exp\left\{\frac{1}{2\epsilon}\frac{\alpha_s(p)}{\pi}L(C_A-\mathbf{T}_t^2)\left[1-\left(\frac{p}{\lambda}\right)^{\epsilon}\right]\right\}\boldsymbol{\Gamma}^{(-)}_{\rm NLL}\left(\alpha_s(\lambda)\right)\mathcal{M}^{(\rm tree)} + \mathcal{O}(\epsilon^0).$$

- By matching we get the soft anomalous dimension to all orders:

$$\boldsymbol{\Gamma}^{(-,\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!}\left(1-R\left(\frac{x}{2}(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\Bigg|_{x^{\ell-1}}\mathbf{T}^2_{s-u}\,,$$

with

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2+10\zeta_6\right)\epsilon^6 + \ldots.$$

Caron-Huot, Gardi, Reichel, LV, 2017

2-REGGEON CUT: INFRARED SINGULARITIES

- Explicitly, for the first few orders we have:

$$\mathbf{\Gamma}^{(-,1)}_{\rm NLL} = i\pi\, \mathbf{T}^2_{s-u}, \qquad \mathbf{\Gamma}^{(-,2)}_{\rm NLL} = 0, \qquad \mathbf{\Gamma}^{(-,3)}_{\rm NLL} = 0,$$

$$\mathbf{\Gamma}^{(-,4)}_{\rm NLL} = -i\pi\, \frac{\zeta_3}{24}\, C_A (C_A - \mathbf{T}^2_t)^2\, \mathbf{T}^2_{s-u},$$

$$\mathbf{\Gamma}^{(-,5)}_{\rm NLL} = -i\pi\, \frac{\zeta_4}{128}\, C_A (C_A - \mathbf{T}^2_t)^3\, \mathbf{T}^2_{s-u},$$

$$\mathbf{\Gamma}^{(-,6)}_{\rm NLL} = -i\pi\, \frac{\zeta_5}{640}\, C_A (C_A - \mathbf{T}^2_t)^4\, \mathbf{T}^2_{s-u},$$

$$\mathbf{\Gamma}^{(-,7)}_{\rm NLL} = i\pi \frac{1}{720}\left[\frac{\zeta_3^2}{16}\, C_A^2 (C_A - \mathbf{T}^2_t)^4 + \frac{1}{32}\left(\zeta_3^2 - 5\zeta_6\right) C_A (C_A - \mathbf{T}^2_t)^5\right] \mathbf{T}^2_{s-u},$$

$$\mathbf{\Gamma}^{(-,8)}_{\rm NLL} = i\pi \frac{1}{5040}\left[\frac{3\zeta_3\zeta_4}{32}\, C_A^2 (C_A - \mathbf{T}^2_t)^5 + \frac{3}{64}\left(\zeta_3\zeta_4 - 3\zeta_7\right) C_A (C_A - \mathbf{T}^2_t)^6\right] \mathbf{T}^2_{s-u}.$$

Caron-Huot, Gardi, Reichel, LV, 2017

- The result can be used as constraint in a bootstrap approach to the soft anomalous dimension.

→

See e.g. **Almelid, Duhr, Gardi, McLeod, White, 2017**

2-REGGEON CUT: INFRARED SINGULARITIES

- Write the soft anomalous dimension as a function $x = L\,\alpha_s/\pi$:

$$\mathbf{\Gamma}^{(-)}_{\rm NLL} = i\pi \frac{\alpha_s}{\pi}\, G\left(\frac{\alpha_s}{\pi} L\right) \mathbf{T}^2_{s-u}\,, \qquad G(x) = \sum_{\ell=1}^{\infty} x^{\ell-1} G^{(\ell)}\,.$$

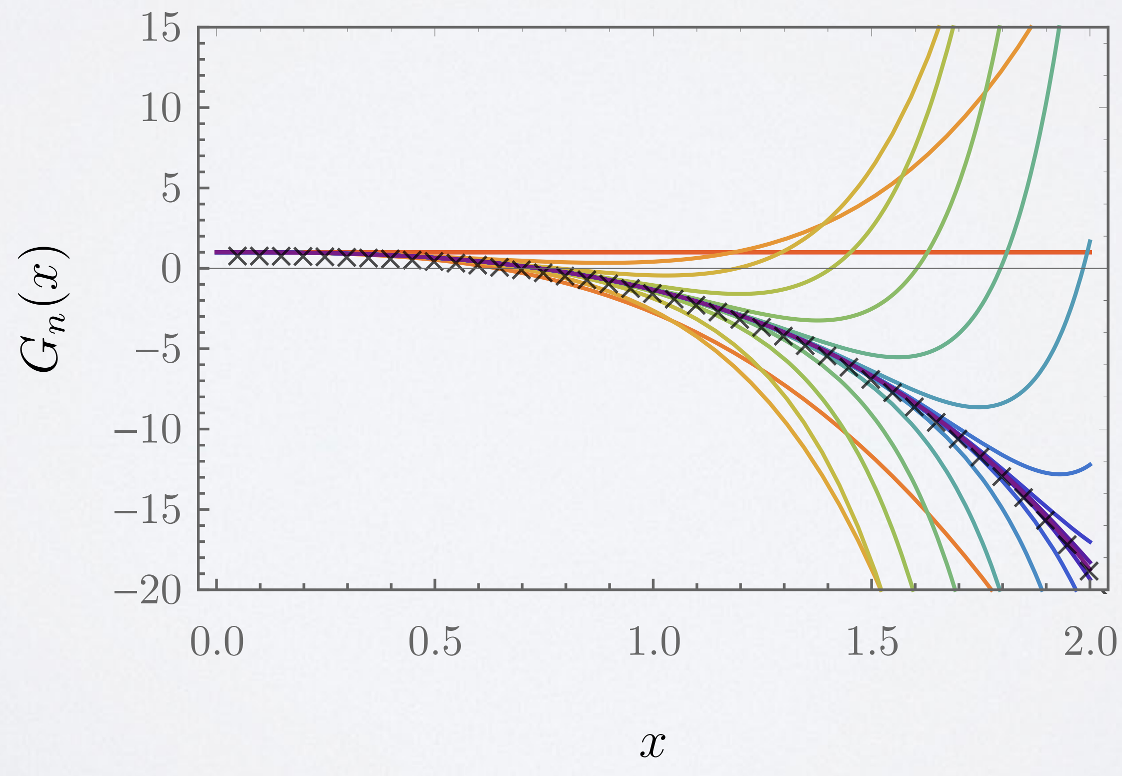
- Write $G(x)$ as the Borel transform of some function $g(1/\eta)$:

$$G(x) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} d\eta\, g\left(\frac{1}{\eta}\right) {\rm e}^{\eta x}\,,$$

$g(1/\eta)$ has isolated singularities away from the origin:

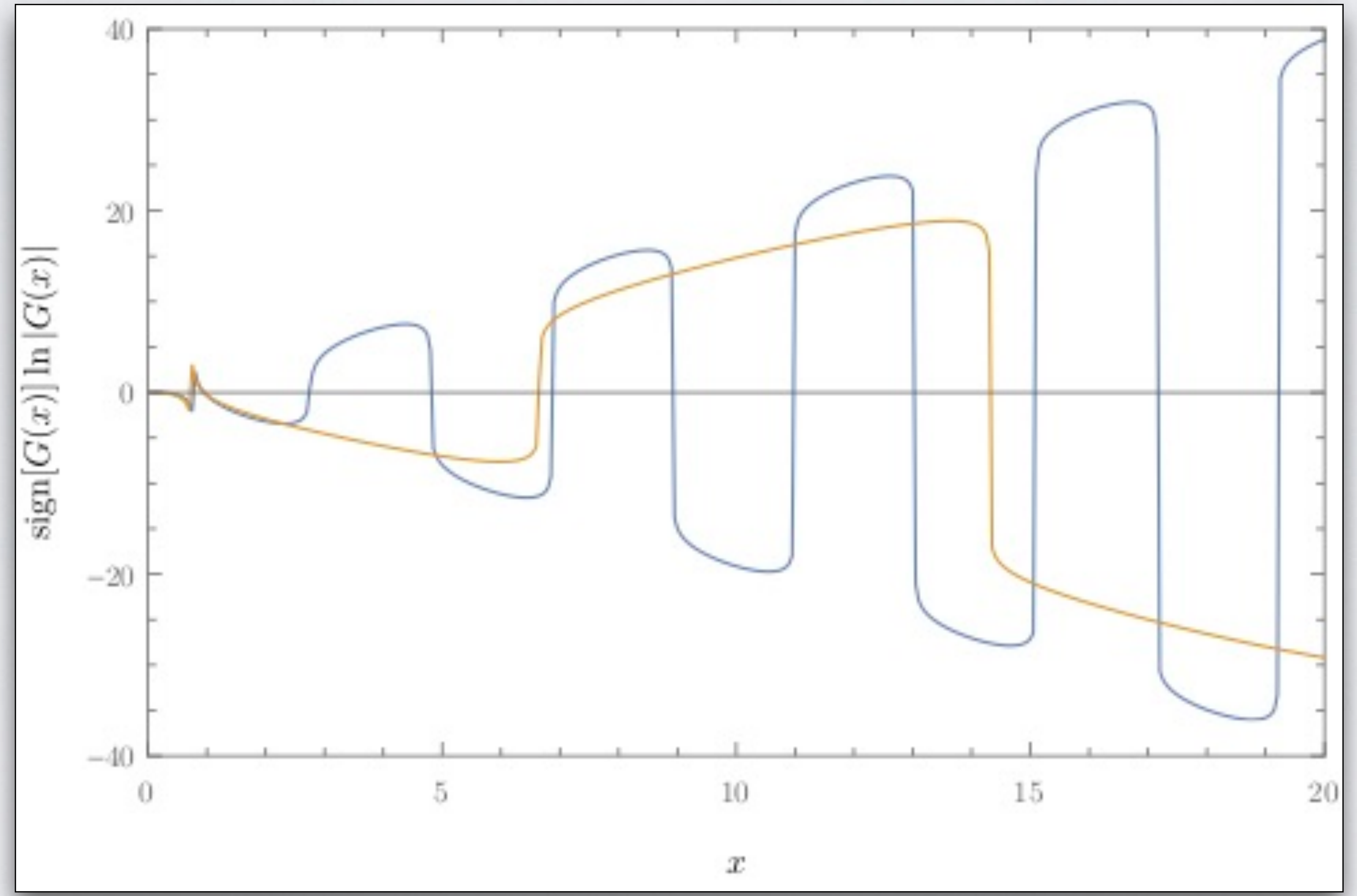
→ $G(x)$ has an infinite radius of convergence.

→ it is an entire function: valid to $x = L\,\alpha_s/\pi \gg 1$, i.e. at asymptotically high energies.



2-REGGEON CUT: INFRARED SINGULARITIES

- Plotting *G(x)* for larger values of *x* reveals oscillations with a constant period and an exponentially growing amplitude.
- Here we plot the logarithm of *|G(x)|* weighted by the sign of *G(x)*:



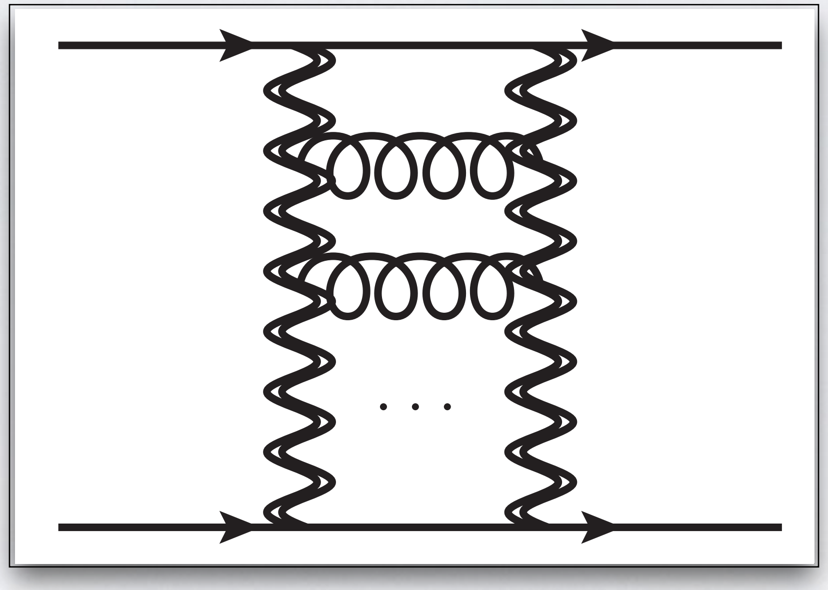
Caron-Huot, Gardi, Reichel, LV, 2017

- The function is well approximated by

$$G(x) \to c\, e^{ax} \cos(bx + d) \ ,$$

	a	b	c	d
1	1.97	1.52	0.25	0.48
27	1.46	0.41	0.58	2.01

FINITE WAVEFUNCTION AND AMPLITUDE



FINITE WAVEFUNCTION AND AMPLITUDE

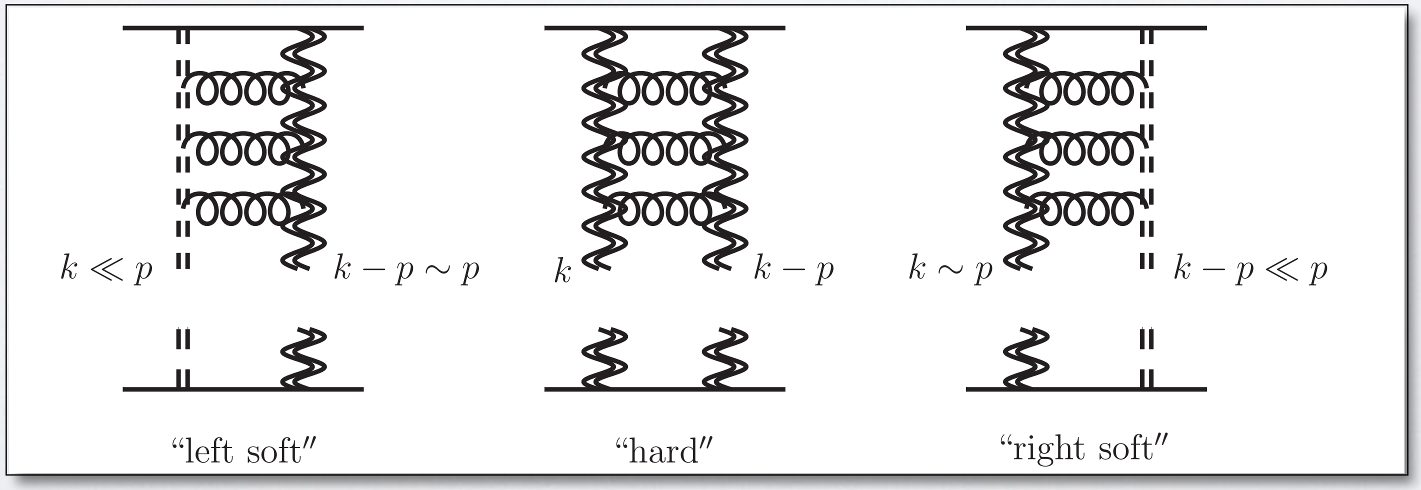
- What about the $O(\epsilon^0)$ contribution?
  - → the soft wavefunction generates all IR singularities in the amplitude:
  - → split the wavefunction into soft and hard, use dimensional regularisation for the soft only:

$$\Omega(p,k) = \Omega_s(p,k) + \Omega_h(p,k)\,,$$

$$\Omega_h^{(2\mathrm{d})}(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\Omega_h = \Omega_{2\mathrm{d}}(z,\bar{z}) - \Omega_s^{(2\mathrm{d})}(z,\bar{z}),$$

- The amplitude then is given by

$$\mathcal{M}^{(+)}_{\mathrm{NLL}}\left(\frac{s}{-t}\right) = -i\pi\Bigg[\underbrace{\int [Dk]\frac{p^2}{k^2(p-k)^2}\Omega_s(p,k)}_{\substack{\text{computable using soft limit}\\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\frac{1}{4\pi}\int\frac{d^2z}{z\bar{z}}\Omega_h^{(2\mathrm{d})}(z,\bar{z})}_{\text{compute in } D=2}\Bigg]\mathbf{T}^2_{s-u}M^{(0)},$$



FINITE WAVEFUNCTION AND AMPLITUDE

- Let us consider the soft contribution first. Recall that we found

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p}{k}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 - \hat{B}_m(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)}\right\},$$

this gives the amplitude

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} (1 + \hat{B}_{-1}) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n}$$

$$\times \prod_{m=0}^{n-2} \left[1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right] \mathbf{T}_{s-u}^2 \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

Caron-Huot, Gardi, Reichel, LV, in preparation

- This expression cannot be summed easily. However, we can modify the soft wavefunction such that it preserve the original symmetry $k \leftrightarrow (p-k)$:

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p}{k}\right)^{n\epsilon} \left(\frac{p^2}{(p-k)^2}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 - \hat{B}_m(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)}\right\}.$$

- With the symmetric wavefunction we obtain an amplitude that can be summed to all orders:

$$\hat{\mathcal{M}}_{\rm NLL,s} = \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \Bigg\{ \left(e^{\frac{B_0}{2\epsilon}(C_A - \mathbf{T}_t^2)x} - 1\right) \frac{B_{-1}(\epsilon)}{B_0(\epsilon)} \left(1 - \hat{B}_{-1}(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)}\right)^{-1} + 1$$

$$- e^{-\gamma_E (2C_A - \mathbf{T}_t^2)\, x} \frac{\Gamma\Big(1 - (C_A - \mathbf{T}_t^2)x\Big)}{\Gamma\Big(1 + (C_A - \mathbf{T}_t^2)x\Big)} \left(\frac{\Gamma\Big(1 + (C_A - \mathbf{T}_t^2)\frac{x}{2}\Big)}{\Gamma\Big(1 - (C_A - \mathbf{T}_t^2)\frac{x}{2}\Big)}\right)^{-\frac{\mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}} \Bigg\} \mathbf{T}_{s-u}^2 \, M^{(0)}.$$

WAVEFUNCTION IN D=2

- Consider now the determination of

$$\Omega_h^{(2\mathrm{d})}(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\Omega_h = \Omega_{2\mathrm{d}}(z,\bar{z}) - \Omega_s^{(2\mathrm{d})}(z,\bar{z}).$$

Introduce complex variables

$$\frac{k}{p} = \frac{z}{z-1}, \qquad \frac{k'}{p} = \frac{w}{w-1}.$$

- BFKL kernel in d=2:

$$\hat{H}_{2\mathrm{d}} = (2C_A - \mathbf{T}_t^2)\hat{H}_{2\mathrm{d,i}} + (C_A - \mathbf{T}_t^2)\hat{H}_{2\mathrm{d,m}}$$

- "Integration" part:

$$\hat{H}_{2\mathrm{d,i}} = \frac{1}{4\pi}\int d^2w\, K(w,\bar{w},z,\bar{z})\Big[\Psi(w,\bar{w}) - \Psi(z,\bar{z})\Big],$$

$$K(w,\bar{w},z,\bar{z}) = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}.$$

- "Multiplication" part:

$$\hat{H}_{2\mathrm{d,m}} = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right]\Psi(z,\bar{z}).$$

WAVEFUNCTION IN D=2

- Translate the action of the BFKL kernel into a set of differential equations, thanks to

$$z\frac{d}{dz}\left[\hat{H}_{\rm 2d,i}\Psi(z,\bar{z})\right] = \hat{H}_{\rm 2d,i}\left[z\frac{d}{dz}\Psi(z,\bar{z})\right].$$

- The full algorithm requires to take care of contact terms,

$$\partial_z\partial_{\bar{z}}\,\log(z\bar{z}) = \pi\,\delta^2(z),$$

and to consider the action of (1-z)d/dz as well.

Brown, 2004, 2013, Schnetz, 2013

- Th 2d wavefunction is computed in terms of single-valued harmonic polylogarithms (SVHPLs): we determine the action of the BFKL kernel on SVHPLs in terms of a set of DEs:

Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2016, …

$$\frac{\rm d}{{\rm d}z}\hat{H}_{\rm 2d,i}\mathcal{L}_{0,\sigma}(z,\bar{z}) = \frac{\hat{H}_{\rm 2d,i}\mathcal{L}_{\sigma}(z,\bar{z})}{z},$$

$$\frac{\rm d}{{\rm d}z}\hat{H}_{\rm 2d,i}\mathcal{L}_{1,\sigma}(z,\bar{z}) = \frac{\hat{H}_{\rm 2d,i}\mathcal{L}_{\sigma}(z,\bar{z})}{1-z} - \frac{1}{4}\frac{\mathcal{L}_{1,\sigma}(z,\bar{z})}{z} - \frac{1}{4}\frac{\mathcal{L}_{0,\sigma}(z,\bar{z}) + 2\mathcal{L}_{1,\sigma}(z,\bar{z}) - [\mathcal{L}_{0,\sigma}(w,\bar{w}) + \mathcal{L}_{1,\sigma}(w,\bar{w})]_{w,\bar{w}\to\infty}}{1-z}.$$

WAVEFUNCTION IN D=2

- An algorithm is set up to iteratively determine the wavefunction to any loop order. In practice, we stop at 13 loops. The first few orders:

$$\Omega^{(1)}_{2\mathrm{d}} = \frac{1}{2} C_2 \left(\mathcal{L}_0 + 2\mathcal{L}_1\right)$$

$$\Omega^{(2)}_{2\mathrm{d}} = \frac{1}{2} C_2^2 \left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4} C_1 C_2 \left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right)$$

$$\begin{aligned}\Omega^{(3)}_{2\mathrm{d}} = \frac{1}{4} C_1 C_2^2 & \left(-2\mathcal{L}_{0,0,1} - 3\mathcal{L}_{0,1,0} - 7\mathcal{L}_{0,1,1} - 2\mathcal{L}_{1,0,0} - 7\mathcal{L}_{1,0,1} - 7\mathcal{L}_{1,1,0}\right. \\ & \left. -14\mathcal{L}_{1,1,1} + 2\zeta_3\right) + \frac{3}{4} C_2^3 \left(\mathcal{L}_{0,0,0} + 2\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + 2\mathcal{L}_{1,0,0}\right. \\ & \left. +4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right) + \frac{1}{16} C_1^2 C_2 \left(\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1}\right. \\ & \left. +\mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right),\end{aligned}$$

where $C_1 = 2C_A - T_t^2$, $C_2 = C_A - T_t^2$ and, e.g.,

Caron-Huot, Gardi, Reichel, LV, in preparation

$$\begin{aligned}\mathcal{L}_{0,0,1,1}(z,\bar{z}) = & \; H_{0,0,1,1}(z) + H_{1,1,0,0}(\bar{z}) \\ & + H_{0,0,1}(z) H_1(\bar{z}) + H_0(z) H_{1,1,0}(\bar{z}) + H_{0,0}(z) H_{1,1}(\bar{z}) - 2\zeta_3 H_1(\bar{z}).\end{aligned}$$

- A closed-form resummed expression is not yet known.

FINITE AMPLITUDE

- We have now all ingredients to obtain

$$\mathcal{M}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right) = -i\pi\Bigg[\underbrace{\int [Dk]\frac{p^2}{k^2(p-k)^2}\Omega_s(p,k)}_{\substack{\text{computable using soft limit}\\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\frac{1}{4\pi}\int\frac{d^2z}{z\bar{z}}\Omega_h^{(2\mathrm{d})}(z,\bar{z})}_{\text{compute in } D=2}\Bigg]\mathbf{T}^2_{s-u}M^{(0)},$$

- Two methods to perform the last integration, and sum consistently soft and hard region.

- First few orders:

$$\hat{\mathcal{M}}^{(1)}|_{\epsilon^0} = 0, \qquad \hat{\mathcal{M}}^{(2)}|_{\epsilon^0} = 0,$$

$$\hat{\mathcal{M}}^{(3)}|_{\epsilon^0} = -i\pi\frac{(B_0)^3}{2!}\left[C_2^2\left(-\frac{11}{4}\zeta_3\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(4)}|_{\epsilon^0} = -i\pi\frac{(B_0)^4}{3!}\left[C_1C_2^2\left(-\frac{3}{16}\zeta_4\right)+C_2^3\left(\frac{3}{16}\zeta_4\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(5)}|_{\epsilon^0} = -i\pi\frac{(B_0)^5}{4!}\left[C_2^4\left(-\frac{717}{16}\zeta_5\right)+C_1C_2^3\left(\frac{333}{16}\zeta_5\right)+C_1^2C_2^2\left(-\frac{5}{2}\zeta_5\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(6)}|_{\epsilon^0} = -i\pi\frac{(B_0)^6}{5!}\left[C_2^5\left(-\frac{2879}{32}\zeta_3^2+\frac{5}{32}\zeta_6\right)+C_1C_2^4\left(\frac{2637}{32}\zeta_3^2-\frac{5}{32}\zeta_6\right)\right.$$
$$\left.+C_1^2C_2^3\left(-\frac{399}{16}\zeta_3^2\right)+C_1^3C_2^2\left(\frac{39}{16}\zeta_3^2\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\ldots$$

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FINITE AMPLITUDE

$$\hat{\mathcal{M}}^{(+,11)}_{\rm NLL} \supset \Big(-\frac{149}{688128000}(2C_A-\mathbf{T}_t^2)^8(C_A-\mathbf{T}_t^2)^2 + \frac{26209}{6193152000}(2C_A-\mathbf{T}_t^2)^7(C_A-\mathbf{T}_t^2)^3 - \frac{14813}{442368000}(2C_A-\mathbf{T}_t^2)^6(C_A-\mathbf{T}_t^2)^4 + \frac{210383}{1548288000}(2C_A-\mathbf{T}_t^2)^5(C_A-\mathbf{T}_t^2)^5 - \frac{7549}{25804800}(2C_A-\mathbf{T}_t^2)^4(C_A-\mathbf{T}_t^2)^6 + \frac{39257}{129024000}(2C_A-\mathbf{T}_t^2)^3(C_A-\mathbf{T}_t^2)^7 - \frac{11}{102400}(2C_A-\mathbf{T}_t^2)^2(C_A-\mathbf{T}_t^2)^8 \Big) \times g_{5,3,3}.$$

- Hard regions contributes only with single-valued ζ_n:
  → consistent with 2d wavefunction made of SVHPLs.
- Finite (hard) amplitude contains $g_{5,3,3}$ at 11 loops, $g_{5,5,3}$, $g_{7,3,3}$ at 13 loops:

$$g_{5,3,3} = -\frac{4}{7}\zeta_2^3\zeta_5 + \frac{6}{5}\zeta_2^2\zeta_7 + 45\zeta_2\zeta_9 + \zeta_{5,3,3}\,.$$

No exponentiation in terms of Γ functions.

Caron-Huot, Gardi, Reichel, LV, in preparation

FINITE AMPLITUDE: RADIUS OF CONVERGENCE

- Focus on the singlet and 27 color representations: with

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \frac{i\pi}{L}\,\Xi^{(+)}_{\rm NLL}\,\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}\,, \qquad\qquad x = \frac{L\alpha_s}{\pi}.$$

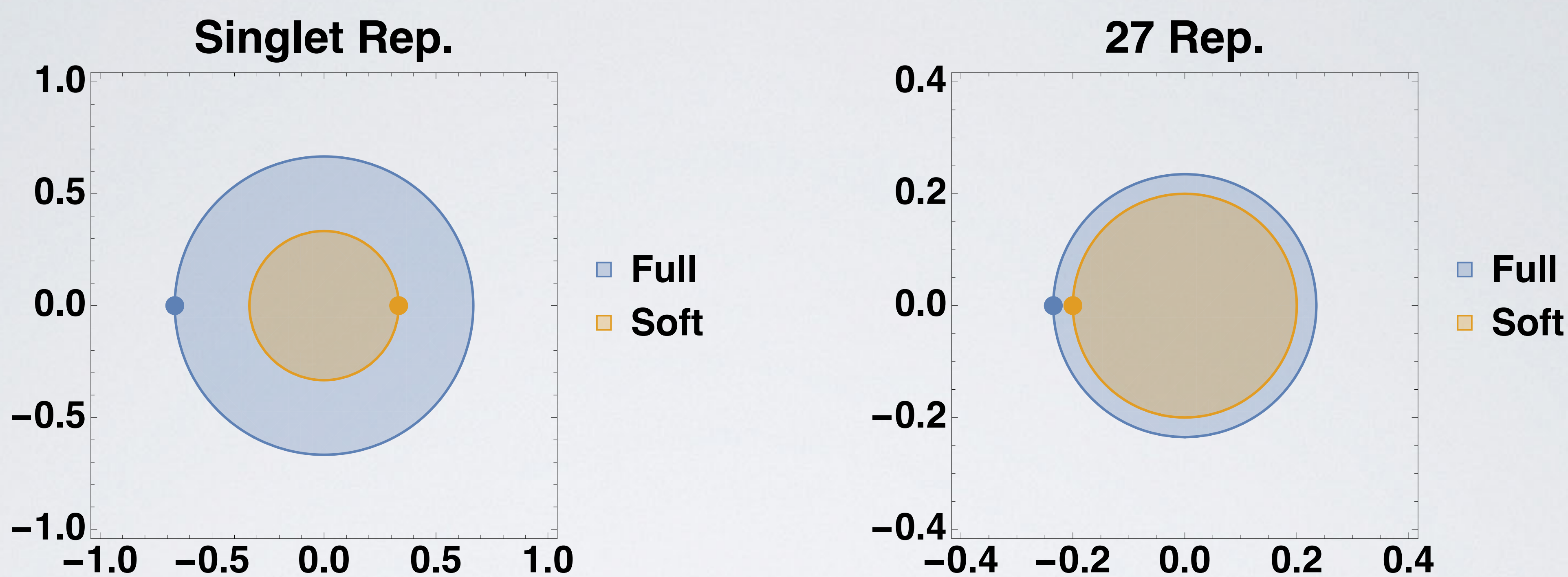
The amplitude reads

$$\begin{aligned}\Xi^{(+)[1]}_{\rm NLL} &= -0.6169\,x^2 - 6.536\,x^3 - 0.8371\,x^4 - 8.483\,x^5 - 1.529\,x^6 - 12.67\,x^7 + 1.610\,x^8 \\ &\quad - 20.62\,x^9 + 16.48\,x^{10} - 35.98\,x^{11} + 46.07\,x^{12} - 74.04\,x^{13} + \mathcal{O}(x^{14}),\end{aligned}$$

$$\begin{aligned}\Xi^{(+)[27]}_{\rm NLL} &= 1.028\,x^2 - 18.16\,x^3 + 2.184\,x^4 - 196.0\,x^5 + 372.3\,x^6 - 2821\,x^7 + 9382\,x^8 \\ &\quad - 46494\,x^9 + 180397\,x^{10} - 797524\,x^{11} + 3.239\times10^6\,x^{12} - 1.374\times10^7\,x^{13} + \mathcal{O}(x^{14}).\end{aligned}$$

FINITE AMPLITUDE: RADIUS OF CONVERGENCE

- Applying Padè approximants we extract the position of the nearest singularity:



- The soft amplitude provide a check on the Padè analysis:

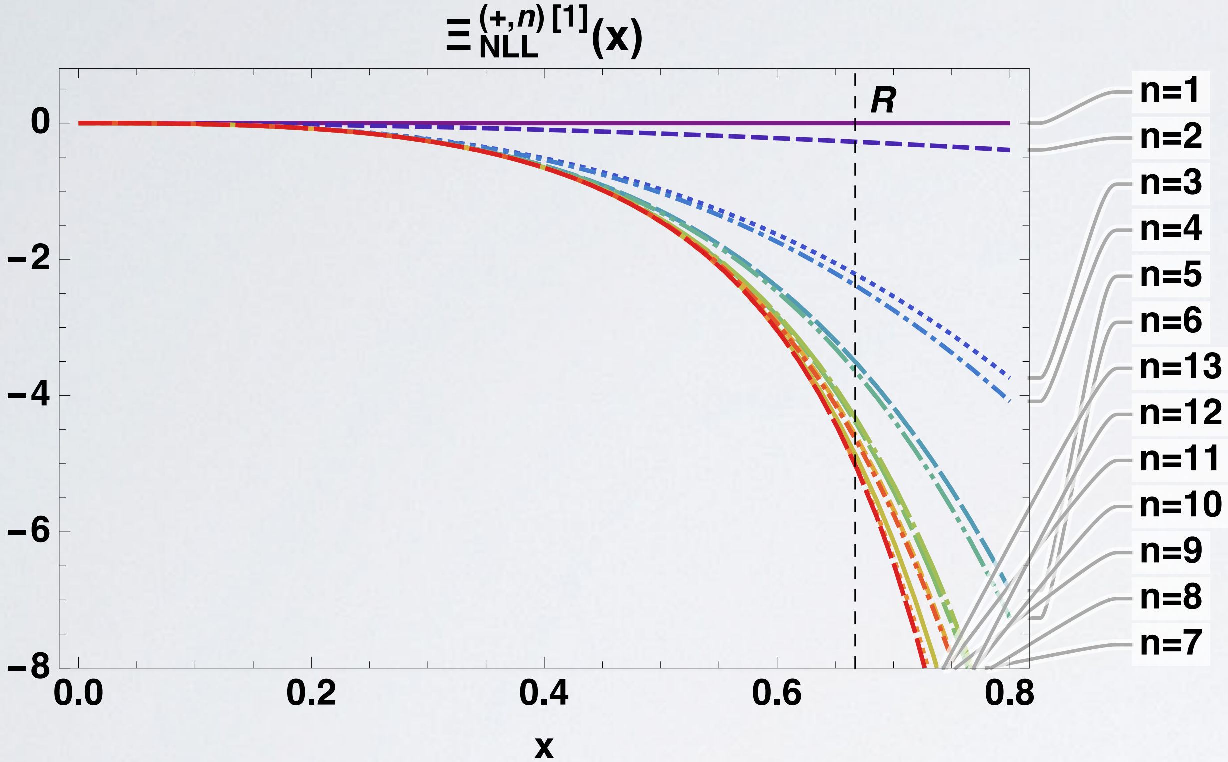
$$\hat{\mathcal{M}}_{\mathrm{NLL,s}} = \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \Bigg\{ \left(e^{\frac{B_0}{2\epsilon}(C_A - \mathbf{T}_t^2)x} - 1 \right) \frac{B_{-1}(\epsilon)}{B_0(\epsilon)} \left(1 - \hat{B}_{-1}(\epsilon) \frac{(2C_A - \mathbf{T}_t^2)}{(C_A - \mathbf{T}_t^2)} \right)^{-1} + 1$$

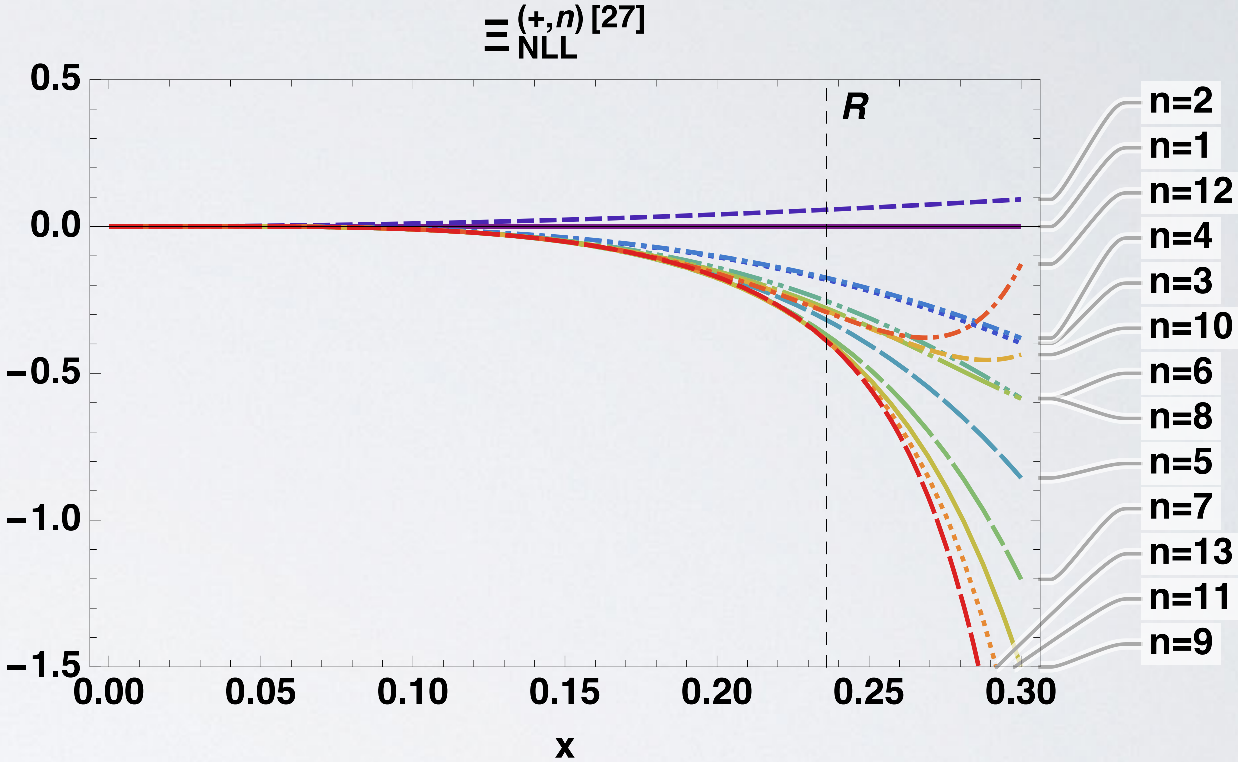
$$- e^{-\gamma_E (2C_A - \mathbf{T}_t^2)\, x} \frac{\Gamma\left(1 - (C_A - \mathbf{T}_t^2)x\right)}{\Gamma\left(1 + (C_A - \mathbf{T}_t^2)x\right)} \left(\frac{\Gamma\left(1 + (C_A - \mathbf{T}_t^2)\frac{x}{2}\right)}{\Gamma\left(1 - (C_A - \mathbf{T}_t^2)\frac{x}{2}\right)} \right)^{-\frac{\mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}} \Bigg\} \mathbf{T}_{s-u}^2 \, M^{(0)}.$$

- Singularity at $(C_A - T_t^2)\, x = 1$ cancels in the full amplitude.
- Soft and hard amplitude are not good approximation of the full amplitude.
- The full amplitude has an additional singularity at $[(C_A - T_t^2) - 3/8(2C_A - T_t^2)]\, x = 1$.

FINITE AMPLITUDE: RADIUS OF CONVERGENCE

- Radius of convergence: $|R| \simeq 0.66$ for singlet, $|R| \simeq 0.24$ for 27 representation.





See also Larkoski, Moult, Neill, 2016 for a similar analysis in the context of non-global logs.

CONCLUSION

- Modern approach to high-energy scattering via Wilson lines:
  - → new theoretical control up to NNLL;
  - → complementary to infrared factorisation.
- $2 \to 2$ amplitude at NLL obtained by iteration of the BFKL kernel.
  - → Solved to all orders in the soft limit;
  - → Soft anomalous dimension in the high-energy limit to all orders;
  - → Algorithm to derive the full amplitude to any given order;
  - → Explicit result to 13 loops.
- Number theory findings:
  - → Soft amplitude given in terms of Gamma functions;
  - → Hard amplitude given in terms of single-valued ζ_n numbers, no exponentiation in terms of Γ functions only.
- Large order behaviour aspects:
  - → The soft anomalous dimension (at NLL in the high-energy limit) is an entire functions: valid to $L\, \alpha_s/\pi \gg 1$;
  - → The finite part has a finite radius of convergence, with asymptotically sign-oscillating coefficients.