BFKL and impact factors at NLO (Rapidity factorization of high-energy scattering processes at NLO)

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Towards Accuracy at small-x
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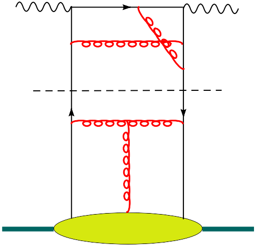
Outline

- Reminder of usual Operator Product Expansion in non local operator.
- High-energy Operator Product Expansion: factorization in rapidity space.
- Evolution equation and background field method.
- NLO impact factor and composite Wilson line operators.
- Application of high-energy OPE at NLO: $\gamma^*\gamma^*$ scattering at NLO.
- Role of the composite Wilson line operators in $\gamma^*\gamma^*$ and DIS amplitudes and the $\gamma \to 1 - \gamma$ symmetry.
- Conclusions and Outlook.

Light-cone expansion and DGLAP evolution equation



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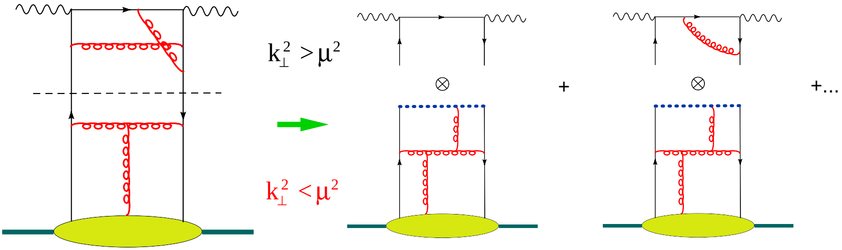

μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions

$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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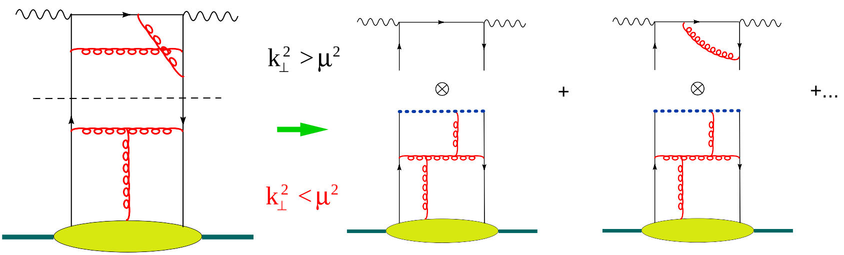

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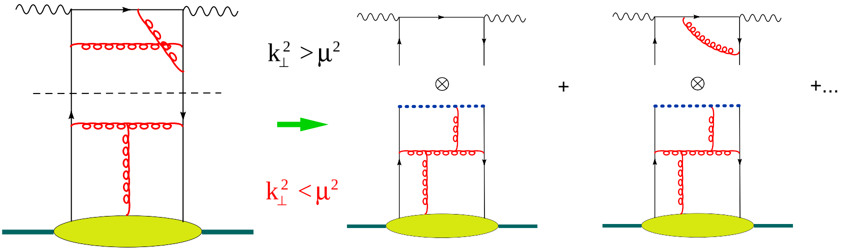
OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)_\xi}{2\pi^2(x-y)^4}\Big[1+\frac{\alpha_s}{\pi}(\ln(x-y)^2\mu^2+C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$$

$[x,y] \;\equiv\; Pe^{ig\int_0^1 du\;(x-y)^\mu A_\mu(ux+(1-u)y)}$ - gauge link

Light-cone expansion and DGLAP evolution equation



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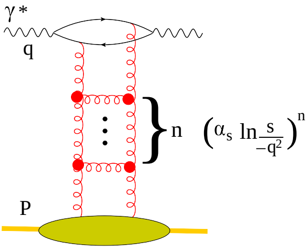
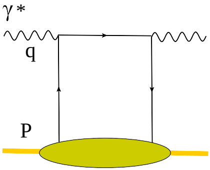
Renorm-group equation for light-ray operators ⇒ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar\psi(x)[x,y]\psi(y) \; = \; K_{\rm LO} \bar\psi(x)[x,y]\psi(y) + \dots$$

electron-proton/nucleus Deep Inelastic Scattering (DIS) $s = (q+P)^2$

$$\langle P|\mathrm{T}\, j^{\mu}(x) j^{\nu}(y)|P\rangle$$

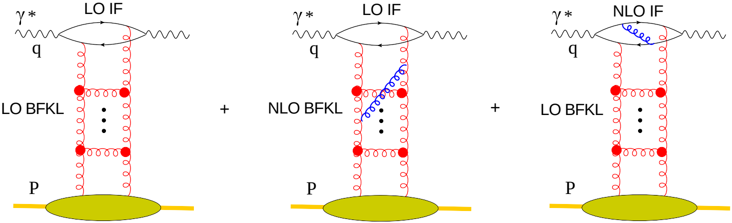




energy suppressed

- BFKL resum $\left(\alpha_s \ln \frac{s}{-q^2}\right)^n$
- Dynamics is linear and it describes proliferation of gluons
  $\Rightarrow$ Violation of Unitarity

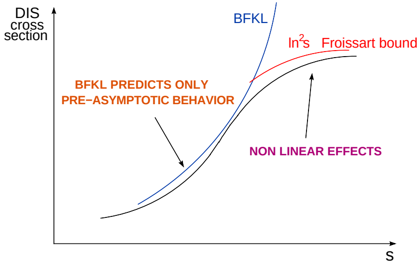
NLO DIS with BFKL dynamics



- NLO BFKL: $\alpha_s \left(\alpha_s \ln \frac{1}{x_B}\right)^n$
- NLO Impact factor contains contributions prop. to $\alpha_s \ln \frac{1}{x_B}$ which can be included in the LLA.
- Need systematics of perturbation theory
  $\Rightarrow$ Operator Product Expansion formalism can be the solution.

DIS cross section

BFKL: Leading Logarithmic Approximation $\alpha_s << 1 \qquad (\alpha_s \ln s)^n \sim 1$



- pQCD at LLA: $A(s,t) \propto s^{\Delta}$

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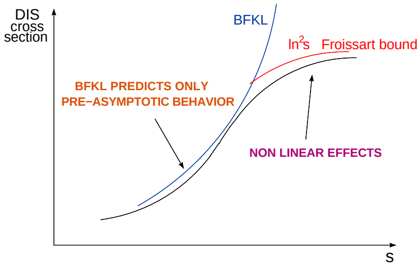

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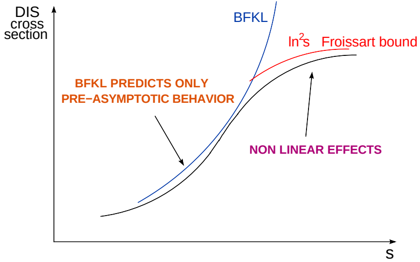

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- At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross section. Unitarity is restored!

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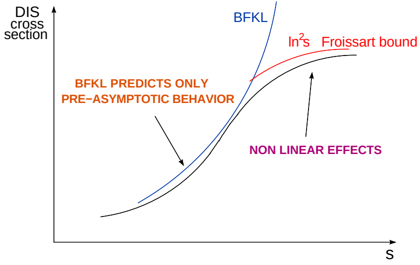

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- At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross section. Unitarity is restored!
- In order to take in to account recombination of gluons the evolution equation for the structure function has to be non-linear.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

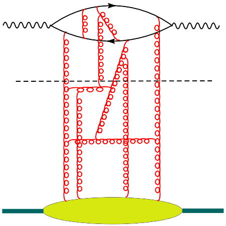
$$[z', z] = Pe^{ig\int_0^1 du(z'-z)^\mu A_\mu(uz'+(1-u)z)} \qquad U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu, \quad p_1^\mu = \sqrt{s/2}(1,0,0,1), \quad p_2^\mu = \sqrt{s/2}(1,0,0,-1)$$

s center-of-mass energy.

Propagation in the shock wave: Wilson line (Spectator frame)

- At high energy all fields are ordered in rapidity $\Rightarrow$ rapidity η is a suitable factorization parameter.



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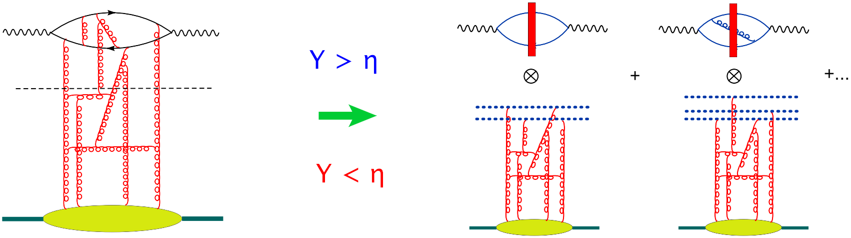
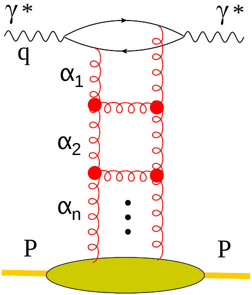


p_1^μ , p_2^μ light-cone vectors $\Rightarrow k^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp^\mu$

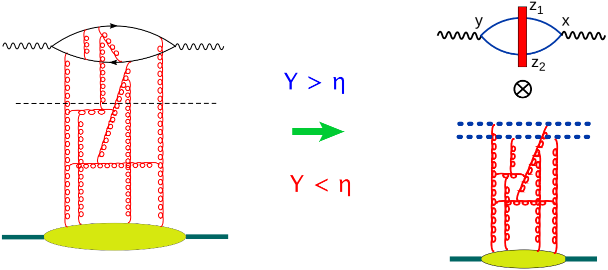
$\alpha_1 \gg \alpha_2 ... \gg \alpha_n$



Fields are ordered in their rapidities $\Rightarrow$

- fast fields are treated as quantum fields
- slow fields are treated as classical fields

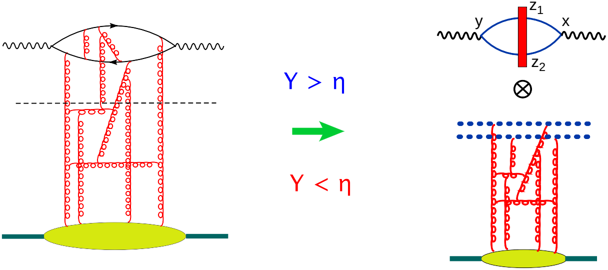
DIS amplitude is factorized in rapidity: η



$|B\rangle$ is the target state.

$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \ldots$$

High-Energy Operator Product Expansion

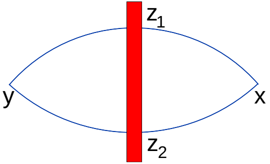


$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \dots$$

- If we use a model to evaluate $\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle$ we can calculate the DIS cross-section.
- If we want to include energy dependence to the DIS cross section, we need to find the evolution of $\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle$ with respect to the rapidity parameter η.

LO Impact Factor

Conformal invariance: $(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$



Conformal vectors:

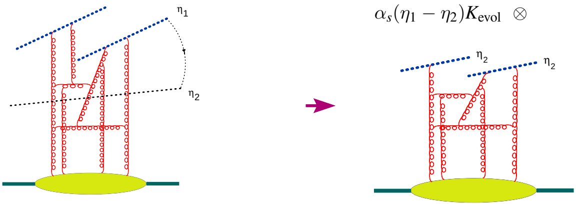
$$\kappa = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1}{s} - x^2 p_2 + x_\perp\right) - \frac{\sqrt{s}}{2y_*}\left(\frac{p_1}{s} - y^2 p_2 + y_\perp\right)$$

$$\zeta_1 = \left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\right), \qquad \zeta_2 = \left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\right)$$

$\sqrt{\frac{s}{2}}x^+ = x_* \equiv x_\mu p_2^\mu$ (similarly for y); $\mathcal{R} = \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$

$$I^{\rm LO}_{\mu\nu}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu \partial y^\nu}\left[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2) - \frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\right]$$

Evolution Equation



- Separate fields in quantum and classical according to low and large rapidity. Formally we may write:

$$\langle B|\mathcal{O}^{\eta_1}|B\rangle \rightarrow \langle \mathcal{O}^{\eta_1}\rangle_A \rightarrow \langle \mathcal{O}'^{\eta_2} \otimes \mathcal{O}'^{\eta_1}\rangle_A$$

- Integrate over the quantum fields and get one-loop rapidity evolution of the operator $\mathcal{O}$

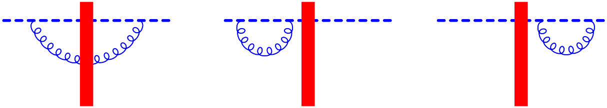
$$\langle \mathcal{O}^{\eta_1}\rangle_A = \alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes \langle \mathcal{O}'^{\eta_2}\rangle_A$$

- Where in principle $\mathcal{O}$ and $\mathcal{O}'$ are different operators.

Non-linear evolution equation

- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\rm evol} \otimes \mathcal{O}^{\eta_2}$

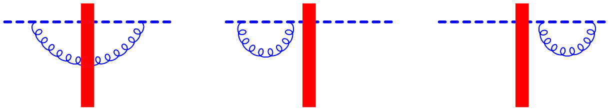
- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\text{evol}} \otimes \mathcal{O}^{\eta_2}$
- Non-linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\text{evol}} \otimes \{\mathcal{O}^{\eta_2}\mathcal{O}^{\eta_2}\}$



$$\langle\{U_x^{\eta_1}\}_{ij}\rangle_A = \frac{\alpha_s}{2\pi^2}\Delta\eta \int \frac{d^2z_\perp}{(x-z)_\perp^2}\left[\langle \mathrm{tr}\{U_x^{\eta_2}U_z^{\eta_2\dagger}\}\{U_z^{\eta_2}\}_{ij}\rangle_A - \langle\frac{1}{N_c}\{U_x^{\eta_2}\}_{ij}\rangle_A\right]$$

$\Delta = \eta_1 - \eta_2$

$$\{U_x^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\eta_1}U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1}U_y^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\dagger\eta_1}U_y^{\dagger\eta_1}\}_{ij}$$



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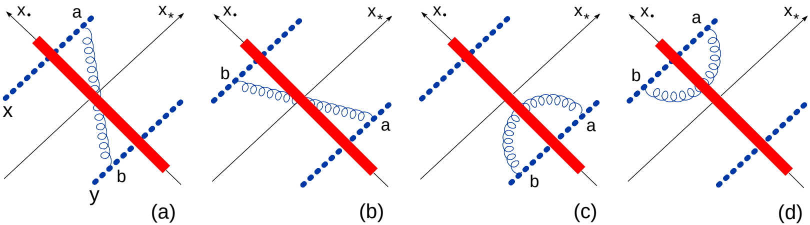
- Obtain a set of rules that allow one to get the LO evolution of any trace or product of traces of Wilson lines
- Hierarchy of evolution equation: B-JIMWLK equation.

Leading order evolution equation

$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non-linear evolution equation: Balitsky-Kovchegov equation

$$U_z^{ab} = 2\mathrm{tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

Non-linear evolution equation: Balitsky-Kovchegov equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: Ian Balitsky (1996), Yu. Kovchegov (1999)

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

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- LLA for DIS in pQCD $\Rightarrow$ BFKL
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- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): describes proliferation of gluons.

- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
  - background field method: describes recombination process.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\cancel{z^4}} \frac{(x-y)^2 \; \cancel{z^4}}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

We calculate the "matrix element" of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

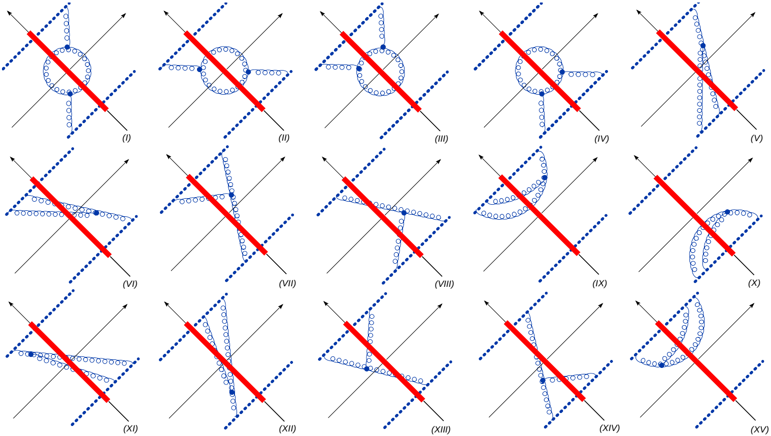
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow \quad \left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

Typical integral

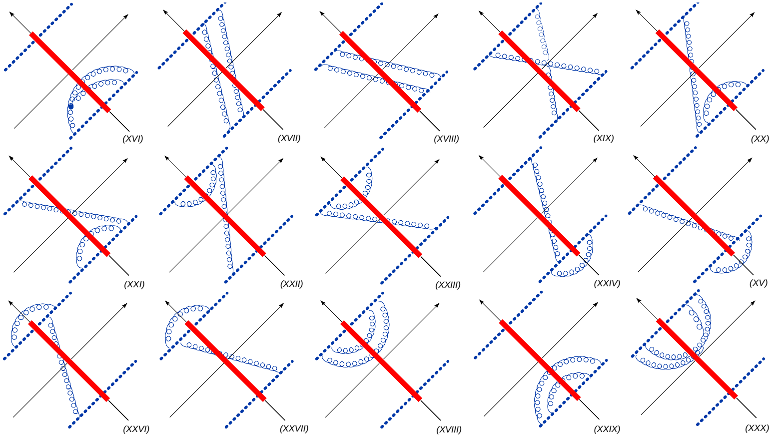
$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Diagrams of the NLO gluon contribution

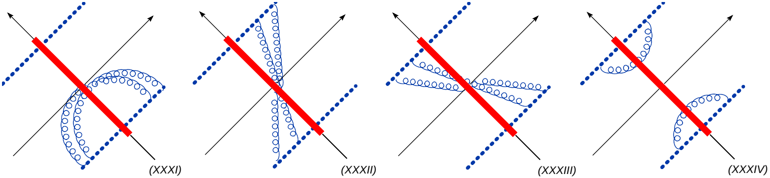
Diagrams with 2 gluons interaction



Diagrams with 2 gluons interaction

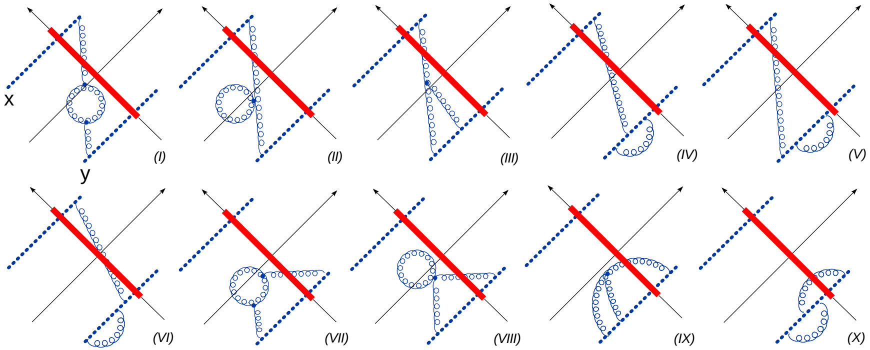


Diagrams with 2 gluons interaction

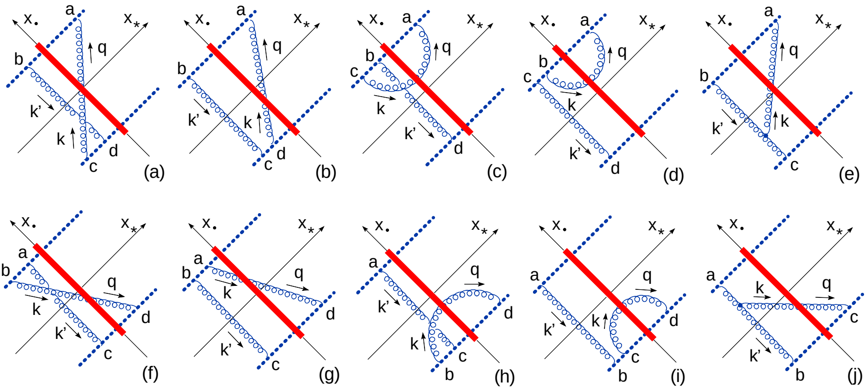


Diagrams of the NLO gluon contribution

"Running coupling" diagrams

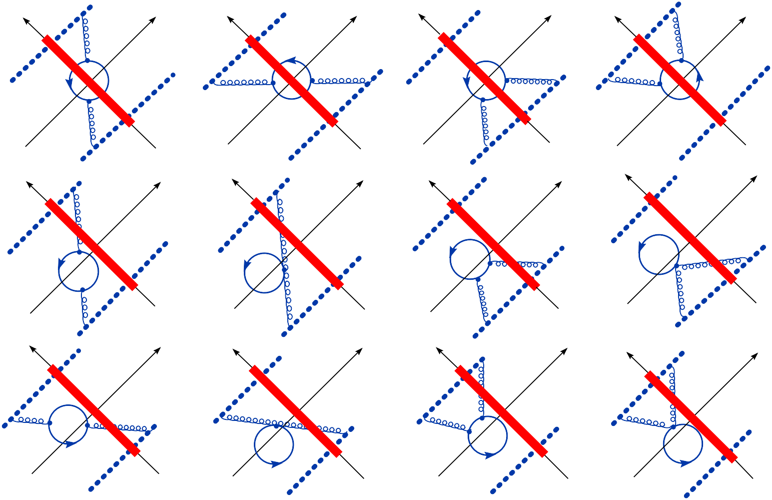


$1 \to 2$ dipole transition diagrams



Diagrams of the NLO gluon contribution

$\mathcal{N} = 4$ SYM diagrams (scalar and gluino loops)



Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$X = x - z, \;\; Y = y - z, \;\; X' = x - z' \;\; Y' = y - z'.$

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\, \Big([\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - N_c\mathrm{Tr}\{U_x U^\dagger_y\}] \\
& \times \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1 + \frac{\alpha_s N_c}{4\pi}(\frac{11}{3}\ln(x-y)^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3})\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} \;-\; \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_{z'}\}\{U_{z'} U^\dagger_y\} - \mathrm{Tr}\{U_x U^\dagger_z U_{z'} U^\dagger_y U_z U^\dagger_{z'}\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2 + \frac{X'^2Y^2 + Y'^2X^2 - 4(x-y)^2(z-z')^2}{2(X'^2Y^2 - Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& + [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_{z'}\}\{U_{z'} U^\dagger_y\} - \mathrm{Tr}\{U_x U^\dagger_{z'} U_z U^\dagger_y U_{z'} U^\dagger_z\} - (z'\to z)] \\
& \times \Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

Our result Agrees with NLO BFKL (Comparing the eigenvalue of the forward kernel) It respects unitarity

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& + [\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

Evolution equation for color dipoles in $\mathcal{N}=4$

(I. Balitsky and G.A.C. 2009)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\ [\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\ \mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

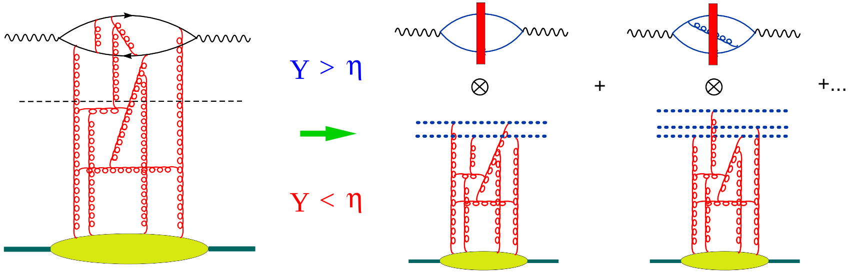
(I. Balitsky and G.A.C. 2009)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant



The high-energy operator expansion is

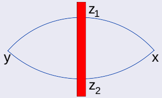
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

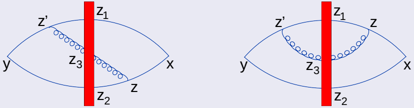
LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



NLO Impact Factor diagrams: $I^{\rm NLO}$



NLO Photon Impact Factor

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3 \left[I_1^{\mu\nu}(z_1,z_2,z_3) + I_2^{\mu\nu}(z_1,z_2,z_3)\right]$$
$$\times[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c {\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

where $I_2^{\mu\nu}(z_1,z_2,z_3)$ is finite and conformal, while

$$I_1^{\mu\nu}(z_1,z_2,z_3) = \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu} \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3}$$

is rapidity divergent.

How to get the NLO Impact factor

$$\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$+ \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}] + \dots$$

$\Rightarrow$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} - \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO}$$

$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO} = \frac{\alpha_s}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]\int_0^{e^\eta}\frac{d\alpha}{\alpha}$$

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3 \; I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]$$
$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3 \; \Big\{ I_2^{\mu\nu}(z_1,z_2,z_3) + \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \Big[\int_0^{+\infty} \frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \Big] \Big\}$$
$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

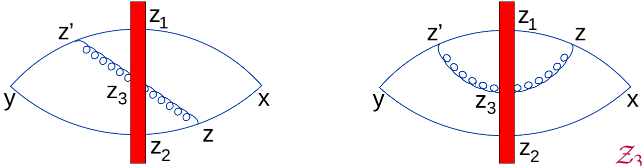
$$\Big[\int_0^{+\infty} \frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \Big] \to -\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C$$

where $\sigma = e^\eta$ and C is the Euler constant

$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$\mathcal{Z}_3$ is not conformal invariant in the transverse 2-d coordinate space, but QCD at tree level has to be conformal invariant.

NLO Impact Factor



$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I_{\mu\nu}^{\rm LO} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

Conformal Composite Operator

Define a composite operator

$$[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

(a - analog of μ^{-2} for usual OPE)
the impact factor becomes conformal at the NLO.

Conformal Composite Operator

Define a composite operator

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

(a - analog of μ^{-2} for usual OPE)
the impact factor becomes conformal at the NLO.

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + BK_{\rm LO}\ln\sqrt{\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2}} + O(\alpha_s^2)$$

Define a composite operator

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\,a\,z_{12}^2}{\sigma^2\,s\,z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

(a - analog of μ^{-2} for usual OPE)
the impact factor becomes conformal at the NLO.

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + BK_{\mathrm{LO}}\ln\sqrt{\frac{4\,a\,z_{12}^2}{\sigma^2\,s\,z_{13}^2z_{23}^2}} + O(\alpha_s^2)$$

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + BK_{\mathrm{LO}}\int_0^{\sqrt{\frac{a\,z_{12}^2}{z_{13}^2z_{23}^2}}}\frac{d\alpha}{\alpha} + O(\alpha_s^2)$$

Conformal Composite Operator

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a,\eta}$$
$$= \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + \frac{\alpha_s}{4\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\,a\,z_{12}^2}{\sigma^2\,s\,z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

choose a rapidity-dependent constant $a \to ae^{-2\eta} \Rightarrow [\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]^{\rm conf}_a$ does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is encoded into a-dependence:

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a}$$
$$= \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + +\frac{\alpha_s}{4\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\,a\,z_{12}^2}{\sigma^2\,s\,z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

Using the leading-order evolution equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$\Rightarrow \frac{d}{d\eta}[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_a = 0$ (with $O(\alpha_s^2)$ accuracy).

$$2a\frac{d}{da}[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = \ - I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z^2_{12}}{z^2_{13}z^2_{23}} \ln\frac{z^2_{12}e^{2\eta}as^2}{z^2_{13}z^2_{23}}\mathcal{Z}^2_3 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$(x-y)^4 T\{\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int\frac{d^2z_1 d^2z_2}{z_{12}^4}\Big\{I^{\mu\nu}_{\rm LO}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][{\rm tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$$

$$+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)} - 2C\Big)I^{\mu\nu}_{\rm LO} + I^{\mu\nu}_2\Big]$$

$$\times[{\rm tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}{\rm tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c{\rm tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}$$

where

$$(I_2)_{\mu\nu}(z_1,z_2,z_3) = \frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)} + \frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}$$

$$+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)} - \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big] + \frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)} - \frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]$$

$$+(\zeta_1\leftrightarrow\zeta_2)\Big\}$$

NLO evolution of composite “conformal” dipoles in QCD

$$2a\frac{d}{da}[\mathrm{tr}\{\hat U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \Big([\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\ \frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\ \Big\{\Big[-2+\frac{z_{14}{}^2 z_{23}^2 + z_{24}{}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}{}^2 z_{23}^2 - z_{24}{}^2 z_{13}^2)}\ln\frac{z_{14}{}^2 z_{23}^2}{z_{24}{}^2 z_{13}^2}\Big]$$

$$\times\ [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\hat U_{z_4}\hat U^\dagger_{z_2}\hat U_{z_3}\hat U^\dagger_{z_4}\} - (z_4\to z_3)]$$

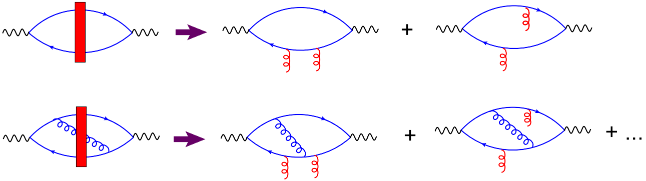
$$+\ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}{}^2}\Big[2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}+\Big(1+\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big)\ln\frac{z_{13}^2 z_{24}{}^2}{z_{14}{}^2 z_{23}^2}\Big]$$

$$\times\ [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_4}\hat U_{z_3}\hat U^\dagger_{z_2}\hat U_{z_4}\hat U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$b = \frac{11}{3}N_c - \frac{2}{3}n_f$ I. Balitsky and G.A.C

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

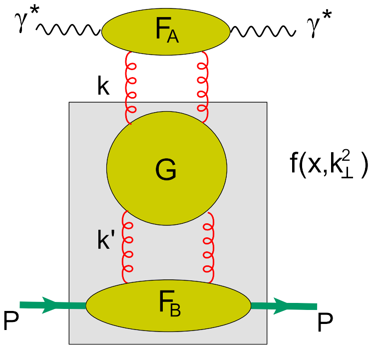
Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).



$$I^{\rm LO}\hat{\mathcal{U}}(x_\perp, y_\perp)$$
$$I^{\rm NLO}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y)\Big\}$$

where $\mathcal{U}(x,y) = 1 - \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}$ and we neglected the non-linear term $\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)$

NLO DIS in the k_T factorization form



- $f(x,k_\perp^2) \propto \int \frac{d^2k'}{k'^2_\perp} F_B(k'^2_\perp)\; k_\perp^2 G(x,k_\perp,k'_\perp)$
- $\mathcal{U}(x,y) = 1 - \frac{1}{N_c}\mathrm{Tr}\{U(x_\perp)U^\dagger(y_\perp)\}$
- $\mathcal{V}(z) \equiv z^{-2}\mathcal{U}(z)$

$$2a\frac{d}{da}\mathcal{V}_a(z) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z' \left[\frac{2\mathcal{V}_a(z')}{(z-z')^2} - \frac{z^2\mathcal{V}_a(z)}{z'^2(z-z')^2}\right]$$

$$\int d^4x\, e^{iqx}\langle p|T\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|p\rangle = \frac{s}{2}\int \frac{d^2k_\perp}{k_\perp^2}\, I_{\mu\nu}(q,k_\perp)\mathcal{V}_{a_m=x_B}(k_\perp)$$

NLO Evolution of the unintegrated gluon distribution

$$
\begin{aligned}
2a\frac{d}{da}\mathcal{V}_a(k) \;=\; & \frac{\alpha_s N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2} - \frac{(k,k')\mathcal{V}_a(k)}{k'^2(k-k')^2}\Big] \\
& \times\Big(1+\ \frac{\alpha_s b}{4\pi}\Big[\ln\frac{\mu^2}{k^2}+\frac{N_c}{b}\Big(\frac{67}{9}-\frac{\pi^2}{3}-\ \frac{10n_f}{9N_c}\Big)\Big]\Big) -\ \frac{b\alpha_s}{4\pi} \\
& \times\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2}\ln\frac{(k-k')^2}{k'^2} - \frac{k^2\mathcal{V}_a(k)}{k'^2(k-k')^2}\ln\frac{(k-k')^2}{k^2}\Big] \\
& + \frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2} +\ F(k,k')+\Phi(k,k')\Big]\ \mathcal{V}_a(k')\Big\} \\
& + 3\frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)\mathcal{V}_a(k)
\end{aligned}
$$

I. Balitsky and G.A.C.

$$\mathcal{V}_{x_B}(z_\perp,\mu) \;=\; \frac{4\pi^2 x_B}{N_c}\alpha_s(\mu)\mathcal{D}(x_B,z_\perp,\mu)$$

where

$$\mathcal{D}(x_B,z_\perp,\mu) \equiv \frac{4}{s^2}\int\frac{dz_*}{\pi x_B}\langle p|\mathrm{Tr}\Big\{\big[\infty p_1+z_\perp,\frac{2}{s}z_*p_1+z_\perp\big]$$
$$\hat{F}_{\bullet\xi}\big(\frac{2}{s}z_*p_1+z_\perp\big)\big[\frac{2}{s}x_*p_1+z_\perp,-\infty p_1+z_\perp\big][-\infty p_1,0]\hat{F}_{\bullet}^{\;\xi}(0)[0,\infty p_1]\Big\}|p\rangle^{a_m=x_B}$$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form I. Balitsky and G.A.C.

$$
\begin{aligned}
& I^{\mu\nu}(q,k_\perp) \\
&= \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu}\left\{\left[\left(\frac{9}{4}+\nu^2\right)\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\right)P_1^{\mu\nu}\right.\right. \\
&\left. +\left(\frac{11}{4}+3\nu^2\right)\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\right)P_2^{\mu\nu}\right] \\
&\left. +\frac{\frac{1}{4}+\nu^2}{2k_\perp^2}\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_3(\nu)\right)\left[\tilde{P}^{\mu\nu}\bar{k}^2+\bar{P}^{\mu\nu}\tilde{k}^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
P_1^{\mu\nu} &= g^{\mu\nu}-\frac{q_\mu q_\nu}{q^2} \\
P_2^{\mu\nu} &= \frac{1}{q^2}\left(q^\mu-\frac{p_2^\mu q^2}{q\cdot p_2}\right)\left(q^\nu-\frac{p_2^\nu q^2}{q\cdot p_2}\right) \\
\bar{P}^{\mu\nu} &= \left(g^{\mu 1}-ig^{\mu 2}-p_2^\mu\frac{\bar{q}}{q\cdot p_2}\right)\left(g^{\nu 1}-ig^{\nu 2}-p_2^\nu\frac{\bar{q}}{q\cdot p_2}\right) \\
\tilde{P}^{\mu\nu} &= \left(g^{\mu 1}+ig^{\mu 2}-p_2^\mu\frac{\tilde{q}}{q\cdot p_2}\right)\left(g^{\nu 1}+ig^{\nu 2}-p_2^\nu\frac{\tilde{q}}{q\cdot p_2}\right)
\end{aligned}
$$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C.

$$\mathcal{F}_{1(2)}(\nu) \;=\; \Phi_{1(2)}(\nu) + \chi_\gamma \Psi(\nu), \qquad \mathcal{F}_3(\nu) \;=\; F_6(\nu) + \Big(\chi_\gamma - \frac{1}{\bar\gamma\gamma}\Big)\Psi(\nu),$$

$$\Psi(\nu) \;\equiv\; \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma),$$

$$F_6(\gamma) \;=\; F(\gamma) - \frac{2C}{\bar\gamma\gamma} - 1 - \frac{2}{\gamma^2} - \frac{2}{\bar\gamma^2} - 3\frac{1+\chi_\gamma - \frac{1}{\gamma\bar\gamma}}{2+\bar\gamma\gamma},$$

$$\Phi_1(\nu) \;=\; F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2}$$

$$\Phi_2(\nu) \;=\; F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma},$$

$$F(\gamma) \;=\; \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}$$

where $\gamma = \frac{1}{2} + i\nu$

No symmetry $\gamma \to 1-\gamma$

two gluons in the dipole $\mathcal{U}(k)$ come with extra g^2 factor $\Rightarrow$

$$\mathcal{L}(k) \;=\; \frac{1}{g^2(k)}\mathcal{V}(k)$$

$$\begin{aligned}
2a\frac{d}{da}\mathcal{L}_a(k) \;=\; & \frac{\alpha_s(k^2)N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{L}_a(k')}{(k-k')^2} \\
& - \frac{(k,k')\mathcal{L}_a(k)}{k'^2(k-k')^2}\Big]\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\Big)\Big] - \\
& - \frac{b\alpha_s}{4\pi}\Big[\frac{\mathcal{L}_a(k')}{(k-k')^2}-\frac{k^2\mathcal{L}_a(k)}{k'^2(k-k')^2}\Big]\ln\frac{(k-k')^2}{k^2} \\
& + \frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2} + F(k,k') + \Phi(k,k')\Big]\,\mathcal{L}_a(k')\Big\} \\
& + 3\frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathcal{L}_a(k)
\end{aligned}$$

NLO Evolution equation for $\mathcal{L}(k)$ coincides with NLO BFKL eq.

BFKL equation in the $\mathcal{N}$=4 SYM case

- In $\mathcal{N} = 4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

$$K(q,k) = \alpha_{\mathrm{SYM}} K^{\mathrm{LO}}(q,k) + \alpha^2_{\mathrm{SYM}} K^{\mathrm{NLO}}(q,k) + \ldots$$

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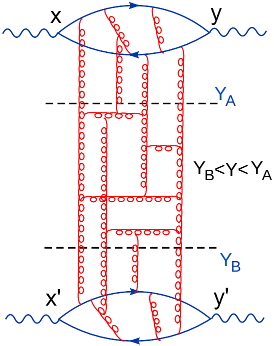
$$K(q,k) = \alpha_{\rm SYM} K^{\rm LO}(q,k) + \alpha^2_{\rm SYM} K^{\rm NLO}(q,k) + \ldots$$

$$\int d^2q\, K(q,k)\,(q^2)^{-\frac{1}{2}+i\nu} = [\alpha_{\rm SYM}\chi_0(\nu) + \alpha^2_{\rm SYM}\chi_1(\nu)\ldots]\,(k^2)^{-\frac{1}{2}+i\nu}$$

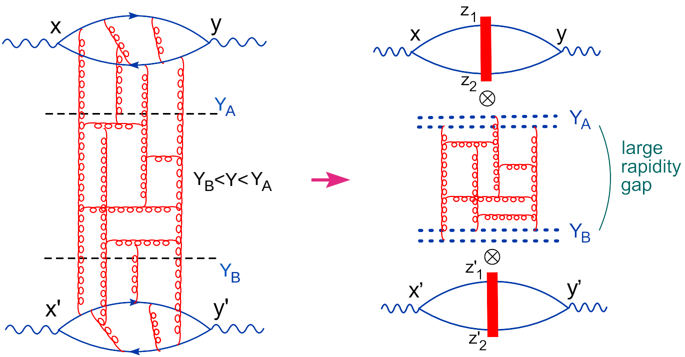
$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2 k\, k'}\, e^{[\alpha_{\rm SYM}\chi_0(\nu)+\alpha^2_{\rm SYM}\chi_1(\nu)\ldots]} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- The eigenvalues $\bar\alpha_\mu\,\chi_0(\nu) + \bar\alpha_\mu^2\,\chi_1^{\rm SYM}(\nu) + \ldots$ are real and symmetric for $\nu \leftrightarrow -\nu$.
- NLO Impact Factor in $\mathcal{N}$=4 is symmetric for $\nu \leftrightarrow -\nu$.

$\gamma^*\gamma^*$ scattering cross-section at LO

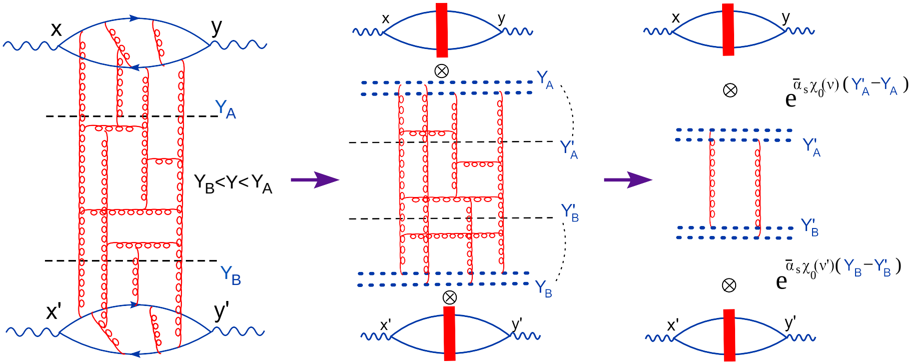


$\gamma^*\gamma^*$ scattering cross-section at LO



$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\, I_B^{\rho\lambda}(x',y';z'_1,z'_2)$$
$$\otimes\langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle$$

$\gamma^*\gamma^*$ scattering cross-section at LO



$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\, I_B^{\rho\lambda}(x',y';z'_1,z'_2)$$
$$\otimes \langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\rangle_A \langle \mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle_A$$

$\gamma^*\gamma^*$ scattering cross-section at LO

$$U_x = \mathrm{Pexp}\left\{ ig \int_{-\infty}^{+\infty} dx^+ A^-(x^+ + x_\perp) \right\}$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1, q_2) \propto i\frac{\alpha_s^2}{Q_1 Q_2} \int d\nu\, I_{\mathrm{LO}}^{\alpha\beta}(\nu)\, I_{\mathrm{LO}}^{\rho\lambda}(\nu) \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar{\alpha}_\mu\, \chi_0(\nu)\,(Y_A - Y_B)}$$

$$Y_A = \frac{1}{2}\ln\frac{s}{Q_1^2}, \qquad Y_B = -\frac{1}{2}\ln\frac{s}{Q_2^2}, \qquad s = (q_1 + q_2)^2$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1, q_2) \propto i\frac{\alpha_s^2}{Q_1 Q_2} \int d\nu\, I_{\mathrm{LO}}^{\alpha\beta}(\nu)\, I_{\mathrm{LO}}^{\rho\lambda}(\nu) \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar{\alpha}_\mu\, \chi_0(\nu)\, \ln\frac{s}{Q_1 Q_2}}$$

What about NLO?

- Can we repit the steps performed at LO also at NLO?
- Problems to be solved:
  - Solve NLO BFKL equation G.A.C and Yu. Kovchegov
  - Calculate NLO Impact Factor I. Balitsky and G.A.C.
  - NLO Impact Factor has to be conformal invariant;
  - $\Rightarrow$ Energy dependence of NLO Impact Factor needs to be eliminated;
  - $\Rightarrow$ Composite Wilson line operators I. Balisky and G.A.C.

BFKL equation at NLO in QCD

$$K^{\mathrm{LO+NLO}}(k,q) \equiv \bar{\alpha}_\mu\, K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2\, K^{\mathrm{NLO}}(k,q)$$

$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\; q^{2\gamma-2} = \left[\bar{\alpha}_\mu\, \chi_0(\gamma) - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\, \frac{\delta(\gamma)}{4}\right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\,N_c - 2\,N_f}{12\,N_c}$$

BFKL equation at NLO in QCD

$$K^{\mathrm{LO+NLO}}(k,q) \equiv \bar{\alpha}_\mu\, K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2\, K^{\mathrm{NLO}}(k,q)$$

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$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\, N_c - 2\, N_f}{12\, N_c}$$

- $-\bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2}$ 1-loop running coupling.
- $\delta(\gamma) = -2\, \beta_2\, \chi_0'(\gamma) + 4\, \chi_1(\gamma)$ Fadin-Lipatov (1998)
- $\chi_1(\gamma)$ Real and symmetric in $\gamma \leftrightarrow 1-\gamma$ $\gamma = \frac{1}{2} + i\nu$.
- $\frac{d}{d\gamma}\chi_0(\gamma) \equiv \chi_0'(\gamma)$ imaginary and NOT symmetric in $\gamma \leftrightarrow 1-\gamma$.

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

- NLO eigenvalues $\quad \Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)$

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1+\bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2}+\frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

- NLO eigenvalues $\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)$

Solution of NLO BFKL equation **G.A.C. and Yu. Kovchegov**

$$G(k,k',Y) = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi^2}\,e^{\left[\bar{\alpha}_\mu\,\chi_0(\nu)+\bar{\alpha}_\mu^2\,\chi_1(\nu)\right]Y}\,H_{\frac{1}{2}+i\nu}(k)\,\left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$

- The perturbative expansion is in both the exponent and in the eigenfunctions (contrary to DGLAP case and $\mathcal{N}$=4 BFKL).

μ-independence of the NLO solution

- The $H_\gamma(k)$ eigenfunctions diagonalize the LO+NLO BFKL kernel

$$\bar{\alpha}_\mu\, K^{\rm LO}(k,q) + \bar{\alpha}_\mu^2\, K^{\rm NLO}(k,q) = \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\,\pi^2\, i}\, \Delta(\gamma)\, H_\gamma(k) H^*_\gamma(q)$$

- LO+NLO BFKL kernel is μ-independent up to $\mathcal{O}(\alpha_\mu^3) \Rightarrow$
- So is its diagonalization through $H_\gamma(k)$ eigenfunctions.

μ-independence of the NLO solution

$$
\begin{aligned}
G(k,k',Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} H_\gamma(k) H_\gamma^*(q) \\
&= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln\frac{kk'}{\mu^2}\right)
\end{aligned}
$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.

μ-independence of the NLO solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} H_\gamma(k) H_\gamma^*(q)$$

$$= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln\frac{kk'}{\mu^2}\right)$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.
- At NLO we may write the solution as (the structure is the same as NLO $\mathcal{N}$=4 SYM)

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\,k')\, \chi_0(\nu) + \bar{\alpha}_s^2(k\,k')\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- At this order the scale $\bar{\alpha}_s^\lambda(k^2)\, \bar{\alpha}_s^\lambda(k'^2) \bar{\alpha}_s^{1-2\lambda}(k\,k')$ (for real λ) works as well.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)+\ldots\right]\,Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- $\chi_2(\nu)$ and higher-order coefficients indicated by the ellipsis in the exponent are the scale-invariant (conformal) $(\nu \leftrightarrow -\nu)$-even (real-valued) parts of the prefactor function generated by the action of the next-to-next-to-leading-order (NNLO) (and higher-order) kernels on the LO eigenfunctions.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)+\ldots\right]\,Y}\,\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- To check the ansatz we have to plug it in the evolution eq. and we need the two-loop beta-function β_3:

$$\mu^2\frac{d\bar{\alpha}_\mu}{d\mu^2} = -\beta_2\bar{\alpha}_\mu^2 + \beta_3\bar{\alpha}_\mu^3$$

and

$$\int d^2q\, K^{\text{LO+NLO+NNLO}}(k,q)\; q^{-1+2i\nu} = \Big\{\bar{\alpha}_\mu\,\chi_0(\nu)\left[1-\bar{\alpha}_\mu\,\beta_2\,\ln\frac{k^2}{\mu^2}+\bar{\alpha}_\mu^2\beta_2^2\,\ln^2\frac{k^2}{\mu^2}+\bar{\alpha}_\mu^2\,\beta_3\,\ln\frac{k^2}{\mu^2}\right]$$
$$+\bar{\alpha}_\mu^2\,\left[\frac{i}{2}\,\beta_2\,\chi_0'(\nu)+\chi_1(\nu)\right]\left[1-2\,\bar{\alpha}_\mu\,\beta_2\,\ln\frac{k^2}{\mu^2}\right]+\bar{\alpha}_\mu^3\,\left[\chi_2(\nu)+i\,\delta_2(\nu)\right]\Big\}\;k^{-1+2i\nu}$$

General Form of the Solution of All-Order BFKL equation

- The ansatz does not work, but it allows us to recover the structure of the NNLO solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k\,k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]Y}\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

$$\times\left\{1+(\bar{\alpha}_\mu\beta_2)^2\left[-\frac{1}{24}(\bar{\alpha}_\mu Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar{\alpha}_\mu Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

provided that the imaginary part $i\delta_2(\gamma)$ is

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

This solution satisfies also the initial condition: the solution is unique so it is the right one.

General Form of the Solution of All-Order BFKL equation

- The ansatz does not work, but it allows us to recover the structure of the NNLO solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k\,k'}\, e^{\left[\bar\alpha_s(k\,k')\,\chi_0(\nu)+\bar\alpha_s^2(k\,k')\chi_1(\nu)+\bar\alpha_s^3(k\,k')\,\chi_2(\nu)\right]Y}\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

$$\times\left\{1+(\bar\alpha_\mu\beta_2)^2\left[-\frac{1}{24}(\bar\alpha_\mu Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar\alpha_\mu Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar\alpha_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

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This solution satisfies also the initial condition: the solution is unique so it is the right one.

- The imaginary part $i\delta_2(\nu)$ has to be confirmed from the explicit calculation of the NNLO eigenfunction.

- Part of $\chi_2(\nu)$ has been already calculated: it is the NNLO BFKL in planar $\mathcal{N}$=4 SYM
  - Gromov, Levkovich-Maslyuka, Sizov (2015); Velizhanin (2015)

General Form of the Solution of All-Order BFKL equation

The eigenfunction of the NNLO BFKL equation is

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 \;+\; \bar{\alpha}_\mu \beta_2 \left(i\frac{\chi_0(\nu)}{2\chi_0'(\nu)}\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right.$$
$$\left. +\; \bar{\alpha}_\mu^2 f_2\left(\frac{k}{\mu},\nu\right) + \ldots\right]$$

- The function $f_2(k/\mu,\nu)$ denotes the NNLO corrections to the eigenfunctions.
- Ansatz for $f_2(k/\mu,\nu)$:

$$f_2(k/\mu,\nu) = c_0^{(2)}(\nu) + c_1^{(2)}(\nu)\ln\frac{k^2}{\mu^2} + c_2^{(2)}(\nu)\ln^2\frac{k^2}{\mu^2} + c_3^{(2)}(\nu)\ln^3\frac{k^2}{\mu^2} + \ldots$$

- We have completely determined the $c_0^{(2)}(\nu), c_1^{(2)}(\nu), c_2^{(2)}(\nu), c_3^{(2)}(\nu)$ and confirmed the ansatz for the structure of the NNLO solution of the NNLO BFKL equation.

Structure of the NNLO BFKL Solution

- From explicit calculation of $f_2(\gamma)$ we not only confirm the imaginary part

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

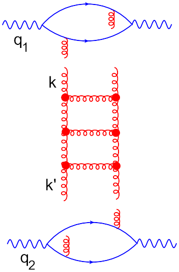
- but also confirm the structure of the NNLO BFKL solution obtained above in an indirect way

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]\,Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

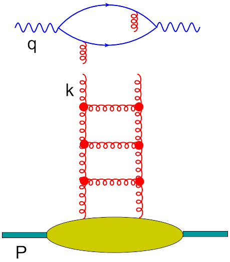
$$\times\left\{1+(\bar{\alpha}_\mu\beta_2)^2\left[-\frac{1}{24}(\bar{\alpha}_\mu\,Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar{\alpha}_\mu\,Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

- It looks like QCD is not just conformal part and running coupling contributions.

BFKL at NNLO and higher orders



VS.

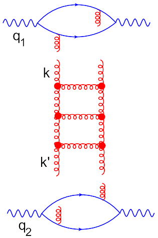


- At NNLO there are Pomeron loop contributions which are inevitable in the symmetric case of γ^*-γ^* case $\Rightarrow$ a simple generalization of BFKL at NNLO (and higher) does not exist unless we consider large N_c limit.

- In the asymmetric case of DIS, there are no pomeron loops. $\Rightarrow$ Linearization (with large N_c limit) of the Balitsky-Kovchegov equation at any order provides a systematic procedure to consistently define a BFKL type of evolution equation at any order.

- BFKL to all order: linearization of all order BK equation in large N_c limit.

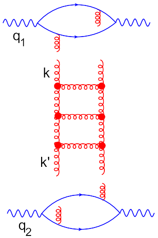
BFKL equation in DIS case



$K_{\rm BFKL}$ is $k \leftrightarrow k'$ symmetric

$Y^{\rm sym} = \ln \frac{s}{k\,k'}$

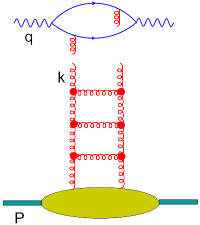
BFKL equation in DIS case



K_{BFKL} is $k \leftrightarrow k'$ symmetric

$Y^{\mathrm{sym}} = \ln \frac{s}{k\,k'}$

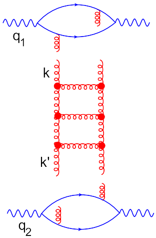
VS.



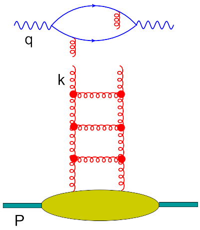
NLO $K^{\mathrm{DIS}}_{\mathrm{BFKL}}$ is not Symmetric

$Y^{\mathrm{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_{\mathrm{B}}}$

BFKL equation in DIS case



VS.



K_{BFKL} is $k \leftrightarrow k'$ symmetric

$Y^{\mathrm{sym}} = \ln \frac{s}{k\,k'}$

NLO $K^{\mathrm{DIS}}_{\mathrm{BFKL}}$ is not Symmetric

$Y^{\mathrm{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_{\mathrm{B}}}$

$$K^{\mathrm{DIS}}_{\mathrm{NLO}} = K^{\mathrm{sym}}_{\mathrm{NLO}} - \frac{1}{2}\int d^{D-2}q'\, K_{\mathrm{LO}}(q_1, q')\ln\frac{q'^2}{q^2}\, K_{\mathrm{LO}}(q, q_2) \qquad \mathrm{Fadin-Lipatov(1998)}$$

- K^{DIS}_{BFKL} is not symmetric $\Rightarrow$ eigenvalues not $\gamma \leftrightarrow 1-\gamma$ symmetric.
- $\Rightarrow$ Eigenvalues get an extra term: $\Delta^{\mathrm{DIS}}(\gamma) = \Delta^{\mathrm{sym}}(\gamma) - \frac{1}{2}\bar{\alpha}^2_\mu\, \chi_0(\gamma)\chi'(\gamma)$
- Reproduced lower order and predicted (and later confirmed) the 3-loop DGLAP anomalous dimension (Fadin-Lipatov (1998))

DGLAP anomalous dimension from NLO BFKL solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]\,(Y^{\rm DIS}+\ln\frac{k}{k'})}\, H_{\frac12+i\nu}(k)\, \left[H_{\frac12+i\nu}(k')\right]^*$$

$$\simeq \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]\,Y^{\rm DIS}}\, H_{\frac12+i\nu}(k)\left[H_{\frac12+i\nu}(k')\right]^*\left(1+\bar\alpha_\mu\chi_0(\nu)\ln\frac{k}{k'}\right)$$

$\Rightarrow$ perform partial integration and exponentiate the $Y^{\rm DIS}$-dependent terms $\Rightarrow$

$$G(k,k',Y^{\rm DIS}) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\left(\chi_1(\nu)+2\,i\,\chi_0(\nu)\chi_0'(\nu)\right)\right]\,Y^{\rm DIS}} H_{\frac12+i\nu}(k)\left[H_{\frac12+i\nu}(k')\right]^*$$

$$\times\left(1+\frac{i}{2}\,\bar\alpha_\mu\,\chi_0'(\nu)\right)$$

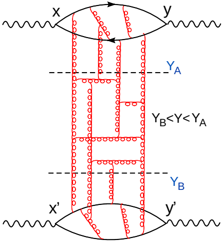
- $\Rightarrow$ $\Delta^{\rm DIS}(\gamma) = \bar\alpha_\mu\,\chi_0(\gamma)+\bar\alpha_\mu^2\,\chi_1(\gamma)-\frac12\,\bar\alpha_\mu^2\,\chi_0(\gamma)\chi_0'(\gamma)$

DGLAP anomalous dimension from NLO BFKL solution

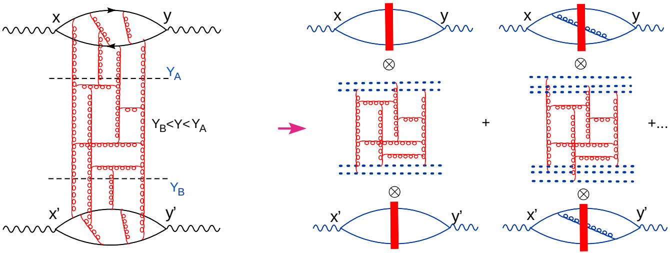
$$G(k,k',Y^{\mathrm{DIS}}) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\left(\chi_1(\nu)+2\,i\,\chi_0(\nu)\chi_0'(\nu)\right)\right]\,Y^{\mathrm{DIS}}} H_{\frac{1}{2}+i\,\nu}(k)\,\left[H_{\frac{1}{2}+i\,\nu}(k')\right]^* \times \left(1+\frac{i}{2}\,\bar\alpha_\mu\,\chi_0'(\nu)\right)$$

- $\Rightarrow \qquad \Delta^{\mathrm{DIS}}(\gamma) = \bar\alpha_\mu\,\chi_0(\gamma) + \bar\alpha_\mu^2\,\chi_1(\gamma) - \frac{1}{2}\,\bar\alpha_\mu^2\,\chi_0(\gamma)\chi_0'(\gamma)$
- Agrees with DGLAP 3-loop anomalous dimension.
- DIS eigenvalues are not symmetric in $\gamma \to 1-\gamma$

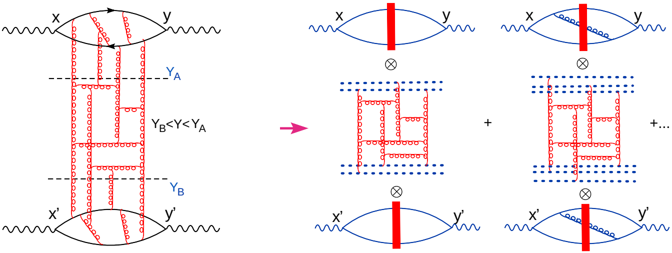
$\gamma^*\gamma^*$ scattering cross-section at NLO



$\gamma^*\gamma^*$ scattering cross-section at NLO



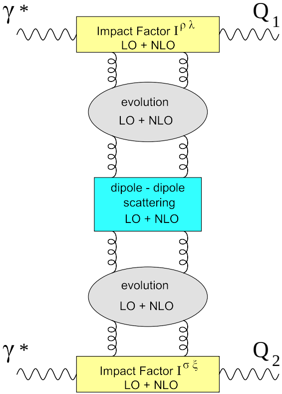
Using high-energy Operator Product Expansion in composite Wilson line operators we get NLO Impact Factor that does not scale with energy.

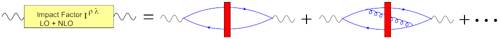


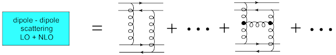
Composite Wilson line operators I. Balitsky and G.A.C.

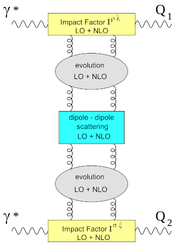
$$[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

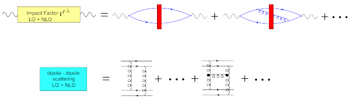
$\gamma^*\gamma^*$ scattering cross-section at NLO   G.A.C. and Yu. Kovchegov









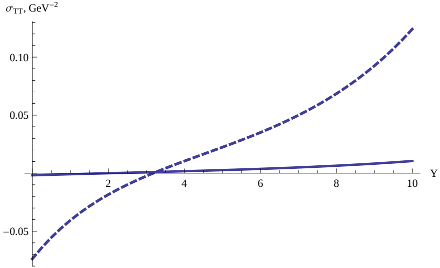


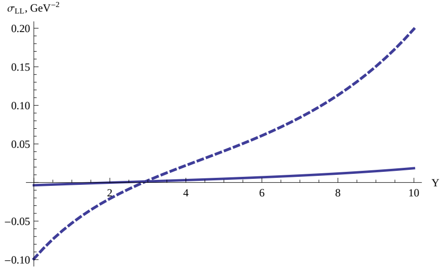
$$
\begin{aligned}
\sigma^{\gamma^*\gamma^*}_{\mathrm{LO+NLO}}(TT) &= \frac{1}{4}\sum_{\lambda_1,\lambda_2=\pm1} 32\pi^6 \varepsilon^{\lambda_1 *}_{\rho_1}(q_1)\varepsilon^{\lambda_1}_{\sigma_1}(q_1)\varepsilon^{\lambda_2 *}_{\rho_2}(q_2)\varepsilon^{\lambda_2}_{\sigma_2}(q_2)\frac{N_c^2-1}{N_c^2}\frac{\alpha_s(Q_1^2)\alpha_s(Q_2^2)}{Q_1Q_2} \\
&\times \int_{-\infty}^{\infty} d\nu \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\left(\frac{s}{Q_1Q_2}\right)^{\bar\alpha_s(Q_1Q_2)\,\chi_0(\nu)+\bar\alpha_s^2(Q_1Q_2)\chi_1(\nu)} \tilde I^{\rho_1\sigma_1}_{\mathrm{LO+NLO}}(q_1,\nu)\tilde I^{\rho_2\sigma_2}_{\mathrm{LO+NLO}}(q_2,-\nu) \\
&\times \Big[1+\bar\alpha_s(Q_1Q_2)\mathrm{Re}[F(\nu)]\Big]
\end{aligned}
$$

$\mathrm{Re}[F(\nu)]$ is the NLO dipole-dipole scattering projected on the LO eigenfunctions.

$$
\begin{aligned}
\sigma^{\gamma^*\gamma^*}_{\mathrm{LO+NLO}}(TT) &= \frac{1}{4}\sum_{\lambda_1,\lambda_2=\pm 1} 32\pi^6 \varepsilon^{\lambda_1 *}_{\rho_1}(q_1)\varepsilon^{\lambda_1}_{\sigma_1}(q_1)\varepsilon^{\lambda_2 *}_{\rho_2}(q_2)\varepsilon^{\lambda_2}_{\sigma_2}(q_2)\frac{N_c^2-1}{N_c^2}\frac{\alpha_s(Q_1^2)\alpha_s(Q_2^2)}{Q_1Q_2} \\
&\times \int_{-\infty}^{\infty} d\nu \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\left(\frac{s}{Q_1Q_2}\right)^{\bar{\alpha}_s(Q_1Q_2)\,\chi_0(\nu)+\bar{\alpha}_s^2(Q_1Q_2)\chi_1(\nu)} \tilde{I}^{\rho_1\sigma_1}_{\mathrm{LO+NLO}}(q_1,\nu)\tilde{I}^{\rho_2\sigma_2}_{\mathrm{LO+NLO}}(q_2,-\nu) \\
&\times\left[1+\bar{\alpha}_s(Q_1Q_2)\mathrm{Re}[F(\nu)]\right]
\end{aligned}
$$

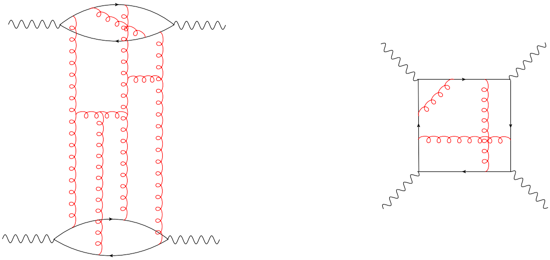
- NLO Impact factor is not symmetric in $\gamma \to 1-\gamma$
  BUT the full amplitude is!





- NLO $\gamma^*\gamma^*$ cross section with $Q_1 = Q_2 = 5$ GeV (dashed line) and $Q_1 = Q_2 = 10$ GeV (solid line) plotted as functions of rapidity Y.

Numerical results



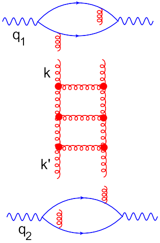
a) b)

- Left panel depicts a Feynman diagram that contributes to the high-energy asymptotics of $\gamma^*\gamma^*$ scattering. The right panel shows an example of the "box" diagram that are suppressed at high energies by a power of energy and, therefore, can be neglected. However, such types of diagrams become relevant, and therefore not anymore negligible at low rapidity where energy is not high enough to suppress such contributions.

NLO DIS with composite Wilson lines operator

- $\Delta^{\rm DIS}(\gamma) = \bar\alpha_\mu\,\chi_0(\gamma) + \bar\alpha_\mu^2\,\chi_1(\gamma) - \frac{1}{2}\,\bar\alpha_\mu^2\,\chi_0(\gamma)\chi_0'(\gamma)$

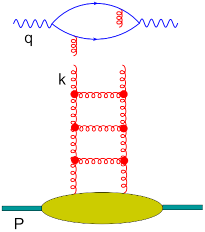
DIS eigenvalues are not symmetric in $\gamma \to 1-\gamma$



$K_{\rm BFKL}$ is $k \leftrightarrow k'$ symmetric

$Y^{\rm sym} = \ln\dfrac{s}{k\,k'}$

VS.



NLO $K^{\rm DIS}_{\rm BFKL}$ is not Symmetric

$Y^{\rm DIS} = \ln\dfrac{s}{k^2} \simeq \ln\dfrac{1}{x_{\rm B}}$

- NLO Impact Factor in front of the composite Wilson lines operator is not symmetric in $\gamma \to 1-\gamma$ either!
- So, evolution equation of composite Wilson lines operator is suitable for both $\gamma^*\gamma^*$ and DIS scattering processes!

Conclusions and Outlook

- At high energy (Regge limit) the dynamics is not linear $\Rightarrow$ need non linear evolution equations.
- Guiding line for the systematics of the high-energy OPE is conformal invariance in 2-d coordinates space order by order.
- The NLO Impact Factor obtained through the OPE in terms of composite Wilson lines operators is conformal invariant and energy independent.
- NLO Impact factor for pomeron exchange in momentum space has been obtained (aloso available in coordinate space and Mellin space).
- To get the DIS cross section at NLO at highe energy one has to convolute the NLO impact factor with the evolution equation of $\mathrm{tr}\{U_{x_1}U^\dagger_{x_2}\}\mathrm{tr}\{U_{x_3}U^\dagger_{x_4}\}$ at LO and the LO Impact factor with NLO BK equation.

Conclusions and Outlook

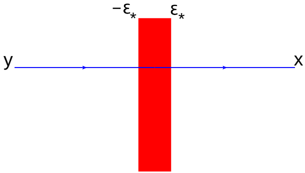
- Is the symmetry $\gamma \to 1-\gamma$ related to conformal invariance? Probably not. It seems to be related to the *intrinsic* symmetry of the amplitude
  - $\gamma^*\gamma^*$ amplitude is upside down symmetric: it has $\gamma \to 1-\gamma$ symmetry.
  - DIS amplitude is not upside down symmetric: it does not have $\gamma \to 1-\gamma$ symmetry.

- Evolution equation of composite Wilson lines operator is suitable for both $\gamma^*\gamma^*$ and DIS scattering processes!

Outlook

- Include spin dynamics at high-energy Wilson line formalism

Shock-wave with finite width



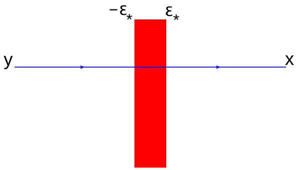
$x_* = \sqrt{\frac{s}{2}}x^+ \qquad x_\bullet = \sqrt{\frac{s}{2}}x^-$

$$
\begin{aligned}
A_\bullet(x_\bullet, x_*, x_\perp) &\rightarrow \lambda A_\bullet(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp)\\
A_*(x_\bullet, x_*, x_\perp) &\rightarrow \lambda^{-1} A_*(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp)\\
A_\perp(x_\bullet, x_*, x_\perp) &\rightarrow A_\perp(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp)
\end{aligned}
$$

λ is the boost parameter

- $p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$
- small α gluons are classical fields large α gluons are quantum fields.
- Longitudinal size classical fields: $\epsilon_* = \frac{\alpha s}{l_\perp^2}$ with $l_\perp$ trans. mom. of classical fileds
- Distance traveled by quantum fields: $z_* = \frac{\alpha s}{k_\perp^2}$ with $k_\perp$ trans. mom. of classical fileds
- We are in the case $l_\perp \sim k_\perp$

Shock-wave with finite width



$x_* = \sqrt{\frac{s}{2}}x^+ \qquad x_\bullet = \sqrt{\frac{s}{2}}x^-$

$$\begin{aligned} A_\bullet(x_\bullet, x_*, x_\perp) &\to \lambda A_\bullet(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_*(x_\bullet, x_*, x_\perp) &\to \lambda^{-1} A_*(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \\ A_\perp(x_\bullet, x_*, x_\perp) &\to A_\perp(\lambda^{-1}x_\bullet, \lambda x_*, x_\perp) \end{aligned}$$

λ is the boost parameter

sub-eikonal terms go like $\frac{1}{\lambda}$

$$\langle x|\frac{i}{\not{P} + i\epsilon}|y\rangle \;\to\; \langle x|\not{P}\,\frac{i}{p^2 + 2\alpha A_\bullet + ig\frac{2}{s}\not{p}_2\gamma^i F_{\bullet i} + \frac{1}{2}F_{ij}\sigma^{ij} + \ldots + i\epsilon}|y\rangle$$

- Note: $[\hat{\alpha}, \hat{A}^{cl}_\mu] = 0$ with $\alpha = \sqrt{\frac{2}{s}}p^+$ and $\not{p}_2 \propto \gamma^+$

$$e^{i\frac{\hat{p}_\perp^2}{\alpha s}z_*}\,\hat{A}_\bullet(z_*)\,e^{-i\frac{\hat{p}_\perp^2}{\alpha s}z_*} \simeq A_\bullet(z_*) - \frac{z_*}{\alpha s}\{p^i, F_{\bullet i}(z_*)\} - \frac{z_*^2}{2\alpha^2 s^2}\{p^j, \{p^i, D_j F_{\bullet i}(z_*)\}\} + \ldots$$

Quark propagator with sub-eikonal corrections

$x_* = \sqrt{\frac{s}{2}}\,x^+ \qquad x_\bullet = \sqrt{\frac{s}{2}}\,x^- \qquad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$

$$\langle x|\frac{i}{\hat{P\!\!\!/} + i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{d\!\!\!{}^-\alpha}{2s\alpha^2}\theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\!\!\!{}^-\alpha}{2s\alpha^2}\theta(y_* - x_*)\right] e^{-i\alpha(x_\bullet - y_\bullet)}\langle x_\perp|\, e^{-i\frac{\hat{p}_\perp^2}{\alpha s}x_*}$$
$$\times\left\{\hat{p\!\!\!/}\,p\!\!\!/_2\,[x_*, y_*]\,\hat{p\!\!\!/} + \hat{p\!\!\!/}\,p\!\!\!/_2\,\hat{\mathcal{O}}_1(x_*, y_*; p_\perp)\,\hat{p\!\!\!/} + \frac{1}{2}\hat{p\!\!\!/}\,p\!\!\!/_2\,\hat{\mathcal{O}}_2(x_*, y_*) - \frac{1}{2}\hat{\mathcal{O}}_2(x_*, y_*)\,p\!\!\!/_2\,\hat{p\!\!\!/}\right\} e^{i\frac{\hat{p}_\perp^2}{\alpha s}y_*}|y_\perp\rangle$$
$$+O(\lambda^{-2})$$

- Leading-eikonal term
- Sub-eikonal terms

Operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ *measure* the deviation from the straight line.

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Quark propagator with sub-eikonal corrections

$$x_* = \sqrt{\tfrac{s}{2}}\,x^+ \qquad x_\bullet = \sqrt{\tfrac{s}{2}}\,x^- \qquad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$\hat{\mathcal{O}}_1(x_*,y_*;p_\perp) = \frac{ig}{2\alpha}\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\Big([x_*,\omega_*]\frac{1}{2}\sigma^{ij}F_{ij}[\omega_*,y_*] + \{\hat{p}^i,[x_*,\omega_*]\,\frac{2}{s}\,\omega_*\,F_{i\bullet}(\omega_*)\,[\omega_*,y_*]\}$$
$$+g\int_{\omega_*}^{x_*} d\frac{2}{s}\,\omega'_*\,\frac{2}{s}(\omega_* - \omega'_*)[x_*,\omega'_*]F^i{}_\bullet[\omega'_*,\omega_*]\,F_{i\bullet}\,[\omega_*,y_*]\Big)$$

$$\hat{\mathcal{O}}_2(x_*,y_*;p_\perp) = \frac{ig}{2\alpha}\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\Big[[x_*,\omega_*]\frac{i}{4}\{(i\not{D}_\perp F_{ij}),\gamma^i\gamma^j\}[\omega_*,y_*] + \{\hat{p}^k,[x_*,\omega_*]iF_{kj}\gamma^j[\omega_*,y_*]\}$$
$$+\int_{\omega_*}^{x_*} d\frac{2}{s}\omega'_*\Big([x_*,\omega'_*]F^k{}_\bullet[\omega'_*,\omega_*]igF_{kj}\gamma^j[\omega_*,y_*] - [x_*,\omega'_*]igF_{kj}\gamma^j[\omega'_*,\omega_*]F^k{}_\bullet[\omega_*,y_*]\Big)$$
$$+\int_{\omega_*}^{x_*} d\frac{2}{s}\omega'_*\Big([x_*,\omega'_*]ig\frac{2}{s}F_{\bullet *}[\omega'_*,\omega_*]\gamma^k F_{k\bullet}[\omega_*,y_*]$$
$$-[x_*,\omega'_*]\gamma^k F_{k\bullet}[\omega'_*,\omega_*]ig\frac{2}{s}F_{\bullet *}[\omega_*,y_*]\Big) + (\hat{\alpha}\not{p}_1 - \hat{\not{p}}_\perp)[x_*,\omega_*]\,i\,\frac{2}{s}F_{\bullet *}[\omega_*,y_*]\Big]$$

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Gluon propagator in the background of gluon fields

$$i\langle A^a_\mu(x)A^b_\nu(y)\rangle \equiv iG^{ab}_{\mu\nu}(x,y) = \langle x|\frac{1}{\Box^{\mu\nu} - P^\mu P^\nu + \frac{1}{w}p_2^\mu p_2^\nu}|y\rangle^{ab}$$

with $\Box^{\mu\nu} = P^2 g^{\mu\nu} + 2i\, g\, F^{\mu\nu}$

$$\begin{aligned}
&\langle A^a_\mu(x)A^b_\nu(y)\rangle_A \\
&= \left[-\int_0^{+\infty}\frac{đ\,\alpha}{2\alpha}\theta(x_* - y_*) + \int_{-\infty}^0\frac{đ\,\alpha}{2\alpha}\theta(y_* - x_*)\right] e^{-i\alpha(x_\bullet - y_\bullet)} \\
&\times\langle x_\perp|e^{-i\frac{\hat{p}_\perp^2}{\alpha s}x_*}\left(\delta^\xi_\mu - \frac{p_{2\mu}}{p_*}p^\xi\right)\mathcal{O}_\alpha(x_*,y_*)\left(g_{\xi\nu} - p_\xi\frac{p_{2\nu}}{p_*}\right)e^{i\frac{\hat{p}_\perp^2}{\alpha s}y_*}|y_\perp\rangle^{ab} + i\langle x|\frac{p_{2\mu}p_{2\nu}}{p_*^2}|y\rangle^{ab} \\
&+ \left[-\int_0^{+\infty}\frac{đ\,\alpha}{2\alpha}\theta(x_* - y_*) + \int_{-\infty}^0\frac{đ\,\alpha}{2\alpha}\theta(y_* - x_*)\right] e^{-i\alpha(x_\bullet - y_\bullet)}\langle x_\perp|e^{-i\frac{\hat{p}_\perp^2}{\alpha s}x_*} \\
&\times\left[\mathfrak{G}^{ab}_{1\mu\nu}(x_*,y_*;p_\perp) + \mathfrak{G}^{ab}_{2\mu\nu}(x_*,y_*;p_\perp) + \mathfrak{G}^{ab}_{3\mu\nu}(x_*,y_*;p_\perp) + \mathfrak{G}^{ab}_{4\mu\nu}(x_*,y_*;p_\perp)\right] \\
&\times e^{i\frac{\hat{p}_\perp^2}{\alpha s}y_*}|y_\perp\rangle + O(\lambda^{-2}),
\end{aligned}$$

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Gluon propagator in the background of gluon fields

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$$\mathcal{O}_\alpha(x_*,y_*) \equiv \quad [x_*,y_*] + \frac{ig}{2\alpha}\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\Big(\{p^i,[x_*,\omega_*]\,\frac{2}{s}\,\omega_*\,F_{i\bullet}(\omega_*)\,[\omega_*,y_*]\}$$
$$+g\int_{\omega_*}^{x_*} d\frac{2}{s}\,\omega'_*\,\frac{2}{s}(\omega_*-\omega'_*)\,[x_*,\omega'_*]F^i{}_\bullet[\omega'_*,\omega_*]\;F_{i\bullet}\,[\omega_*,y_*]\Big)\,.$$

$$\mathfrak{G}^{ab}_{1\mu\nu}(x_*,y_*;p_\perp) = -\frac{g\,p_{2\mu}p_{2\nu}}{s^2\alpha^3}\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\Big[4p^i[x_*,\omega_*]F_{ij}[\omega_*,y_*]p^j$$
$$+ig\int_{\omega'_*}^{x_*} d\frac{2}{s}\omega'_*\,\frac{2}{s}(\omega'_*-\omega_*)[x_*,\omega'_*]iD^iF_{i\bullet}[\omega'_*,\omega_*]iD^jF_{j\bullet}[\omega_*,y_*]\Big]^{ab},$$

$$\mathfrak{G}^{ab}_{2\mu\nu}(x_*,y_*;p_\perp) = -\frac{g}{\alpha}\delta^i_\mu\delta^j_\nu\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\big([x_*,\omega_*]F_{ij}[\omega_*,y_*]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{3\mu\nu}(x_*,y_*;p_\perp) = \frac{g}{\alpha^2 s}\Big(\delta^j_\mu p_{2\nu}+\delta^j_\nu p_{2\mu}\Big)\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\,\big([x_*,\omega_*]iD^iF_{ij}[\omega_*,y_*]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{4\mu\nu}(x_*,y_*;p_\perp) = -\frac{2g^2}{\alpha^2 s}\int_{y_*}^{x_*} d\frac{2}{s}\omega_*\int_{\omega_*}^{x_*} d\frac{2}{s}\omega'_*\Big(\delta^j_\nu p_{2\mu}[x_*\omega'_*]F^i{}_\bullet[\omega'_*,\omega_*]F_{ij}[\omega_*,y_*]$$
$$+\delta^j_\mu p_{2\nu}[x_*\omega'_*]F_{ij}[\omega'_*,\omega_*]F^i{}_\bullet[\omega_*,y_*]\Big)^{ab}.$$

Gluon propagator in the background of quark fields

$$
\begin{aligned}
&\langle A^a_\mu(x)A^b_\nu(y)\rangle_{\psi,\bar\psi} \\
&= \Big[-\int_0^{+\infty}\!\! d\!\!\!{}^-\alpha\,\theta(x_*-y_*) + \int_{-\infty}^0\!\! d\!\!\!{}^-\alpha\,\theta(y_*-x_*)\Big]e^{-i\alpha(x_\bullet-y_\bullet)} \\
&\times\frac{g^2}{2\alpha^2 s^2}\int_{y_*}^{x_*}\! dz_{1*}\int_{y_*}^{z_{1*}}\! dz_{2*}\Bigg[\langle x_\perp|\,e^{-i\frac{\hat p_\perp^2}{\alpha s}x_*}\Big(g^\xi_{\perp\mu}-\frac{p_{2\mu}}{p_*}p^\xi_\perp\Big) \\
&\times\bar\psi(z_{1*})\gamma^\perp_\xi\not{p}_1\,[z_{1*},x_*]t^a[x_*,y_*]t^b[y_*,z_{2*}]\gamma^\sigma_\perp\psi(z_{2*})\Big(g^\perp_{\sigma\nu}-p^\perp_\sigma\frac{p_{2\nu}}{p_*}\Big)e^{i\frac{\hat p_\perp^2}{\alpha s}y_*}|y_\perp\rangle \\
&+\langle y_\perp|\,e^{-i\frac{\hat p_\perp^2}{\alpha s}y_*}\Big(g^\xi_{\perp\nu}-\frac{p_{2\nu}}{p_*}p^\xi_\perp\Big)\bar\psi(z_{2*})\gamma^\perp_\xi\not{p}_1[z_{2*},y_*]t^b[y_*,x_*]t^a[x_*,z_{1*}]\gamma^\sigma_\perp\psi(z_{1*}) \\
&\times\Big(g^\perp_{\sigma\mu}-p^\perp_\sigma\frac{p_{2\mu}}{p_*}\Big)e^{i\frac{\hat p_\perp^2}{\alpha s}x_*}|x_\perp\rangle\Bigg]+O(\lambda^{-2})
\end{aligned}
$$

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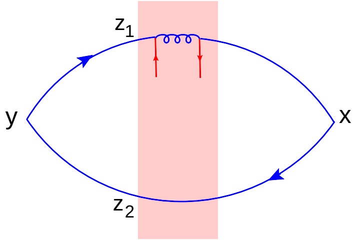
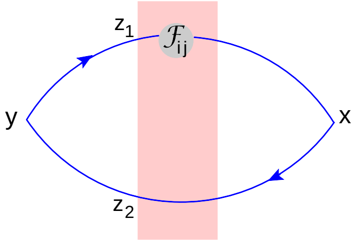
Quark propagator with sub-eikonal corrections





Let $|P\rangle$ be proton or nuclear target

$$\langle P|J^\mu(x)J^\nu(y)|P\rangle \;\rightarrow\; \langle J^\mu(x)J^\nu(y)\rangle_{A\,\psi\,\bar\psi} = \mathrm{Tr}\{\gamma^\mu\langle x|\frac{1}{\not{P}+i\epsilon}|y\rangle\,\gamma^\nu\langle y|\frac{1}{\not{P}+i\epsilon}|y\rangle\}$$

Quark propagator with sub-eikonal corrections: $\Rightarrow$ New evolution equations

OPE in Wilson lines with sub-eikonal corrections

$$\langle P|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|P\rangle \simeq \int d^2z_1 d^2z_2 \Bigg[I^{eikonal}_{\mu\nu}(z_{1\perp}, z_{2\perp}; x, y)\langle P|\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}|P\rangle$$
$$+ I^{sub\,eik}_{1\,\mu\nu}(z_1, z_2; x, y)\langle P|\Big(Q_1(z_{1\perp})\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\} - \frac{1}{N_c}\mathrm{Tr}\{\tilde{Q}_1(z_{1\perp})U^\dagger_{z_2}\}\Big) + h.c|P\rangle$$
$$+ I^{sub\,eik}_{5\,\mu\nu}(z_{1\perp}, z_2; x, y)\langle P|\Big(Q_5(z_{1\perp})\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\} - \frac{1}{N_c}\mathrm{Tr}\{\tilde{Q}_5(z_{1\perp})U^\dagger_{z_2}\Big) + h.c|P\rangle$$
$$+ 2\,C_F\, I^{sub\,eik}_{\mathcal{F}\,\mu\nu}(z_{1\perp}, z_2; x, y)\langle P|\mathrm{Tr}\{\mathcal{F}_{z_1}U^\dagger_{z_2}\} + h.c|P\rangle \Bigg]$$

OPE in Wilson lines with sub-eikonal corrections

$\not{p}_1 = \sqrt{\frac{s}{2}}\gamma^- \qquad x_* = \sqrt{\frac{s}{2}}x^+$

$$Q_1(x_\perp) = g^2\int_{-\infty}^{+\infty}dz_*\int_{-\infty}^{z'_*}dz'_*\,\bar{\psi}\,i\not{p}_1(z_*,x_\perp)[z_*,z'_*]_x\psi(z'_*)$$

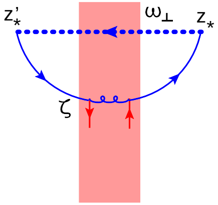
$$Q_5(x_\perp) = g^2\int_{-\infty}^{+\infty}dz_*\int_{-\infty}^{z'_*}dz'_*\,\bar{\psi}\gamma^5\not{p}_1(z_*,x_\perp)[z_*,z'_*]_x\psi(z'_*)$$

$$\tilde{Q}_1(x_\perp) = g^2\int_{-\infty}^{+\infty}dz_*\int_{-\infty}^{z_*}dz'_*[\infty p_1,z_*]_x\mathrm{tr}\{\psi(z_*,x_\perp)\bar{\psi}(z'_*,x_\perp)\,i\not{p}_1\}[z'_*,-p_1\infty]$$

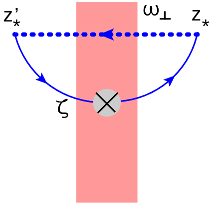
$$\tilde{Q}_5(x_\perp) = g^2\int_{-\infty}^{+\infty}dz_*\int_{-\infty}^{z_*}dz'_*[\infty p_1,z_*]_x\mathrm{tr}\{\psi(z_*,x_\perp)\bar{\psi}(z'_*,x_\perp)\,\gamma^5\not{p}_1\}[z'_*,-p_1\infty]$$

$$\mathcal{F}(x_\perp) = \frac{ig}{2C_F}\,\frac{s}{2}\int_{-\infty}^{+\infty}dz_*\,[\infty,z_*]_x\,\epsilon^{ij}F_{ij}(z_*,x_\perp)[z_*,-\infty]_x$$

Sample of diagrams for the quark operator Q_1 or Q_5



a)



b)

Evolution equation flavor non-singlet case

Unpolarized case

$$\langle Q_{1\,\omega}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2\zeta\,\frac{1}{(\omega-\zeta)^2_\perp}\Big(\frac{1}{2}Q_{1\,\zeta}\mathrm{Tr}\{U_\zeta U^\dagger_\omega\} - \frac{1}{2N_c}\mathrm{Tr}\{U^\dagger_\omega \tilde{Q}_{1\,\zeta}\}\Big)$$

$$\langle \mathrm{Tr}\{U^\dagger_z \tilde{Q}_{1\,\omega}\}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2\zeta\,\Big[\frac{1}{(\omega-\zeta)^2_\perp}\Big(\frac{1}{2}\mathrm{Tr}\{U^\dagger_z U_\zeta\}Q_{1\,\zeta} - \frac{1}{2N_c}\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{1\,\zeta}\}\Big)$$
$$+\frac{(\omega-z)^2_\perp}{(\omega-\zeta)^2_\perp(\zeta-z)^2_\perp}\Big(\mathrm{Tr}\{U_\zeta U^\dagger_z\}\mathrm{Tr}\{U^\dagger_\zeta \tilde{Q}_{1\,\omega}\} - (\frac{1}{N_c}+C_F)\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{1\,\omega}\}\Big)\Big]$$

Polarized case

$$\langle Q_{5\,\omega}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2\zeta\,\frac{1}{(\omega-\zeta)^2_\perp}\Big(\frac{1}{2}\mathrm{Tr}\{U^\dagger_\omega U_\zeta\}\,Q_{5\,\zeta} - \frac{1}{2N_c}\mathrm{Tr}\{U^\dagger_\omega \tilde{Q}_{5\,\zeta}\}\Big)$$

$$\langle \mathrm{Tr}\{U^\dagger_z \tilde{Q}_{5\,\omega}\}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2\zeta\,\Big[\frac{1}{(\omega-\zeta)^2_\perp}\Big(\frac{1}{2}\mathrm{Tr}\{U^\dagger_z U_\zeta\}Q_{5\,\zeta} - \frac{1}{2N_c}\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{5\,\zeta}\}\Big)$$
$$+\frac{(\omega-z)^2_\perp}{(\omega-\zeta)^2_\perp(\zeta-z)^2_\perp}\Big(\mathrm{Tr}\{U_\zeta U^\dagger_z\}\mathrm{Tr}\{U^\dagger_\zeta \tilde{Q}_{5\,\omega}\} - (\frac{1}{N_c}+C_F)\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{5\,\omega}\}\Big)\Big]$$

Work in progress