Parton Distribution Functions and LHC phenomenology

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Towards accuracy at small x, Edinburgh, 10 September 2019

Outline

- Motivations (from theory and experiments)
- Theory overview
- Small-x resummation in the evolution
- Small-x resummation in DIS
- PDF fits
- Small-x resummation in LHC observables
- Phenomenology at LHC and future colliders

I will also keep an eye on desired future developments

Context: collinear factorization

Collinear factorization: $$\sigma(x,Q^2) = \int_x^1 \frac{dz}{z}\, C_i\big(z,\alpha_s(Q^2)\big)\, f_i\Big(\frac{x}{z},Q^2\Big)$$

DGLAP evolution: $$\mu^2 \frac{d}{d\mu^2} f_i(x,\mu^2) = \int_x^1 \frac{dz}{z}\, P_{ij}(z,\alpha_s(\mu^2))\, f_j\Big(\frac{x}{z},\mu^2\Big)$$

Heavy-quark matching: $$f_i^{[n_f+1]}(x,\mu_m^2) = \int_x^1 \frac{dz}{z}\, A_{ij}(z,\alpha_s(\mu_m^2))\, f_j^{[n_f]}\Big(\frac{x}{z},\mu_m^2\Big)$$

Any object with a perturbative expansion can exhibit a logarithmic enhancement:

- coefficient functions $C(x,\alpha_s)$ (observable)
- splitting functions $P(x,\alpha_s)$ and matching conditions $A(x,\alpha_s)$ (evolution)

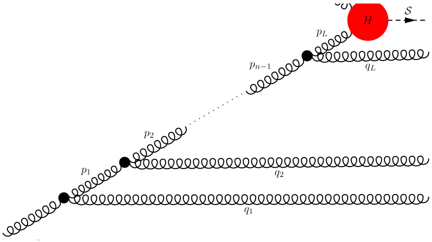
Small-x logarithms: single logs $\alpha_s^n \frac{1}{x} \log^k \frac{1}{x} \quad (0 \le k \le n-1)$

When $\alpha_s \log \frac{1}{x} \sim 1$ perturbativity is spoiled $\to$ **all-order resummation needed**

In $\overline{\text{MS}}$ and related schemes, both coefficient $C(x,\alpha_s)$ and splitting $P(x,\alpha_s)$ functions, and also matching conditions $A(x,\alpha_s)$, are logarithmically enhanced at small x (in the singlet sector).

Why singlet?

Small-x logarithms appear from integration of t-channel gluon exchanges $\to$ needs an intermediate gluon ladder



I can focus on gluon and quark-singlet channels
(exception: subtraction of collinear logarithms/singularities $\to$ more later)

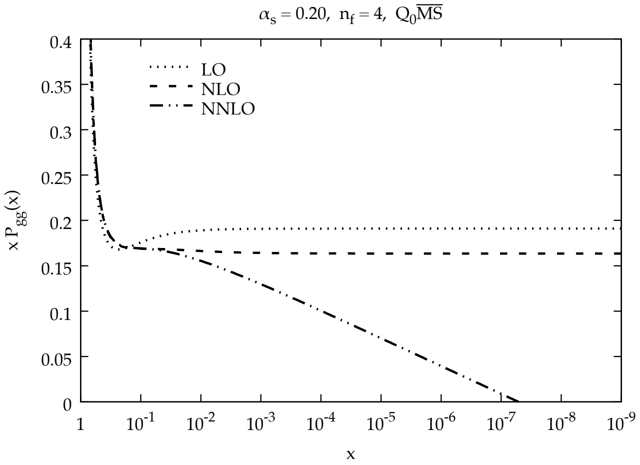
Singlet DGLAP evolution couples gluon and quark-singlet PDFs:

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} f_{\mathsf{g}} \\ f_{\mathsf{q}} \end{pmatrix} = \begin{pmatrix} P_{\mathsf{gg}} & P_{\mathsf{gq}} \\ P_{\mathsf{qg}} & P_{\mathsf{qq}} \end{pmatrix} \otimes \begin{pmatrix} f_{\mathsf{g}} \\ f_{\mathsf{q}} \end{pmatrix}$$

$$f_{\mathsf{q}} = \sum_i (f_{q_i} + f_{\bar{q}_i})$$

Small-x logarithms in the splitting functions

Only **singlet sector** affected: $\boldsymbol{P_{gg}}$, P_{gq}, P_{qg}, P_{qq}



fixed-order $xP_{gg}(x, \alpha_s)$ splitting function at small x:

LO:
$\alpha_s \times \text{const}$

NLO:
$\alpha_s^2 \left(\cancel{\ln \frac{1}{x}} + \text{const}\right)$

NNLO:
$\alpha_s^3 (\cancel{\ln^2 \frac{1}{x}} + \ln \frac{1}{x} + \text{const})$

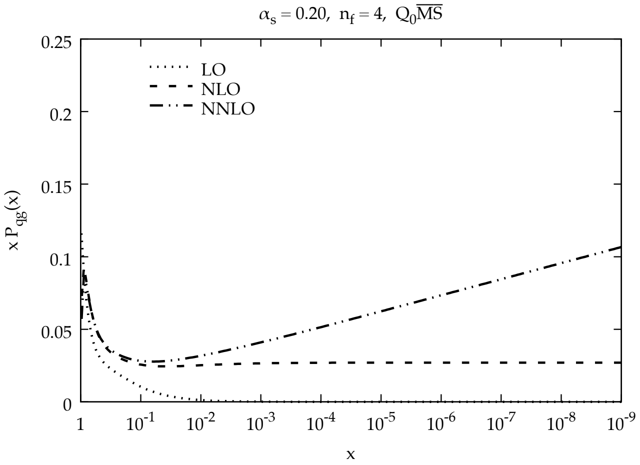
N^{3}LO:
$\alpha_s^4 (\ln^3 \frac{1}{x} + \ln^2 \frac{1}{x} + \ln \frac{1}{x} +$

accidental zeros!

Logarithms start to grow for $x \lesssim 10^{-2}$ (for $Q \sim 5\text{GeV}$)
$\to$ very large resummation region!

Small-x logarithms in the splitting functions

Only **singlet sector** affected: P_{gg}, P_{gq}, $\boldsymbol{P}_{\mathrm{qg}}$, P_{qq}



fixed-order $xP_{\mathrm{qg}}(x,\alpha_s)$ splitting function at small x:

LO:
$\alpha_s \times 0$

NLO:
$\alpha_s^2 \times \mathrm{const}$

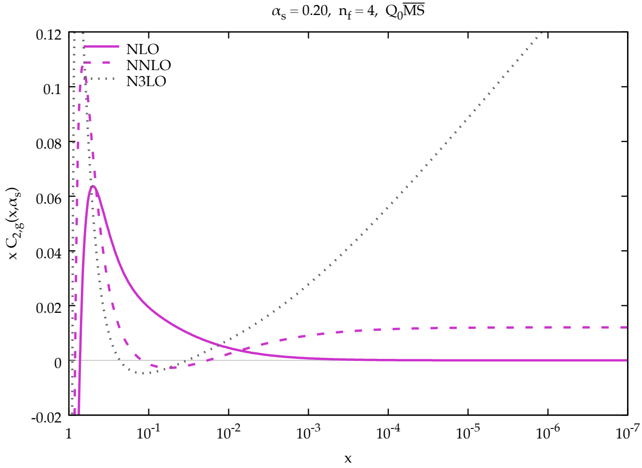
NNLO:
$\alpha_s^3 \left(\ln\frac{1}{x} + \mathrm{const}\right)$

N^{3}LO:
$\alpha_s^4 \left(\ln^2\frac{1}{x} + \ln\frac{1}{x} + \mathrm{const}\right)$

P_{qg} and P_{qq} are NLL quantities, while P_{gg} and P_{gq} are LL
Small-x resummation is gluon-driven

Small-x logarithms in DIS coefficient functions

Only **singlet sector** affected: $\boldsymbol{C_{a,g}}$, $C^{\mathrm{S}}_{a,q}$, $a = \mathbf{2}, L, 3$



fixed-order $xC_{2,g}(x, \alpha_s)$ splitting function at small x:

LO:
0

NLO:
$\alpha_s \times 0$

NNLO:
$\alpha_s^2 \times \text{const}$

N^{3}LO:
$\alpha_s^3 \left(\ln \frac{1}{x} + \text{const}\right)$

DIS coefficient functions are NLL quantities

Small-x logarithms in DIS coefficient functions

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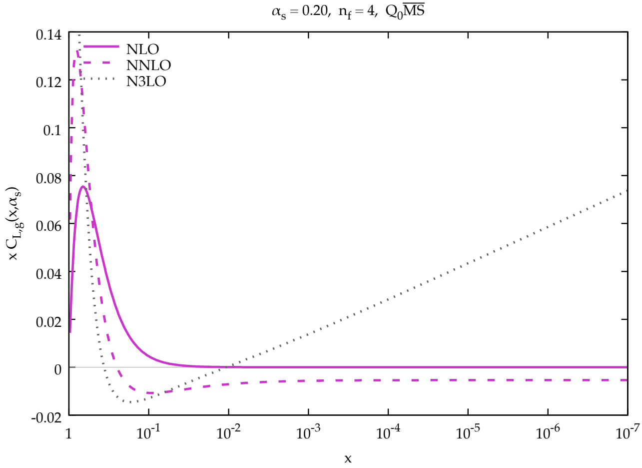


fixed-order $xC_{L,g}(x, \alpha_s)$ splitting function at small x:

LO:
0

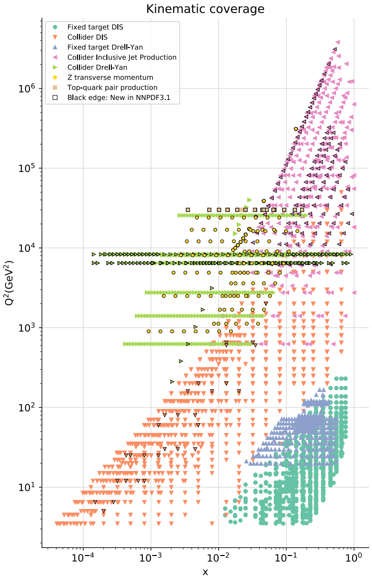
NLO:
$\alpha_s \times 0$

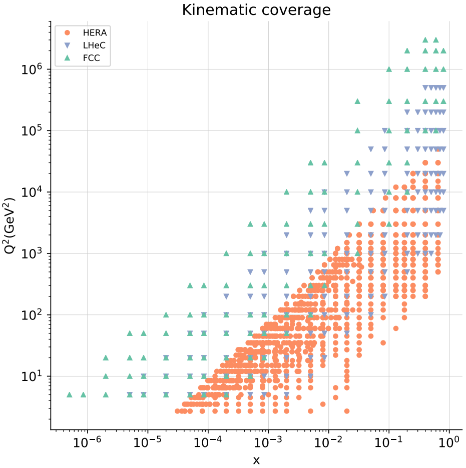
NNLO:
$\alpha_s^2 \times \text{const}$

N^{3}LO:
$\alpha_s^3 \left(\ln \frac{1}{x} + \text{const}\right)$

DIS coefficient functions are NLL quantities

Do we reach those x values?

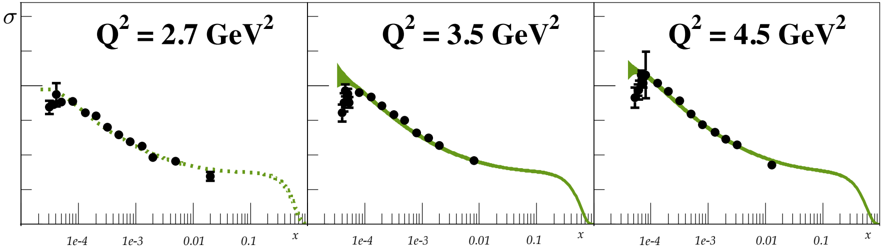




Experimental motivation: Low x data at HERA

Deep-inelastic scattering (DIS) data from HERA extend down to $x \sim 3 \times 10^{-5}$

Tension between HERA data at low Q^2 and low x with fixed-order theory



Also leads to a deterioration of the χ^2 when including low-Q^2 data in PDF fits

Attempts to explain this deviation with higher twists, saturation models, ... do we really need them?

Maybe it's just missing resummation of small-x logarithms...

The theory of small-x resummation

Warming up: Mellin space

It is convenient to work in Mellin space

$$F(N) = \int_0^1 dx\, x^N\, F(x) \qquad \text{unusual convention!}$$

In this way, small-x logs correspond to poles in $N = 0$, more precisely

$$\int_0^1 dx\, x^N\, \frac{1}{x} \log^k \frac{1}{x} = \frac{k!}{N^{k+1}}$$

The log structure translates into

$$\alpha_s^n \frac{1}{x} \log^k \frac{1}{x} \quad 0 \le k \le n-1 \qquad \leftrightarrow \qquad \frac{\alpha_s^n}{N^k} \quad 0 \le k \le n$$

which leads to a resummed expansion of the form

$$F(N, \alpha_s) = \underbrace{F_s\left(\frac{\alpha_s}{N}\right)}_{\text{LL}} + \underbrace{\alpha_s F_{ss}\left(\frac{\alpha_s}{N}\right)}_{\text{NLL}} + \underbrace{\alpha_s^2 F_{sss}\left(\frac{\alpha_s}{N}\right)}_{\text{NNLL}} + \ldots$$

namely, a perturbative expansion in powers of α_s at fixed α_s/N

Ingredients for resummation

Small-x resummation is based on the interplay of

Collinear factorization: $$\sigma(x,Q^2) = \int_x^1 \frac{dz}{z}\, C_i\big(z,\alpha_s(Q^2)\big)\, f_i\Big(\frac{x}{z},Q^2\Big)$$

DGLAP evolution: $$\mu^2\frac{d}{d\mu^2} f_i(x,\mu^2) = \int_x^1 \frac{dz}{z}\, P_{ij}\big(z,\alpha_s(\mu^2)\big)\, f_j\Big(\frac{x}{z},\mu^2\Big)$$

with

k_t factorization: $$\sigma(x,Q^2) = \int_x^1 \frac{dz}{z}\int_0^\infty dk_t^2\, \mathcal{C}_i\big(z,k_t^2,\alpha_s\big)\, \mathcal{F}_i\Big(\frac{x}{z},k_t^2\Big)$$

BFKL evolution: $$-x\frac{d}{dx}\mathcal{F}_{\rm g}(x,k_t^2) = \int_0^\infty \frac{dq_t^2}{k_t^2}\, \mathcal{K}\Big(\frac{k_t^2}{q_t^2},\alpha_s(\cdot)\Big)\, \mathcal{F}_{\rm g}\big(x,q_t^2\big)$$

- $\mathcal{F}_i\big(x,k_t^2\big)$: unintegrated ($k_t$-dependent) PDF
- $\mathcal{C}_i\big(z,k_t^2,\alpha_s\big)$: off-shell coefficient function

Consistency between equations allows to resum small-x logs:

- DGLAP + BFKL eqns $\to$ resum $P(x,\alpha_s)$ *(Stefano's talk)*
- collinear + k_t factorizations $\to$ resum $C(x,\alpha_s)$ and heavy quark matching $A(x,\alpha_s)$

The key: relation between unintegrated and integrated PDFs (1)

The relation between standard and unintegrated PDFs is [is it valid beyond (N)LL?]

$$\mathcal{F}_{\mathsf{g}}(N, k_t^2) = R(N, \alpha_s) \frac{d}{dk_t^2} f_{\mathsf{g}}(N, k_t^2)$$

where R is a scheme-dependent function. In $\overline{\mathsf{MS}}$ we have

$$R_{\overline{\mathsf{MS}}}(N, \alpha_s) = 1 + \mathcal{O}(\alpha_s^3)$$

In small-x theory, one usually defines the $Q_0\overline{\mathsf{MS}}$ scheme by

$$R_{Q_0\overline{\mathsf{MS}}}(N, \alpha_s) = 1,$$

which is equal to $\overline{\mathsf{MS}}$ up to N^{3}LO, such that

$$\mathcal{F}_{\mathsf{g}}(N, k_t^2) = \frac{d}{dk_t^2} f_{\mathsf{g}}^{Q_0\overline{\mathsf{MS}}}(N, k_t^2)$$

From now on we work in $Q_0\overline{\mathsf{MS}}$ and drop the label

more on Francesco's talk

Evolution in greater detail (1)

DGLAP in Mellin space (splitting functions $P_{ij} \to$ anomalous dimensions γ_{ij})

$$\mu^2 \frac{d}{d\mu^2}\begin{pmatrix} f_\mathrm{g}(N,\mu^2) \\ f_\mathrm{q}(N,\mu^2) \end{pmatrix} = \begin{pmatrix} \gamma_\mathrm{gg}(N,\alpha_s(\mu^2)) & \gamma_\mathrm{gq}(N,\alpha_s(\mu^2)) \\ \gamma_\mathrm{qg}(N,\alpha_s(\mu^2)) & \gamma_\mathrm{qq}(N,\alpha_s(\mu^2)) \end{pmatrix} \begin{pmatrix} f_\mathrm{g}(N,\mu^2) \\ f_\mathrm{q}(N,\mu^2) \end{pmatrix}$$

can be diagonalised

$$\mu^2 \frac{d}{d\mu^2}\begin{pmatrix} f_+(N,\mu^2) \\ f_-(N,\mu^2) \end{pmatrix} = \begin{pmatrix} \gamma_+(N,\alpha_s(\mu^2)) & 0 \\ 0 & \gamma_-(N,\alpha_s(\mu^2)) \end{pmatrix} \begin{pmatrix} f_+(N,\mu^2) \\ f_-(N,\mu^2) \end{pmatrix}$$

through a transformation matrix $T(N,\alpha_s(\mu^2))$ as

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = T\begin{pmatrix} f_\mathrm{g} \\ f_\mathrm{q} \end{pmatrix}, \qquad \begin{pmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{pmatrix} = T\begin{pmatrix} \gamma_\mathrm{gg} & \gamma_\mathrm{gq} \\ \gamma_\mathrm{qg} & \gamma_\mathrm{qq} \end{pmatrix} T^{-1} + \mu^2 \frac{dT}{d\mu^2} T^{-1}$$

Fact: in suitable factorisation schemes γ_- **is not singular in** $\boldsymbol{N = 0}$

$$\gamma_-(N,\alpha_s) = \underbrace{\alpha_s \times \mathcal{O}(N^0)}_{\text{NLL}} + \underbrace{\alpha_s^2 \times \mathcal{O}(N^0)}_{\text{NNLL}} + \ldots$$

In such schemes **only the "+" eigenvalue/vector is enhanced at small** x**!**
$\to$ only γ_+ needs to be resummed, using duality with BFKL *(Stefano's talk)*

Evolution in greater detail (2)

$$\begin{pmatrix} \gamma_{\mathsf{gg}} & \gamma_{\mathsf{gq}} \\ \gamma_{\mathsf{qg}} & \gamma_{\mathsf{qq}} \end{pmatrix} \overset{\overline{\mathsf{MS}}\text{-like}}{=} \underbrace{\begin{pmatrix} \gamma_s & \frac{C_F}{C_A}\gamma_s \\ 0 & 0 \end{pmatrix}}_{\mathsf{LL}} + \underbrace{\alpha_s \begin{pmatrix} \gamma_{\mathsf{gg},ss} & \gamma_{\mathsf{gq},ss} \\ \gamma_{\mathsf{qg},ss} & \frac{C_F}{C_A}\left[\gamma_{\mathsf{qg},ss} - \frac{n_f}{3\pi}\right] \end{pmatrix}}_{\mathsf{NLL}} + \ldots$$

A diagonalising matrix T is ($t_\pm \neq 0$ to guarantee invertibility)

$$T = \begin{pmatrix} t_+ & \frac{C_F}{C_A} t_+ \\ 0 & t_- \end{pmatrix} + \alpha_s \begin{pmatrix} 0 & \frac{\gamma_{\mathsf{gq},ss} - \frac{C_F}{C_A}\gamma_{\mathsf{gg},ss} - \left(\frac{C_F}{C_A}\right)^2 \frac{n_f}{3\pi}}{\gamma_s} t_+ \\ -\frac{\gamma_{\mathsf{qg},ss}}{\gamma_s} t_- & 0 \end{pmatrix} + \ldots$$

giving

$$f_+ = t_+\left[f_{\mathsf{g}} + \frac{C_F}{C_A} f_{\mathsf{q}} + \ldots\right] \qquad \gamma_+ = \gamma_s + \alpha_s\left[\gamma_{\mathsf{gg},ss} + \frac{C_F}{C_A}\gamma_{\mathsf{qg},ss} - \beta_0\alpha_s \frac{d}{d\alpha_s}\log t_+\right] + \ldots$$

$$f_- = t_-[f_{\mathsf{q}} + \ldots] \qquad \gamma_- = \alpha_s\left[-\frac{C_F}{C_A}\frac{n_f}{3\pi} - \beta_0\alpha_s \frac{d}{d\alpha_s}\log t_-\right] + \ldots$$

Comments:

- at LL only gluon contributes, resummation encoded in a single function γ_s
- freedom: normalization $t_\pm$ of $f_\pm$, equivalent to scheme choice
- $t_\pm$ affects $\gamma_\pm$ at NLL, unless $t_\pm = \text{const}_\pm$
- the scheme change from $\overline{\mathsf{MS}}$ to $Q_0\overline{\mathsf{MS}}$ is achieved by diagonalising with $t_+ = R_{\overline{\mathsf{MS}}}$ and then transforming back to the flavour basis with $t_+ = 1$, and vice versa.

Relation between unintegrated and integrated PDFs (2)

Starting from $Q_0\overline{\text{MS}}$, we diagonalise with $t_+ = 1, t_- = -\frac{C_F}{C_A}$ and get

$$\begin{aligned}\mathcal{F}_g(N, k_t^2) &= \frac{d}{dk_t^2}\left[f_+(N, k_t^2) + f_-(N, k_t^2)\right] \qquad \text{(valid up to NLL at least)} \\ &= \frac{d}{dk_t^2}\left[U_+(N, k_t^2, \mu^2) f_+(N, \mu^2) + U_-(N, k_t^2, \mu^2) f_-(N, \mu^2)\right] \\ &= \frac{d}{dk_t^2} U_+(N, k_t^2, \mu^2) f_+(N, \mu^2) + \delta(k_t^2) f_-(N, \mu^2) + \mathcal{O}(\alpha_s N^0)\end{aligned}$$

where $U_\pm$ is the solution of the diagonal DGLAP evolution

$$U_\pm(N, k_t^2, \mu^2) = \exp \int_{\mu^2}^{k_t^2} \frac{d\nu^2}{\nu^2} \gamma_\pm(N, \alpha_s(\nu^2))$$

and having used

$$U_-(N, k_t^2, \mu^2) = 1 + \mathcal{O}(\alpha_s N^0)$$

Since f_- is not enhanced at small x, it's now clear that BFKL is dual to DGLAP for f_+, and that the duality can be used to resum γ_+

Resummation of the DGLAP anomalous dimensions γ_{ij}:

- resum γ_+ through DGLAP-BFKL duality *(Stefano's talk)*
- resum $\gamma_{\rm qg}$ *(see later)*
- take γ_- at fixed order
- transform back to the flavour basis

Missing ingredient: $\gamma_{{\rm gq},ss}$ How can I get it?

At LL, we have a color-charge relation

$$\gamma_{{\rm gq},s} = \frac{C_F}{C_A}\gamma_{{\rm gg},s} \qquad (\gamma_{{\rm gg},s} = \gamma_s)$$

but beyond LL it is violated. At the moment we do not know how to resum it.

The state of the art is

$$\begin{pmatrix} \gamma_{\rm gg} & \gamma_{\rm gq} \\ \gamma_{\rm qg} & \gamma_{\rm qq} \end{pmatrix} = \begin{pmatrix} \text{LL+NLL} & \text{LL} \\ \text{NLL} & \text{NLL} \end{pmatrix}$$

Only for $\gamma_{\rm gg}$ two non-trivial logarithmic orders are known....
How about the (resummed) perturbative stability of the other entries?

Recent developments on the resummation of splitting functions

After approximately 10 years from the publication of stable NLO+NLL results, we have recently revived the small-x business

[MB,Marzani,Peraro 1607.02153][MB,Marzani,Muselli 1708.07510][MB,Marzani 1805.06460]

Many improvements to the Altarelli,Ball,Forte procedure and new developments, most notably:

- various technical improvements [e.g. we now use a (physical) collinear approximation to perform the resummation of running coupling contributions, while originally a (unphysical) quadratic approximation was used]
- estimate of the uncertainty from subleading logs
- match resummation to NNLO, allowing **NNLO+NLL phenomenology**
- made a prediction of the yet unknown **N^{3}LO** splitting functions
- computed all ingredients to match resummation to N^{3}LO
- we released (and keep developing) a public code
  `HELL: High-Energy Large Logarithms` `www.ge.infn.it/~bonvini/hell`
  which delivers resummed splitting functions and coefficient functions
- `HELL` interfaced to `APFEL` (`apfel.hepforge.org`) $\to$ PDF fits

How HELL works

HELL is actually a family of three codes:

1. preHELL (private)

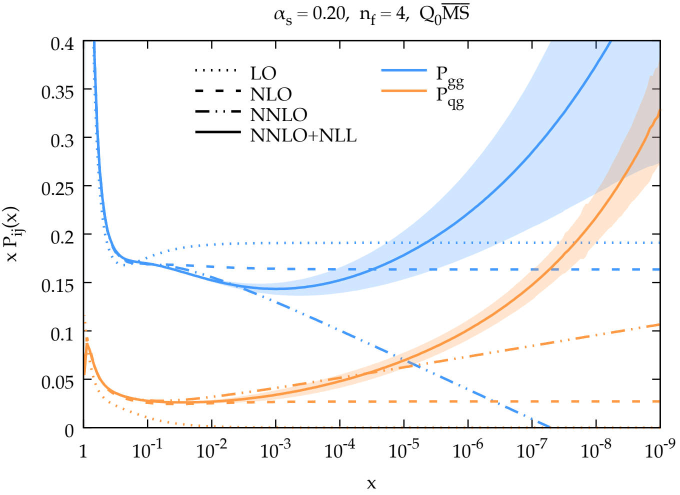
   resummation of the eigenvalue $\gamma_+(N)$, along the contour for Mellin inversion
   - duality with BFKL evolution kernel
   - symmetry of the BFKL kernel
   - momentum conservation
   - resummation of subleading (but dominant) running coupling effects

2. HELL (public, slow, uses results from preHELL)

   resummation of everything else and Mellin inversion
   - $\gamma_{\rm qg}$
   - coefficient functions (for each considered process)
   - heavy-quark matching functions

3. HELL-x (public, interpolates pre-tabulated functions, fast, used in APFEL)
   - construction of the splitting function matrix
   - matching to fixed order

Some representative HELL results: splitting functions



All-order behaviour rather different from fixed order (especially for P_{gg})

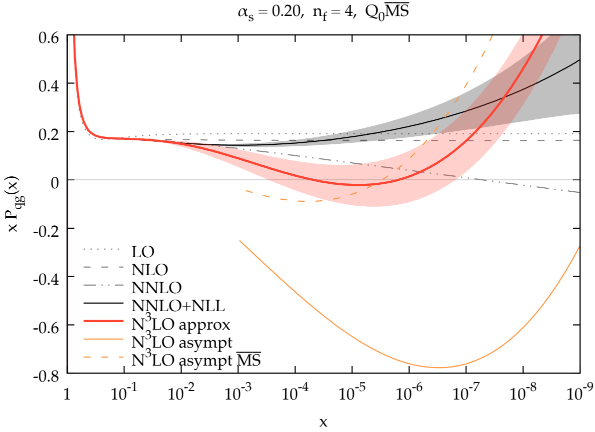
$P_{gg} > P_{qg}$ at resummed level (at NNLO they swap at some x)

Towards N^{3}LO evolution

Recent impressive progress towards N^{3}LO splitting functions

[Davies,Vogt,Ruijl,Ueda,Vermaseren 1610.07477] [Moch,Ruijl,Ueda,Vermaseren,Vogt 1707.08315]

At small x, approximate predictions from NLL resummation [MB,Marzani 1805.06460]

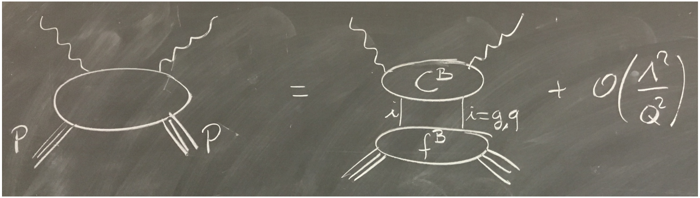


Large uncertainties from subleading logs

N^{3}LO splitting functions are much more unstable at small x $\rightarrow$ need resummation!

Small-x resummation of coefficient functions

Consider factorization of short- and long-distance parts in DIS at bare level



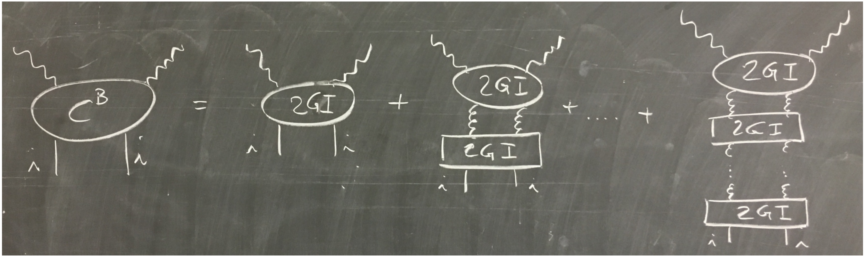
$$\sigma = C_i^B f_i^B + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

Collinear singularities can be factorized (à la Curci,Furmanski,Petronzio) expanding C^B in 2PI kernels, projecting the collinear singularities of each kernel and collecting them into a collinear counterterm Γ_{ki}

$$C_i^B = C_k \Gamma_{ki}$$

$$\begin{aligned} \sigma &= C_k \Gamma_{ki} f_i^B + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right) \\ &= C_k f_k + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right) \end{aligned}$$

The *bare* coefficient function can be expanded in 2GI kernels



Since small-x logarithms originate from integration over t-channel exchanged gluons, each 2GI blob is finite at small x. With suitable "high-energy projectors" one can isolate small-x contributions, and resum them into a "Green function" $\mathcal{G}^B_{ji}$

$$C_i^B = \int dk_t^2\, \mathcal{C}_j(k_t^2)\, \mathcal{G}^B_{ji}(k_t^2) \overset{\text{LL}}{=} \int dk_t^2 \Big[\mathcal{C}_{\mathsf{g}}(k_t^2)\, \mathcal{G}^B_{gi}(k_t^2) + \delta_{iq}\mathcal{C}_{\mathsf{q}}(k_t^2)\delta(k_t^2)\Big]$$

Since these projectors are more general than the "collinear" ones, it is still possible to factorize collinear singularities afterwards

$$\mathcal{G}^B_{ji}(k_t^2) = \mathcal{G}_{jk}(k_t^2)\, \Gamma_{ki}$$

Comparison of factorizations

$$
\begin{aligned}
\sigma(N,Q^2) &= C_j\big(N,\alpha_s(Q^2)\big)\, f_j\big(N,Q^2\big) && \text{(collinear factorization)}\\
&= C_i^B\big(N,\alpha_s(Q^2)\big)\, f_i^B\big(N,Q^2\big) && \text{(one step back: bare)}\\
&= \int_0^\infty dk_t^2\, \mathcal{C}_j\big(N,k_t^2,\alpha_s\big)\, \mathcal{G}_{ji}^B(N,k_t^2)\, f_i^B\big(N,Q^2\big) && (k_t \text{ factorization, bare})\\
&= \int_0^\infty dk_t^2\, \mathcal{C}_j\big(N,k_t^2,\alpha_s\big)\, \mathcal{G}_{jk}(N,k_t^2)\, f_k\big(N,Q^2\big) && (k_t\text{+coll factorization})\\
&= \int_0^\infty dk_t^2\, \mathcal{C}_j\big(N,k_t^2,\alpha_s\big)\, \mathcal{F}_j\big(N,k_t^2\big) && (k_t \text{ factorization})
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{F}_{\mathrm{g}}\big(N,k_t^2\big) &= \mathcal{G}_{\mathrm{gg}}(N,k_t^2)\, f_{\mathrm{g}}\big(N,Q^2\big) + \mathcal{G}_{\mathrm{gq}}(N,k_t^2)\, f_{\mathrm{q}}\big(N,Q^2\big)\\
\mathcal{F}_{\mathrm{q}}\big(N,k_t^2\big) &= \delta(k_t^2)\big[f_{\mathrm{q}}\big(N,Q^2\big) + \mathrm{NLL}\big]
\end{aligned}
$$

By comparison, for the collinear coefficient functions one gets

$$
\begin{aligned}
C_{\mathrm{g}}(N,\alpha_s) &= \int_0^\infty dk_t^2\, \mathcal{C}_{\mathrm{g}}\big(N,k_t^2,\alpha_s\big)\mathcal{G}_{\mathrm{gg}}(N,k_t^2) + \mathcal{C}_{\mathrm{q}}\big(N,0,\alpha_s\big) \times \mathrm{NLL}\\
C_{\mathrm{q}}(N,\alpha_s) &= \int_0^\infty dk_t^2\, \mathcal{C}_{\mathrm{g}}\big(N,k_t^2,\alpha_s\big)\mathcal{G}_{\mathrm{gq}}(N,k_t^2) + \mathcal{C}_{\mathrm{q}}\big(N,0,\alpha_s\big) \times (1+\mathrm{NLL})
\end{aligned}
$$

Comparison of unintegrated PDFs

We have found

$$\mathcal{F}_{\rm g}\big(N,k_t^2\big) = \mathcal{G}_{\rm gg}(N,k_t^2)\, f_{\rm g}\big(N,Q^2\big) + \mathcal{G}_{\rm gq}(N,k_t^2)\, f_{\rm q}\big(N,Q^2\big)$$

In the ladder approximation, one can prove that

$$\mathcal{G}_{\rm gq}(N,k_t^2) = \frac{C_F}{C_A}\left[\mathcal{G}_{\rm gg}(N,k_t^2) - \delta(k_t^2)\right] \qquad\qquad \mathcal{G}_{\rm gg}(N,k_t^2) = \frac{d}{dk_t^2}\left(\frac{k_t^2}{Q^2}\right)^{\gamma_{\rm gg}(N,\alpha_s)}$$

leading to

$$\mathcal{F}_{\rm g}\big(N,k_t^2\big) = \frac{d}{dk_t^2}\left(\frac{k_t^2}{Q^2}\right)^{\gamma_{\rm gg}}\left[f_{\rm g}\big(N,Q^2\big) + \frac{C_F}{C_A} f_{\rm q}\big(N,Q^2\big)\right] - \delta(k_t^2)\frac{C_F}{C_A} f_{\rm q}\big(N,Q^2\big)$$

Before we had found

$$\mathcal{F}_{\rm g}\big(N,k_t^2\big) = \frac{d}{dk_t^2} U_+(N,k_t^2,Q^2) f_+(N,Q^2) + \delta(k_t^2) f_-(N,Q^2)$$

which is equivalent when restricting to LL and fixed coupling, but valid also beyond

Running coupling effects in coefficient functions

As for the splitting functions *(Stefano's talk)*, running coupling effects are important also in the resummation of coefficient functions [Ball NPB796]

Using the full evolutor

$$U_+(N, k_t^2, Q^2) = \exp \int_{Q^2}^{k_t^2} \frac{d\nu^2}{\nu^2}\, \gamma_+(N, \alpha_s(\nu^2))$$

we can incorporate them as [Bonvini,Marzani,Peraro 1607.02153]

$$C_{\rm g}(N, \alpha_s) = \int_0^\infty dk_t^2\, \mathcal{C}_{\rm g}\big(N, k_t^2, \alpha_s\big) \frac{d}{dk_t^2} U_+(N, k_t^2, Q^2) + \mathcal{C}_{\rm q}(N, 0, \alpha_s) \times \text{NLL}$$

which can be proved to be equivalent to the result of [Ball NPB796]

Note: this formulation is very convenient numerically wrt the ABF one

But what about collinear subtractions?

Explicit collinear factorization for the bare gluon coefficient

$$C_{\mathsf{g}}^{B}\big(N,\alpha_s(Q^2)\big) = C_{\mathsf{g}}\big(N,\alpha_s(Q^2)\big)\,\Gamma_{\mathsf{gg}} + C_{\mathsf{q}}\big(N,\alpha_s(Q^2)\big)\,\Gamma_{\mathsf{qg}}$$

derive wrt $\ln Q^2$

$$\frac{dC_{\mathsf{g}}^{B}(N,\alpha_s)}{d\ln Q^2} \stackrel{\epsilon\to 0}{=} [C_{\mathsf{g}}(N,\alpha_s)\gamma_{\mathsf{gg}}(N,\alpha_s) + C_{\mathsf{q}}(N,\alpha_s)\gamma_{\mathsf{qg}}(N,\alpha_s)]\Gamma_{\mathsf{gg}}$$
$$\stackrel{\mathsf{LL}}{=} \int_0^\infty dk_t^2\, \mathcal{C}_{\mathsf{g}}\big(N,k_t^2,\alpha_s\big)\,\frac{d}{dk_t^2}U_+(N,k_t^2,Q^2)\,\gamma_{\mathsf{gg}}(N,\alpha_s)\,\Gamma_{\mathsf{gg}}$$

since Γ_{gg} is a common factor and everything else is finite, at LL we get

$$C_{\mathsf{g}}(N,\alpha_s) \stackrel{\mathsf{LL}}{=} \int_0^\infty dk_t^2\, \mathcal{C}_{\mathsf{g}}\big(N,k_t^2,\alpha_s\big)\,\frac{d}{dk_t^2}U_+(N,k_t^2,Q^2) - C_{\mathsf{q}}(N,\alpha_s)\,\frac{\gamma_{\mathsf{qg}}(N,\alpha_s)}{\gamma_{\mathsf{gg}}(N,\alpha_s)}$$

Running coupling effects have been introduced in [Altarelli,Ball,Forte NPB799]

Is there a more direct way to construct collinear subtractions?

Our idea consists in rotating to the diagonal basis

$$\begin{aligned}\sigma(N,Q^2) &= C_{\mathsf{g}}\big(N,\alpha_s(Q^2)\big)\, f_{\mathsf{g}}\big(N,Q^2\big) + C_{\mathsf{q}}\big(N,\alpha_s(Q^2)\big)\, f_{\mathsf{q}}\big(N,Q^2\big) \\ &= C_+\big(N,\alpha_s(Q^2)\big)\, f_+\big(N,Q^2\big) + C_-\big(N,\alpha_s(Q^2)\big)\, f_-\big(N,Q^2\big)\end{aligned}$$

and taking a derivative

$$\begin{aligned}\frac{d\sigma(N,Q^2)}{d\ln Q^2} &= \left(\frac{dC_{\mathsf{g}}}{d\ln Q^2} + C_{\mathsf{g}}\gamma_{\mathsf{gg}} + C_{\mathsf{q}}\gamma_{\mathsf{qg}}\right) f_{\mathsf{g}}(N,Q^2) + \left(\frac{dC_{\mathsf{q}}}{d\ln Q^2} + C_{\mathsf{q}}\gamma_{\mathsf{qq}} + C_{\mathsf{g}}\gamma_{\mathsf{gq}}\right) f_{\mathsf{q}}(N,Q^2) \\ &= \left(\frac{dC_+}{d\ln Q^2} + C_+\gamma_+\right) f_+(N,Q^2) + \left(\frac{dC_-}{d\ln Q^2} + C_-\gamma_-\right) f_-(N,Q^2).\end{aligned}$$

Neglecting subleading contributions one can solve the equation and get

$$C_{\mathsf{g}}(N,\alpha_s) = \int_0^\infty dk_t^2\, \mathcal{C}_{\mathsf{g}}\big(N,k_t^2,\alpha_s\big)\, \frac{d}{dk_t^2} U_+(N,k_t^2,Q^2) - \mathcal{C}_{\mathsf{q}}(N,0,\alpha_s) S_{\mathsf{qg}}(N,Q^2)$$

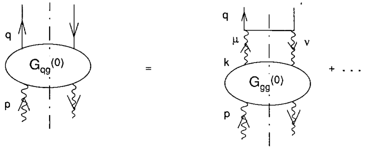
with

$$S_{\mathsf{qg}}(N,Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2} \gamma_{\mathsf{qg}}(N,\alpha_s(k^2)) U_+(N,k^2,Q^2)$$

being $\gamma_{\mathsf{qg}}(N,\alpha_s(q^2))$ the resummed qg anomalous dimension

Reduces to [Catani,Hautmann NPB427] at fixed coupling, and is equivalent to [Ball NPB796] with running coupling

The quark green function can be expanded in 2GI kernels as done for coefficient functions



$$G^B_{\text{qg}}(k_t^2) = K_{\text{qg}}(k_t^2) G^B_{\text{gg}}(k_t^2)$$

Factorization of collinear singularities leads to

$$G^B_{\text{qg}}(k_t^2) = G_{\text{qg}}(k_t^2)\Gamma_{\text{gg}} + \Gamma_{\text{qg}} \tag{1}$$

with

$$\Gamma_{\text{qg}}(N,\alpha_s) = \frac{1}{\epsilon}\int \frac{d\alpha_s}{\alpha_s}\gamma_{\text{qg}}(N,\alpha_s)\Gamma_{\text{gg}}(N,\alpha_s)$$

Matching the singular parts of the LHS and RHS of Eq. (1) one can construct γ_{qg} order by order

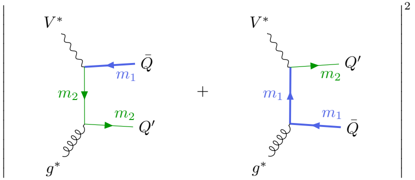
However, no closed all-order form found yet!

All-order approx including running coupling effects obtained in [Altarelli,Ball,Forte NPB799]

Recent developments in the resummation of coefficient functions

In a recent work, a number of developments in the resummation of coefficient functions have been presented [MB,Marzani,Muselli 1708.07510]

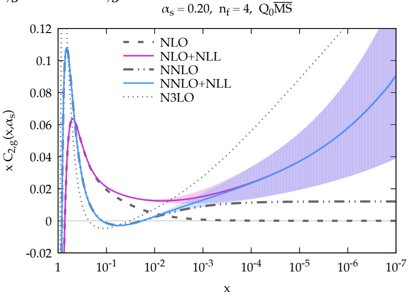
- resummation of all DIS structure functions F_2, F_L and F_3, both in the massless limit and including mass effects, both for neutral and charged currents
- resummation of heavy flavour matching conditions
- implementation of VFNS (FONLL/S-ACOT/ACOT)
- matching of resummation to NNLO and N^{3}LO
- everything available from the HELL code

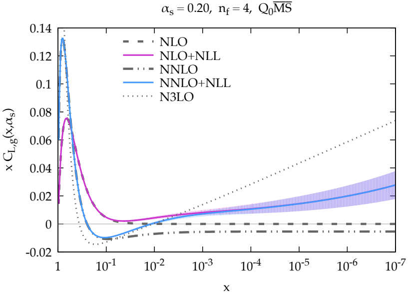


All these developments make possible a PDF fit to DIS data at NNLO+NLL

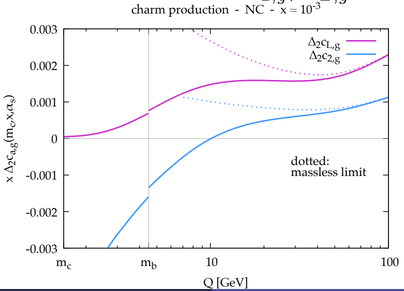
Some representative HELL results: DIS coefficient functions

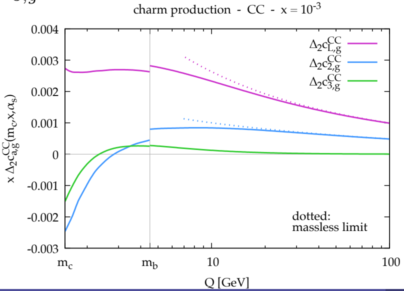
$F_{2,g}$ and $F_{L,g}$ massless DIS coefficient functions



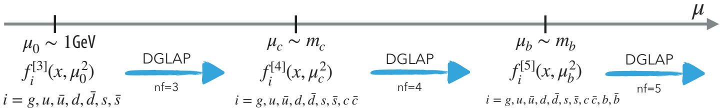


resummed contribution to $F_{2,g}$, $F_{L,g}$ and $F_{3,g}$ DIS coeff.functs with mass effects



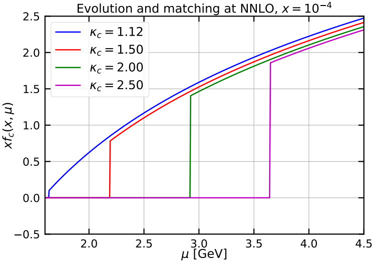


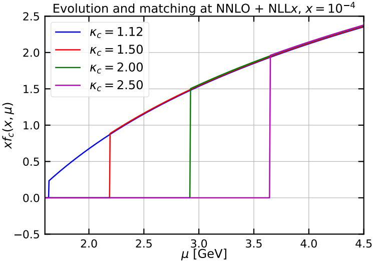
The number n_f of "active" flavours changes during the evolution (factorization scheme choice to resum large collinear mass logarithms from heavy quark pair production)



Matching relation between PDFs in schemes with different n_f

$$f_i^{[n_f+1]}(\mu^2) = \sum_{j=\text{light}} A_{ij}(m^2/\mu^2) \otimes f_j^{[n_f]}(\mu^2) \qquad A_{ij} = \text{perturbative matching coefficients}$$



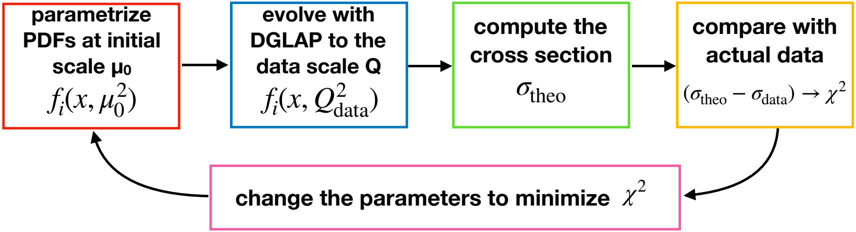


The perturbatively generated charm PDF is much less dependent on the (unphysical) matching scale when small-x resummation is included!

PDF fits with small-x resummation

PDF determination: general strategy

Fit $f_i(x, \mu_0^2)$ by comparison with (many) data



Such fitted PDFs depend unavoidably on the accuracy on the perturbative ingredients $P_{ij}(x, \alpha_s)$, $C_{ij}(x, ..., \alpha_s)$

NNPDF and xFitter fits with small-x resummation

APFEL+HELL $\rightarrow$ make possible a PDF fit with small-x resummation

NNPDF3.1sx [1710.05935]	xFitter [1802.00064]
NeuralNet parametrization of PDFs	polynomial paramterization
MonteCarlo uncertainty	Hessian uncertainty
VFNS: FONLL	VFNS: FONLL
charm PDF is fitted	charm PDF perturbatively generated
DIS+tevatron+LHC (~ 4000 datapoints)	only HERA data (~ 1200 datapoints)
NLO, NLO+NLL, NNLO, NNLO+NLL	NNLO, NNLO+NLL

One interesting difference in the HERA data we include:

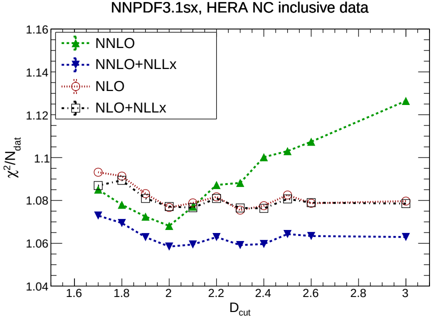
Lowest Q^2 HERA bins	NNPDF3.1/HERAPDF2.0	NNPDF3.1sx/xFitter
$Q^2 = 3.5$ GeV^2	included	included
$Q^2 = 2.7$ GeV^2	excluded	included
$Q^2 = 2.0$ GeV^2	excluded	excluded

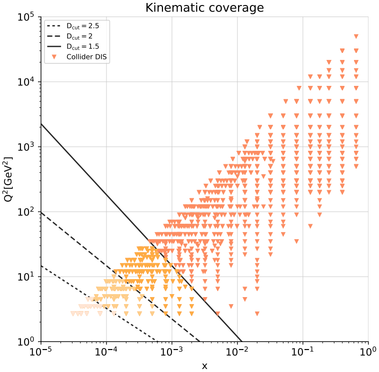
lower $Q^2 \rightarrow$ lower x

$\chi^2/N_{\rm dat}$	NLO	NLO+NLLx	NNLO	NNLO+NLLx
xFitter			**1.23**	**1.17**
NNPDF3.1sx	**1.117**	**1.120**	**1.130**	**1.100**
	these are similar		largest	smallest

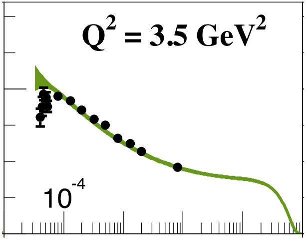
Hierarchy as expected from splitting function behaviour!

Mostly due to HERA data: we study the $\chi^2/N_{\rm dat}$ profile as we cut out HERA data at small x small Q^2





Fit results: description of the HERA data



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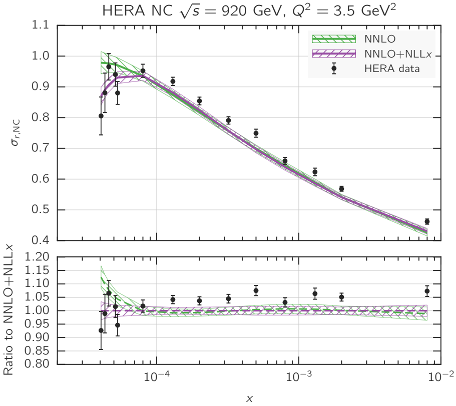
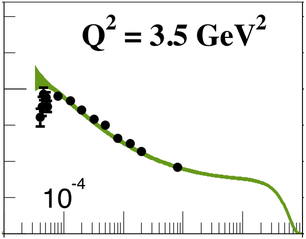
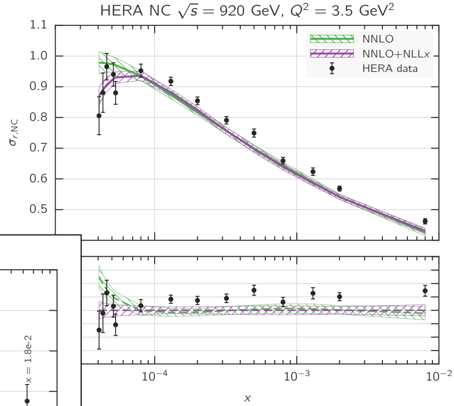
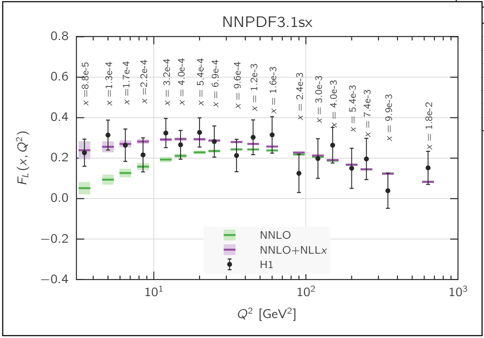




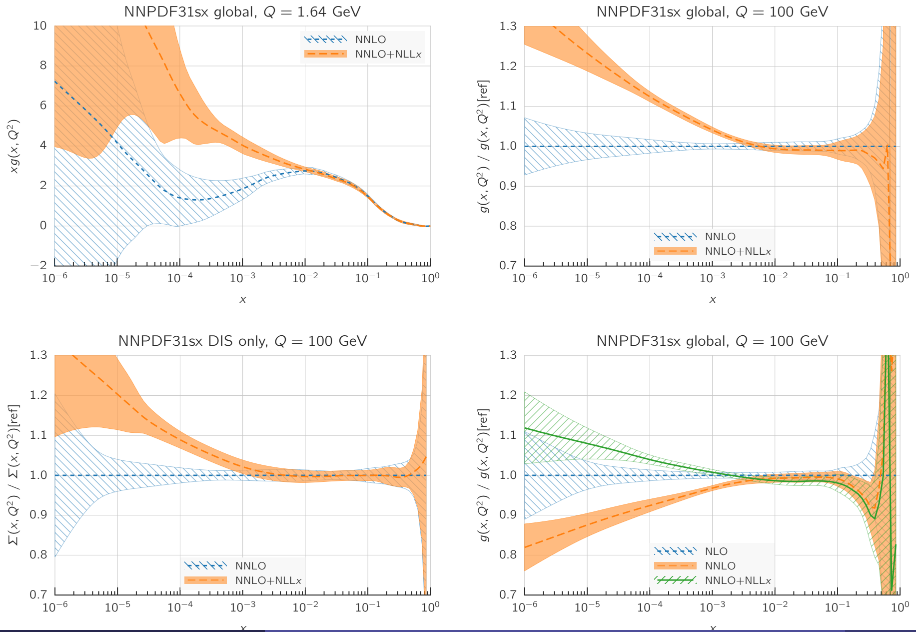




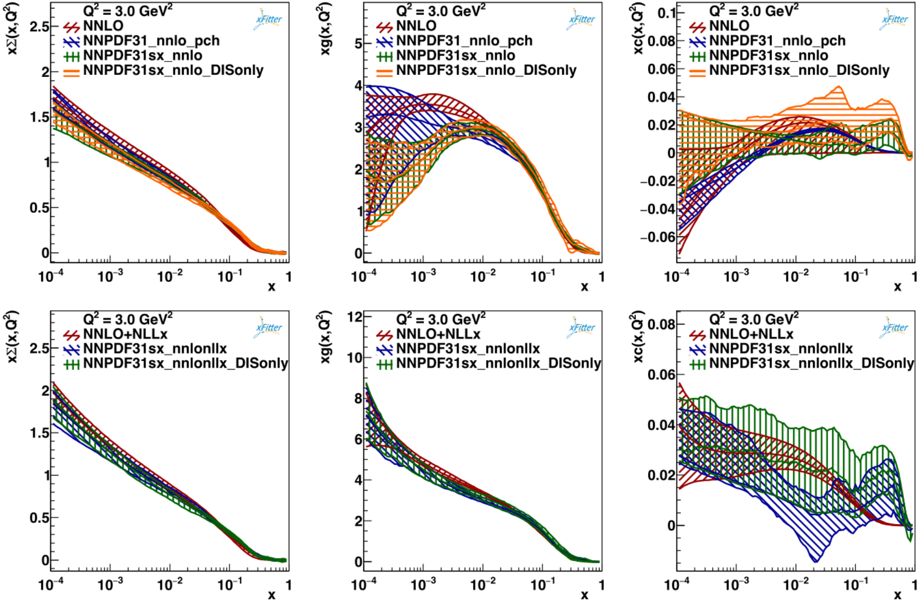
The better description mostly comes from F_L $\quad \left(y = \frac{Q^2}{xs}\right)$

$$\sigma_{r,\mathrm{NC}} = F_2(x, Q^2) - \frac{y^2}{Y_+} F_L(x, Q^2)$$

Fit results: impact on PDFs



xFitter comparison

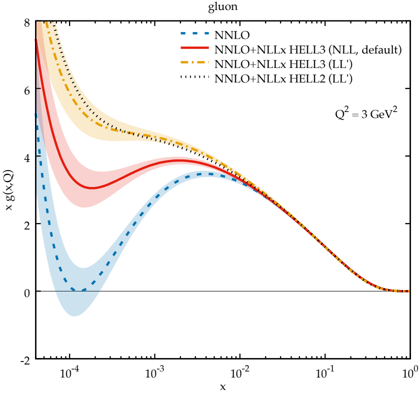


A new parametrization, more flexible at small x, proposed in [MB,Giuli 1902.11125]

$$xf(x,\mu_0^2) = A\,x^B(1-x)^C\left[1 + Dx + Ex^2 + F\log x + G\log^2 x + H\log^3 x\right]$$

Improved description of the low x data even at fixed order
Adding resummation is anyway useful: χ^2 reduced from 1312 to 1284 with 1127 dof

First fit with `HELL 3.0` (contains a new choice for subleading logs)

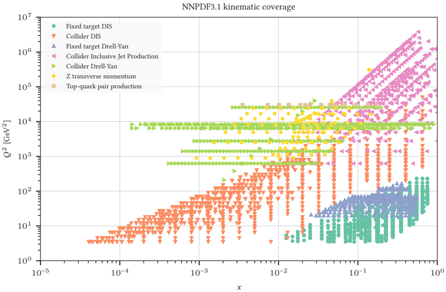


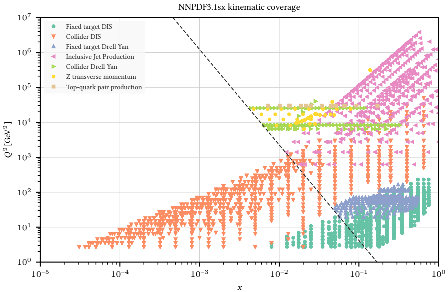
The global NNPDF fit in greater detail

We have full resummation for DGLAP evolution and DIS structure functions, but not hadron-hadron collider observables (yet)

We cut those hadronic data potentially sensitive to small-x resummation, i.e.

$$\alpha_s(Q^2)\log\frac{1}{x} > H_{\rm cut} = 0.6 \quad \text{(default cut)}$$

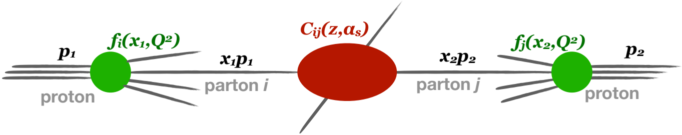




The most important missing observable is Drell-Yan
Would provide a very important validation of the fit (low x but high Q^2)

Work in progress to include it in `HELL` *(Tommaso's talk)*

LHC phenomenology



Differential cross section

$y = Y - \frac{1}{2}\log\frac{x_1}{x_2}$

$$\frac{d\sigma}{dQ^2 dY\ldots} = \int_0^1 dx_1\, dx_2\, f_i\left(x_1, Q^2\right) f_j\left(x_2, Q^2\right) \frac{dC_{ij}}{dy\ldots}\left(\frac{\tau}{x_1 x_2}, y, \ldots, \alpha_s\right)$$
$$= \int_\tau^1 \frac{dz}{z} \int d\hat{y}\, \mathscr{L}_{ij}\left(\frac{\tau}{z}, \hat{y}, Q^2\right) \frac{dC_{ij}}{dy\ldots}(z, Y - \hat{y}, \ldots, \alpha_s)$$

with $\tau = Q^2/s$ and

$$\mathscr{L}_{ij}\left(x, \hat{y}, Q^2\right) = f_i(\sqrt{x}e^{\hat{y}}, Q^2)\, f_j(\sqrt{x}e^{-\hat{y}}, Q^2)\, \theta(e^{-2|\hat{y}|} - x)$$

When differential in rapidity Y more directly sensitive to the PDFs at a given x

PDFs only sensitive to longitudinal variables x and $\hat{y}$

LHC observables in k_t factorization

An obvious extension of k_t factorization is given by

$$\frac{d\sigma}{dQ^2 dY\ldots} = \int_\tau^1 \frac{dz}{z} \int d\hat{y} \int_0^\infty dk_1^2 \int_0^\infty dk_2^2\, \mathcal{L}_{ij}\left(\frac{\tau}{z}, \hat{y}, k_1^2, k_2^2\right) \frac{d\mathcal{C}_{ij}}{dy\ldots}(z, Y-\hat{y}, k_1^2, k_2^2, \ldots, \alpha_s)$$

with

$$\mathcal{L}_{ij}(x, \hat{y}, k_1^2, k_2^2) = \mathcal{F}_i(\sqrt{x}e^{\hat{y}}, k_1^2)\, \mathcal{F}_j(\sqrt{x}e^{-\hat{y}}, k_2^2)\, \theta(e^{-2|\hat{y}|} - x)$$

This can be derived from the results of [Caola,Forte,Marzani 1010.2743] [Muselli 1710.09376]

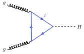
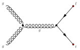
Important: transverse final-state variables do not play a role (except in kinematic constraints in the off-shell coefficient)

Mellin-Fourier convolution diagonalised in Mellin-Fourier space

$$\frac{d\sigma}{dQ^2 dY\ldots} = \int_0^\infty dk_1^2 \int_0^\infty dk_2^2\, \mathcal{F}_i\left(N + \frac{ib}{2}, k_1^2\right) \mathcal{F}_j\left(N - \frac{ib}{2}, k_2^2\right) \frac{d\tilde{\mathcal{C}}_{ij}}{dy\ldots}(N, b, k_1^2, k_2^2, \ldots, \alpha_s)$$

Resummation can proceed as before, exploiting the relation between unintegrated and standard PDFs, and comparing with collinear factorization

Resummed LHC observables without collinear subtractions

Consider a gluon-gluon initiated process (e.g. Higgs  or $t\bar{t}$  production)

The result, with running coupling effects, is

$$\frac{d\tilde{C}_{\rm gg}}{dy...}(N,b,...) = \int_0^\infty dk_1^2 \int_0^\infty dk_2^2 \, \frac{d}{dk_1^2} U_+\left(N+\frac{ib}{2},k_1^2,Q^2\right) \frac{d}{dk_1^2} U_+\left(N-\frac{ib}{2},k_2^2,Q^2\right) \times \frac{d\tilde{\mathcal{C}}_{\rm gg}}{dy...}(N,b,k_1^2,k_2^2,...,\alpha_s)$$

Neglecting the N dependence of the off-shell coefficient (subleading log), we can Mellin-Fourier invert and get

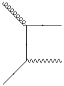
$$\frac{dC_{\rm gg}}{dy...}(z,y,...) = \int_0^\infty dk_1^2 \int_0^\infty dk_2^2 \int d\hat{y} \, \frac{d}{dk_1^2} U_+\left(\sqrt{z}e^{\hat{y}},k_1^2,Q^2\right) \frac{d}{dk_1^2} U_+\left(\sqrt{z}e^{-\hat{y}},k_2^2,Q^2\right) \times \frac{d\mathcal{C}_{\rm gg}}{dy...}(N=0,y-\hat{y},k_1^2,k_2^2,...,\alpha_s)$$

with

$$U_+(N,k^2,Q^2) = \int_0^1 dx\, x^N U_+(x,k^2,Q^2)$$

This form is very convenient, as it uses the same x-space evolutor used for DIS in HELL

Consider now a quark-quark initiated process, e.g. Drell-Yan *(more in Tommaso's talk)*

The first log appears in the qg channel  which contains a collinear singularity

The quark line can be considered on-shell, so the treatment is identical to DIS

The result in Mellin-Fourier space is

$$\frac{d\tilde{C}_{\mathsf{qg}}}{dy\ldots}(N,b,\ldots) = \int_0^\infty dk^2\, \frac{d}{dk^2} U_+\left(N-\frac{ib}{2},k^2,Q^2\right) \frac{d\tilde{C}_{\mathsf{qg}}}{dy\ldots}(N,b,k^2,\ldots,\alpha_s)$$
$$- S_{\mathsf{qg}}\left(N-\frac{ib}{2},Q^2\right) \frac{d\tilde{C}_{\mathsf{q\bar{q}}}}{dy\ldots}(N,b,\ldots)$$

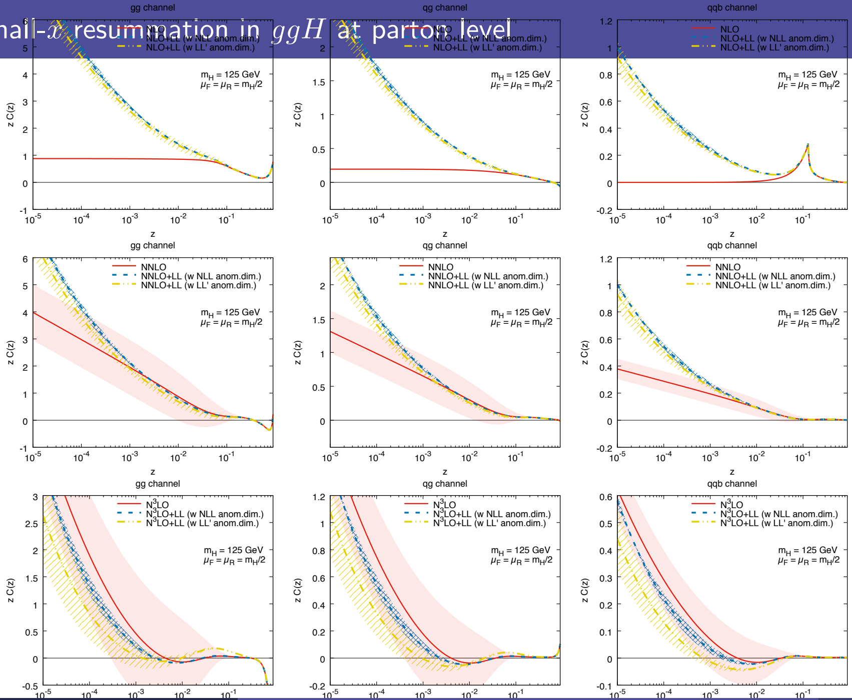
which is different if we consider a differential distribution in rapidity only or also in p_t

$$\frac{dC_{\mathsf{q\bar{q}}}}{dy}(z,y,\ldots) = \delta(1-z)\delta(y) \qquad \Leftrightarrow \qquad \frac{d\tilde{C}_{\mathsf{q\bar{q}}}}{dy}(N,b,\ldots) = 1$$

$$\frac{dC_{\mathsf{q\bar{q}}}}{dy\,dp_t}(z,y,p_t,\ldots) = \delta(1-z)\delta(y)\delta(p_t) \qquad \Leftrightarrow \qquad \frac{d\tilde{C}_{\mathsf{q\bar{q}}}}{dy\,dp_t}(N,b,p_t,\ldots) = \delta(p_t)$$

The collinear subtraction is ineffective in the p_t spectrum for $p_t > 0$

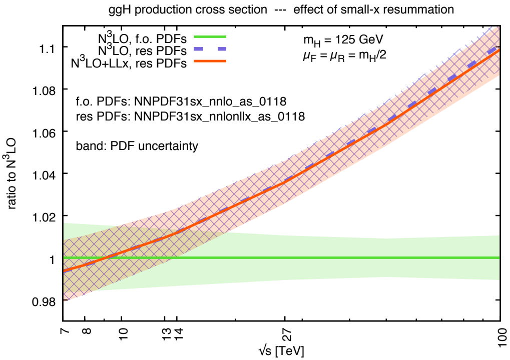
Small-x resummation in ggH at parton level



Impact of resummation in ggH at LHC and future colliders

Hadron-hadron collider processes in `HELL 3.x`:

- Drell-Yan: work in progress
- $gg \to H$ inclusive cross section: done [MB,Marzani 1802.07758] [MB 1805.08785]

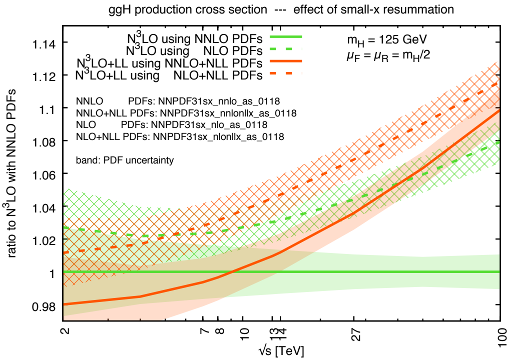


ggH cross section at FCC-hh can be $\sim 10\%$ larger than expected with NNLO PDFs!

At LHC $+1\%$ effect (plus another 1% from threshold resummation)

Hadron-hadron collider processes in `HELL 3.x`:

- Drell-Yan: work in progress
- $gg \to H$ inclusive cross section: done [MB,Marzani 1802.07758] [MB 1805.08785]



ggH cross section at FCC-hh can be $\sim 10\%$ larger than expected with NNLO PDFs!

At LHC $+1\%$ effect (plus another 1% from threshold resummation)

Why is the effect of resummation mostly driven by the PDFs?

Let's consider again the collinear factorization formula

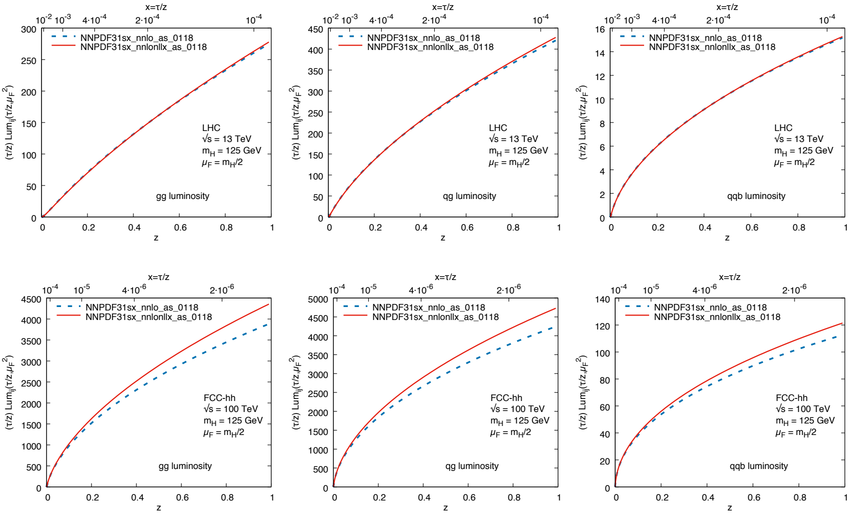
$$\frac{d\sigma}{dQ^2dY...} = \int_\tau^1 \frac{dz}{z} \int d\hat{y}\, \mathscr{L}_{ij}\left(\frac{\tau}{z}, \hat{y}, Q^2\right) \frac{dC_{ij}}{dy...}(z, Y-\hat{y}, ..., \alpha_s)$$

$$\mathscr{L}_{ij}\left(x, \hat{y}, Q^2\right) = f_i(\sqrt{x}e^{\hat{y}}, Q^2)\, f_j(\sqrt{x}e^{-\hat{y}}, Q^2)\, \theta(e^{-2|\hat{y}|} - x)$$

The small z integration region, where large logs in C are important, is weighted by the luminosity at large x, where both PDFs are forced to be at large momentum fractions

Since PDFs die fast at large x, especially the gluon, the small-z region is suppressed!

Rather, the large z region is enhanced by the gluon-gluon luminosity
In that region, the difference between fixed-order and resummed PDFs is large

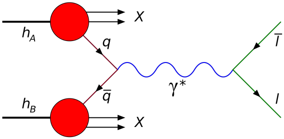


$$\sigma(\tau, Q^2) = \sigma_0(Q^2) \sum_{i,j=g,q} \int_\tau^1 \frac{dz}{z}\, C_{ij}(z, \alpha_s)\, \mathscr{L}_{ij}\left(\frac{\tau}{z}, \mu_F^2\right) \qquad \tau = \frac{m_H^2}{s}$$

Work in progress

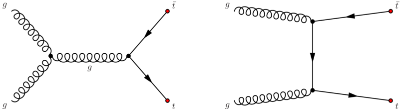
Other processes:

- Drell-Yan at fully differential level [MB,Gauld,Giani,Marzani (in progress)]



Very precise data, also in region sensitive to small-x resummation (forward rapidities and low mass, especially from LHCb) (Tommaso's talk)

- Heavy-quark pair production at differential level [MB,Silvetti,... (master thesis)]



LHCb data sensitive to very small x useful to constrain the PDFs

Summary: state of the art

Evolution known at

$$\begin{pmatrix} \gamma_{\rm gg} & \gamma_{\rm gq} \\ \gamma_{\rm qg} & \gamma_{\rm qq} \end{pmatrix} = \begin{pmatrix} \text{LL+NLL} & \text{LL} \\ \text{NLL} & \text{NLL} \end{pmatrix}$$

k_t factorization / ladder expansion

$$\sigma(x, Q^2) = \int_x^1 \frac{dz}{z} \int_0^\infty dk_t^2\, \mathcal{C}_i(z, k_t^2, \alpha_s)\, \mathcal{F}_i\left(\frac{x}{z}, k_t^2\right)$$

known at LL (meaning the lowest non-vanishing logarithmic order)

Relation between unintegrated and integrated PDF

$$\mathcal{F}_{\rm g}(N, k_t^2) = \frac{d}{dk_t^2}\left[U_+(N, k_t^2, \mu^2) f_+(N, \mu^2) + U_-(N, k_t^2, \mu^2) f_-(N, \mu^2)\right]$$

known at LL (maybe valid also at NLL? and beyond?)

Outlook and open questions

Theory desires:

- resum γ_{gq} at NLL
- closed form for γ_{qg} at NLL
- resum γ_{qg} and γ_{qq} at NNLL
- resum γ_{gg} at NNLL ?
- resum coefficient functions (and matching conditions) beyond LL

Pheno aspirations:

- resum other processes in `HELL`: DY, $c\bar{c}$ at LHC, ...
- study in greater detail the impact of subleading terms
- global PDF fit
- applications: (HE)-LHC, FCC, ultra high energy astrophysics, ...

Backup slides

Small-x resummation of DGLAP evolution (1)

Let's focus on the gluon PDF, which is appropriate at LL. In Mellin space we have

DGLAP: $$\mu^2 \frac{d}{d\mu^2} f(N,\mu^2) = \gamma\big(N, \alpha_s(\mu^2)\big) f(N,\mu^2)$$

BFKL: $$N \mathcal{F}(N,k^2) = \int_0^\infty \frac{dk'^2}{k^2} \mathcal{K}\left(\frac{k^2}{k'^2}, \alpha_s(k^2), \alpha_s(k'^2)\right) \mathcal{F}(N,k'^2)$$

DGLAP evolution can be solved explicitly

$$f(N,\mu^2) = f(N,\mu_0^2) \exp \int_{\mu_0^2}^{\mu^2} \frac{d\nu^2}{\nu^2} \gamma(N, \alpha_s(\nu^2))$$

and using the relation

$$\mathcal{F}(N,k^2) = \frac{d}{dk^2} f(N,k^2) = \frac{1}{k^2} \gamma\big(N, \alpha_s(k^2)\big)\, f(N,k^2)$$

we obtain

$$N \gamma\big(N, \alpha_s(k^2)\big) \cancel{f(N,k^2)}$$
$$= \int_0^\infty \frac{dk'^2}{k'^2} \mathcal{K}\left(\frac{k^2}{k'^2}, \alpha_s(k^2), \alpha_s(k'^2)\right) \gamma\big(N, \alpha_s(k'^2)\big) \exp\left[\int_{k^2}^{k'^2} \frac{d\nu^2}{\nu^2} \gamma(N, \alpha_s(\nu^2))\right] \cancel{f(N,k^2)}$$

Duality Relation

Small-x resummation of DGLAP evolution (2)

If the strong coupling is frozen

$$N = \int_0^\infty \frac{dk'^2}{k'^2} \mathcal{K}\left(\frac{k^2}{k'^2}, \alpha_s\right) \left(\frac{k'^2}{k^2}\right)^{\gamma(N,\alpha_s)}$$
$$\equiv \chi(\gamma(N,\alpha_s),\alpha_s)$$

where $\chi(M,\alpha_s) = \int \frac{dk'^2}{k'^2} \left(\frac{k'^2}{k^2}\right)^M \mathcal{K}\left(\frac{k^2}{k'^2}, \alpha_s\right)$ is the BFKL kernel in M Mellin space

Running coupling corrections can be carried out perturbatively

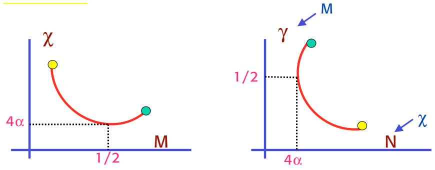
The duality relation **at fixed coupling** is an inverse-function relation in double-Mellin space

$$\chi(\gamma(N,\alpha_s),\alpha_s) = N \qquad \leftrightarrow \qquad \gamma(\chi(M,\alpha_s),\alpha_s) = M$$

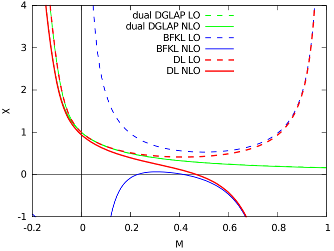
For $\chi(M,\alpha_s) = \alpha_s \chi_0(M)$
the dual γ contains all orders in α_s/N

$\alpha_s^k \frac{1}{x} \log^{k-1} \frac{1}{x} \leftrightarrow \left(\frac{\alpha_s}{N}\right)^k$ LL terms

BFKL at NLO $\rightarrow$ NLL resummation



Small-x resummation of DGLAP evolution (3)



What do we get?

- LL: strong growth at small x (not observed)
- NLL: no enhancement at small x (!!)

Totally unstable,
due to perturbative instability of the BFKL kernel

ABF solution [Altarelli,Ball,Forte 1995,...,2008]

other groups used similar approaches [Catani,Ciafaloni,Colferai,Hautmann,Salam,Stasto] [Thorne,White]

- use duality to resum BFKL kernel
- exploit symmetry $M \to 1 - M$ of χ (broken by running of α_s)
- impose momentum conservation

The BFKL kernel is now perturbatively stable!

Finally

- reuse duality to get resummed anomalous dimensions
- resum running coupling contributions (changes the nature of the small-N singularity: branch-cut to pole)

Small-x resummation of DGLAP evolution (4)

The "traditional" (ABF) way of resumming RC effects works in M Mellin space, promoting α_s to an operator:

$$\mathcal{K}\left(\frac{k^2}{k'^2}, \alpha_s(k^2)\right) \quad \to \quad \chi(M, \hat{\alpha}_s), \qquad \hat{\alpha}_s = \frac{\alpha_s}{1 - \alpha_s \beta_0 \frac{d}{dM}}$$

The BFKL in double-Mellin space becomes

$$N \mathcal{F}(N, M) = \chi(M, \hat{\alpha}_s)\, \mathcal{F}(N, M)$$

In a linear approximation of the $\hat{\alpha}_s$ dependence of χ one gets a first-order differential equation

An analytic solution can be found for simple M dependencies (quadratic [ABF] or collinear [BMM])

From the solution one can derive the anomalous dimension

$$\gamma_{\mathrm{qg}} = \sum_{k=0}^{\infty} h_k \gamma^k, \qquad \gamma = \gamma_+ = \text{the eigenvalue anomalous dimension}$$

$$\gamma_{\mathrm{qq}} = \frac{C_F}{C_A}\gamma_{\mathrm{qg}} + \text{NNLLx} \qquad \gamma_{\mathrm{gq}} = \frac{C_F}{C_A}\gamma_{\mathrm{gg}} + \text{NLLx}$$

Running coupling effects included computing

$$\gamma_{\mathrm{qg}} = \sum_{k=0}^{\infty} h_k \left[\gamma^k\right]$$

with

$$\left[\gamma^{k+1}\right] = \left(\gamma + k\frac{\dot\gamma}{\gamma}\right)\left[\gamma^k\right], \qquad [\gamma] = \gamma, \qquad \dot\gamma = -\beta_0\alpha_s^2\frac{d\gamma}{d\alpha_s}$$

A guess for the gq splitting function

At fixed order, the following structure is found at NLL,

$$(2\pi)^{k+1} N^k \left[\frac{C_A}{C_F}\gamma_{\mathrm{gq}}^{(k)}(N) - \gamma_{\mathrm{gg}}^{(k)}(N)\right] = \frac{1}{3}\left[C_A^{k-1}(C_A - 2C_F)n_f + \delta_k\right] + \mathcal{O}(N)$$

with

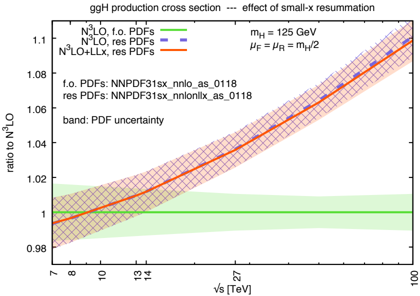
$$\begin{aligned}
\delta_0 &= C_A + 2n_f\frac{C_F}{C_A}, \\
\delta_1 &= 2C_A^2, \\
\delta_2 &= 0
\end{aligned}$$

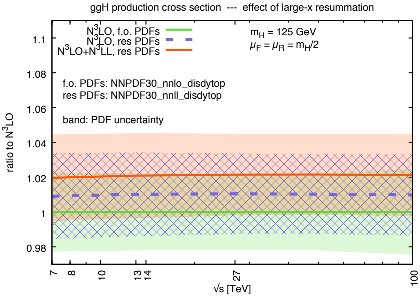
Maybe the coefficients δ_k for $k \geq 1$ are proportional to the coefficients of the LL part of γ_{gg} shifted by a unity, which are accidentally zero at NLO and NNLO? If so, one could guess

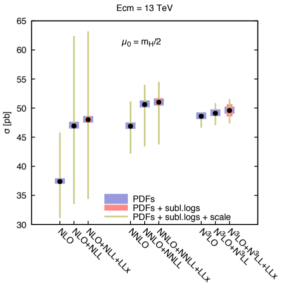
$$\frac{C_A}{C_F}\gamma_{\mathrm{gq}}(N,\alpha_s) - \gamma_{\mathrm{gg}}(N,\alpha_s) = \frac{\alpha_s}{6\pi}\left[\frac{C_A - 2C_F}{C_A}\frac{n_f}{1-\gamma_s(\alpha_s/N)/2} + C_A\gamma_s(\alpha_s/N) + \delta_0\right] + \text{NNLL}$$

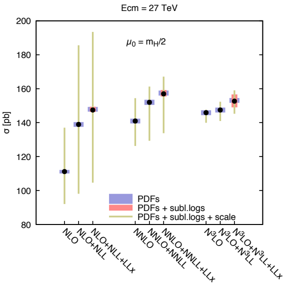
It's just a guess...

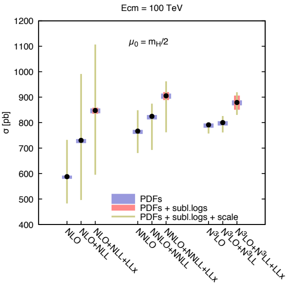
Resummations in Higgs production



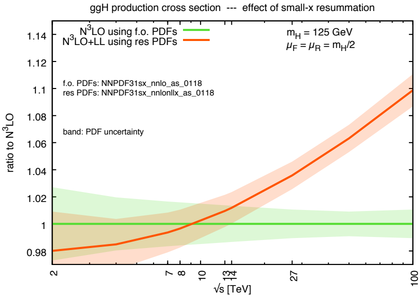


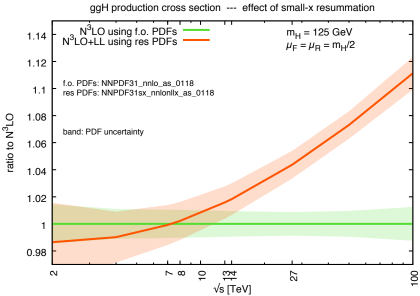






Small-x resummation in Higgs production: PDF dependence





Singlet:

$$P(x,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{n} a_{nk} \frac{\log^k x}{x} + \sum_{k=0}^{2n} b_{nk} \log^k x + \dots \right]$$

$$\int_0^1 dx\, x^N P(x,\alpha_s) \equiv \gamma(N,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{n} \frac{a_{nk}}{N^{k+1}} + \sum_{k=0}^{2n} \frac{b_{nk}}{(N+1)^{k+1}} + \dots \right]$$

Single log enhancement at leading small x, in the singlet sector

$$P_{\text{singlet}} = \begin{pmatrix} P_{\mathsf{gg}} & P_{\mathsf{gq}} \\ P_{\mathsf{qg}} & P_{\mathsf{qq}} \end{pmatrix} = \begin{pmatrix} \text{LL} & \text{LL} \\ \text{NLL} & \text{NLL} \end{pmatrix}$$

eigenvalues: P_+ is LL, P_- is not enhanced at small x.

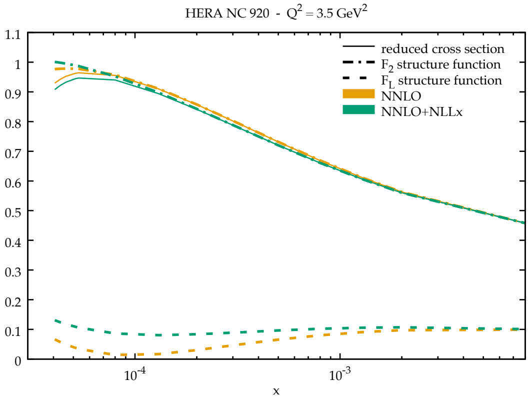
Non-singlet:

$$P^{\text{NS}}(x,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{2n} b_{nk}^{\text{NS}} \log^k x + \dots \right]$$

$$\gamma^{\text{NS}}(N,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{2n} \frac{b_{nk}^{\text{NS}}}{(N+1)^{k+1}} + \dots \right]$$

is double log enhanced but subleading.

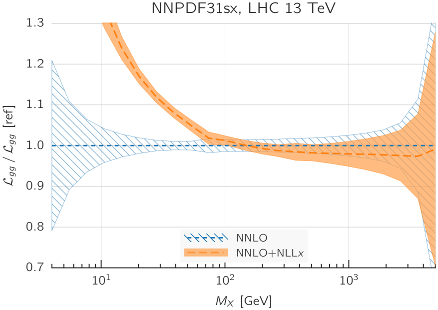
The good agreement obtained at fixed order with the low x HERA data is achieved in a different way wrt the resummed case [MB,Giuli 1902.11125]

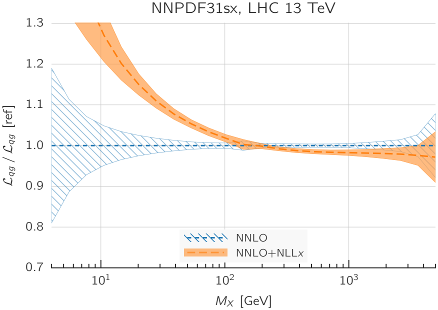


At resummed level, both F_L and F_2 grow

At fixed order, F_L grows below $x \sim 10^{-4}$ and F_2 decreases, due to the sudden growth of the gluon PDF

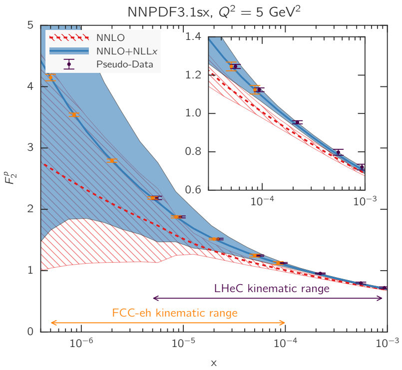
Parton luminosities (gg and qg)

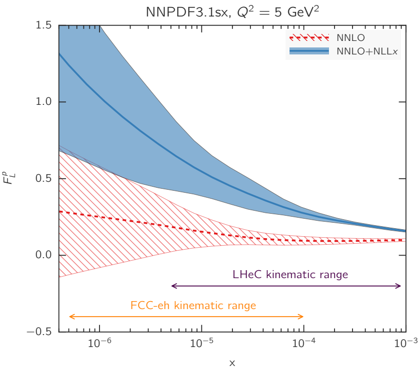




Significant impact for $M_X \lesssim 100$ GeV, but also non-negligible impact above.

The role of FCC-eh (and LHeC)



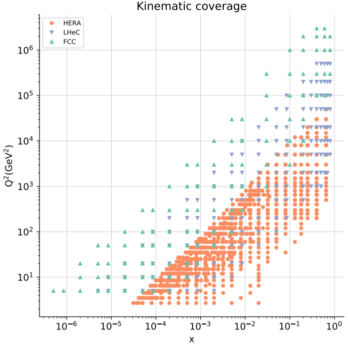


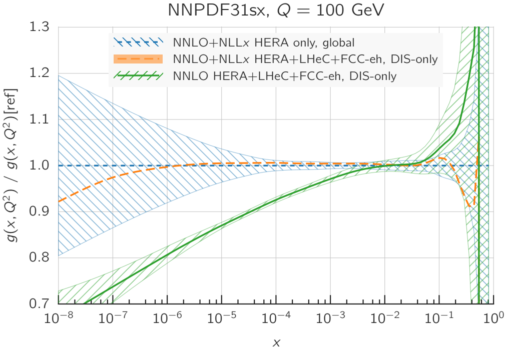
Prediction in the LHeC and FCC-eh kinematic regions for F_2 and F_L

Uncertainties are large (extrapolation region)

Pseudo data show a small error — significant constraining power!

We performed a fit using LHeC and FCC-eh pseudodata





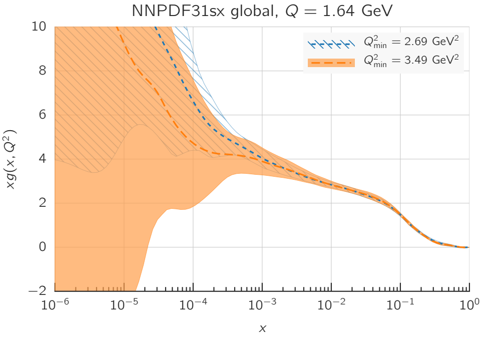
	$N_{\rm dat}$	$\chi^2/N_{\rm dat}$ NNLO	$\chi^2/N_{\rm dat}$ NNLO+NLLx	$\Delta\chi^2$
HERA I+II incl. NC	922	1.22	1.07	-138
LHeC incl. NC	148	1.71	1.22	-73
FCC-eh incl. NC	98	2.72	1.34	-135
Total	**1168**	**1.407**	**1.110**	**-346**

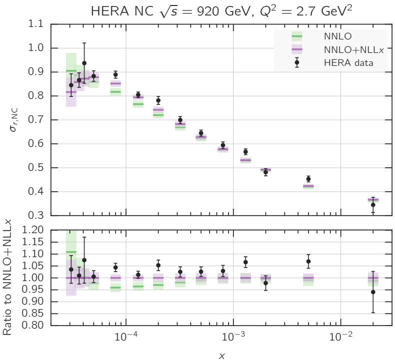
Small-x resummation will be crucial!

Breakdown of χ^2 for each dataset

	$\chi^2/N_{\rm dat}$ NLO	$\chi^2/N_{\rm dat}$ NLO+NLLx	$\Delta\chi^2$	$\chi^2/N_{\rm dat}$ NNLO	$\chi^2/N_{\rm dat}$ NNLO+NLLx	$\Delta\chi^2$
NMC	1.35	1.35	$+1$	1.30	1.33	$+9$
SLAC	1.16	1.14	-1	0.92	0.95	$+2$
BCDMS	1.13	1.15	$+12$	1.18	1.18	$+3$
CHORUS	1.07	1.10	$+20$	1.07	1.07	-2
NuTeV dimuon	0.90	0.84	-5	0.97	0.88	-7
HERA I+II incl. NC	1.12	1.12	-2	1.17	1.11	-62
HERA I+II incl. CC	1.24	1.24	-	1.25	1.24	-1
HERA $\sigma_c^{\rm NC}$	1.21	1.19	-1	2.33	1.14	-56
HERA F_2^b	1.07	1.16	$+3$	1.11	1.17	$+2$
DY E866 $\sigma^d_{\rm DY}/\sigma^p_{\rm DY}$	0.37	0.37	-	0.32	0.30	-
DY E886 σ^p	1.06	1.10	$+3$	1.31	1.32	-
DY E605 σ^p	0.89	0.92	$+3$	1.10	1.10	-
CDF Z rap	1.28	1.30	-	1.24	1.23	-
CDF Run II k_t jets	0.89	0.87	-2	0.85	0.80	-4
D0 Z rap	0.54	0.53	-	0.54	0.53	-
D0 $W \to e\nu$ asy	1.45	1.47	-	3.00	3.10	$+1$
D0 $W \to \mu\nu$ asy	1.46	1.42	-	1.59	1.56	-
ATLAS total	1.18	1.16	-7	0.99	0.98	-2
ATLAS W, Z 7 TeV 2010	1.52	1.47	-	1.36	1.21	-1
ATLAS HM DY 7 TeV	2.02	1.99	-	1.70	1.70	-
ATLAS W, Z 7 TeV 2011	3.80	3.73	-1	1.43	1.29	-1
ATLAS jets 2010 7 TeV	0.92	0.87	-4	0.86	0.83	-2
ATLAS jets 2.76 TeV	1.07	0.96	-6	0.96	0.96	-
ATLAS jets 2011 7 TeV	1.17	1.18	-	1.10	1.09	-1
ATLAS Z p_T 8 TeV (p_T^{ll}, M_{ll})	1.21	1.24	$+2$	0.94	0.98	$+2$
ATLAS Z p_T 8 TeV (p_T^{ll}, y_{ll})	3.89	4.26	$+2$	0.79	1.07	$+2$
ATLAS σ_{tt}^{tot}	2.11	2.79	$+2$	0.85	1.15	$+1$
ATLAS $t\bar{t}$ rap	1.48	1.49	-	1.61	1.64	-
CMS total	0.97	0.92	-13	0.86	0.85	-3
CMS Drell-Yan 2D 2011	0.77	0.77	-	0.58	0.57	-
CMS jets 7 TeV 2011	0.88	0.82	-9	0.84	0.81	-3
CMS jets 2.76 TeV	1.07	0.98	-7	1.00	1.00	-
CMS Z p_T 8 TeV (p_T^{ll}, y_{ll})	1.49	1.57	$+1$	0.73	0.77	-
CMS σ_{tt}^{tot}	0.74	1.28	$+2$	0.23	0.24	-
CMS $t\bar{t}$ rap	1.16	1.19	-	1.08	1.10	-
Total	**1.117**	**1.120**	$\mathbf{+11}$	**1.130**	**1.100**	$\mathbf{-121}$

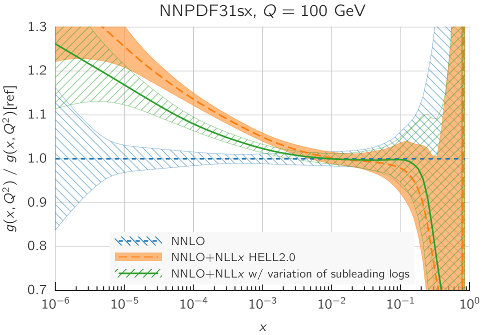
The small-Q^2 HERA bin

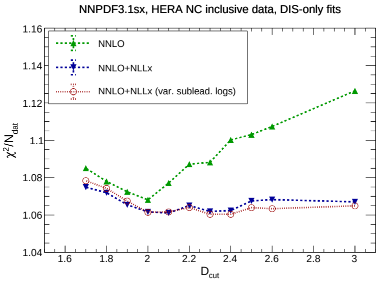




Including the $Q^2 = 2.7$ GeV^2 bin reduces the uncertainty on the small-x gluon, and does not deteriorate the fit quality — we perfectly describe those data

Impact of subleading logarithmic contributions



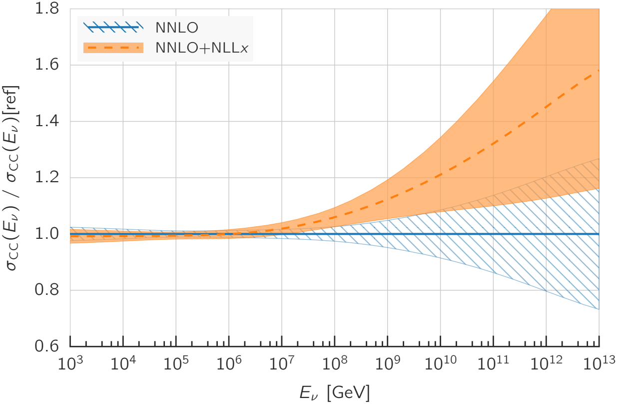


Changing by subleading logarithmic contributions the resummation results can change slightly quantitatively, but not qualitatively

Interestingly, it's clear that the PDF uncertainty cannot cover theoretical uncertainty

Ultra high energy astrophysics

UHE neutrino cross section, relevant for IceCube and KM3NET experiments



Sizeable impact, though uncertainties still large

Recent study with small-x resummation [Bertone,Gauld,Rojo 1808.02034]