Coulomb gluons and factorization

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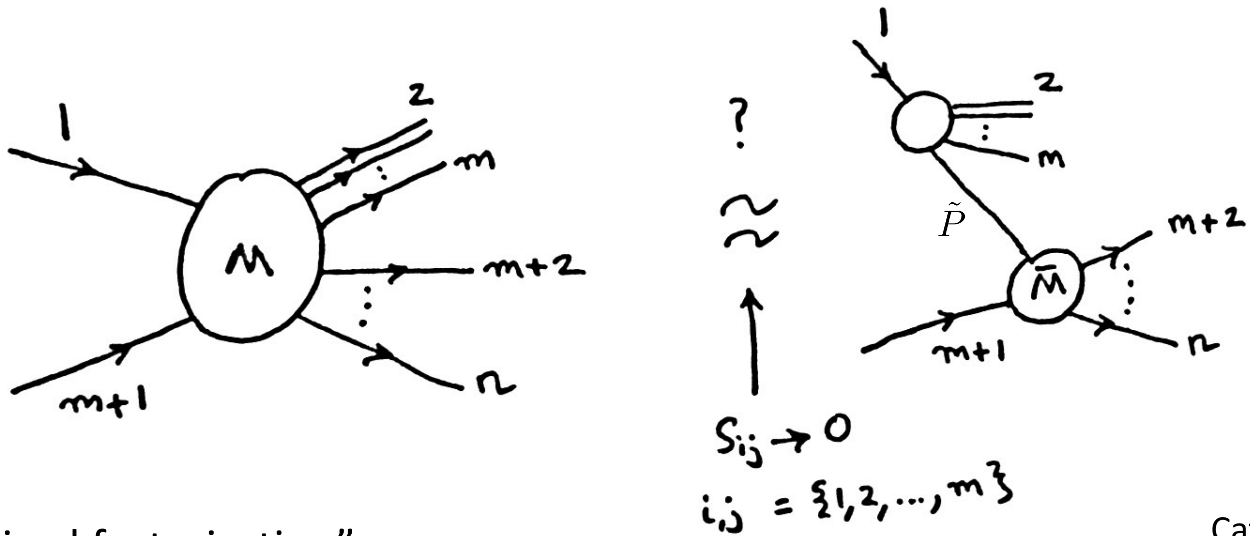
1. A quick recap of the basics
2. Factorization breaking in general
3. A specific example of the manifest failure of DGLAP
4. All orders soft gluon evolution and a remarkable result

Matrix elements factorize in the infra-red



"Generalized factorization":

Catani, de Florian, Rodrigo
arXiv:1112.4405

$$|M\rangle \approx \boldsymbol{Sp}(p_1, \ldots, p_m; \tilde{P}; p_{m+1} \ldots, p_n)|\overline{M}\rangle$$

Corrections are $\sqrt{s_{ij}}$ suppressed.

Even at one-loop it is **not** the case that

$$|M\rangle \approx \boldsymbol{Sp}(p_1, \ldots, p_m; \tilde{P})|\overline{M}\rangle$$

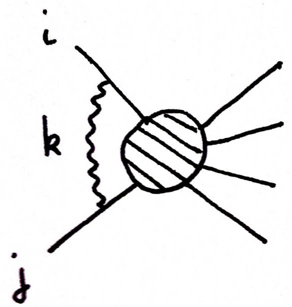
So factorization into PDFs is not true at amplitude level

But the factorization breaking terms cancel in $\langle M|M\rangle$.

A quick recap on how infra-red poles cancel, leaving behind potentially large logarithms

One-loop correction to a hard process

$$|M^{(1)}\rangle = \boldsymbol{I}_V^{(1)}|M^{(0)}\rangle$$



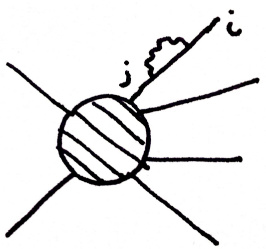
soft

$$\mathrm{Re}\ \boldsymbol{I}^{(1)}_{\mathrm{V,soft}} = \sum_{i\neq j}\frac{\alpha_s}{4\pi}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\int_0^{s_{ij}}\frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}}\mu^{2\epsilon}\int_{\frac{k_\perp^2}{s_{ij}}}^1\frac{dz}{z}$$

$$\mathrm{Im}\ \boldsymbol{I}^{(1)}_{\mathrm{V,soft}} = \sum_{i\neq j}\frac{\alpha_s}{4\pi}\boldsymbol{T}_s^2\int_0^{s_{ij}}\frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}}\mu^{2\epsilon}\ 2\pi$$

Coulomb/Glauber

$$\boldsymbol{T}_s = \boldsymbol{T}_1 + \boldsymbol{T}_{m+1} = -\sum_{i\neq 1,m+1}^{n}\boldsymbol{T}_i$$



hard collinear

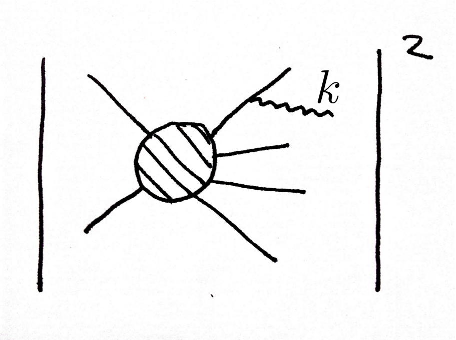
$$\boldsymbol{I}^{(1)}_{\mathrm{V,hard}} = \sum_{i}\frac{\alpha_s}{4\pi}\int_0^{\mu_i^2}\frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}}\mu^{2\epsilon}\sum_j\int_0^1 dz\,P_{ij}(z)$$

$$\begin{aligned}\text{e.g.}\quad P_{qq} &= -\boldsymbol{T}_q^2\,(1+z)\\ P_{qq}^{\mathrm{full}} &= \boldsymbol{T}_q^2\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right)\\ &= \boldsymbol{T}_q^2\left(\frac{2}{(1-z)_+}+\frac{3}{2}\delta(1-z)+P_{qq}\right)\end{aligned}$$

$$P_{gq} = -\boldsymbol{T}_q^2\left(\frac{1+(1-z)^2}{z}-\frac{2}{z}\right)$$

$$d\sigma_V^{(1)} = \langle M^{(0)}|I_V^{(1)} + I_V^{(1)\dagger}|M^{(0)}\rangle\,d(PS)_n$$

One-emission correction to a hard process



$$d\sigma_R^{(1)} = -\langle M^{(0)} | \boldsymbol{Sp}^{(0)\dagger} \boldsymbol{Sp}^{(0)} | M^{(0)} \rangle \, \Phi(\{p_i\}) \, d(PS)_{n+1}$$

$$\Phi(\{p_i\}) = 1 \rightarrow \text{perfect cancellation} = \text{KLN/Bloch-Nordsieck}$$

$$\int \boldsymbol{Sp}^{(0)\dagger} \boldsymbol{Sp}^{(0)} \, d(PS)_k = I_V^{(1)} + I_V^{(1)\dagger}$$

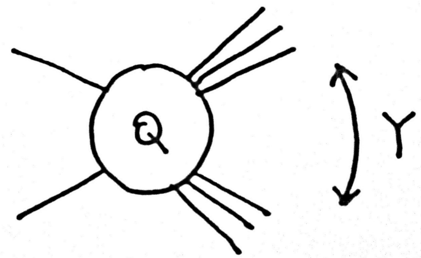
$$(I_V^{(1)} + I_V^{(1)\dagger}) \otimes \Phi(\{p_i\}) = \frac{\alpha_s}{4\pi} \mu^{2\epsilon} \int \frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}} \frac{dz}{z} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j + zP(z)) \, \Phi(\{p_i\}))$$

For a general observable only the poles need to cancel (IRC safety).

Generally, the remnant of this cancellation is uncancelled logarithms.

<u>Note</u>: the imaginary part of the loops **needs a different mechanism** to cancel.

An example: dijet production with a veto



Veto in-gap real emission with $k_\perp > Q_0$

Emitted "out of the gap"

$$(I_V^{(1)} + I_V^{(1)\dagger}) \otimes \Phi(\{p_i\}) \quad \sim \quad 2\mu^{2\epsilon} \int_0^{Q^2} \frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}} \int_{k_\perp^2/Q^2}^{e^{-Y/2}} \frac{dz}{z} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

$$+ \quad 2\mu^{2\epsilon} \int_0^{Q_0^2} \frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}} \int_{e^{-Y/2}}^{1} \frac{dz}{z} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) + \cdots$$

Emitted "in the gap"

Putting $s_{ij} \sim Q^2 > Q_0^2$ for i and j on opposite sides of the gap.

Adding the loop correction gives

$$d\sigma_V^{(1)} + d\sigma_R^{(1)} = \langle M^{(0)} | \boldsymbol{T}_L \cdot \boldsymbol{T}_R | M^{(0)} \rangle \, d(PS)_n \, \frac{\alpha_s}{2\pi} Y \log \frac{Q^2}{Q_0^2}$$

This is simply the loop correction integrated over the region of "phase-space" where the real emission is vetoed.

Factorization breaking

$$|M\rangle \approx \boldsymbol{Sp}(p_1, ..., p_m; \tilde{P}; p_{m+1} ..., p_n)|\overline{M}\rangle$$

One loop

$$\langle M^{(0)}|M^{(1)}\rangle + \text{h.c.} = \langle \overline{M}^{(0)}|\boldsymbol{Sp}^{(0)\dagger}\,\boldsymbol{Sp}^{(1)}|\overline{M}^{(0)}\rangle + \text{h.c.}$$
$$+ \langle \overline{M}^{(0)}|\boldsymbol{Sp}^{(0)\dagger}\,\boldsymbol{Sp}^{(0)}|\overline{M}^{(1)}\rangle + \text{h.c.}$$

The one-loop splitting operator is

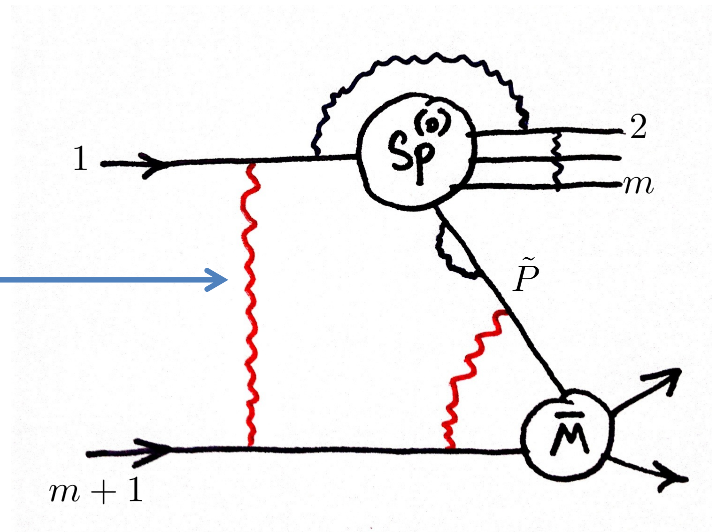
$$\boldsymbol{Sp}^{(1)} = \boldsymbol{I}_C^{(1)}\,\boldsymbol{Sp}^{(0)} \;,$$

$$\boldsymbol{I}_C^{(1)} = \boldsymbol{I}^{(1)} - \overline{\boldsymbol{I}}^{(1)} .$$

$$\boldsymbol{I}_C^{(1)} = \frac{\alpha_s}{2\pi}\frac{1}{2}\Bigg\{\left(\frac{1}{\epsilon^2}C_{\widetilde{P}} + \frac{1}{\epsilon}\gamma_{\widetilde{P}}\right) - \sum_{i=1}^{m}\left(\frac{1}{\epsilon^2}C_i + \frac{1}{\epsilon}\gamma_i - \frac{2}{\epsilon}C_i \ln|z_i|\right) - \frac{i\pi}{\epsilon}\left(C_{\widetilde{P}} - C_1 + \sum_{i=2}^{m} C_i\right)$$
$$- \frac{1}{\epsilon}\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{m} \boldsymbol{T}_i\cdot\boldsymbol{T}_\ell \ln\frac{|s_{i\ell}|}{|z_i|\,|z_\ell|\,\mu^2}\Bigg\} + \widetilde{\boldsymbol{\Delta}}_C^{(1)},$$

with

$$\widetilde{\boldsymbol{\Delta}}_C^{(1)} = \frac{\alpha_s}{2\pi}\left\{2\times\frac{i\pi}{\epsilon}\boldsymbol{T}_{m+1}\cdot(\boldsymbol{T}_1 - \boldsymbol{T}_{\widetilde{P}})\right\}.$$

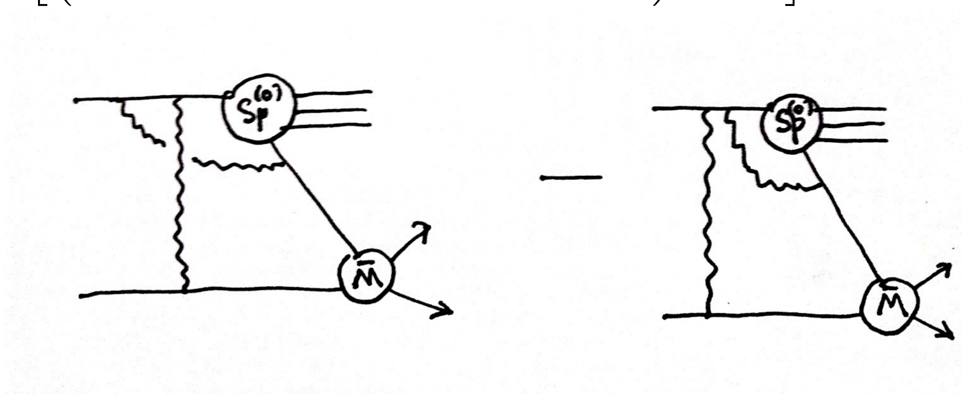


Coulomb exchange in the initial state breaks amplitude-level factorization at one-loop level. It **cancels at the cross-section level.** Since $\widetilde{\boldsymbol{\Delta}}_C^{(1)}$ is anti-Hermitian

Two loops

$$\langle \overline{M}^{(0)} | \mathbf{P}^{(2)}_{\text{n.f.}} | \overline{M}^{(0)} \rangle$$

$$\mathbf{P}^{(2)}_{\text{n.f.}} = \frac{1}{2} \boldsymbol{Sp}^{(0)\dagger} \left[\left(\overline{\boldsymbol{I}}^{(1)} + \overline{\boldsymbol{I}}^{(1)\dagger} + \boldsymbol{I}_C^{(1)\text{fact.}} + \boldsymbol{I}_C^{(1)\text{fact.}\dagger} \right), \widetilde{\boldsymbol{\Delta}}_C^{(1)} \right] \boldsymbol{Sp}^{(0)}$$



The factorization breaking still **cancels at the cross-section** level in pure QCD processes.

$$\text{Tr}\Big[\left(|M^{(0)}\rangle \langle M^{(0)}| \right) \; [\boldsymbol{I}, \widetilde{\boldsymbol{\Delta}}_C^{(1)}] \Big] = 0$$

$[\boldsymbol{I}, \widetilde{\boldsymbol{\Delta}}_C^{(1)}]$ is hermitian and a colour basis exists in which it is **anti-symmetric**.

$\mathbf{H} = |M^{(0)}\rangle \langle M^{(0)}|$ is real and **symmetric** in the same basis.

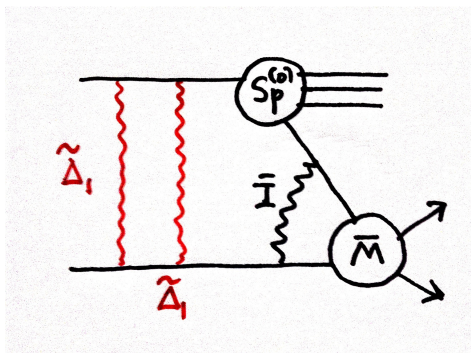
Seymour & Sjödahl

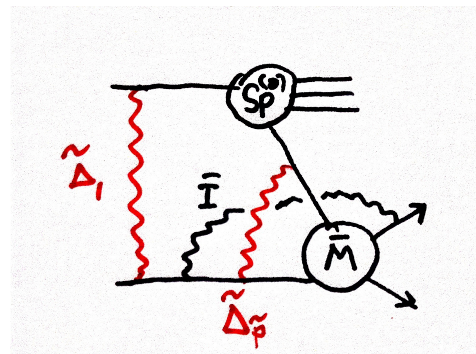
arXiv:0810.5756

Three loops: no escape!

$$\langle \overline{M}^{(0)} | \mathbf{P}^{(3)}_{\text{n.f.}} | \overline{M}^{(0)} \rangle$$

$$\begin{aligned}\mathbf{P}^{(3)}_{\text{n.f.}} \sim \quad & \frac{1}{6} \boldsymbol{Sp}^{(0)\dagger} \left(\left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{1}, \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{1}, \overline{\boldsymbol{I}}^{(1)} + \overline{\boldsymbol{I}}^{(1)\dagger} \right] \right] - \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{\widetilde{P}}, \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{\widetilde{P}}, \overline{\boldsymbol{I}}^{(1)} + \overline{\boldsymbol{I}}^{(1)\dagger} \right] \right] \right) \boldsymbol{Sp}^{(0)} \\ & + \frac{1}{2} \boldsymbol{Sp}^{(0)\dagger} \left(\left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{\widetilde{P}}, \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{\widetilde{P}}, \overline{\boldsymbol{I}}^{(1)} + \overline{\boldsymbol{I}}^{(1)\dagger} \right] \right] - \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{\widetilde{P}}, \left[\widetilde{\boldsymbol{\Delta}}^{(1)}_{1}, \overline{\boldsymbol{I}}^{(1)} + \overline{\boldsymbol{I}}^{(1)\dagger} \right] \right] \right) \boldsymbol{Sp}^{(0)}.\end{aligned}$$



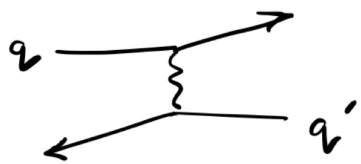


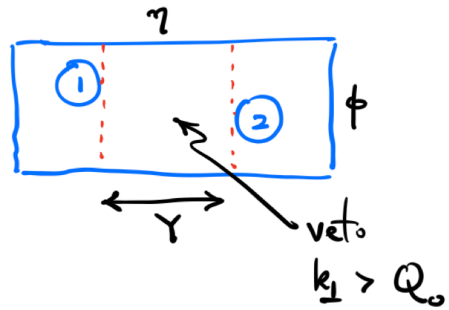
This gives a **non-zero** contribution to the cross-section in QCD.

JRF, Seymour, Siodmok
arXiv:1206.6363

The example of gaps-between-jets or how the plus prescription fails

Consider $qq' \to qq'$

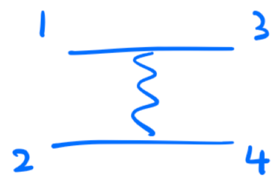




The original calculation of Oderda & Sterman

Originally (pre-NGLs) calculated by Oderda & Sterman (1998 & 2000)

$$|M\rangle = V_{Q_0,Q}\,|M_0\rangle$$



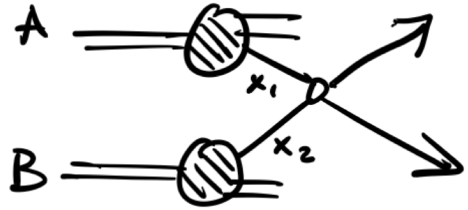
$$V_{a,b} = \exp\left[\frac{\alpha_s}{\pi}\int_a^b \frac{dk_\perp}{k_\perp}\,\Gamma\right]$$

$$T_t^2 = (T_1 + T_3)^2$$

$$\Gamma = \int_{\text{gap}} dy\,\frac{d\phi}{2\pi}\sum_{i<j} T_i \cdot T_j\,\omega_{ij} - 2\pi i\,T_1 \cdot T_2 = -Y\,T_t^2 - i\pi\,T_s^2$$

(+ diagonal)

$$\omega_{ij} = \frac{k_\perp^2}{2}\,\frac{P_i \cdot P_j}{P_i \cdot k\;P_j \cdot k}$$



$$H = |M_0\rangle\langle M_0|$$

$$\frac{d\Sigma_{OS}}{dx_1\,dx_2} = f_A(x_1,Q)\, f_B(x_2,Q)\, \mathrm{Tr}\left(V_{Q_0,Q}\, H\, V^{\dagger}_{Q_0,Q}\right)$$

$$= \left[f_A(x_1,Q_0) + \frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int_{x}^{1-k_\perp^2/Q^2}\frac{dz}{z}\, P_{qq}(z) \right.$$

$$P_{qq}(z) = C_F\,\frac{1+z^2}{1-z}$$

$$\left. \times \left\{ f_A\!\left(\frac{x_1}{z},Q_0\right) - z^2 f_A(x_1,Q_0)\right\} + \cdots \right] .$$

$$\times\; f_B(x_2,Q)\, \mathrm{Tr}\left(V_{Q_0,Q}\, H\, V^{\dagger}_{Q_0,Q}\right)$$

It is WRONG

It should be.....

$$\frac{d\Sigma_1}{dx_1\, dx_2} = \frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int_{x_1}^{1-k_\perp^2/Q^2}\frac{dz}{z}\, P_{qq}(z)\ \times$$

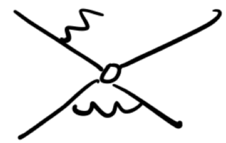
$$\Big[f_A\left(\frac{x_1}{z}, Q_0\right) \frac{1}{T_1^2} \mathrm{Tr}\left(V_{Q_0,k_\perp} T_1 V_{k_\perp,Q} H V^{\dagger}_{k_\perp,Q} T_1^{\dagger} V^{\dagger}_{Q_0,k_\perp} \right)$$

$$- z^2 f_A(x_1, Q_0)\, \mathrm{Tr}\left(V_{Q_0,Q} H V^{\dagger}_{Q_0,Q} \right) \Big] f_B(x_2, Q_0)$$

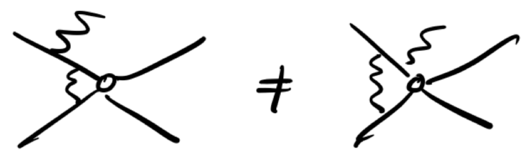
The Coulomb exchanges are spoiling the plus prescription and destroying our ability to factorize the collinear logarithms into PDFs

$$\Gamma = -\gamma\, T_t^2 - i\pi\, T_s^2$$

$$[T_1, T_t^2] = 0$$

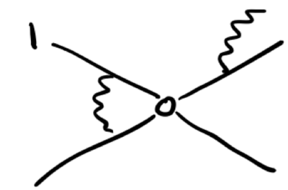


$$[T_1, T_s^2] \neq 0$$

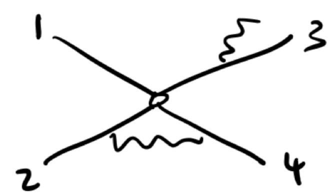


Note: final state collinear case does lead to zero. Since

$$[T_3 , T_s^2] = [T_3 , T_t^2] = 0$$



$T_s^2 \propto T_1 \cdot T_2$



$T_t^2 \propto T_2 \cdot T_4$

$$\frac{d\Sigma_1}{dx_1\,dx_2} = \frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int_{x_1}^{1-k_\perp^2/Q^2}\frac{dz}{z}\,P_{qq}(z)\times$$

$$\Big[f_A\!\left(\frac{x_1}{z},Q_0\right)\frac{1}{T_1^2}\,\mathrm{Tr}\left(V_{Q_0,k_\perp}T_1 V_{k_\perp,Q}\,H\,V^\dagger_{k_\perp,Q}\,T_1^\dagger V^\dagger_{Q_0,k_\perp}\right)$$

$$- z^2 f_A(x_1,Q_0)\,\mathrm{Tr}\left(V_{Q_0,Q}\,H\,V^\dagger_{Q_0,Q}\right)\Big] f_B(x_2,Q_0)$$

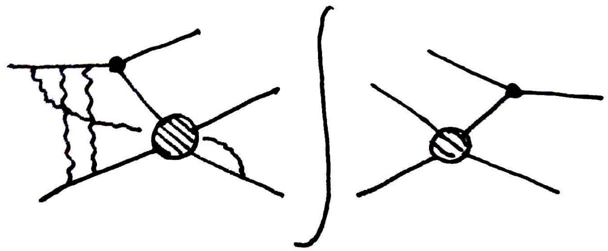
The non-vanishing of [….] at z=1 induces "super-leading" logs

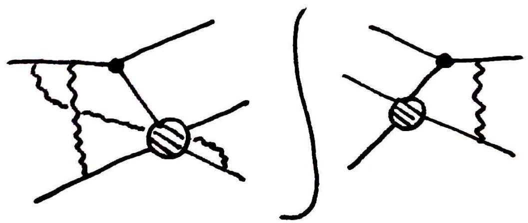
$$\frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int_{x}^{1-k_\perp^2/Q^2}\frac{dz}{z}\,P_{qq}(z)\times[\sim]$$

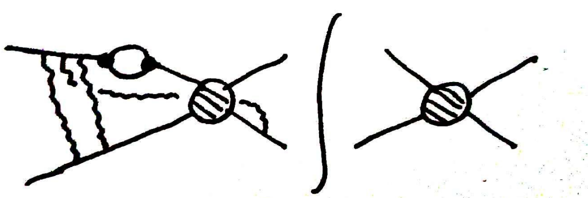
$$\approx \frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int^{1-k_\perp^2/Q^2}\frac{dz}{1-z}\,2C_F\,[\sim] \approx \frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\log\frac{Q^2}{k_\perp^2}\,C_F\,[\sim]$$

Kyrieleis, Seymour, JRF
arXiv:hep-ph/0604094

One collinear splitting contribution to gaps-between-jets







$$\sigma_1 = -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\left(2\ln\frac{Q}{k_T}\right)\left\langle m_0\right| \; e^{-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{t}_t^2 - i\pi\mathbf{t}_1\cdot\mathbf{t}_2\right)}$$

$$\left\{\mathbf{t}_1^2 e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{t}_t^2 - i\pi\mathbf{t}_1\cdot\mathbf{t}_2\right)} e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{t}_t^2 + i\pi\mathbf{t}_1\cdot\mathbf{t}_2\right)}\right.$$

$$\left. -\mathbf{t}_1^{a\dagger} e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{T}_t^2 - i\pi\mathbf{T}_1\cdot\mathbf{T}_2\right)} e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{T}_t^2 + i\pi\mathbf{T}_1\cdot\mathbf{T}_2\right)}\mathbf{t}_1^a\right\}$$

$$e^{-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\left(\frac{1}{2}Y\mathbf{t}_t^2 + i\pi\mathbf{t}_1\cdot\mathbf{t}_2\right)}\left|m_0\right\rangle.$$

$$\left\{\;\right\}_2 = \left(\frac{i\pi Y}{2}\right)\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\right)^2\left\{\mathbf{t}_1^2\left[\mathbf{t}_t^2, \mathbf{t}_1\cdot\mathbf{t}_2\right] - \mathbf{t}_1^{a\dagger}\left[\mathbf{T}_t^2, \mathbf{T}_1\cdot\mathbf{T}_2\right]\mathbf{t}_1^a\right\}$$

$$\left\{\;\right\}_3 \equiv -\frac{Y\pi^2}{6}\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\right)^3\left\{\mathbf{t}_1^2\Big[\left[\mathbf{t}_t^2, \mathbf{t}_1\cdot\mathbf{t}_2\right], \mathbf{t}_1\cdot\mathbf{t}_2\Big]\right.$$

$$\left. -\mathbf{t}_1^{a\dagger}\Big[\left[\mathbf{T}_t^2, \mathbf{T}_1\cdot\mathbf{T}_2\right], \mathbf{T}_1\cdot\mathbf{T}_2\Big]\mathbf{t}_1^a\right\}.$$

This is the failure of factorization we anticipated earlier

Kyrieleis, Seymour, JRF
arXiv:0808.1269

Note: SLL occurs at lowest possible order

$$\alpha_s^2 (i\pi)^2 \times \alpha_s \times \alpha_s Y \times \log^5 Q/Q_0$$

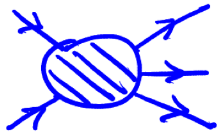
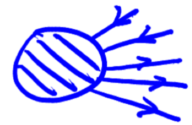
- $\alpha_s^2 (i\pi)^2$ ↑ Coulomb ($\sim T_s^2$)
- α_s ↑ Collinear
- $\alpha_s Y$ ↑ non-Coulomb ($\sim T_t^2$)

This physics is not specific to non-global observables

Soft gluon evolution

Amplitude-level evolution is needed to go beyond leading colour, see Zoltan’s talk and his work with Dave Soper.

Soft gluon evolution

$|H\rangle =$  $=$ 

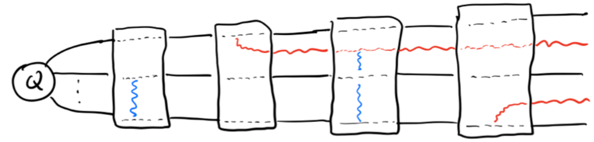
$|H\rangle\langle H| = H =$ 

$$D_i = \sum_j T_j \, E_i \frac{p_j}{p_j \cdot q_i}$$

$$V_{a,b} = \exp\left[\int_a^b \frac{dE}{E}\,\Gamma\right]$$

$$\Gamma = \frac{\alpha_s}{\pi}\sum_{i<j}\left(-T_i\cdot T_j\right)\left\{\int \frac{d\Omega_k}{4\pi}\, w_{ij}(\hat{k}) - i\pi\,\tilde{\delta}_{ij}\right\}$$

$$w_{ij}(\hat{k}) = E_k^2 \frac{p_i \cdot p_j}{p_i \cdot k \; p_j \cdot k}$$



e.g. $A_1 =$

$$d\sigma_n = \mathrm{Tr}\, A_n(\mu)\, d\Pi_n$$

$\mu = 0$ always
& $\mu = Q_0$ if inclusive
for $E < Q_0$

$$\Sigma = \sum_n \int d\sigma_n \; u_n(q_1, \dots, q_n)$$

← measurement function

$$A_n(E) = V_{E,E_n} D_n A_{n-1}(E_n) D_n^\dagger V_{E,E_n}^\dagger \, \Theta(E \le E_n)$$

The leading N_c part of this hierarchy = BMS equation = dual to the BK equation.

Is this hierarchy dual to JIMWLK? (Weigert Nucl.Phys. B685 (2004) 321, hep-ph/0312050)

Collinearly improved in
Forshaw, Holguin, Plätzer
JHEP 1908 (2019) 145
arXiv:1905.0868

Ángeles Martínez, de Angelis, JF, Plätzer, Seymour
JHEP 05(2018) 044
arXiv:1802.08531

The ordering variable often does not matter.

(It does for the super-leading logs, where it must be transverse momentum.)

However, we have additional insight from the work of Catani & Grazzini and from the PhD thesis of René Ángeles-Martínez.

Catani & Grazzini: Nucl. Phys.B591, 435 (2000), hep-ph/0007142

Catani-Grazzini: all-orders amplitude-level factorization in the soft limit

$$|M_N\rangle = (g_s \mu^\epsilon)^N \, \mathbf{J}(q_N) \cdots \mathbf{J}(q_1) \; |M_0\rangle$$

At one-loop:

$$\mathbf{J}(q) = \mathbf{J}^{(0)}(q) + \mathbf{J}^{(1)}(q)$$

$$|M_0\rangle = |M_0^{(0)}\rangle + |M_0^{(1)}\rangle$$

$$\mathbf{J}^{(1)}(q_{m+1}) = \frac{1}{2} \sum_{j=1}^{n+m} \sum_{k=1}^{n+m} \mathbf{d}_{jk}^{(1)}(q_{m+1})$$

$$|M_0^{(1)}\rangle = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \mathbf{I}_{ij}(0, Q) \; |M_0^{(0)}\rangle$$

$$\mathbf{d}_{ij}^{(1)}(q_a) = \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon^2} \, \mathbf{T}_{n+a} \cdot \mathbf{T}_i \left(\frac{(q_a^{(ij)})^2 \; e^{-i\pi \tilde{\delta}_{i(n+a)}} \; e^{-i\pi \tilde{\delta}_{j(n+a)}}}{4\pi \mu^2 \, e^{-i\pi \tilde{\delta}_{ij}}} \right)^{-\epsilon} \mathbf{d}_{ij}(q_a)$$

$$\mathbf{d}_{ij}(q) = \mathbf{T}_j \left(\frac{p_j \cdot \varepsilon}{p_j \cdot q} - \frac{p_i \cdot \varepsilon}{p_i \cdot q} \right)$$

$$\sum_j \mathbf{d}_{ij}(q) = \sum_j \mathbf{T}_j \frac{p_j \cdot \varepsilon}{p_j \cdot q} = \mathbf{J}^{(0)}(q)$$

Remarkably this can be re-written

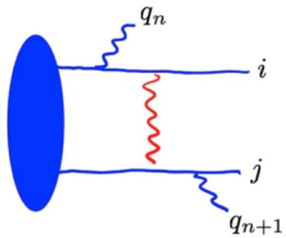
$$
\begin{aligned}
|M_N^{(1)}\rangle \; &= \; \sum_{m=0}^{N}\sum_{i=2}^{p}\sum_{j=1}^{i-1} (g_s\mu^\epsilon)^{N-m}\, \mathbf{J}^{(0)}(q_N)\cdots\mathbf{J}^{(0)}(q_{m+1})\, \mathbf{I}_{ij}(\tilde{q}_{m+1},\tilde{q}_m)\; |M_m^{(0)}\rangle \\
&+ \; \sum_{m=1}^{N}\sum_{j=1}^{n+m-1}\sum_{k=1}^{n+m-1} (g_s\mu^\epsilon)^{N-m}\, \mathbf{J}^{(0)}(q_N)\cdots\mathbf{J}^{(0)}(q_{m+1})\, \mathbf{I}_{n+m,j}(\tilde{q}_{m+1},q_m^{(jk)})\, \mathbf{d}_{jk}(q_m)\, |M_{m-1}^{(0)}\rangle \;\; ,
\end{aligned}
$$

$$
|M_m^{(0)}\rangle = (g_s\mu^\epsilon)^m\, \mathbf{J}^{(0)}(q_m)\, \mathbf{J}^{(0)}(q_{m-1})\cdots\mathbf{J}^{(0)}(q_1)\, |M_0^{(0)}\rangle
$$

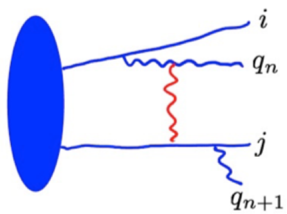
$$
\tilde{q} = \; (q^{(ij)})^2 = \frac{2\, q\cdot p_i\; q\cdot p_j}{p_i\cdot p_j}
$$

$$
\begin{aligned}
\mathbf{I}_{ij}(a,b) &= \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j \left[\left(\frac{b^2}{4\pi\mu^2}\right)^{-\epsilon}\left(1+i\pi\epsilon\,\tilde{\delta}_{ij}-\epsilon\ln\frac{2p_i\cdot p_j}{b^2}\right)-\left(\frac{a^2}{4\pi\mu^2}\right)^{-\epsilon}\left(1+i\pi\epsilon\,\tilde{\delta}_{ij}-\epsilon\ln\frac{2p_i\cdot p_j}{a^2}\right)\right] \\
&= \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[-\frac{1}{2}\ln^2\frac{2p_i\cdot p_j}{b^2}+\frac{1}{2}\ln^2\frac{2p_i\cdot p_j}{a^2}-i\pi\tilde{\delta}_{ij}\ln\frac{b^2}{a^2}\right] \qquad a^2,b^2>0
\end{aligned}
$$

Ángeles-Martínez , JRF, Seymour: arXiv:1602.00623

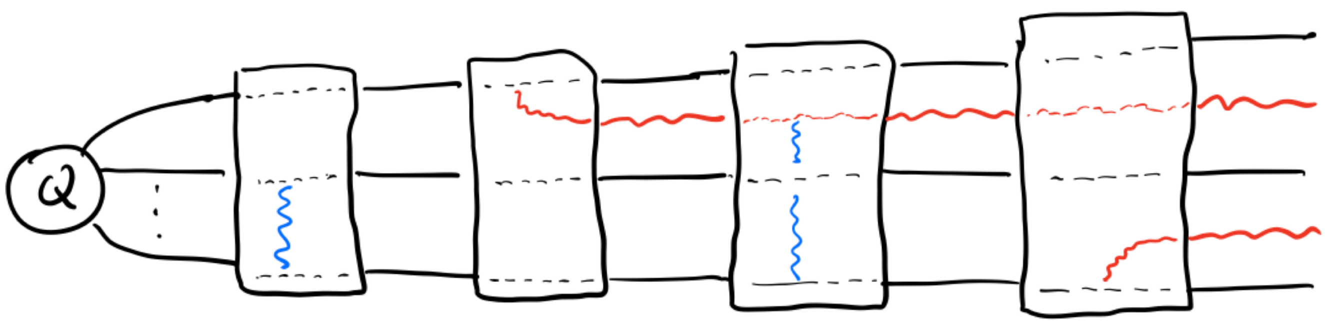


$$\int_{q_{n+1}^{(ij)}}^{q_n^{(ij)}} \frac{\mathrm{d}k_T}{k_T}$$



$$\int_{q_{n+1}^{([n]j)}}^{q_n^{(ij)}} \frac{\mathrm{d}k_T}{k_T}$$

exchange between latest emission & any other



Conclusions

- Coulomb gluons wreck collinear factorization as soon as they are able to (though factorization always holds below the ”inclusivity” scale, which means Collins-Soper-Sterman factorization of the collinear poles works)
- Phenomenology of this?
- We have the means to go beyond leading colour in general purpose event generators.
- It is remarkable how QCD selects a special ordering variable (not easy to see this in SCET?).
- Can we make more precise the link between soft-gluon evolution and JIMWLK? Why is there even any link at all?