

Carroll Approximations of GR and Near-Singularity Dynamics

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Introduction

Kasner geometries

$$ds^2 = - dt^2 + t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2$$

homogeneous and anisotropic solution to Einstein equations

Adding spatial curvature or matter:

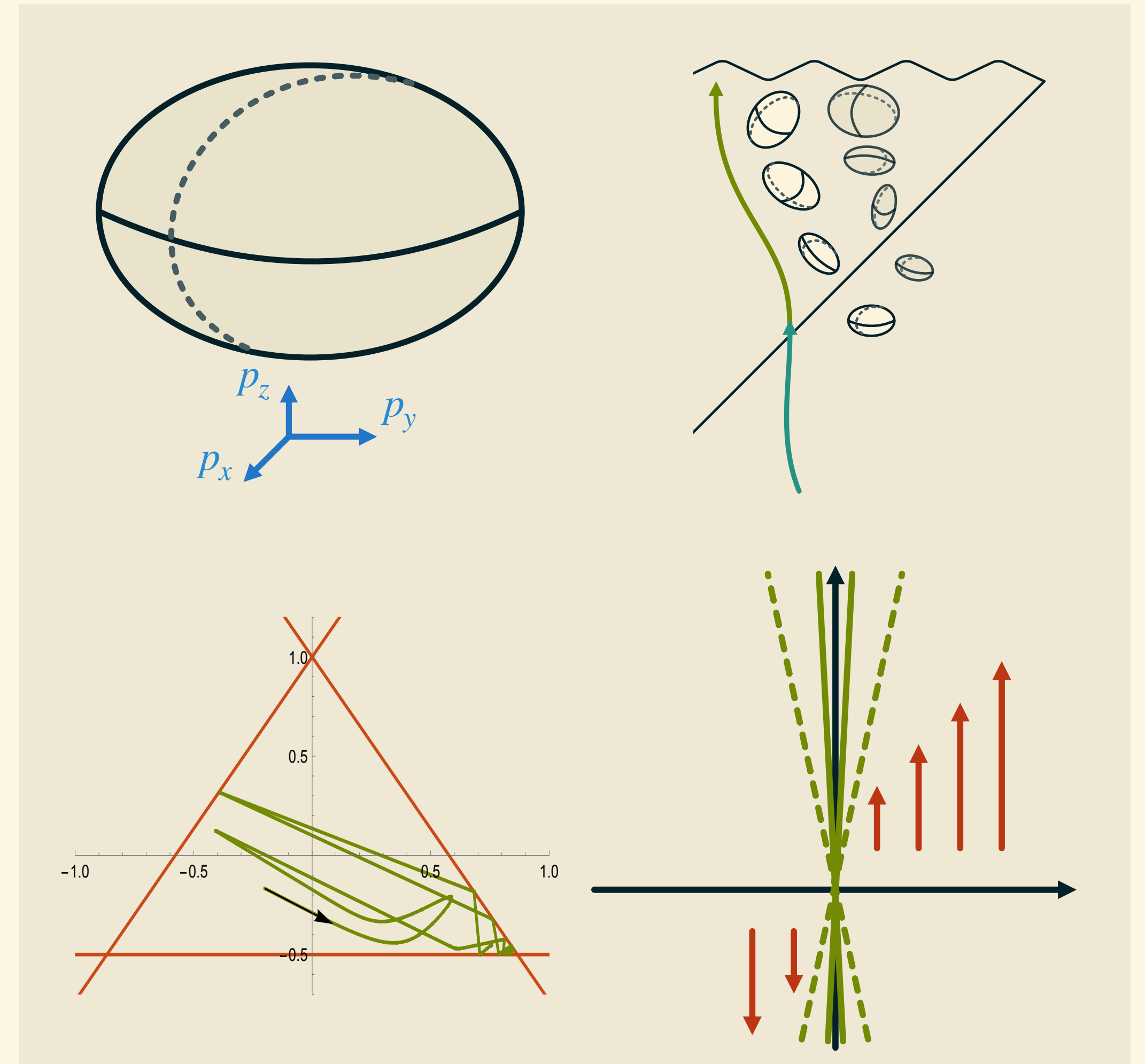
rich and possibly chaotic dynamics in space of allowed p_i

Belinski–Khalatnikov–Lifshitz (BKL) conjecture:

near spacelike singularities, the generic behavior of GR is given by ultra-local chaotic dynamics of this type

Motivations:

- Difficult regime in GR, study using Carroll approximation?
- How does AdS/CFT encode singularity and BKL dynamics?



Outline

- Kasner and BKL in gravity
- Carroll limits and geometry
- Carroll approximation of general relativity
- Mixmaster from Carroll gravity

Kasner geometries in GR

Take planar AdS black hole and zoom in **behind horizon**, $f(z) = 1 - (z/z_H)^3 \approx - (z/z_H)^3$

$$ds^2 = \frac{1}{z^2} \left[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right]$$

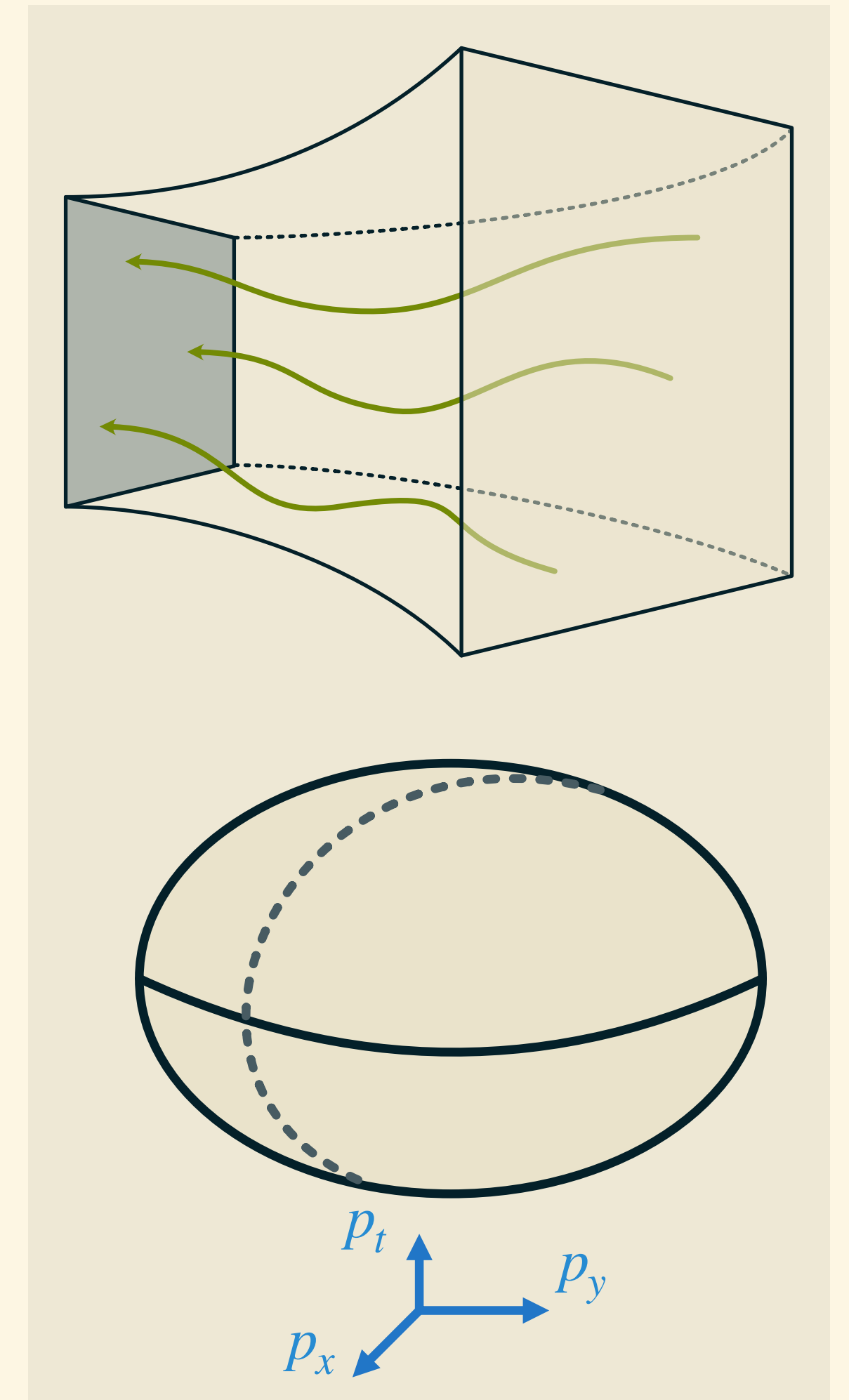
$$\approx + \# \frac{dt^2}{\tau^{2/3}} - d\tau^2 + \# \tau^{4/3} (dx^2 + dy^2)$$

where $\tau = \tau(z)$ is interior 'radial time'

Example of **Kasner geometry** with $p_t = -1/3$ and $p_x = p_y = 2/3$

$$ds^2 = -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} dx^2 + \tau^{2p_y} dy^2$$

Solution of vacuum Einstein equations if $\sum p_i = 1$ and $\sum (p_i)^2 = 1$



Kasner geometries in GR

Parametrize Kasner solutions using lapse $\alpha(t)$ and scaling exponents $\beta_i(t)$

$$ds^2 = -e^{-2\alpha(t)} dt^2 + e^{2\beta_x(t)} dx^2 + e^{2\beta_y(t)} dy^2 + e^{2\beta_z(t)} dz^2$$

Vacuum Einstein equations then give

$$0 = E_{tt} = \dot{\beta}^T \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \dot{\beta} = -\left(\ddot{\beta}_1\right)^2 + \left(\ddot{\beta}_2\right)^2 + \left(\ddot{\beta}_3\right)^2$$

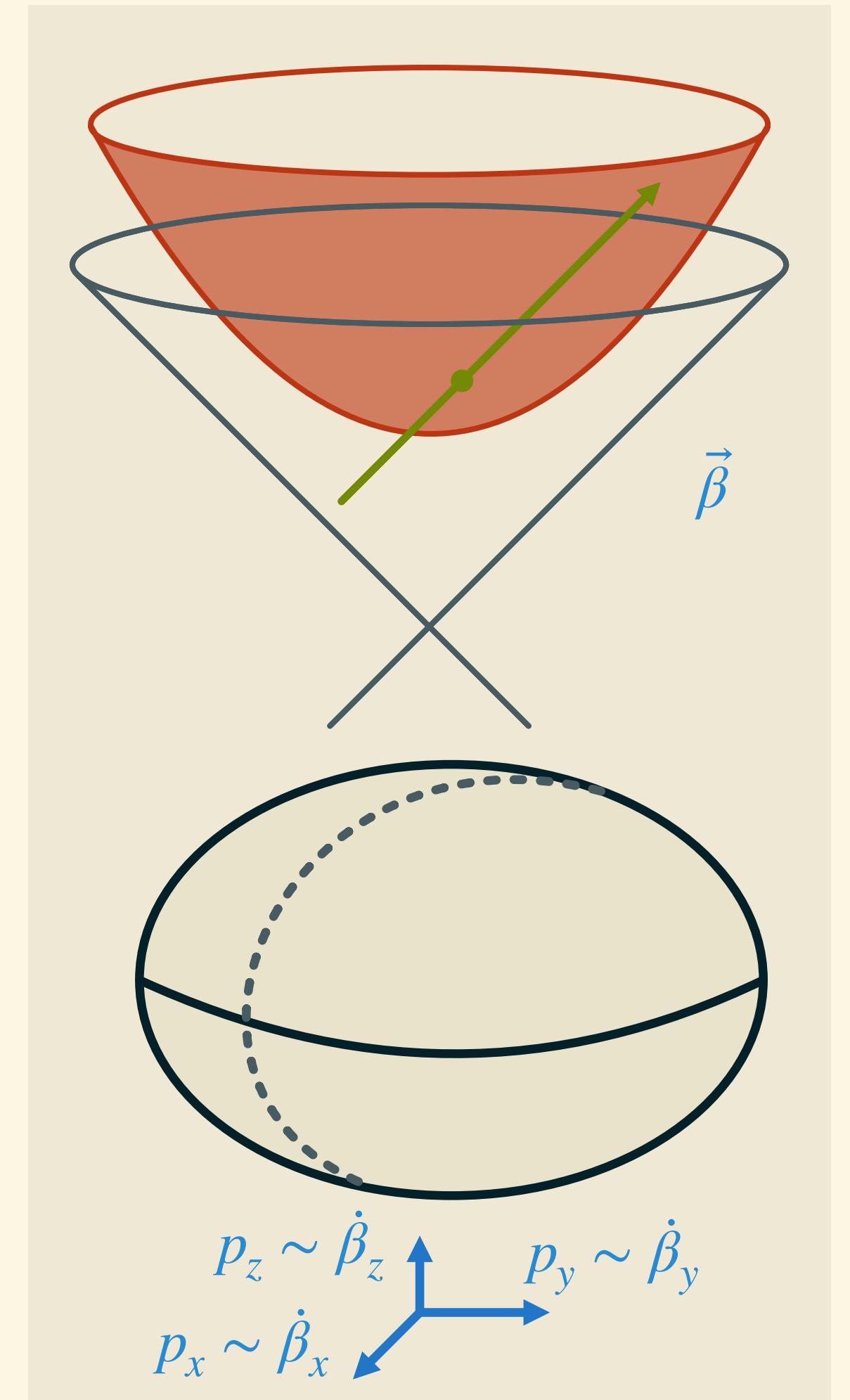
Interpret $\dot{\beta}_i(t)$ as **null vector** in Minkowski **superspace!** [Chitre] [Damour, Henneaux, Nicolai]

After shifting lapse and reparametrizing $t = t(\tau)$, spatial components $E_{ii} = 0$ give

$$0 = \ddot{\beta}_i(\tau) \quad \implies \quad \beta_i = \beta_i^{(0)} + v_i \tau$$

so $\beta_i(\tau)$ parametrizes **null geodesic** in $\mathbb{R}^{1,2}$ superspace!

Maps to (so far simple) **particle motion** on **future hyperboloids** $\mathbb{H}^2 \subset \mathbb{R}^{1,2}$



Mixmaster dynamics in GR

Interesting dynamics from spatial curvature and/or matter coupling

Example: $SO(3)$ group manifold ('mixmaster/Bianchi IX') [Misner]

Homogeneous isotropic metric from Maurer-Cartan forms μ^i

$$d\sigma^2 = \delta_{ij} \mu^i \mu^j = d\varphi^2 + 2 \cos \theta d\varphi d\psi + d\theta^2$$

Homogeneous **anisotropic** space using scaling exponent matrix $\beta_{ij}(t) = \beta_i(t) \delta_{ij}$

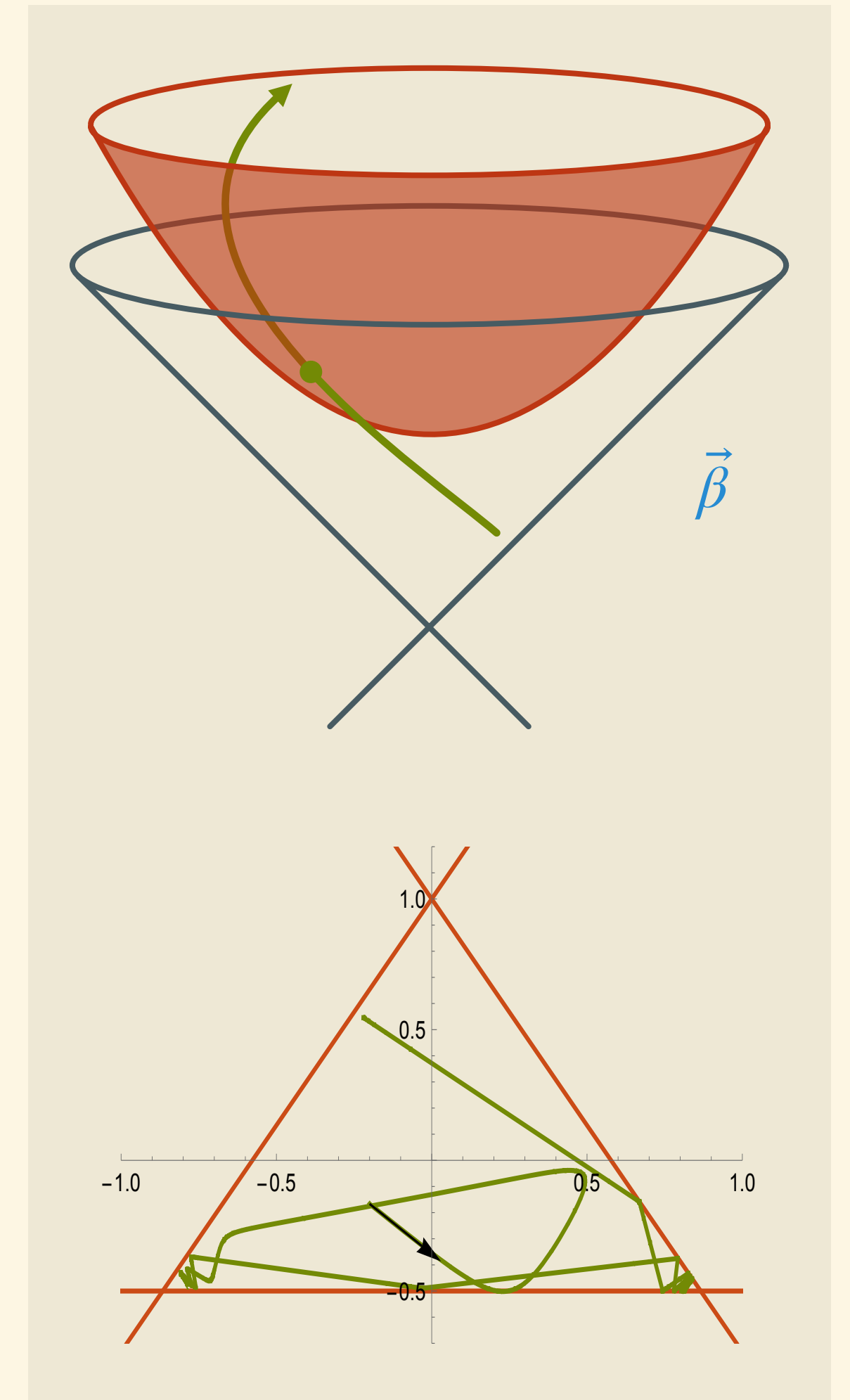
$$ds^2 = -e^{-2\alpha} dt^2 + (e^{2\beta})_{ij} \mu^i \mu^j$$

$$R^{(3)} = \frac{1}{2} e^{-2 \operatorname{tr} \beta} (\operatorname{tr}(\beta)^2 - 2 \operatorname{tr}(\beta^2)) \quad \Longrightarrow \quad V(\beta) = e^{4\beta_x} + e^{4\beta_y} + e^{4\beta_z}$$

Potential modifies null geodesics, bounce around in **hyperbolic triangle** [Misner] [Chitre]

BKL: near spacelike singularities, 4D GR reduces to ultra-local chaotic dynamics of this kind

[Belinskii, Khalatnikov, Lifshitz] [Damour, Henneaux, Nicolai]



Kasner geometries in holography

Planar AdS-RN black hole with charged massive scalar (holographic superconductor)

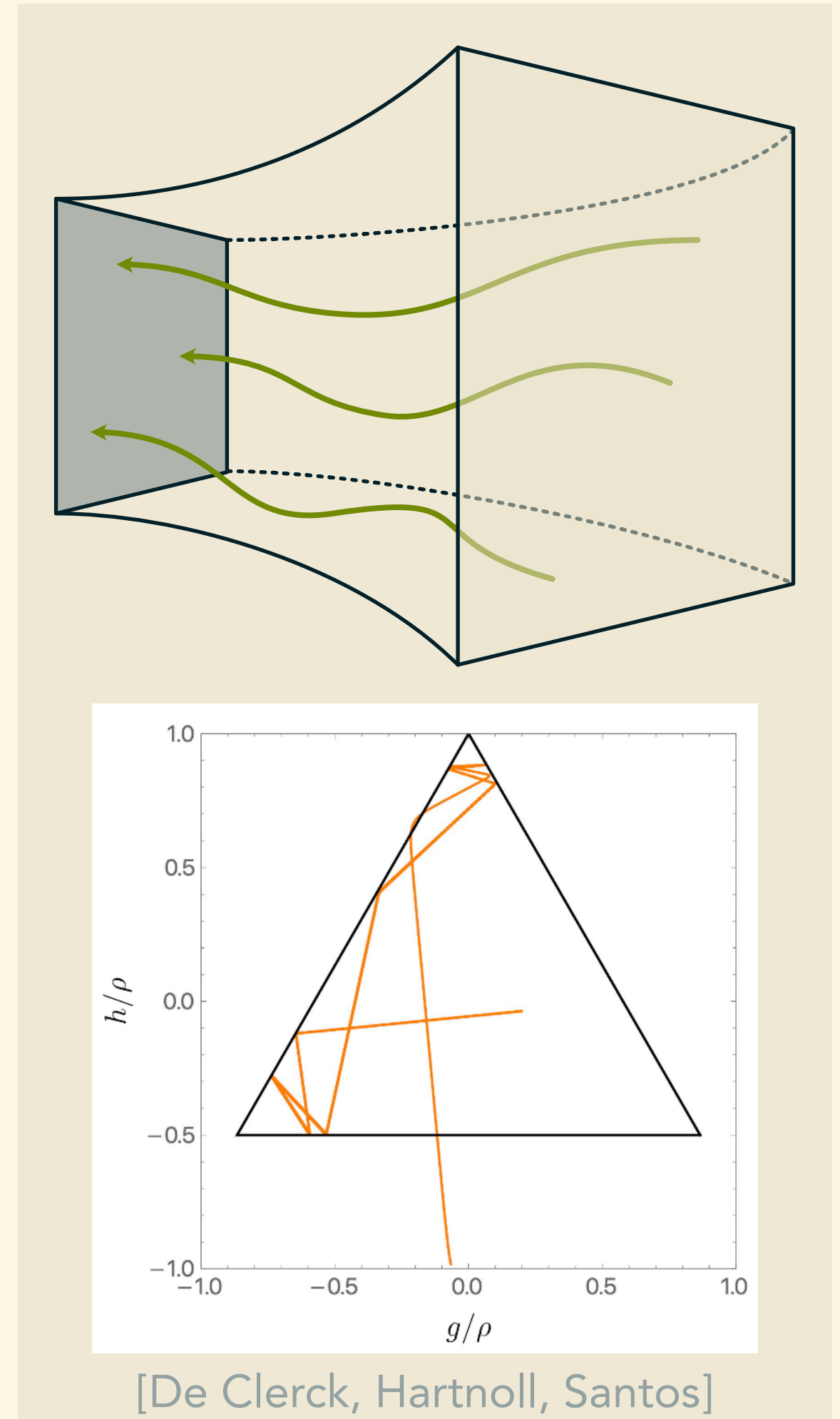
$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (R + 6) - \frac{1}{2} \int d^4x \sqrt{-g} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + D^\mu \phi \bar{D}_\nu \phi + m^2 \phi^2 \right)$$

- Nontrivial dynamics **behind horizon!** [Hartnoll, Horowitz, Kruthoff, Santos]
- Different **Kasner epochs**, but eventually reaches final state [Henneaux]

Mixmaster-style chaotic behavior obtained from three gauge fields [De Clerck, Hartnoll, Santos]

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (R + 6) - \frac{1}{2} \int d^4x \sqrt{-g} \left(\frac{1}{2} F_{(i)}^{\mu\nu} F_{\mu\nu}^{(i)} + \mu_{(i)}^2 A_{(i)}^2 \right)$$

- Give same **hyperbolic triangle** dynamics in interior, with AdS asymptotics
- Interpretation in terms of **holographic description** of black holes?



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Carroll geometry

From 'relativistic' Lorentz boosts

$$t \rightarrow t + \beta x, \quad x \rightarrow x + \beta t$$

get Carroll boosts in ultra-local $c \rightarrow 0$ limit, [Levy-Leblond] [Sen Gupta]

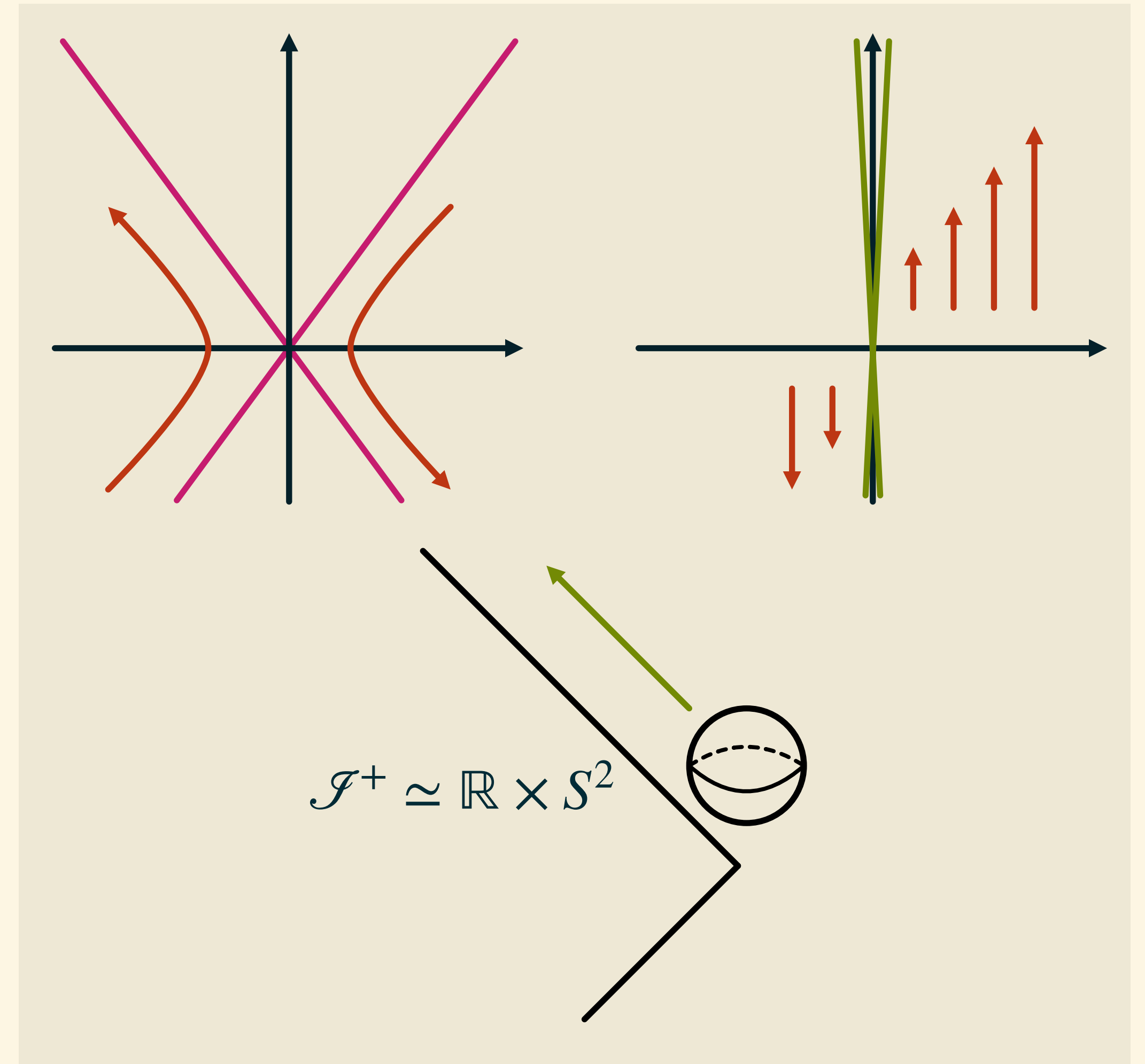
$$t \rightarrow t + \lambda x, \quad x \rightarrow x \quad \text{and} \quad \partial_t \rightarrow \partial_t, \quad \partial_x \rightarrow \partial_x + \lambda \partial_t$$

Less obviously physical than non-relativistic $c \rightarrow \infty$ limit, but:

- appears in Lorentzian geometry on null surfaces such as \mathcal{I}^+
- BMS asymptotic symmetries as conformal Carroll symmetries

[see many other talks in this meeting!]

Here instead use ultra-local limit in bulk to describe BKL dynamics



Carroll geometry

Compare: Lorentzian geometry

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_2$$

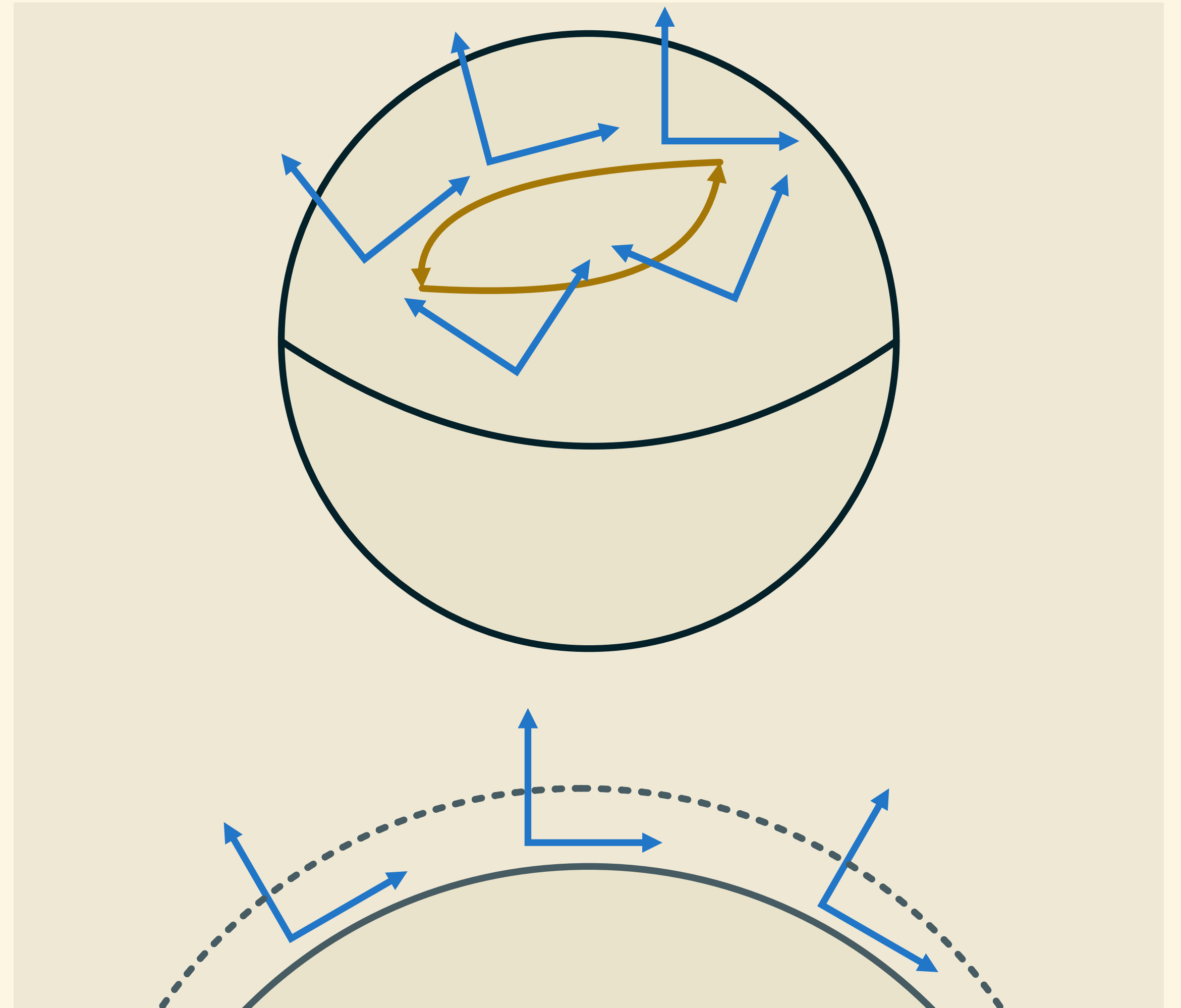
Compatible torsion-free connection $\nabla_\rho g_{\mu\nu} = 0$

defines curvature $[\nabla_\mu, \nabla_\nu] X^\sigma = -R_{\mu\nu\rho}{}^\sigma X^\rho$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{AB} \begin{pmatrix} \sqrt{f} & 0 \\ 0 & 1/\sqrt{f} \end{pmatrix}_\mu \begin{pmatrix} \sqrt{f} & 0 \\ 0 & 1/\sqrt{f} \end{pmatrix}_\nu^B$$
$$= \eta_{AB} e_\mu^A e_\nu^B$$

metric has local Minkowski structure

Goal: mirror this for local Carroll structures



Carroll geometry

Carroll boost

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

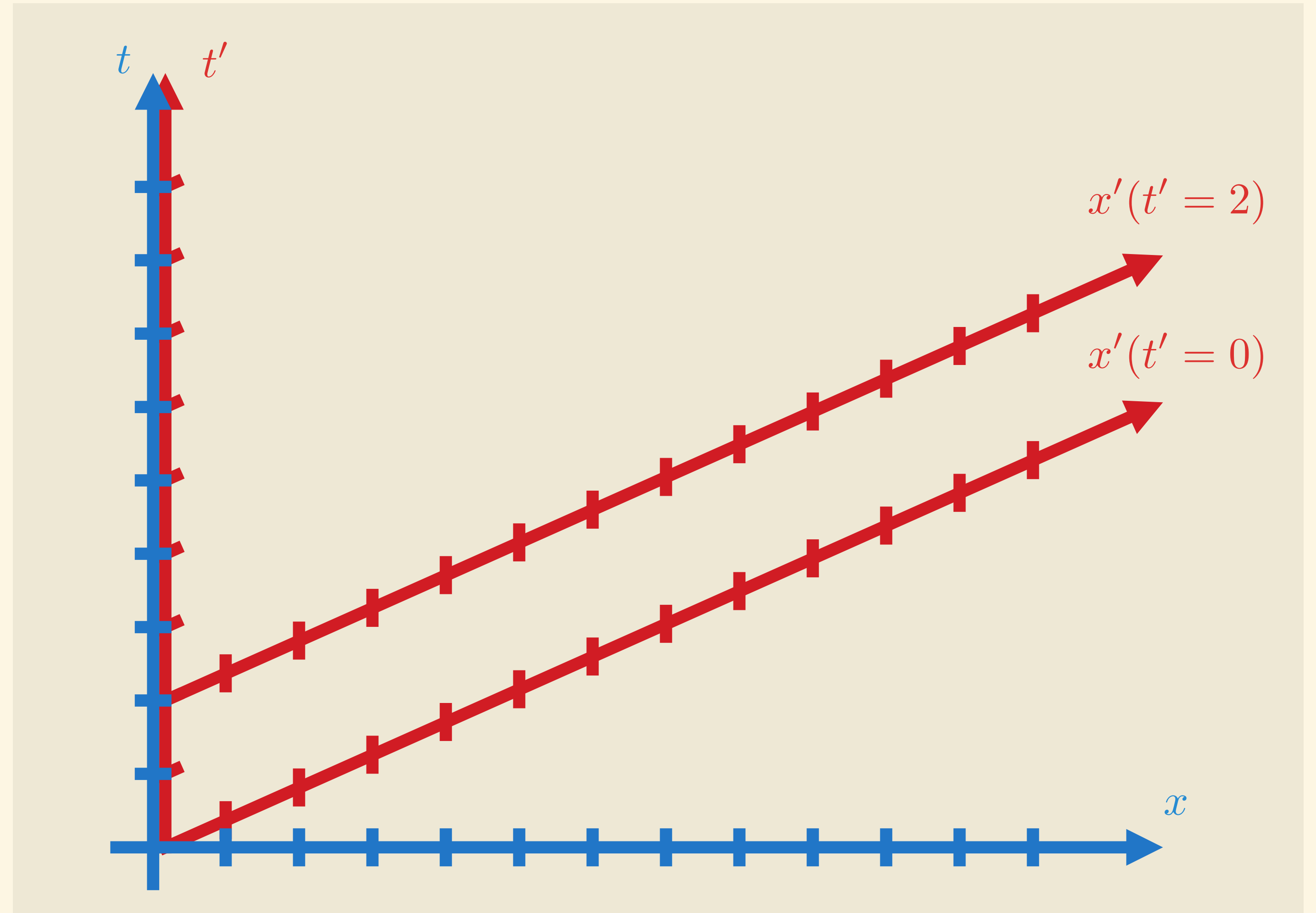
preserves

- time direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- space coordinate $\begin{pmatrix} 0 & 1 \end{pmatrix}$

For curved geometry:

- time vector field $v^\mu \partial_\mu \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- spatial metric $h_{\mu\nu} dx^\mu dx^\nu \sim \text{twice } \begin{pmatrix} 0 & 1 \end{pmatrix}$

gives general **Carroll** geometry



Carroll geometry

Curved Carroll geometry is specified by

time vector field $v^\mu(x^\rho)$ and

spatial 'metric' $h_{\mu\nu}(x^\rho)$

Example: Kasner-type geometry

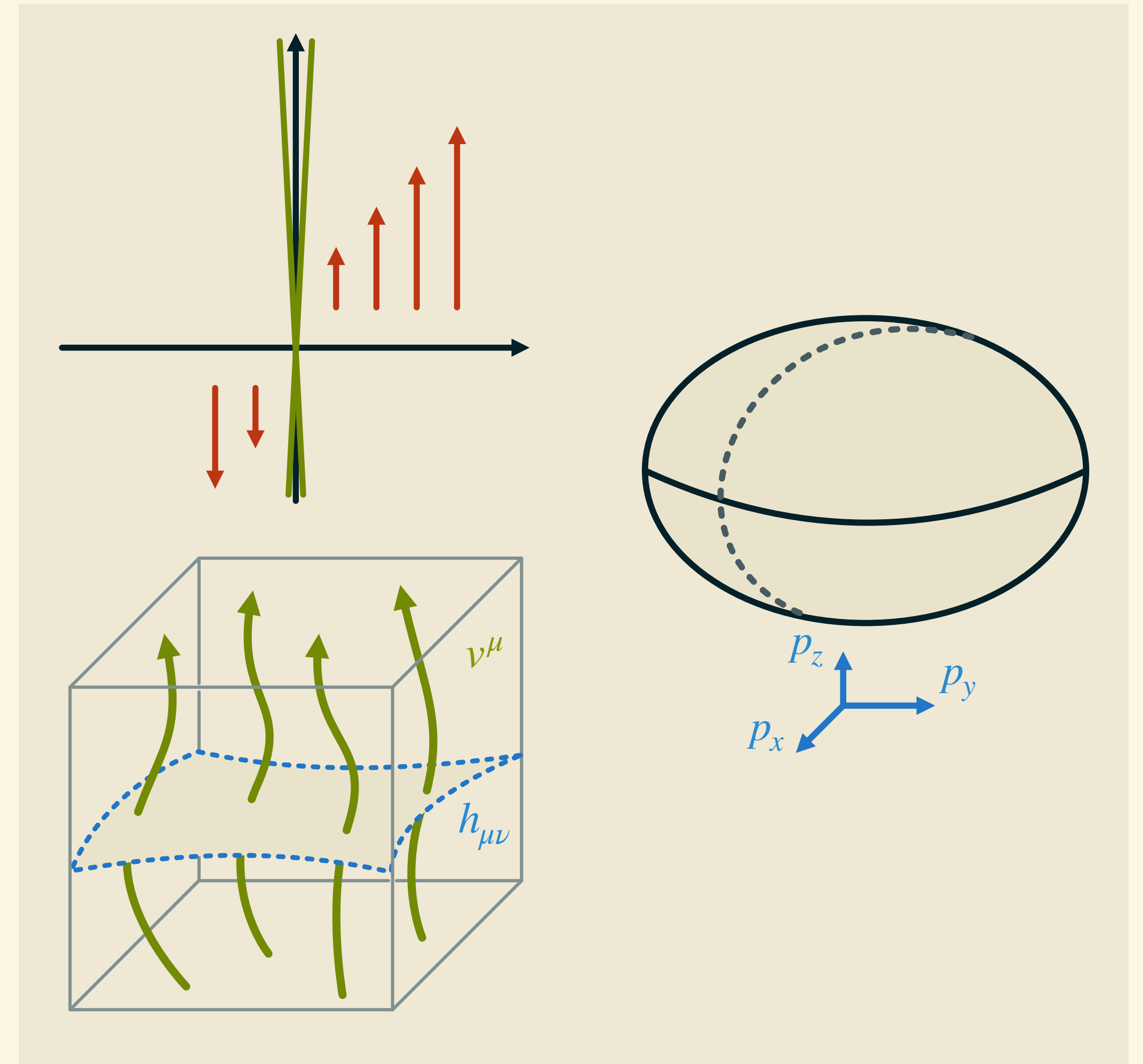
$$v^\mu \partial_\mu = -\partial_t, \quad h_{\mu\nu} dx^\mu dx^\nu = t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2$$

Extrinsic curvature $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}$

Compatible connection $\tilde{\nabla}_\rho v^\mu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$ [other choices too!]

curvature $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] X^\sigma = -\tilde{R}_{\mu\nu\rho}{}^\sigma X^\rho - 2\tilde{\Gamma}_{[\mu\nu]}^\rho \tilde{\nabla}_\rho X^\sigma$

torsion $2\tilde{\Gamma}_{[\mu\nu]}^\rho = 2h^{\rho\sigma} \tau_{[\mu} K_{\nu]\sigma}$



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Carroll from Lorentzian

From Lorentzian geometry get Carroll plus corrections by **expanding around $c \rightarrow 0$**

Two-step process: **rewrite** and **expand** [Hansen, Obers, GO, Søgaard]

Rewrite: Choose time vector V^μ and rewrite

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \quad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}$$

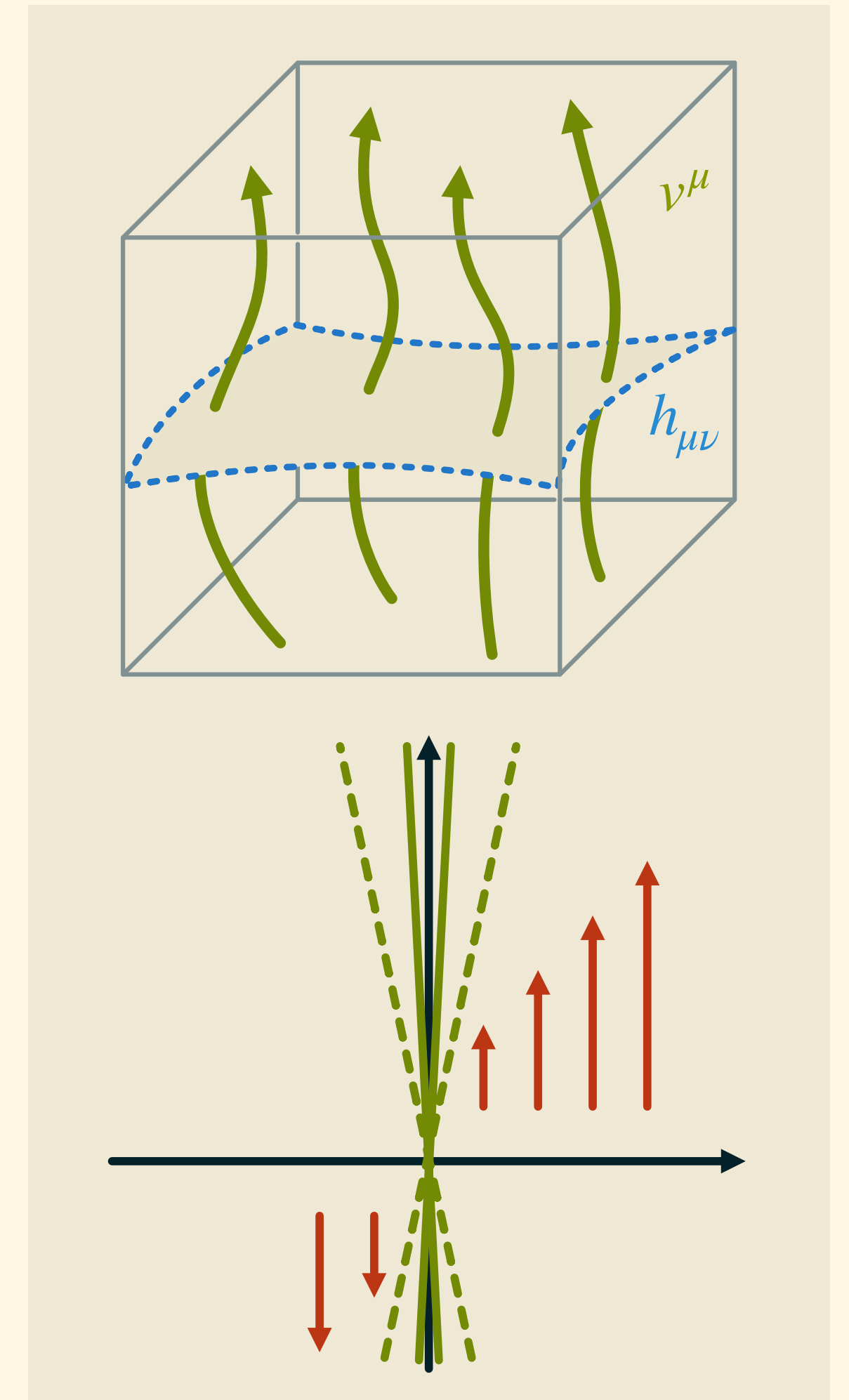
exposes overall factors of c^2 in the metric

Expand: Carroll geometry at leading order in c^2 expansion

$$\begin{aligned} V^\mu &= v^\mu + c^2 M^\mu + \dots, & T_\mu &= \tau_\mu + \dots \\ \Pi^{\mu\nu} &= h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \dots, & \Pi_{\mu\nu} &= h_{\mu\nu} + \dots \end{aligned}$$

local Lorentz boosts \rightarrow local Carroll boosts + corrections

[Bergshoeff, Izquierdo, Ortín, Romano] [Gomis, Kleinschmidt, Palmkvist, Salgado-Rebolledo] ...



Carroll from Lorentzian

Carroll connection $\tilde{\Gamma}_{\mu\nu}^{\rho}$ and curvature $\tilde{R}_{\mu\nu\rho}^{\sigma}$ from Levi-Civita

Rewrite Levi-Civita with explicit factors of c^2

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{c^2} S_{(-2)}^{\rho}{}_{\mu\nu} + \bar{C}_{\mu\nu}^{\rho} + S_{(0)}^{\rho}{}_{\mu\nu} + c^2 S_{(2)}^{\rho}{}_{\mu\nu},$$

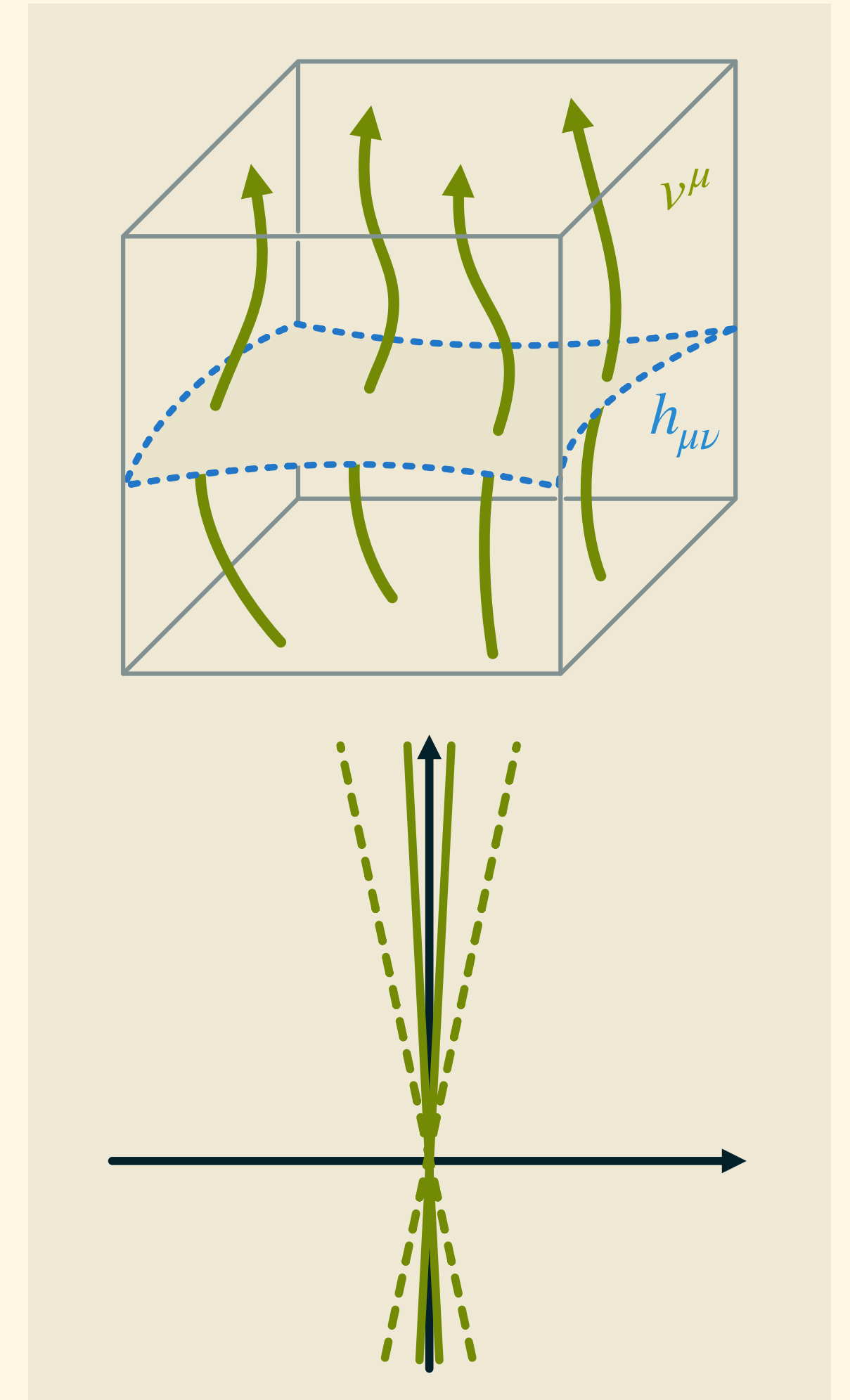
where $S_{(n)}^{\rho}{}_{\mu\nu}$ are known tensors. **Expand** to get $\bar{C}_{\mu\nu}^{\rho} = \tilde{\Gamma}_{\mu\nu}^{\rho} + \dots$

Obtain **connection** $\tilde{\Gamma}_{\mu\nu}^{\rho}$ so that $\tilde{\nabla}_{\mu} v^{\nu} = 0$ and $\tilde{\nabla}_{\rho} h_{\mu\nu} = 0$ [or any other choice of connection!]

$$\tilde{\Gamma}_{\mu\nu}^{\rho} = -v^{\rho} \partial_{(\mu} \tau_{\nu)} - v^{\rho} \tau_{(\mu} \mathcal{L}_{v} \tau_{\nu)} + \frac{1}{2} h^{\rho\sigma} \left(\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\sigma\mu} - \partial_{\sigma} h_{\mu\nu} \right) - h^{\rho\sigma} \tau_{\nu} K_{\mu\sigma}$$

Non-zero **torsion** $\tilde{T}^{\rho}{}_{\mu\nu} = 2h^{\rho\sigma} \tau_{[\mu} K_{\nu]\sigma}$

Rewrite $\sqrt{-g} = cE$ where $E = \det(T_{\mu}, \Pi_{\mu\nu})$ and **expand** $E = e + \dots$ where $e = \det(\tau_{\mu}, h_{\mu\nu})$



Carroll from Lorentzian

Rewrite the Einstein-Hilbert action, $\mathcal{K}_{\mu\nu} = -\frac{1}{2}\mathcal{L}_V\Pi_{\mu\nu} = K_{\mu\nu} + \dots$ is extrinsic curvature

$$S = \frac{c^3}{16\pi G} \int_M R \sqrt{-g} d^d x$$

$$\approx \frac{c^2}{16\pi G} \int_M \left[\left(\mathcal{K}^{\mu\nu} \mathcal{K}_{\mu\nu} - \mathcal{K}^2 \right) + c^2 \Pi^{\mu\nu} \bar{R}_{\mu\nu} + c^4 \Pi^{\mu\rho} \Pi^{\nu\sigma} \partial_{[\mu} T_{\nu]} \partial_{[\rho} T_{\sigma]} \right] E d^d x$$

From Lorentzian point of view a strange thing to do!

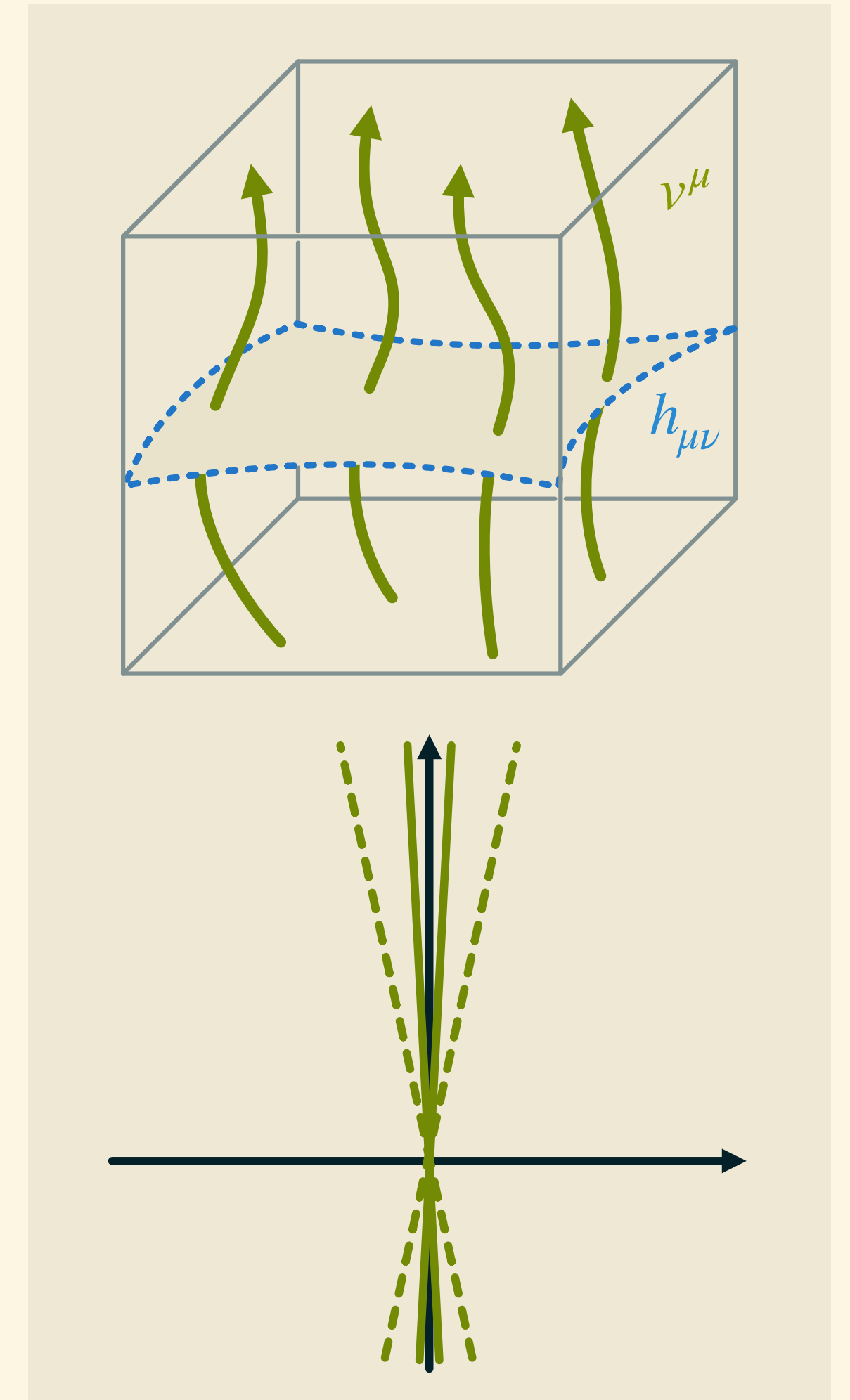
$(\bar{C}^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + \dots$ is neither flat nor Lorentz-metric-compatible nor torsion-free)

But enables us to expand the action in c^2 , **Carroll geometric expansion!**

$$S = c^2 S_{\text{LO}} + c^4 S_{\text{NLO}} + c^6 S_{\text{NNLO}} + \dots$$

At leading order get timelike (or electric) Carroll gravity action

$$S_{\text{LO}} = \frac{1}{16\pi G} \int_M \left(K^{\mu\nu} K_{\mu\nu} - K^2 \right) e d^d x$$



Carroll limit of GR

Expanding GR in $c \rightarrow 0$ gives **Carroll gravity** at LO [Hansen, Obers, GO, Søgaard]

$$S_{\text{EH}} = \frac{c^2}{2\kappa} \int_M \left(K^{\mu\nu} K_{\mu\nu} - K^2 \right) e d^d x + \dots$$

similar actions found in [Henneaux] [Hartong] [Henneaux, Salgado-Rebodello]

EOM split into **constraint** and **evolution equations** [Hansen, Obers, GO, Søgaard] [Dautcourt]

$$0 = K^{\mu\nu} K_{\mu\nu} - K^2$$

$$0 = -R^{(3)} + K^{\mu\nu} K_{\mu\nu} - K^2$$

$$0 = h^{\rho\sigma} \tilde{\nabla}_\rho (K_{\sigma\mu} - Kh_{\sigma\mu})$$

$$0 = h^{\rho\sigma} \nabla_\rho^{(3)} (K_{\sigma\mu} - Kh_{\sigma\mu})$$

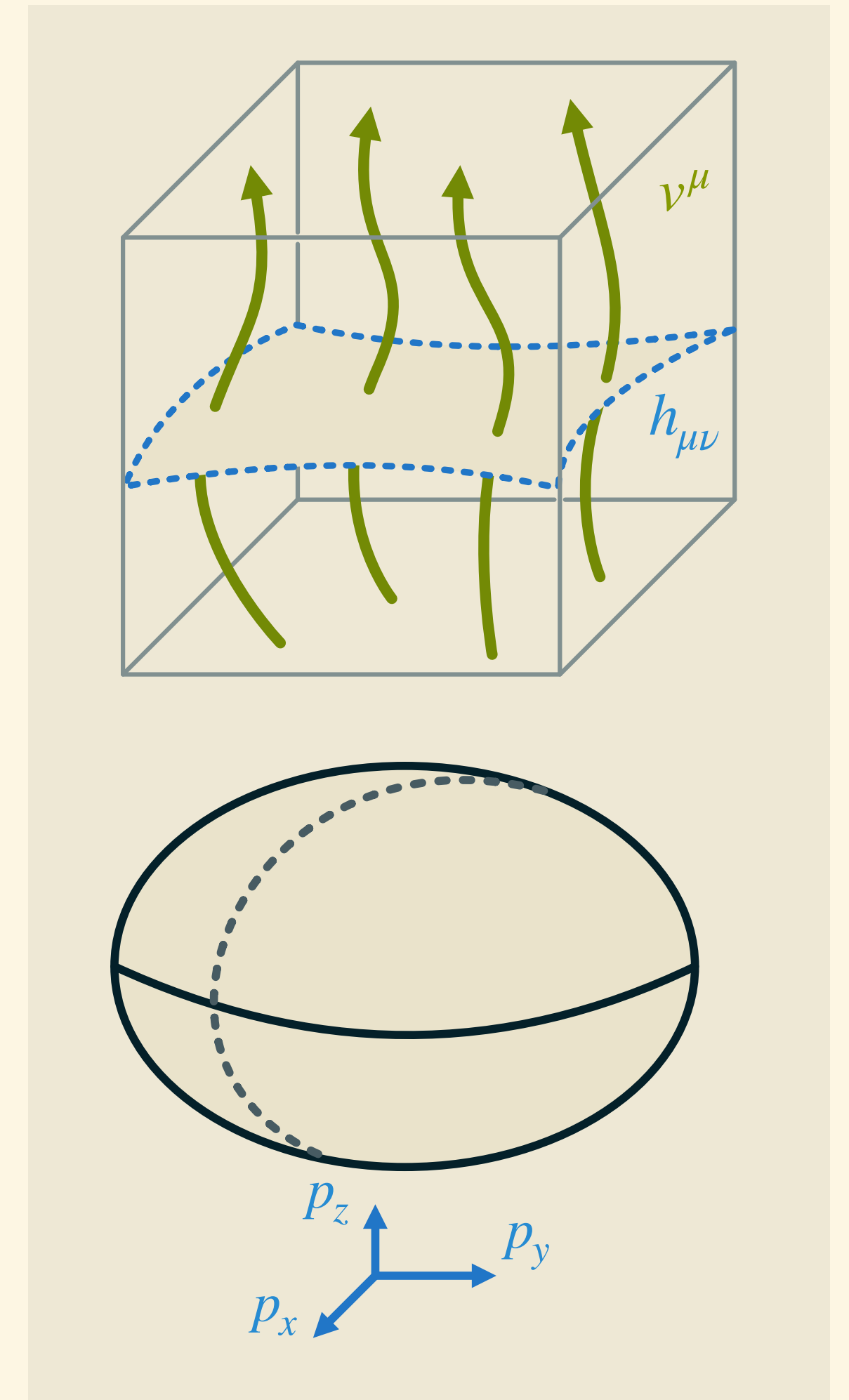
$$\mathcal{L}_v K_{\mu\nu} = -2K_\mu^\rho K_{\rho\nu} + KK_{\mu\nu}$$

$$\mathcal{L}_v K_{\mu\nu} = R_{\mu\nu}^{(3)} - 2K_\mu^\rho K_{\rho\nu} + KK_{\mu\nu} - \nabla_\mu^{(3)} a_\nu - a_\mu a_\nu$$

Evolution equations are now just **ODEs!**

Solutions include **Kasner geometry** $v^\mu \partial_\mu = -\partial_t$, $h_{\mu\nu} dx^\mu dx^\nu = t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2$

[Henneaux] [Søgaard] [De Boer, Hartong, Obers, Sybesma, Vandoren]



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Kasner solutions in Carroll gravity

Vacuum Carroll LO 'electric' gravity theory **has Kasner solutions**

$$v^\mu \partial_\mu = -\partial_t, \quad h_{\mu\nu} dx^\mu dx^\nu = t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2$$

$$\text{Solves EOM for } S_{\text{LO}} = \frac{1}{2\kappa} \int_M \left[K^{\mu\nu} K_{\mu\nu} - K^2 \right] e d^d x$$

Again can reformulate in terms of β variables

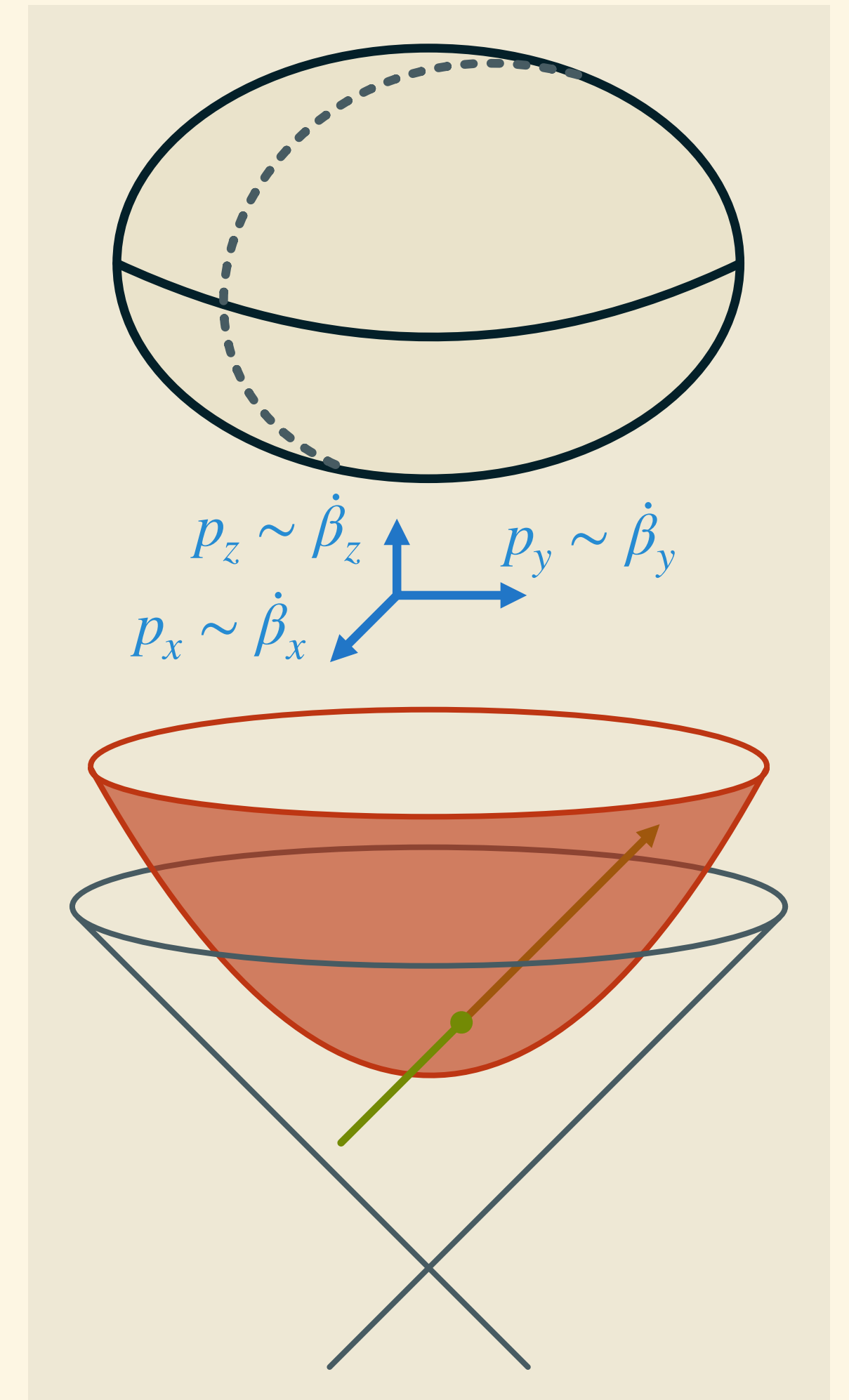
$$v^\mu \partial_\mu = -e^{\alpha(t)} \partial_t, \quad h_{\mu\nu} dx^\mu dx^\nu = (e^{2\beta(t)})_{ij} dx^i dx^j$$

Same picture of **null geodesics** in **hyperbolic superspace**

$$\dot{\beta}_i \dot{\beta}^i = 0, \quad \beta_i(\tau) = \beta_i^{(0)} + v_i \tau$$

How to get richer dynamics? **Spatial curvature** and/or **matter coupling**!

But: **no** $R^{(3)}$ in LO theory! No 'gravity walls'!



Mixmaster in LO Carroll gravity

Mimic AdS setup! Leading-order Carroll gravity coupled to electric $U(1)^3$ Carroll YM

$$\frac{1}{2\kappa} \int d^d x e \left(K^{\mu\nu} K_{\mu\nu} - K^2 \right) + \frac{1}{2g} \int d^d x e h^{\mu\nu} E_\mu^{(i)} E_\nu^{(i)}$$

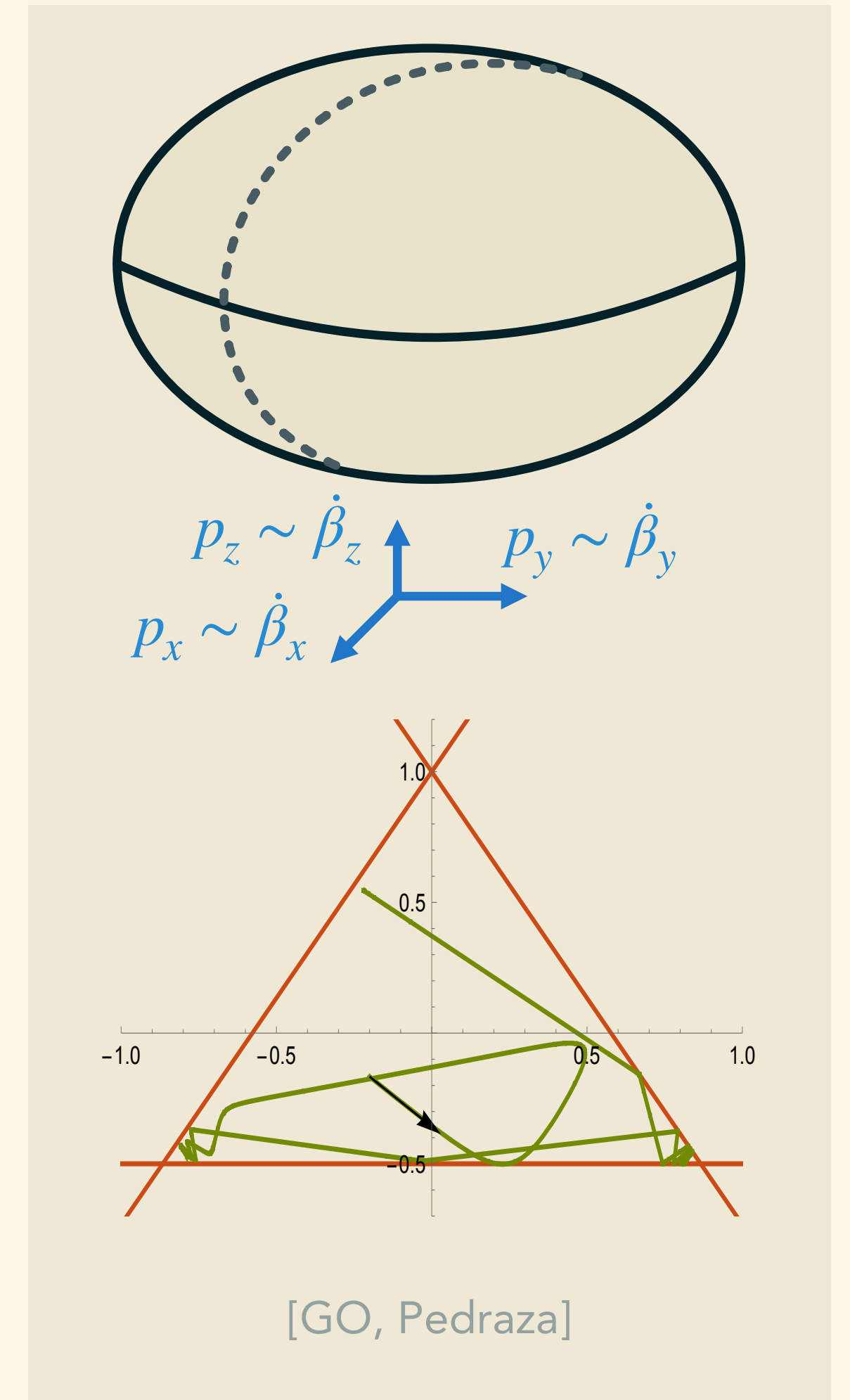
Gauge field ansatz $A^{(i)} = f_i(t) dx^i$ gives 'matter walls' for exponents in hyperbolic space

$$\ddot{\beta}_i = \frac{\kappa}{2g} \left(1 - \partial_{\beta_i} \right) V(\beta)$$

where potential gives mixmaster triangle motion for exponents [GO, Pedraza]

$$V(\beta) = (\phi_x)^2 e^{\sqrt{2/3}(\bar{\beta}_0 - \bar{\beta}_1 - \sqrt{3}\bar{\beta}_2)} + (\phi_y)^2 e^{\sqrt{2/3}(\bar{\beta}_0 - \bar{\beta}_1 + \sqrt{3}\bar{\beta}_2)} + (\phi_z)^2 e^{\sqrt{2/3}(\bar{\beta}_0 + 2\bar{\beta}_1)}$$

Reproduce holographic setup of [De Clerck, Hartnoll, Santos '23] using LO Carroll gravity



Summary and outlook

Close relation between BKL/mixmaster and Carroll expansion of GR

More models easily accessible from Carroll limits/expansion

Separate ultralocality and strong gravity approximations

Work in progress:

- Explicit BKL models with spatial inhomogeneity?
- 'Standard' mixmaster from $R^{(3)}$ in NLO Carroll gravity?

Future:

- Subleading corrections to BKL dynamics?
- Tractable (Gowdy) models for 'spikes'?
- Probes for near-singularity chaos in AdS/CFT?

