# S-matrix Bootstrap and Non-invertible Symmetries

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University of Amsterdam

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Laboratory: 2D QFTs

→ *simpler kinematics, connection to 2D CFTs and integrable models, exact non-perturbative results!*

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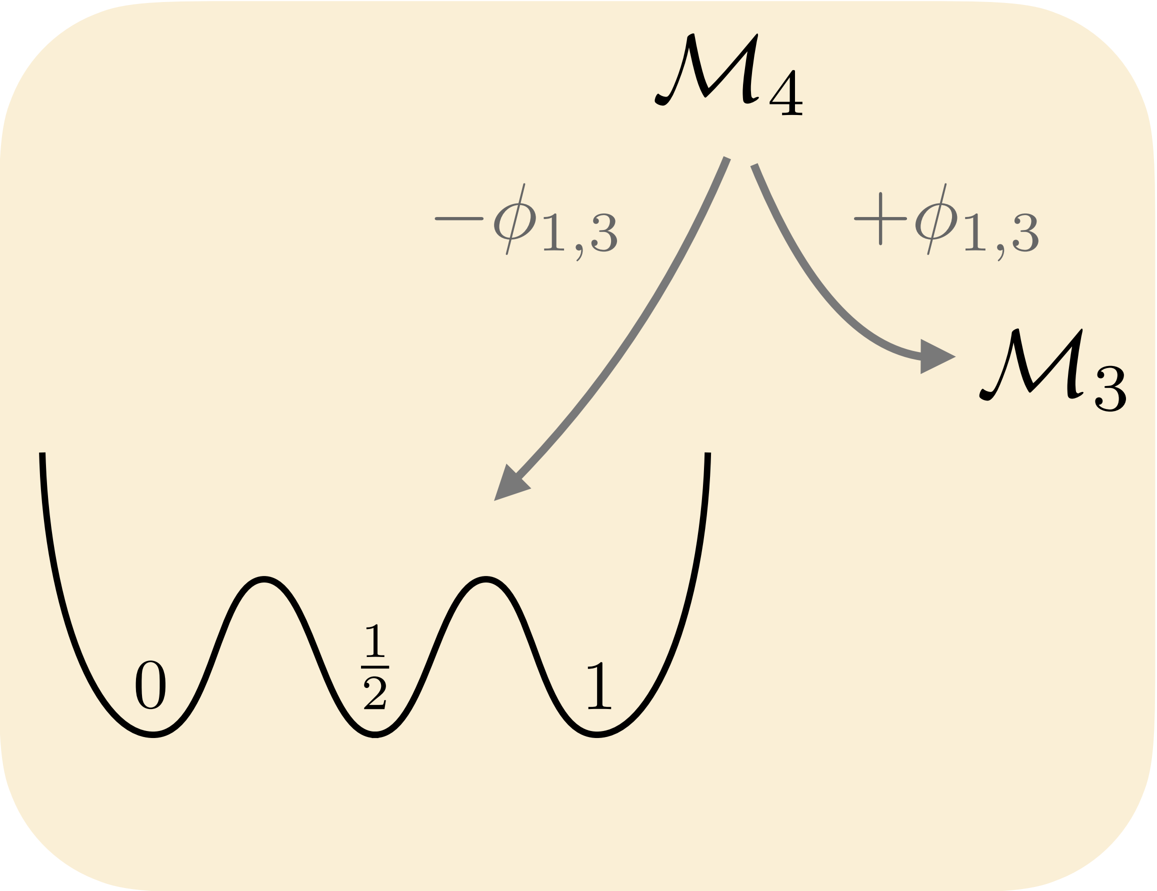

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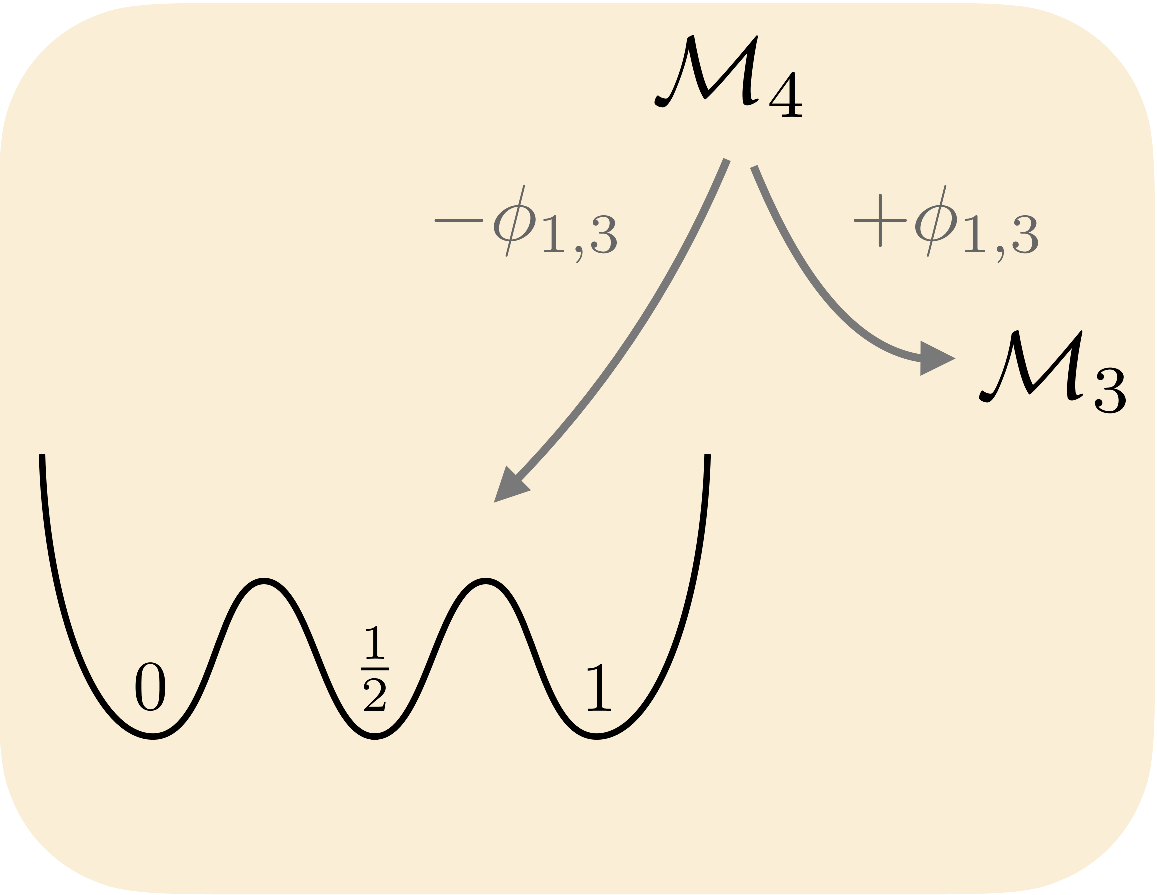
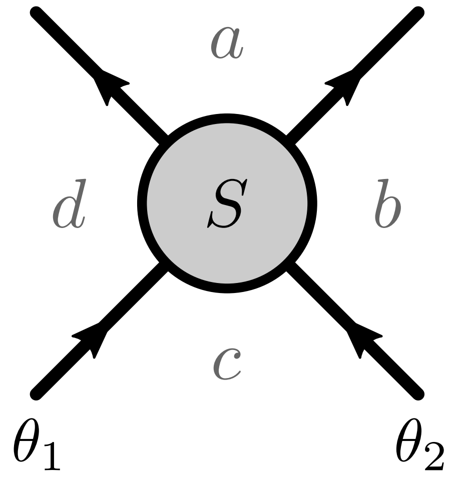



Scattering kinks interpolating between vacua $K_{ab}$

$S^{ab}_{dc}(\theta) =$ 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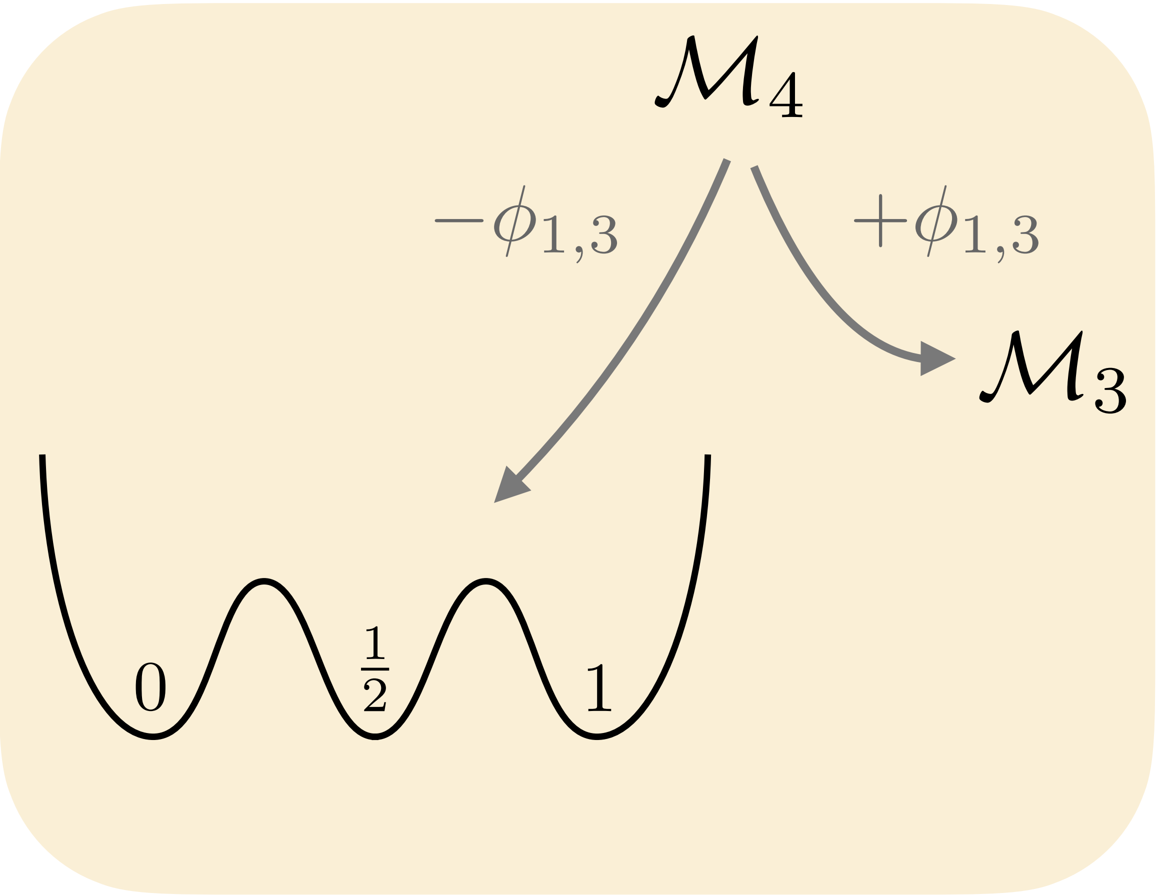


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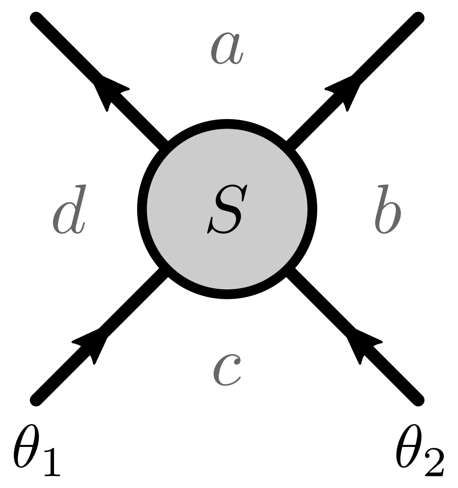
$S^{ab}_{dc}(\theta) =$ 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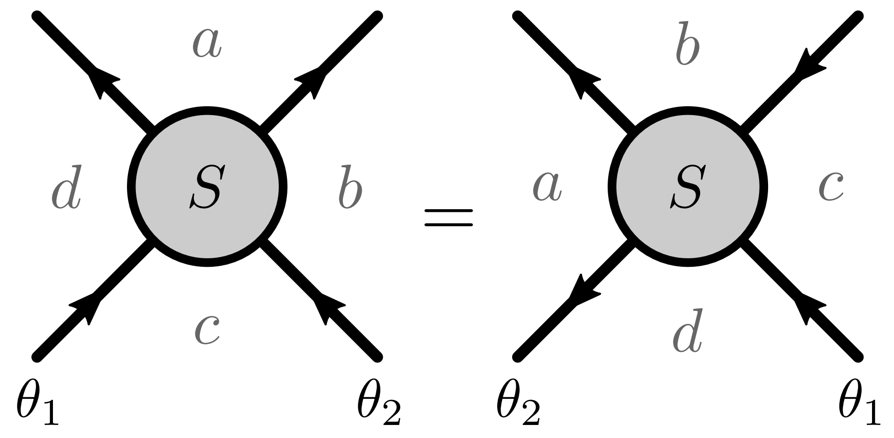
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$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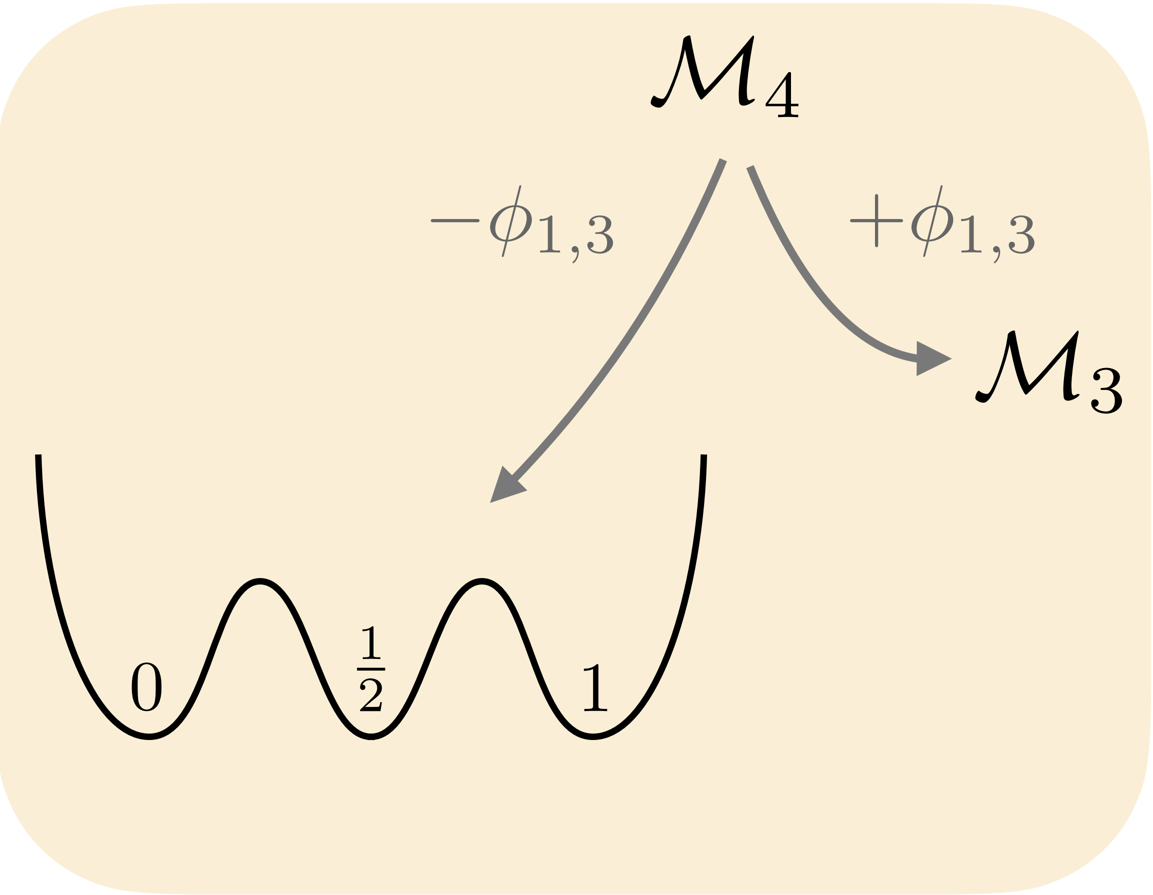


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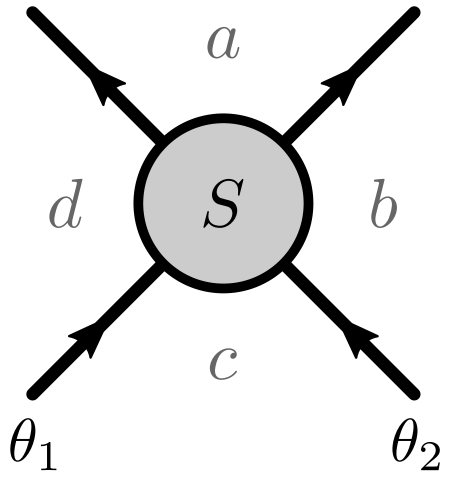
$S^{ab}_{dc}(\theta) =$ 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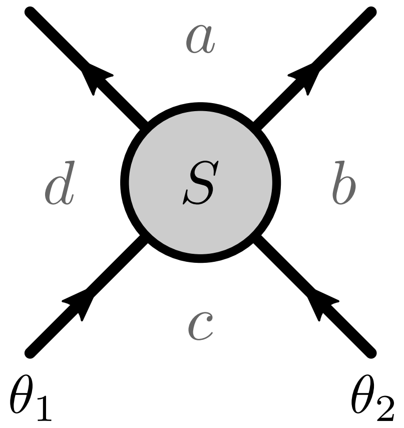
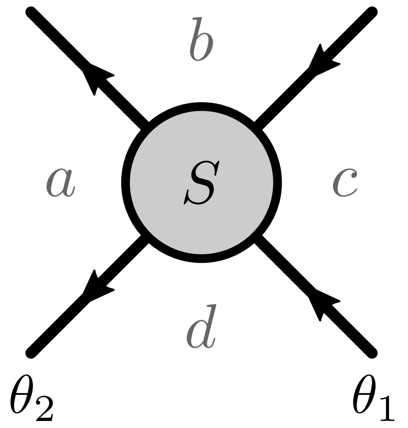
$S^{ab}_{dc}(\theta) =$  $=$  $= S^{bc}_{ad}(i\pi - \theta)$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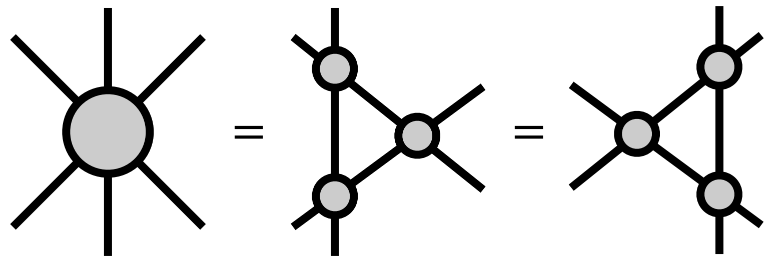

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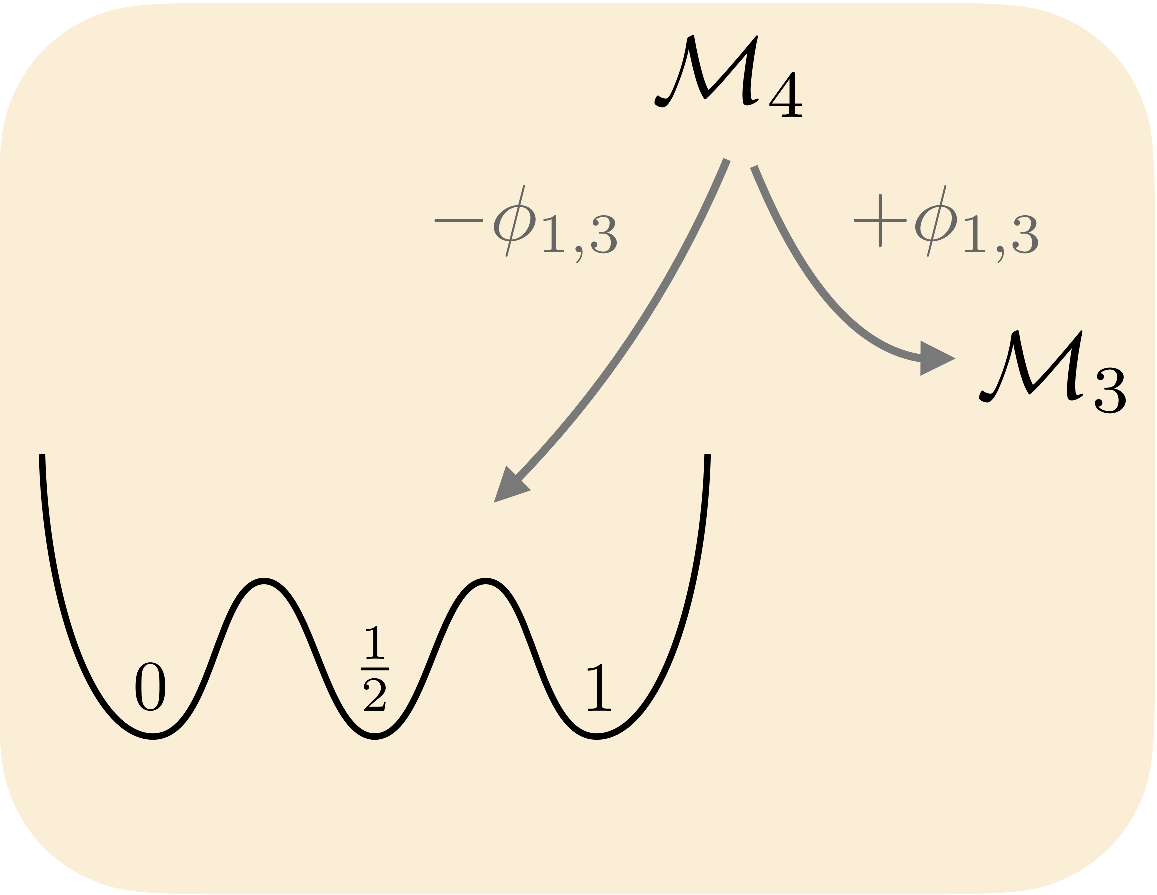


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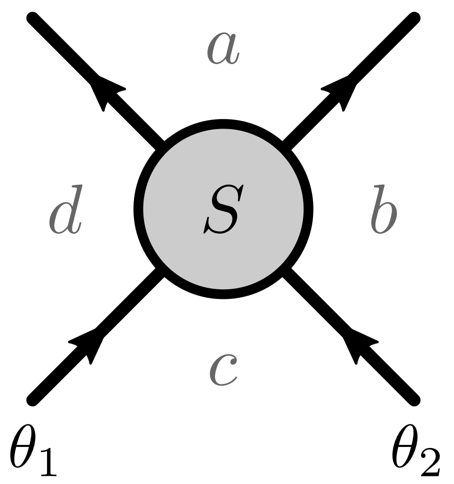
$S^{ab}_{dc}(\theta) =$ 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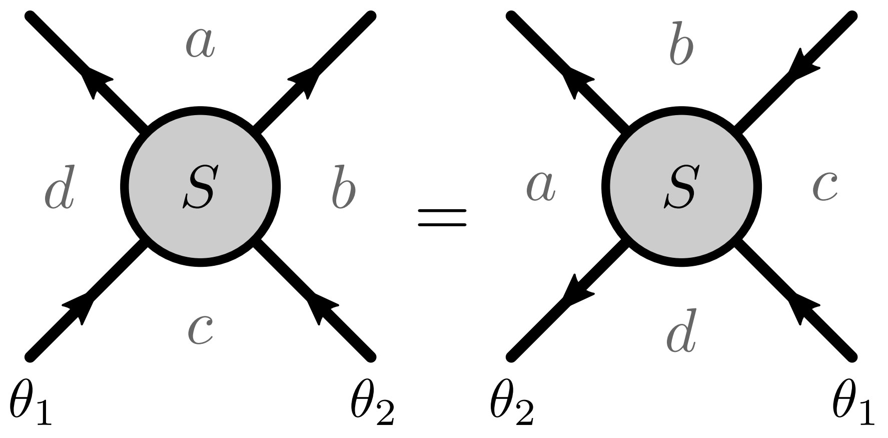
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$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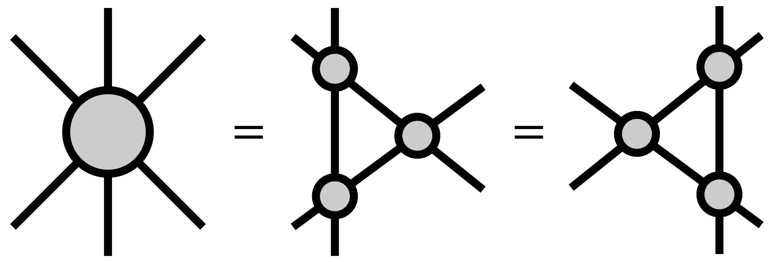
- S-matrix bootstrapped from unitarity+crossing+integrability 

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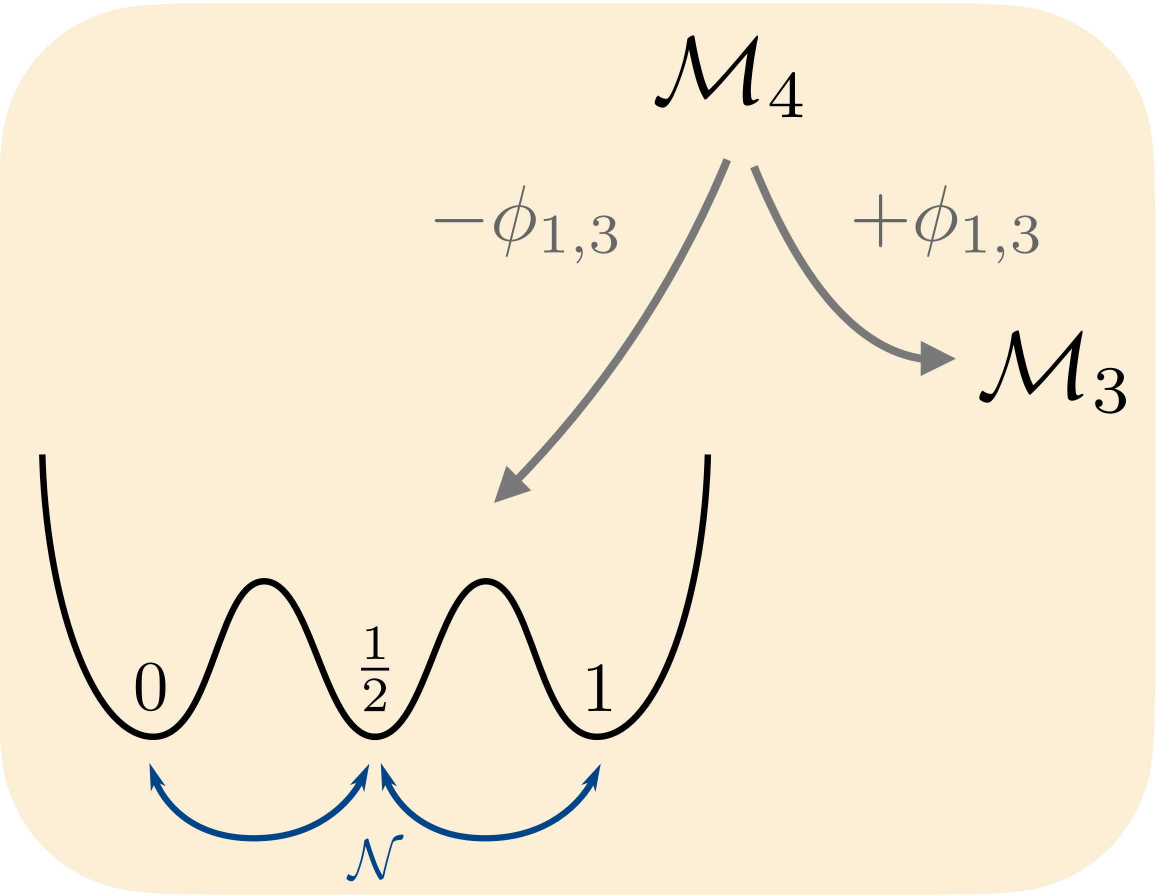
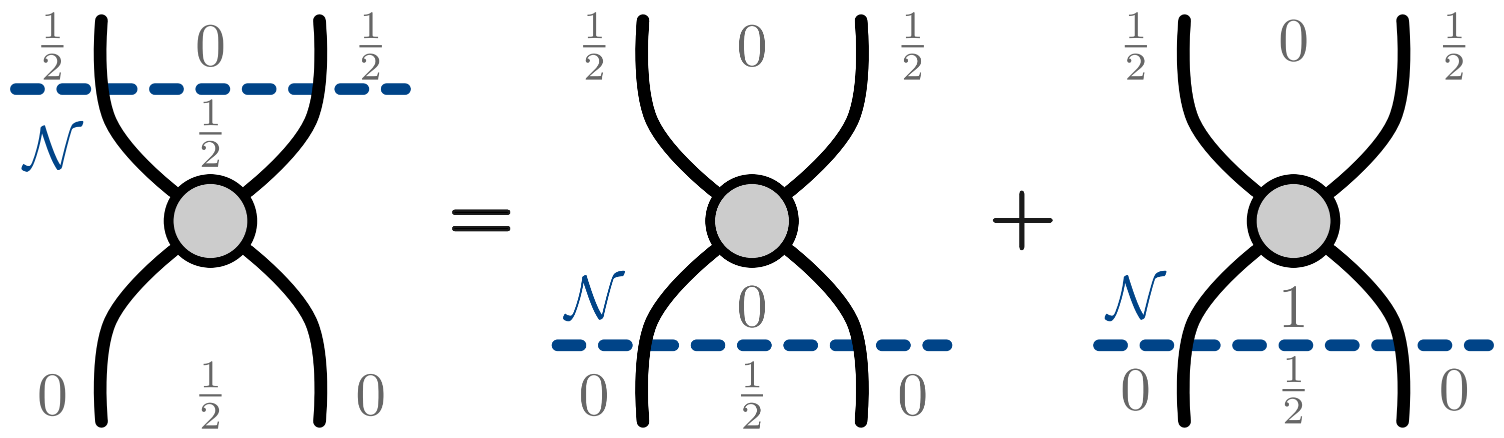



Non-invertible symmetry $\mathcal{N}$

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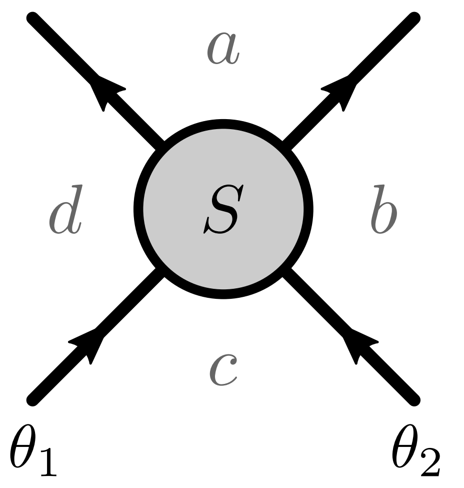


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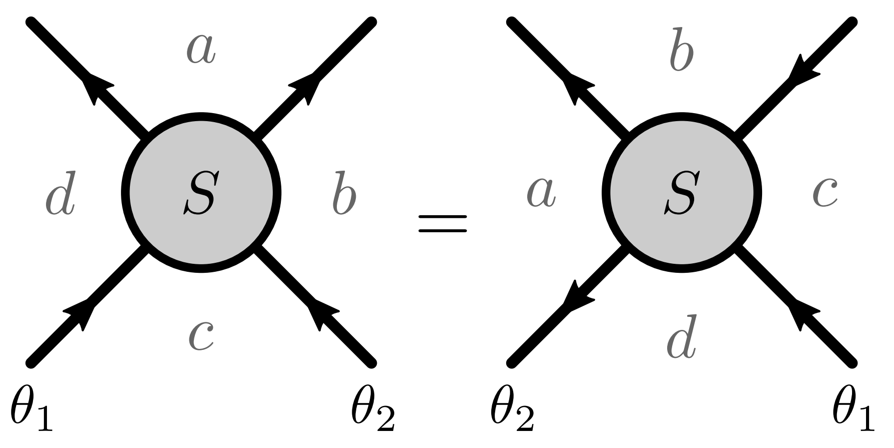


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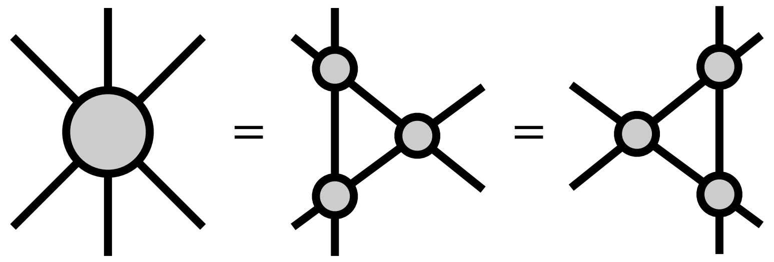


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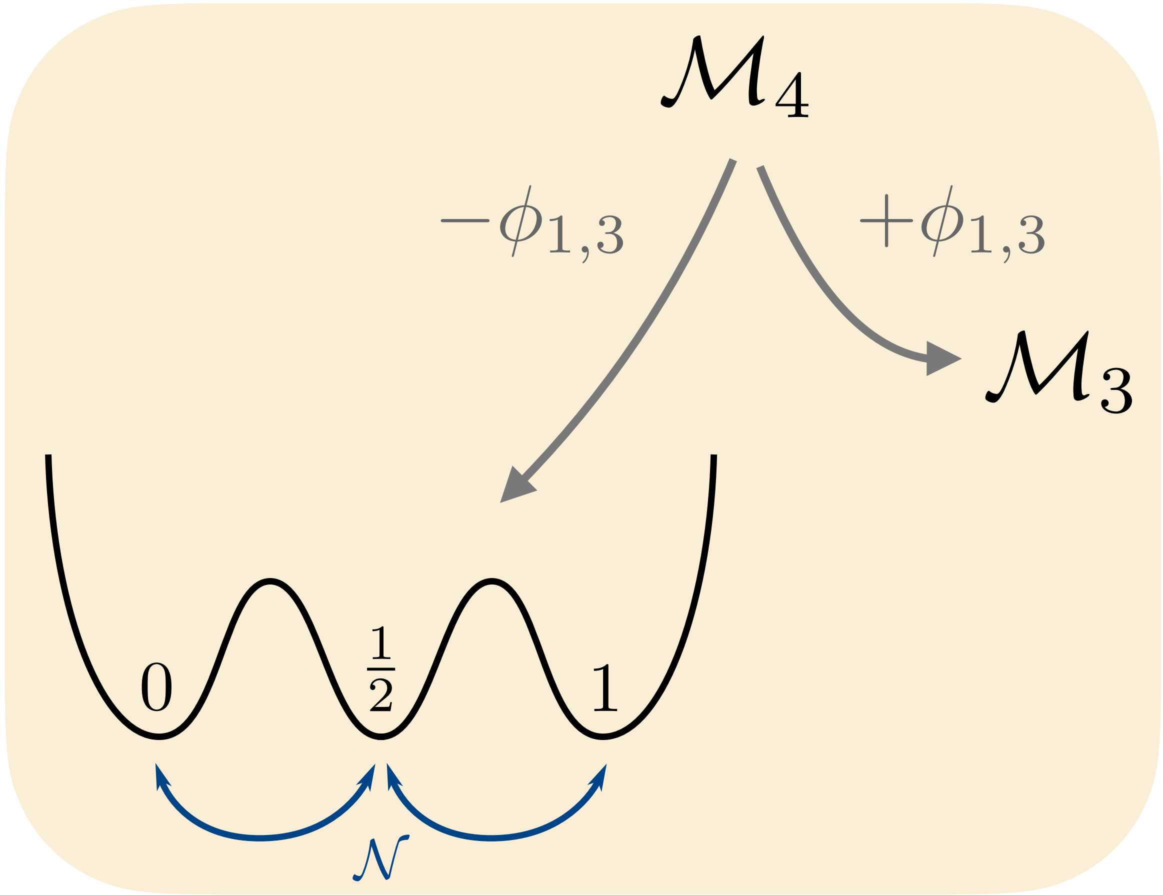


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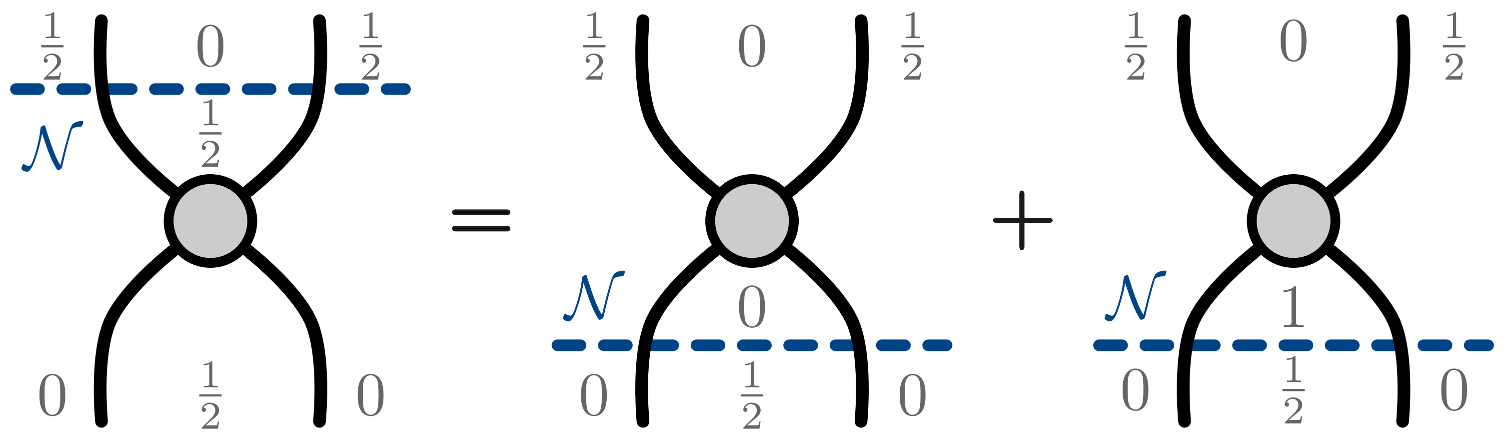


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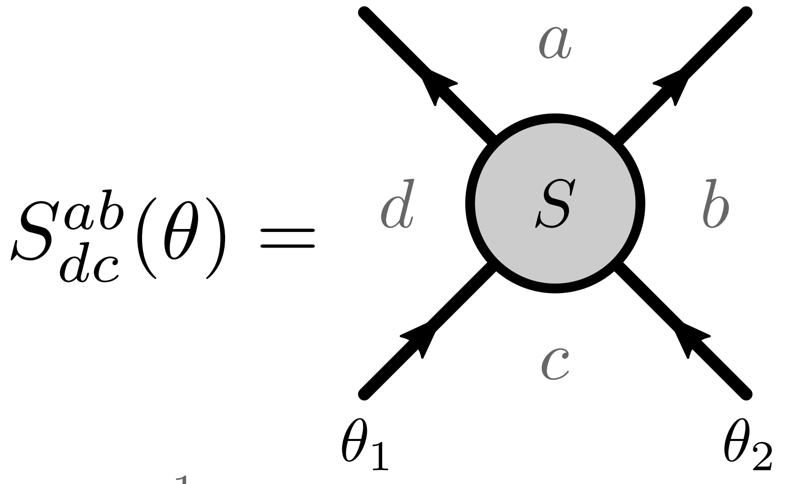


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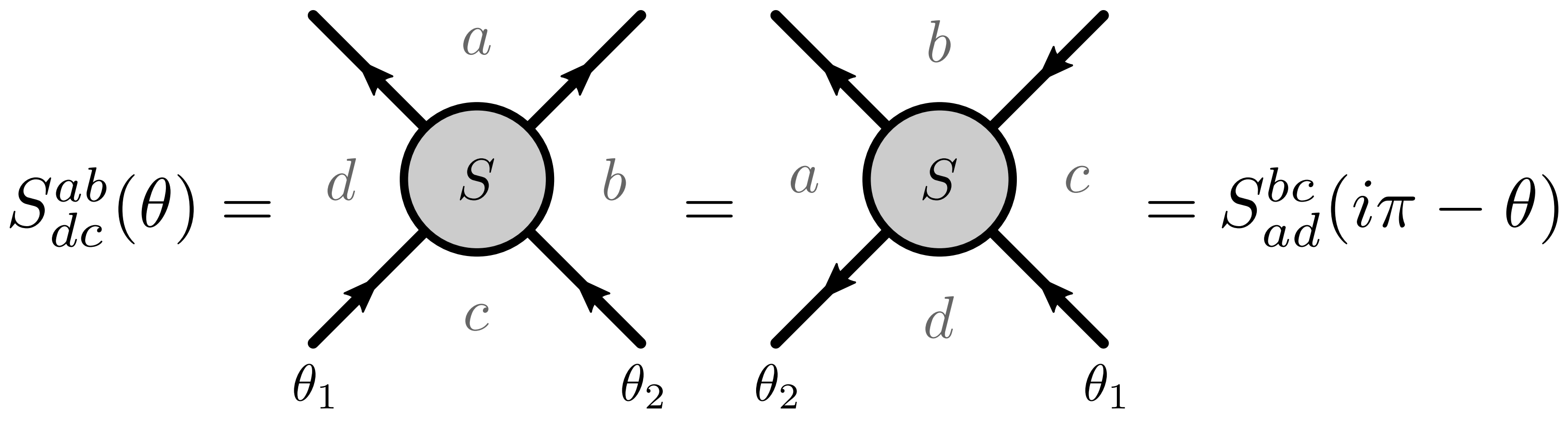


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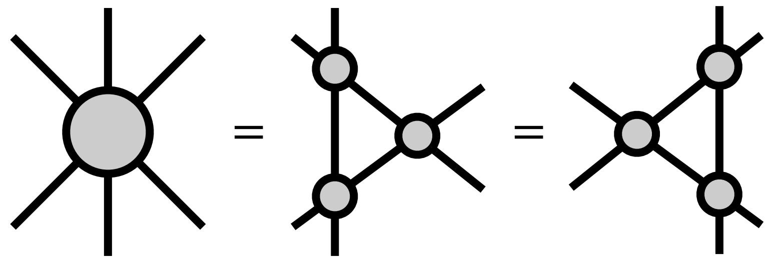

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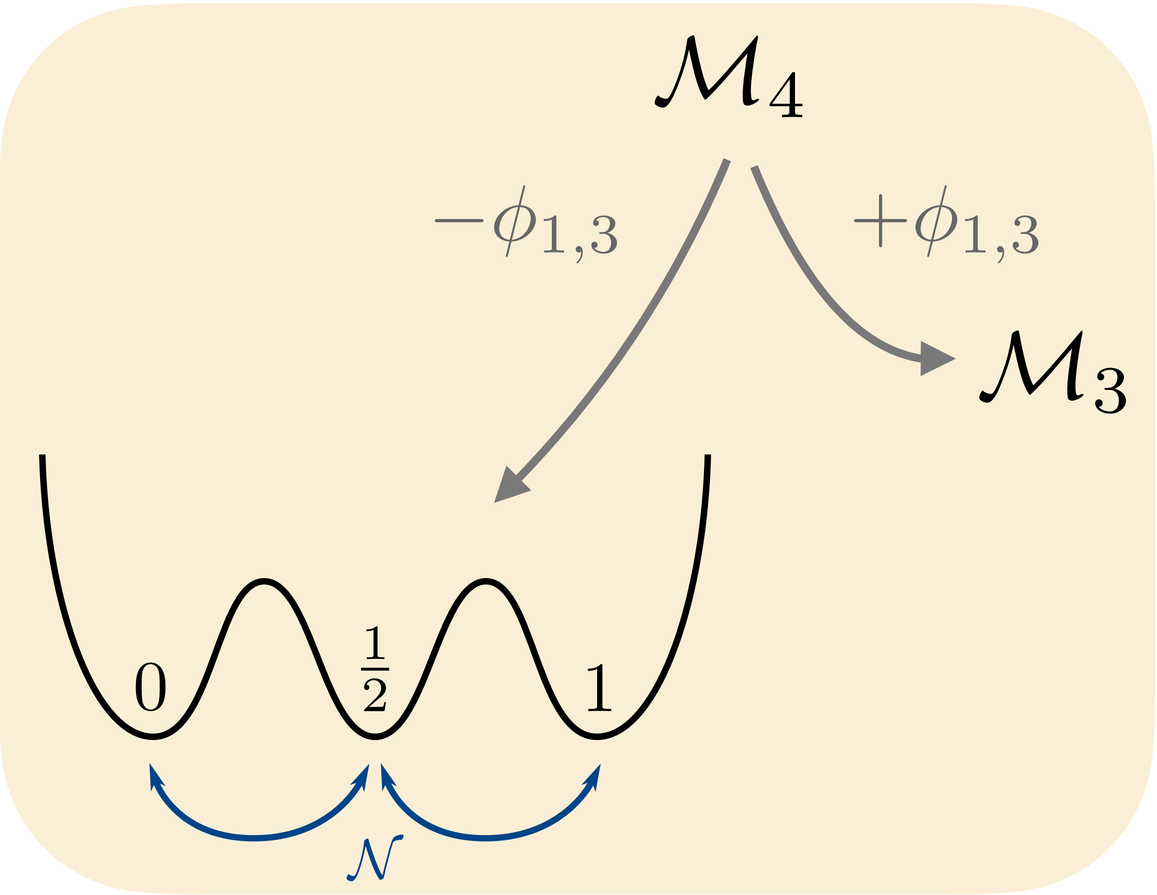


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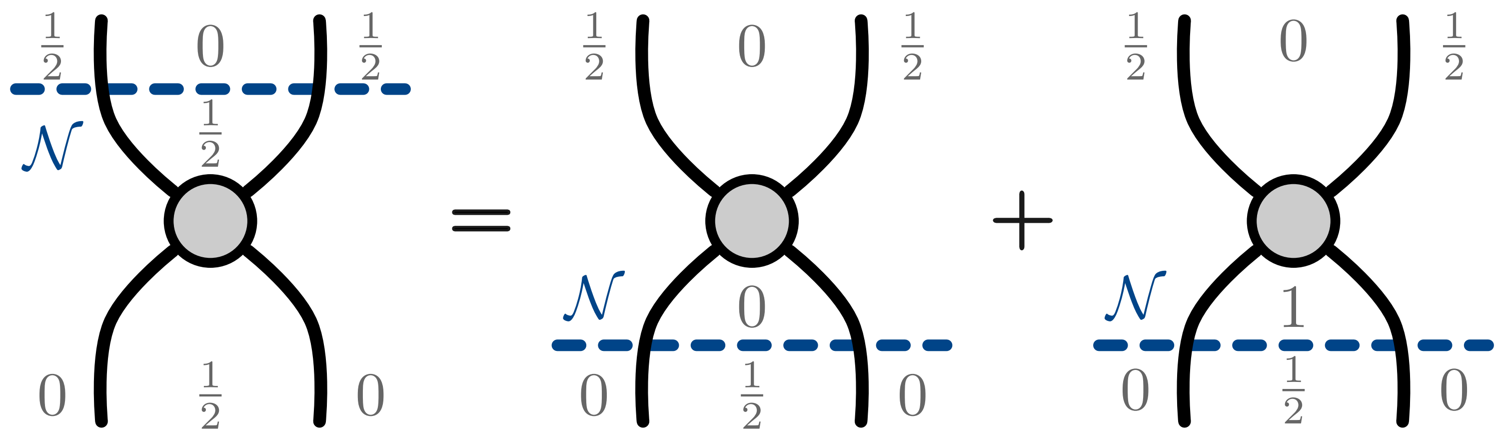


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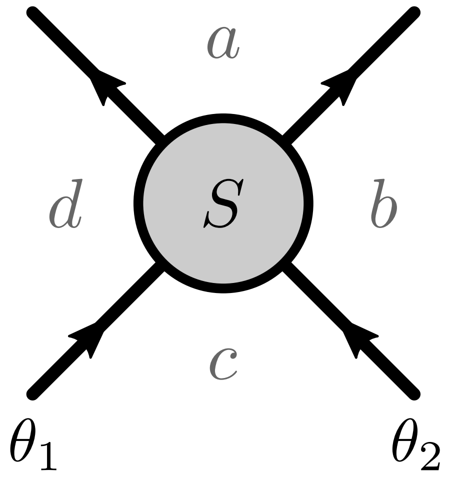
$S^{ab}_{dc}(\theta) =$ 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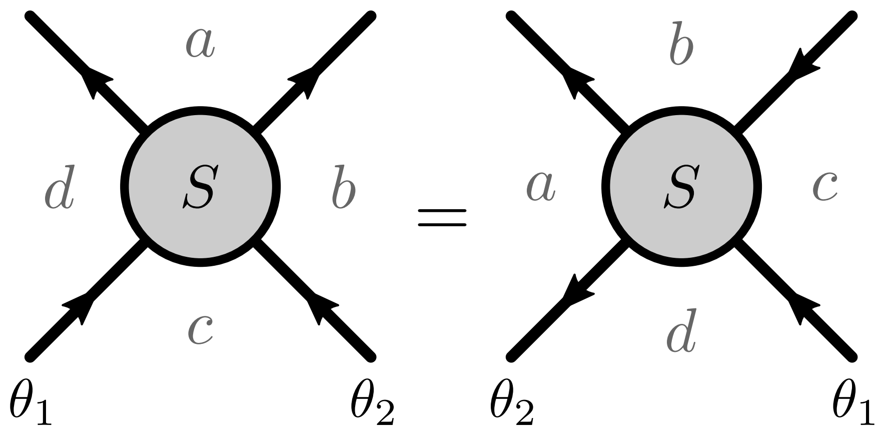
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$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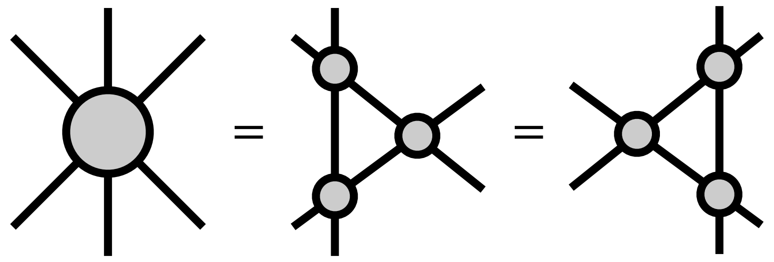
- S-matrix bootstrapped from unitarity+crossing+integrability 

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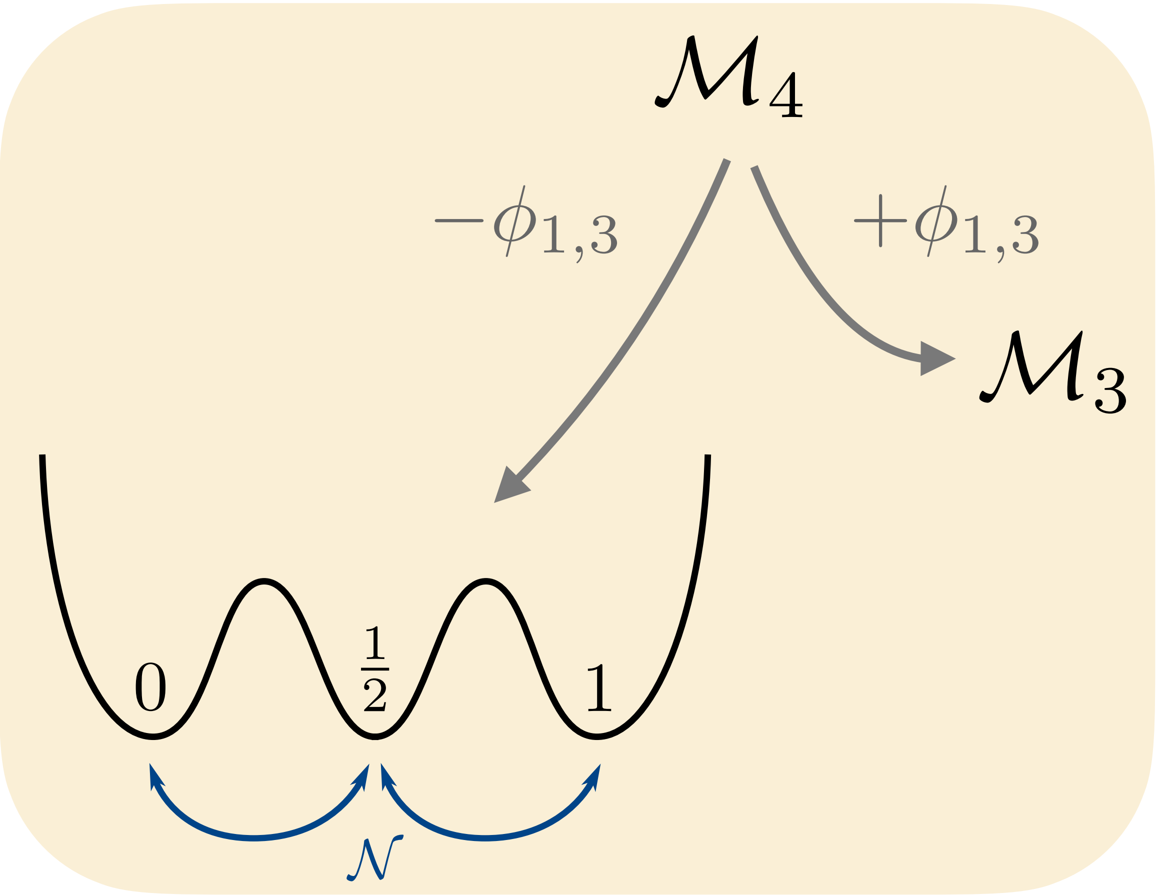


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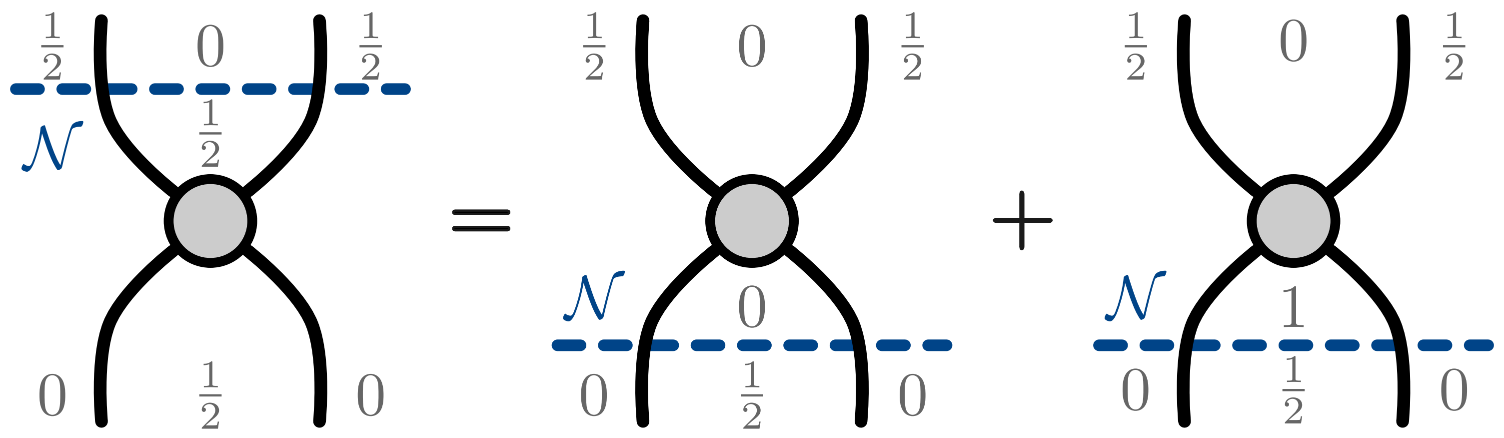


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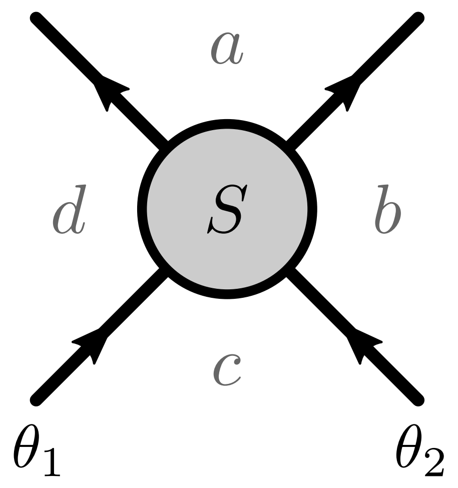
$S^{ab}_{dc}(\theta) =$ 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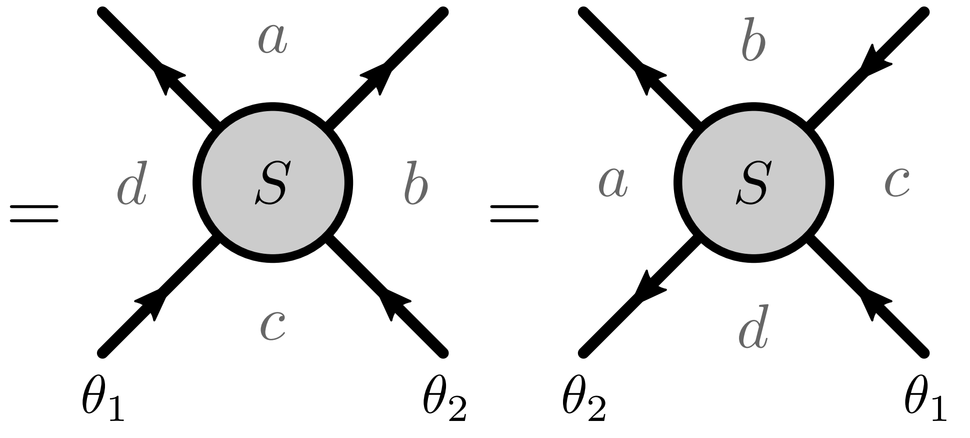
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$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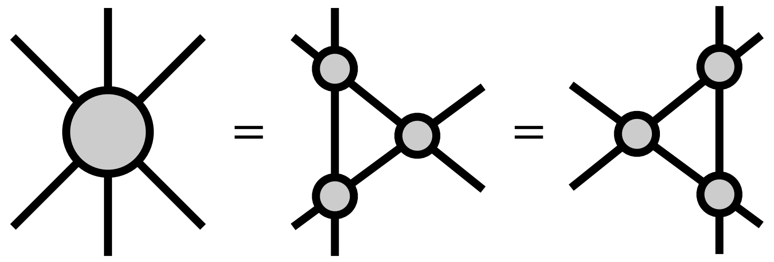
- S-matrix bootstrapped from unitarity+crossing+integrability 
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→ **Modified crossing!**

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

# Outline

1. Fusion Categories and Minimal Models
2. Scattering Amplitudes and Modified Crossing
3. S-matrix Bootstrap and Non-invertible Syms
4. Final Remarks

# Fusion Categories and Minimal Models

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- (global) Symmetries in QFT ↔ Topological operators

[Gaiotto, Kapustin, Seiberg, Willett '14]

*Move in $t$ → charge conservation*

*e.g. from Noether current* $j_\mu$: $U = exp\left(ia\oint d^{d-1}x\, j_0(x)\right)$

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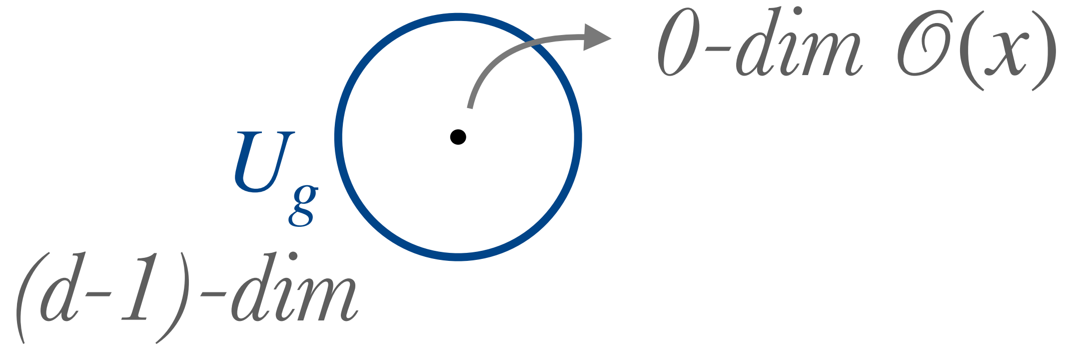


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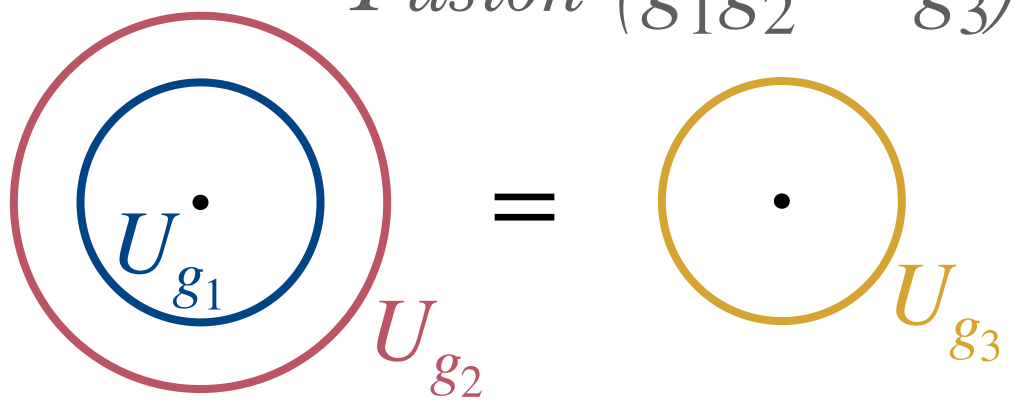
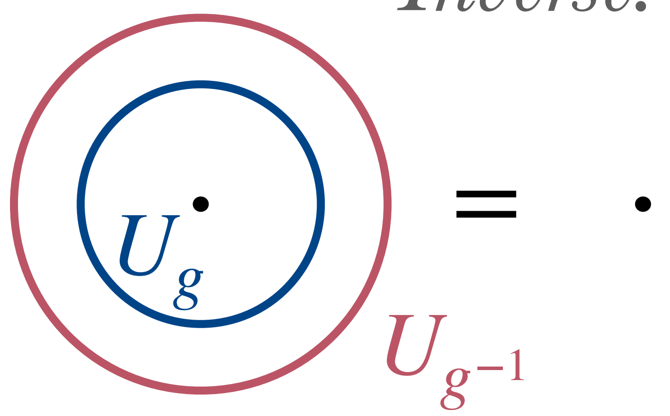



*Inverse:* $g^{-1}g = 1$



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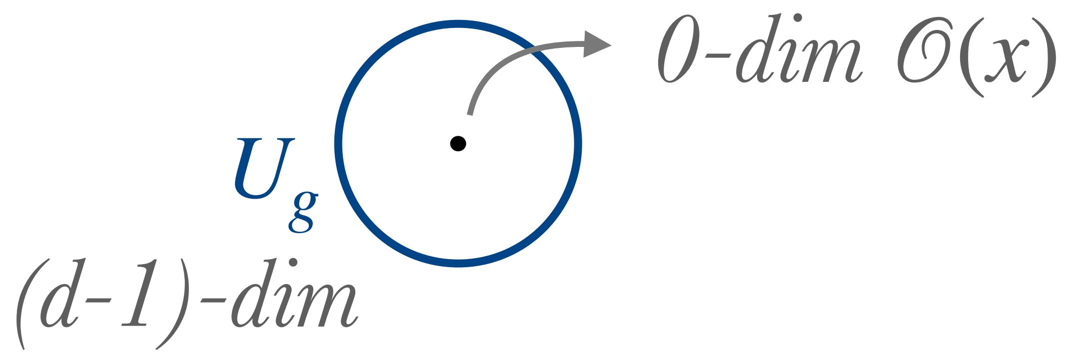


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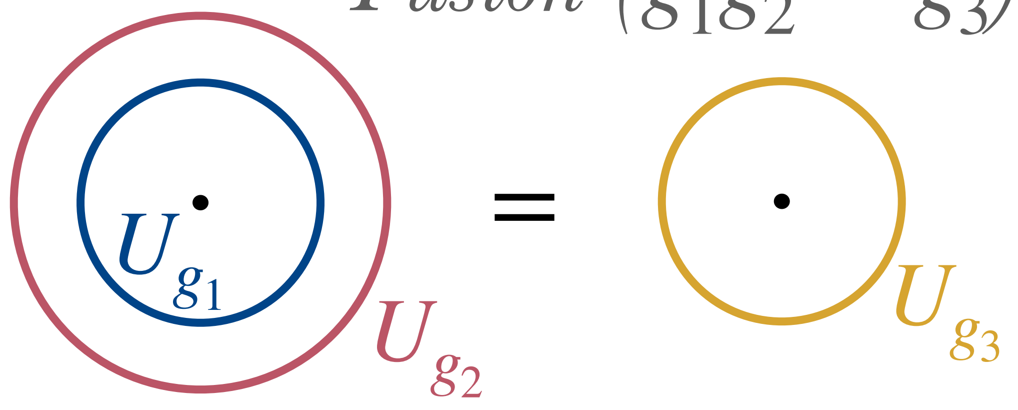


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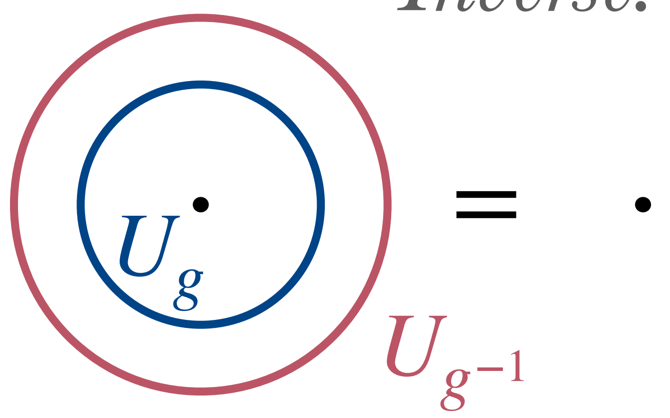


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- (global) Symmetries in QFT ↔ Topological operators

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*Move in t → charge conservation*

*e.g. from Noether current* $j_\mu$: $U = \exp\left( i\alpha \oint d^{d-1}x\, j_0(x) \right)$


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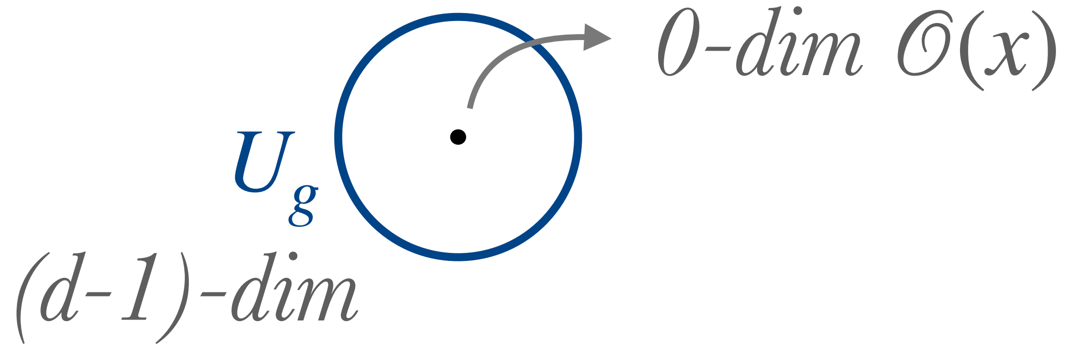


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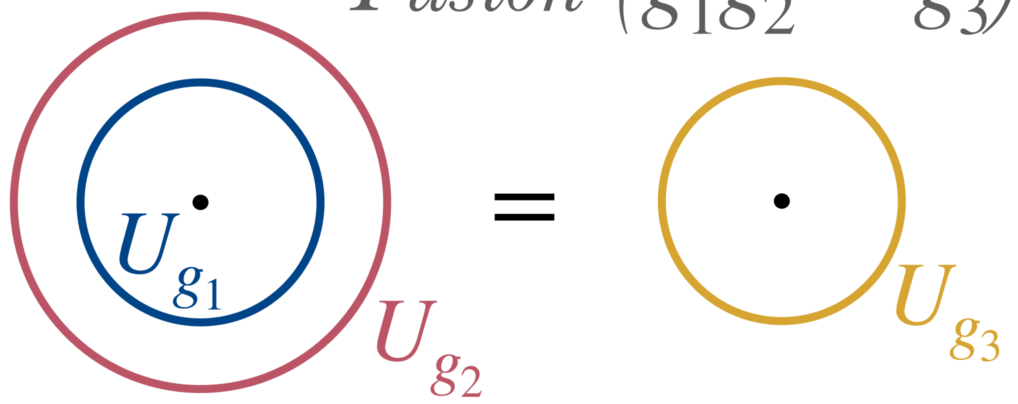


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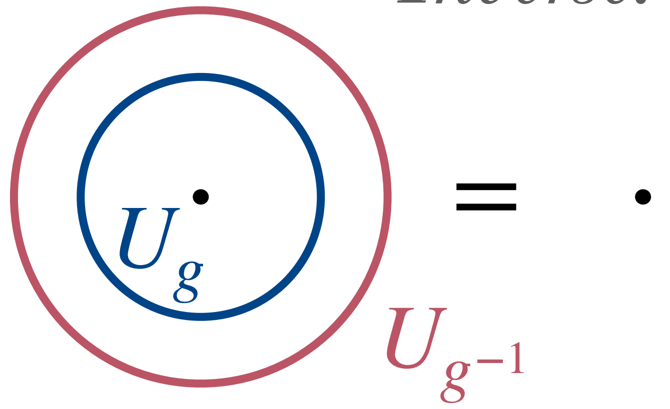


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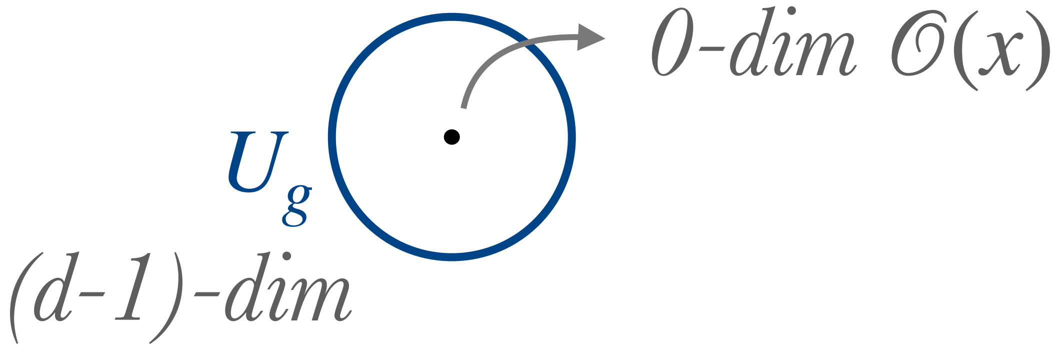


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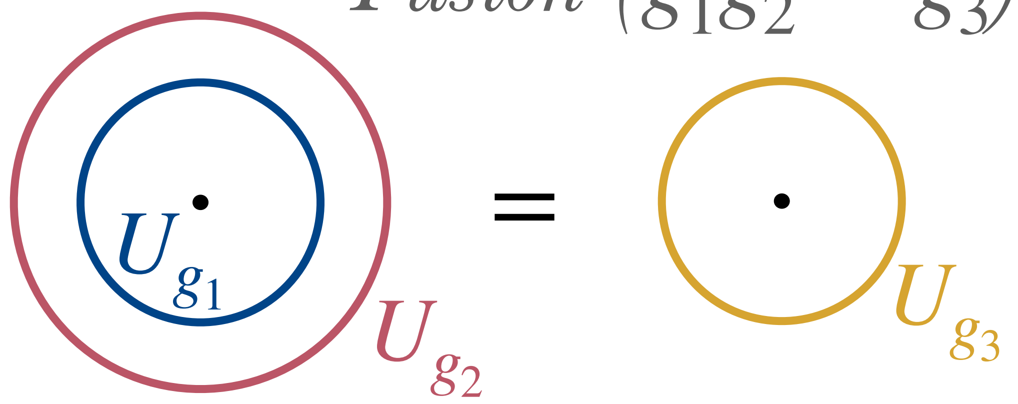


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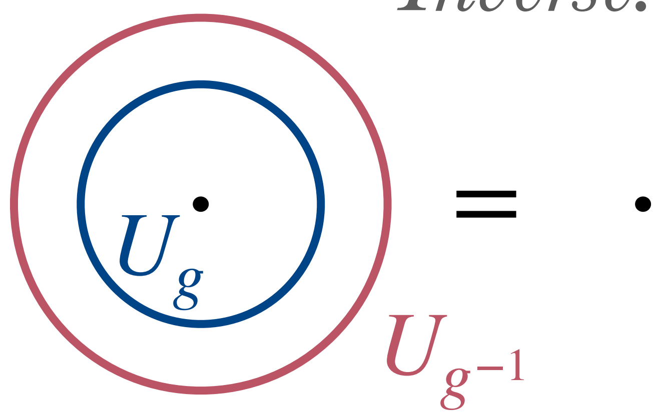


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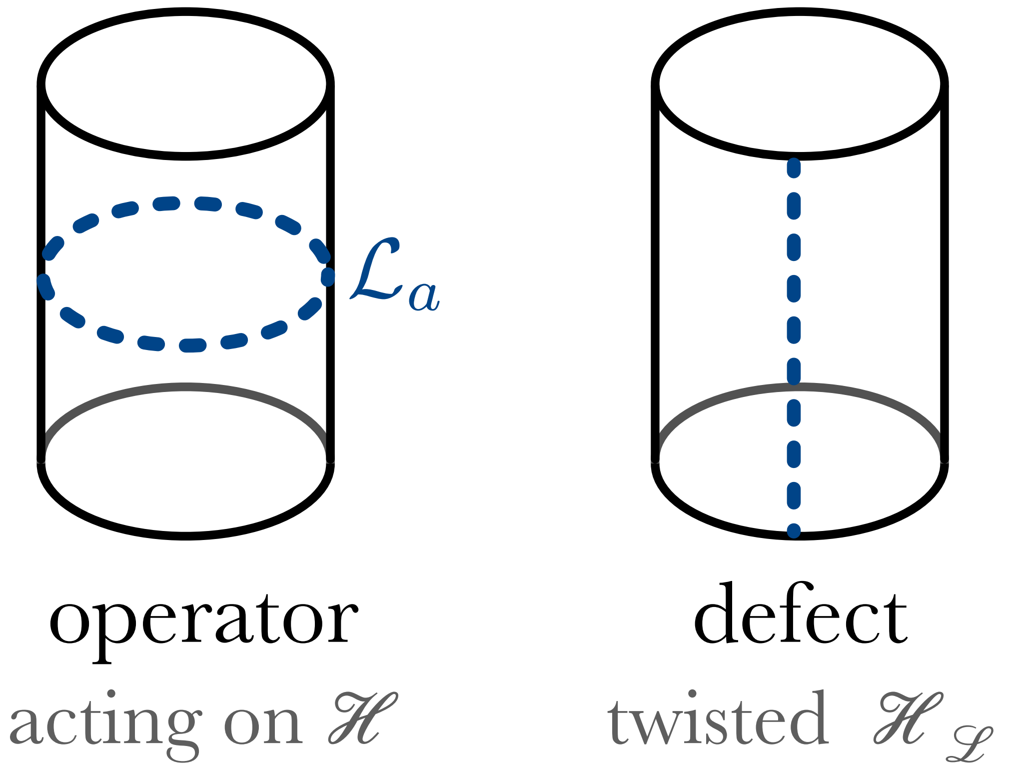

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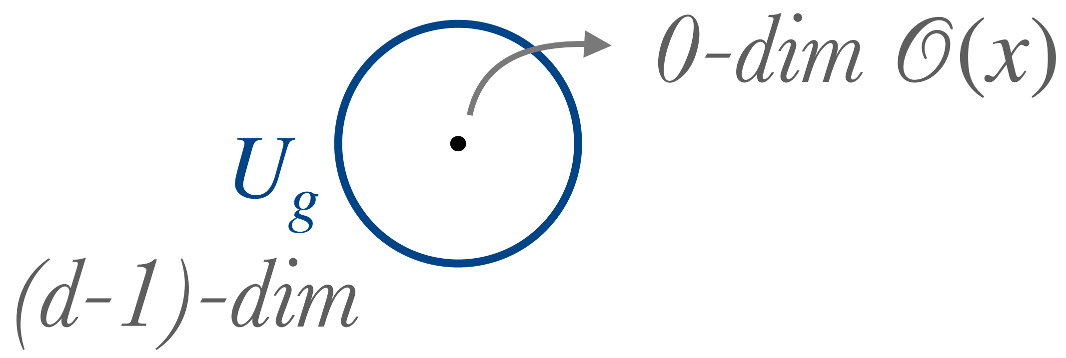


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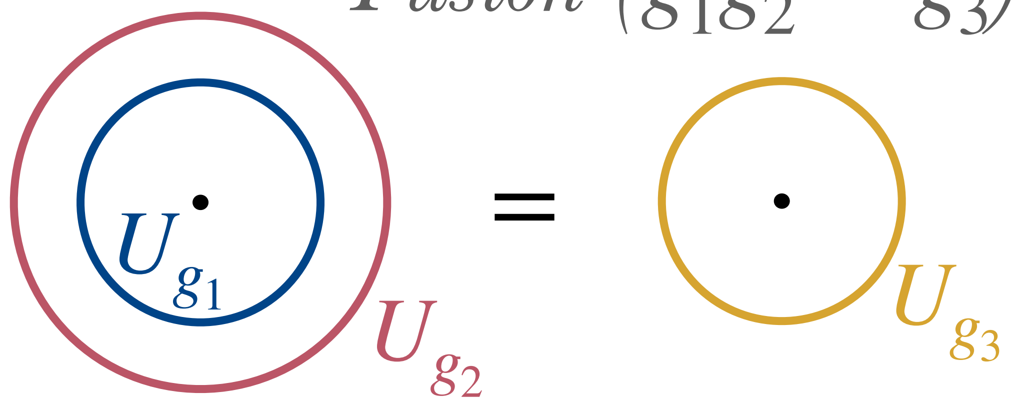


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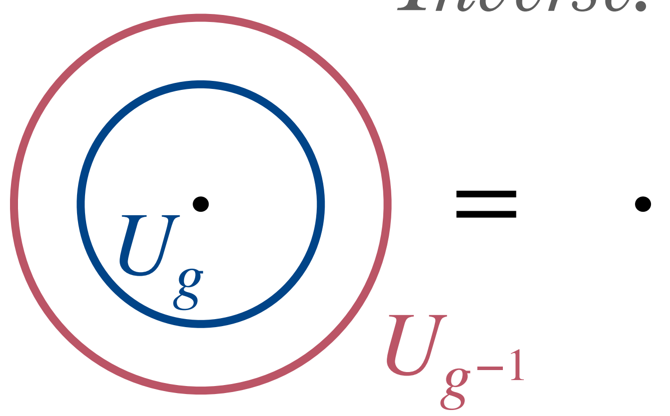


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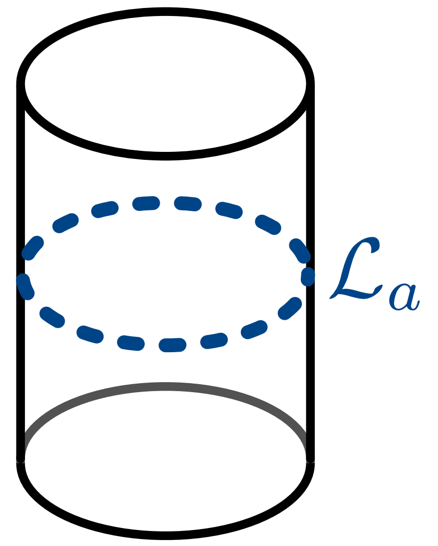
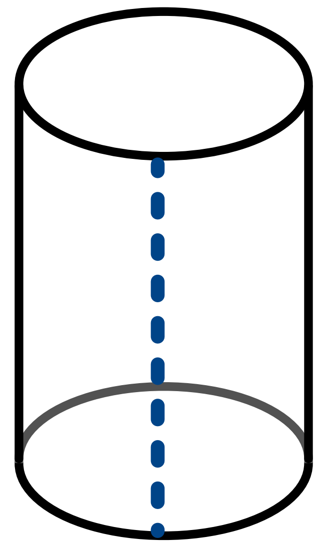



operator
acting on $\mathscr{H}$



defect
twisted $\mathscr{H}_{\mathscr{L}}$

# Fusion Categories

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- **Objects** *(1, finite number)*

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- **Fusion coefficients**

- **F-symbols**

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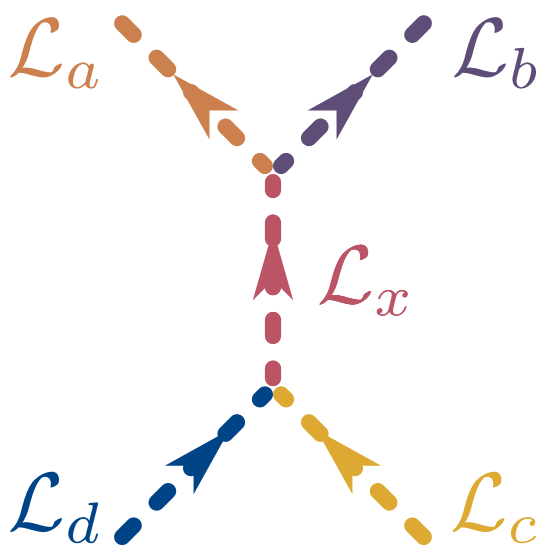


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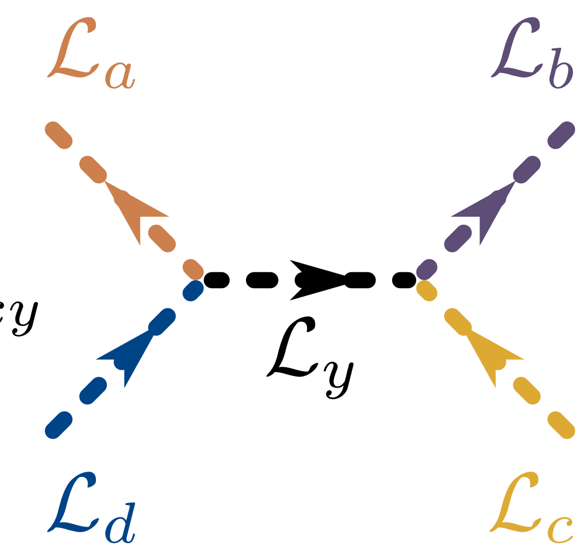


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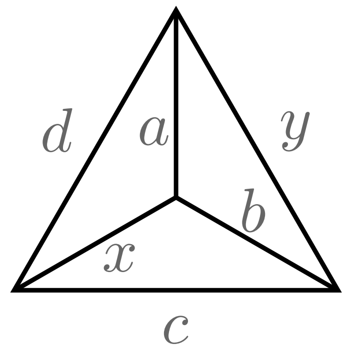


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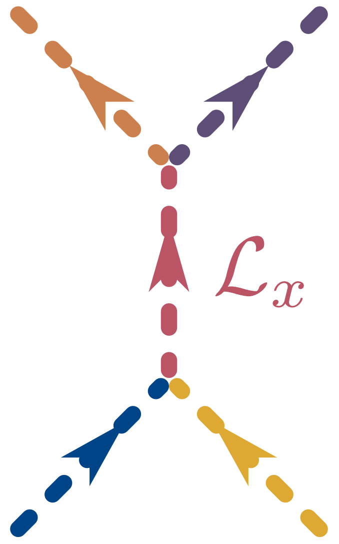



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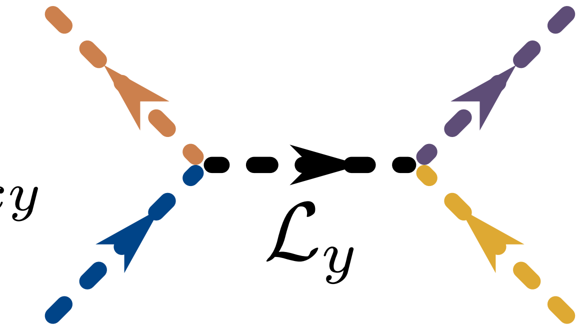



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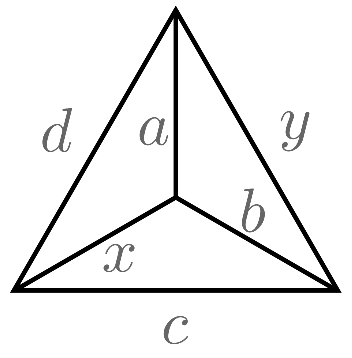
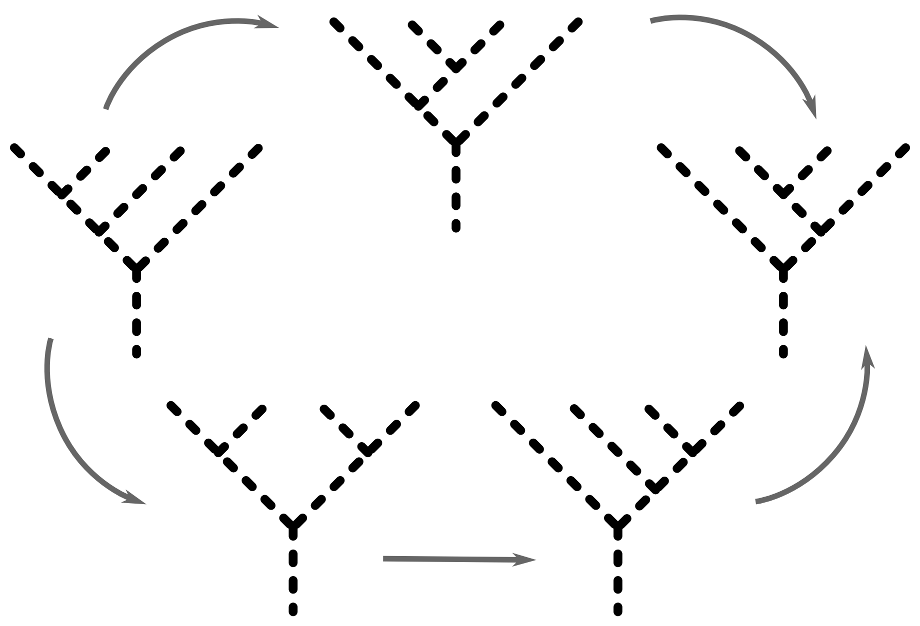




Solutions to
Pentagon equation



*'FF = FFF'*

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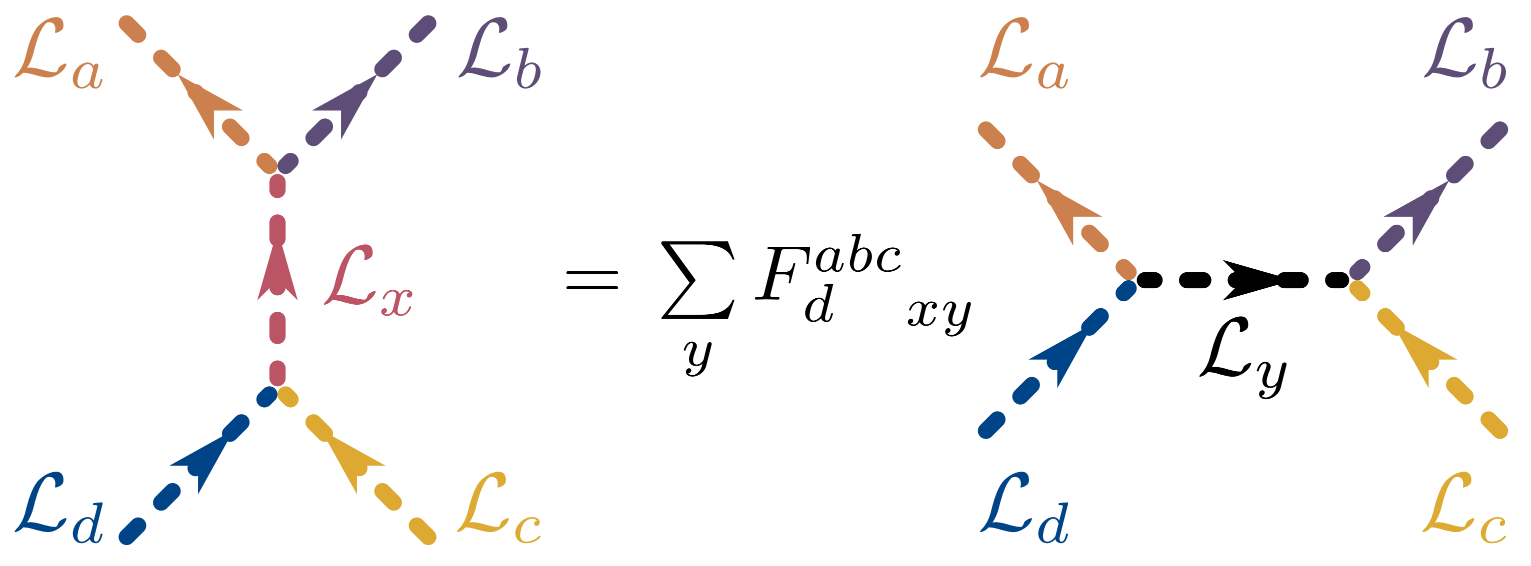



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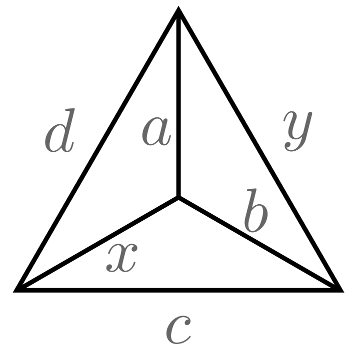



## Examples

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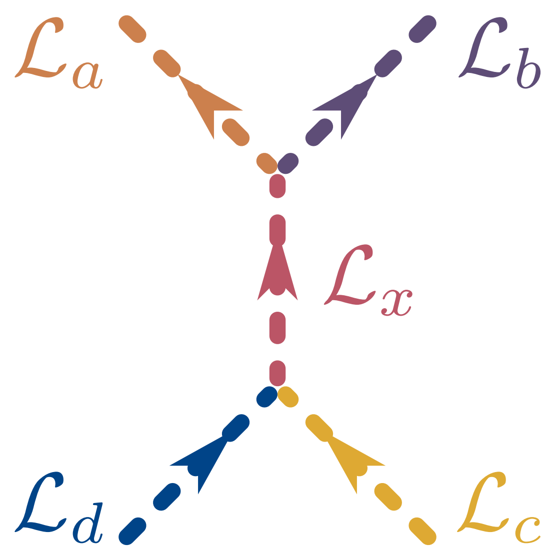


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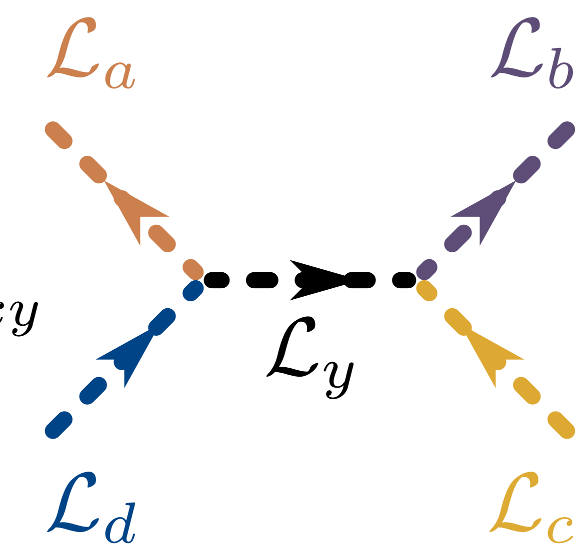


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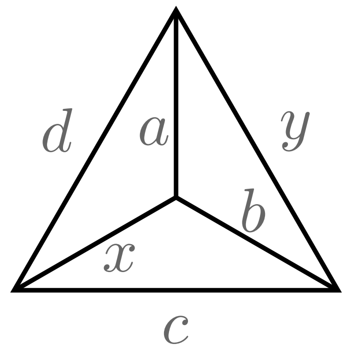


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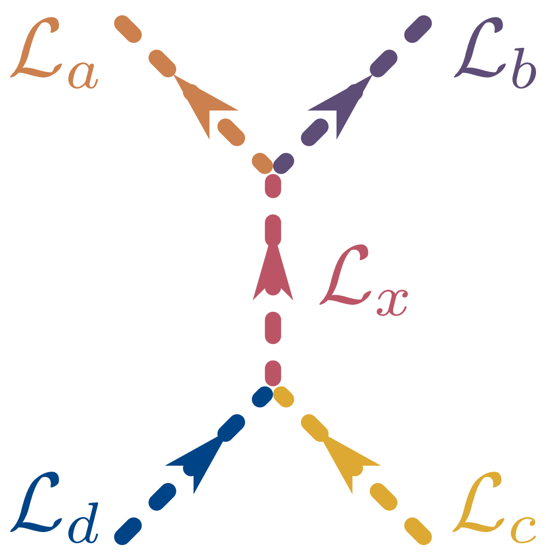



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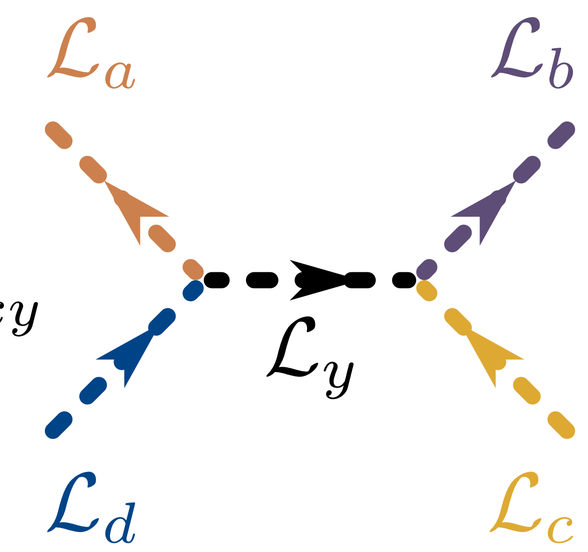



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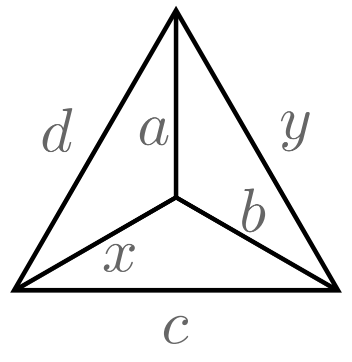



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  Here: RG flows with anomalous symmetries leading to **degenerate vacua** in IR

# Minimal Models and deformations

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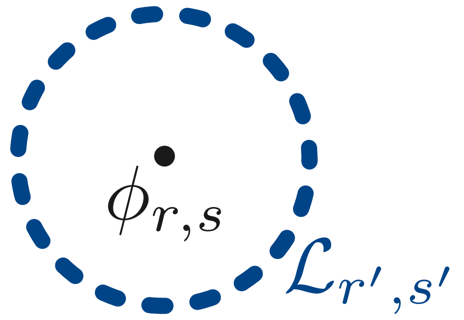


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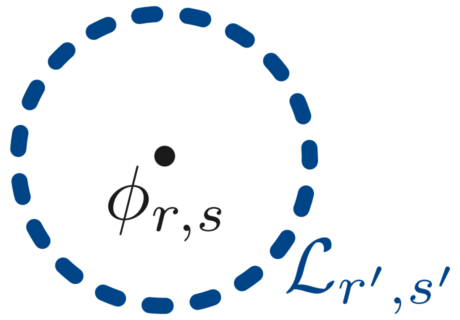


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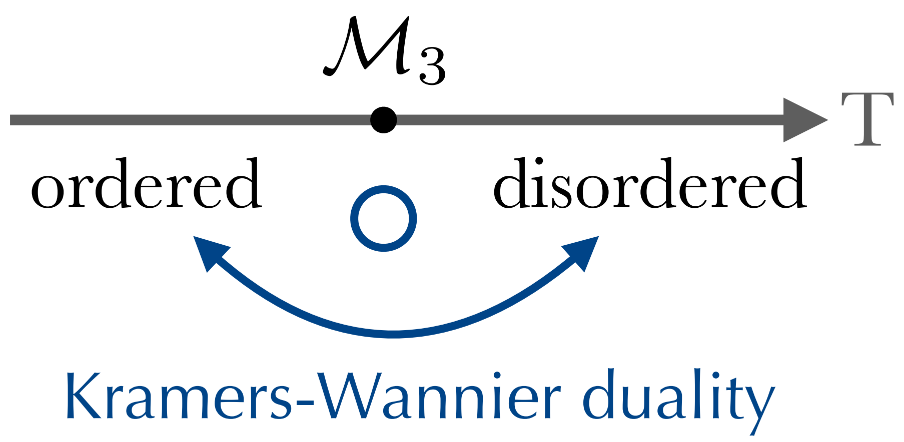

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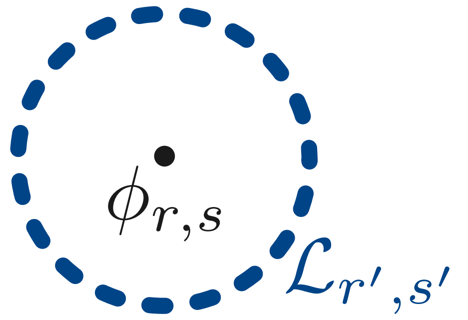


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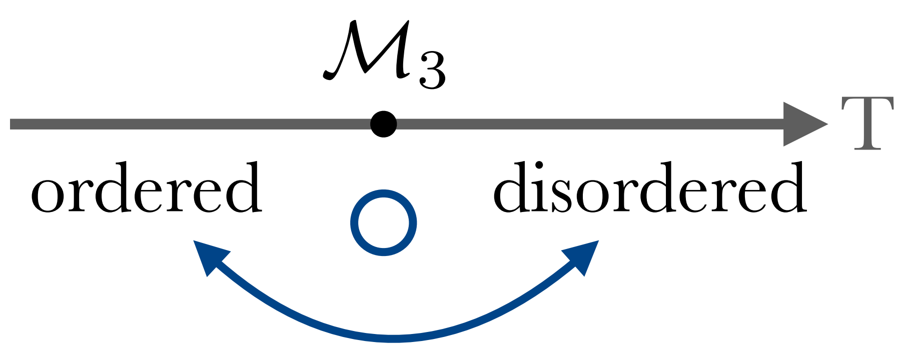
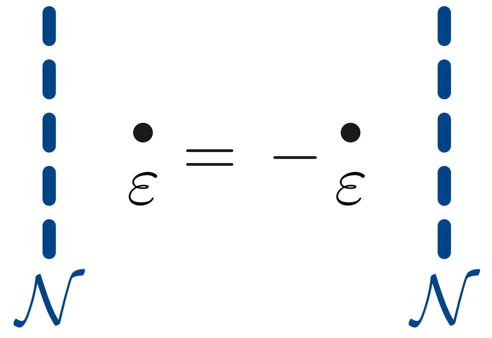
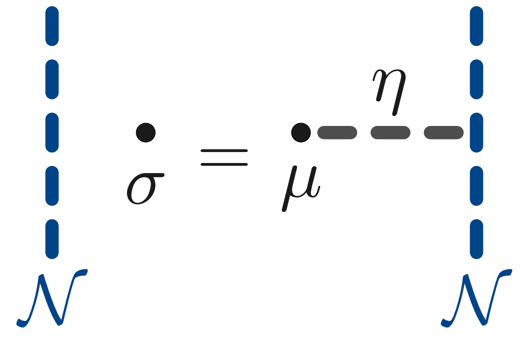



Kramers-Wannier duality




$\rightarrow \langle \varepsilon\varepsilon\varepsilon \rangle_{\mathscr{M}_3} = 0$




$\sigma$ (local) $\leftrightarrow$ $\mu$ (twisted)

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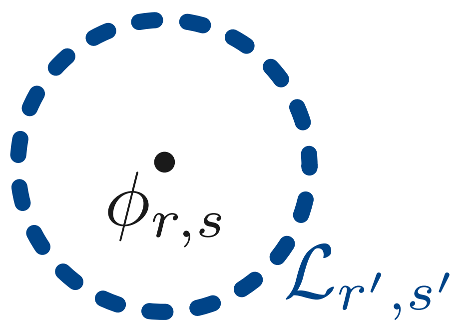


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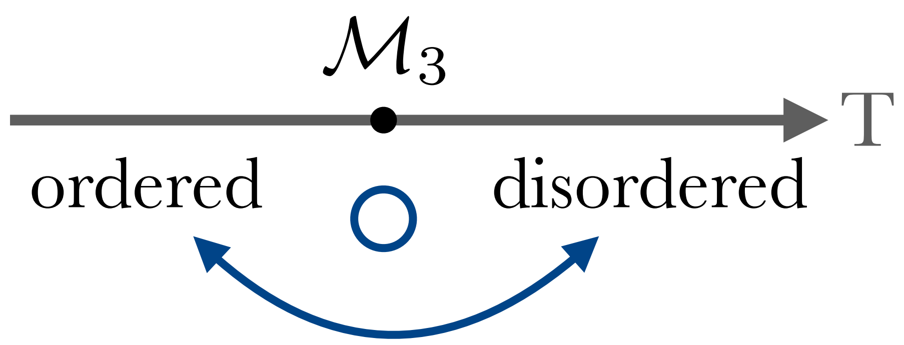


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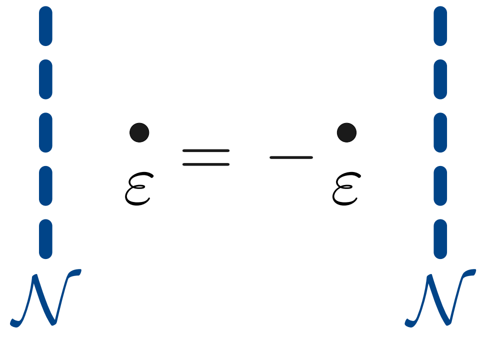
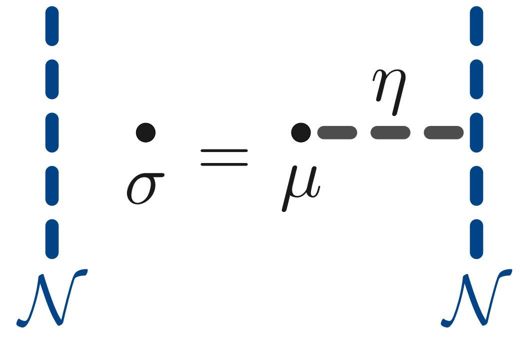


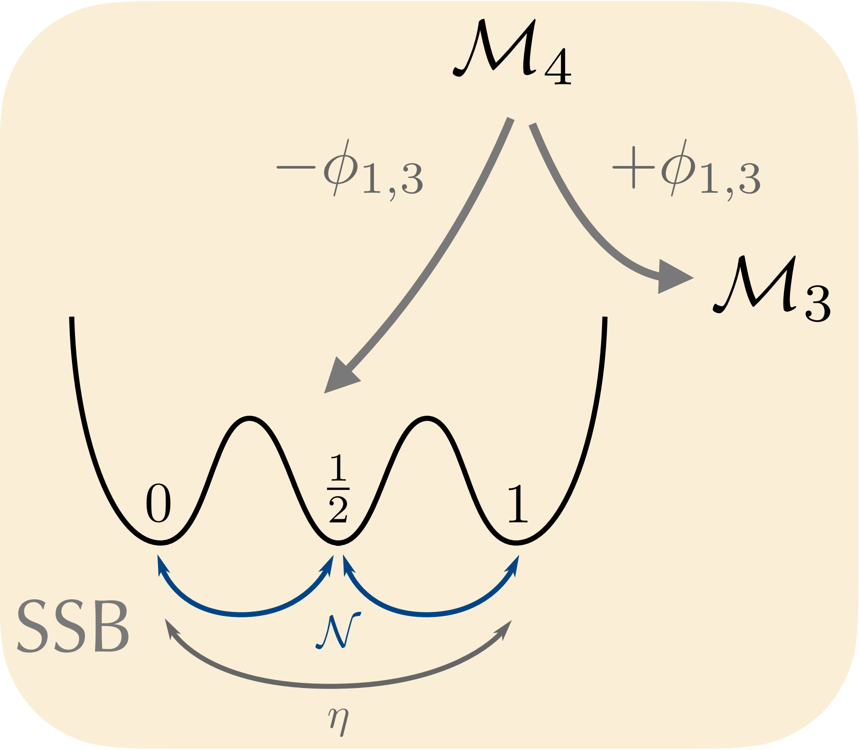

$\to \langle \varepsilon\varepsilon\varepsilon \rangle_{\mathscr{M}_3} = 0$




$\sigma$ (local) $\leftrightarrow$ $\mu$ (twisted)

- **Tricritical Ising** $\mathscr{M}_4$

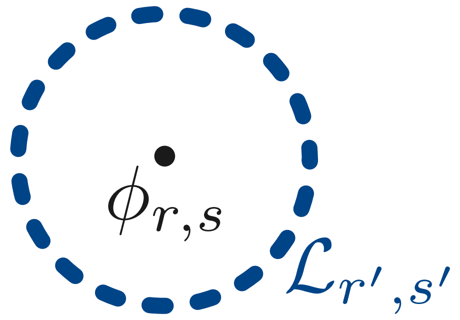
$$\begin{array}{rl} & \overbrace{\qquad\qquad\qquad}^{\mathbb{Z}_2\ \text{TY}\ \ (\phi_{1,3} = \varepsilon'\ \text{def.})} \\ \text{Fibonacci} & \left\{\begin{array}{ccc} 1 & \eta & \mathscr{N} \\ W & W' & Z \end{array}\right. \\ (\phi_{2,1} = \sigma'\ \text{def.}) & \end{array}$$



# Minimal Models and deformations

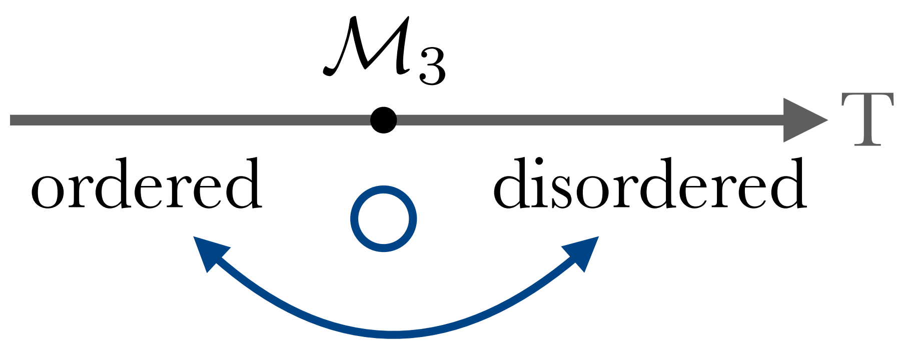
- **Verlinde lines** $\mathscr{L}_{r,s}$ : Topological lines with same fusion algebra as primaries $\phi_{r,s}$

  [Petkova, Zuber '00; Chang, Lin, Shao, Wang, Yin '18]




$= \dfrac{\mathcal{S}_{r',s';\,r,s}}{\mathcal{S}_{1,1;\,r,s}} \; \phi_{r,s}$ $\qquad \mathcal{S}_{r,s}$: modular S-matrix

- **Critical Ising** $\mathscr{M}_3$ $\quad \to$ sym. lines: $\{\mathscr{L}\} = 1, \eta, \mathscr{N}$. $\quad \mathscr{N}^2 = 1+\eta$. $\quad$ ($\mathbb{Z}_2$ Tambara-Yamagami)

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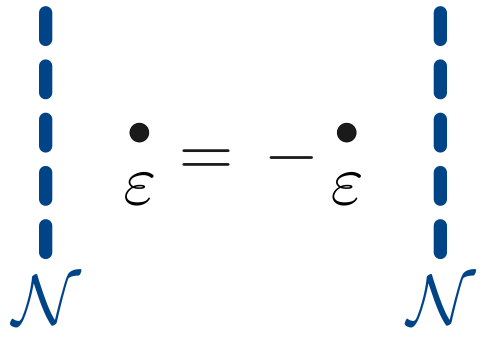


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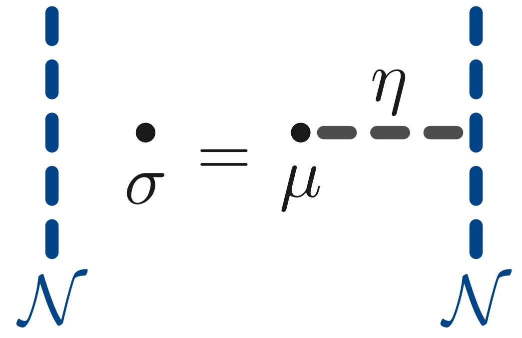


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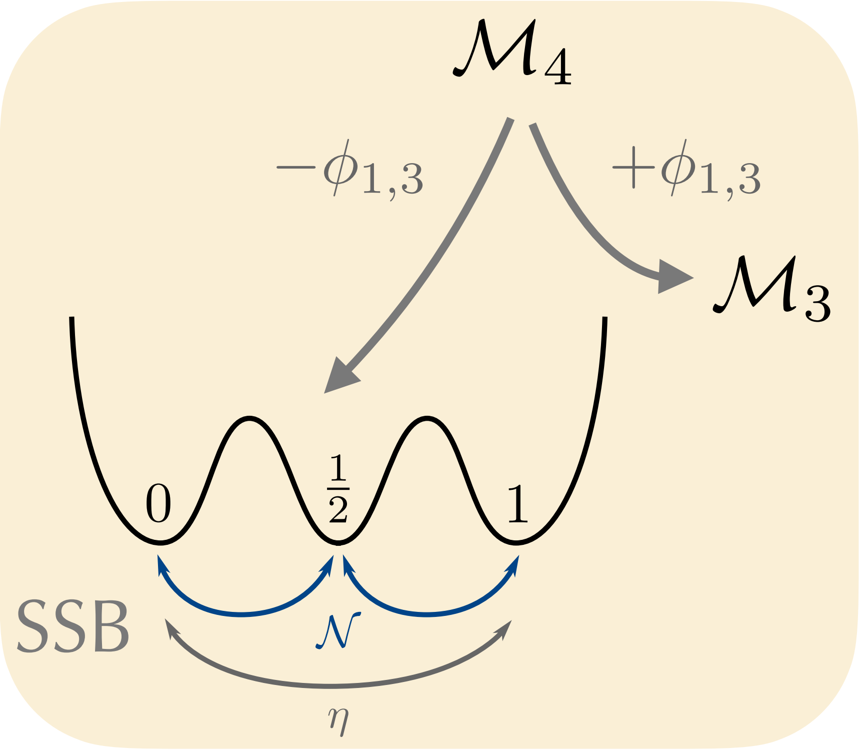
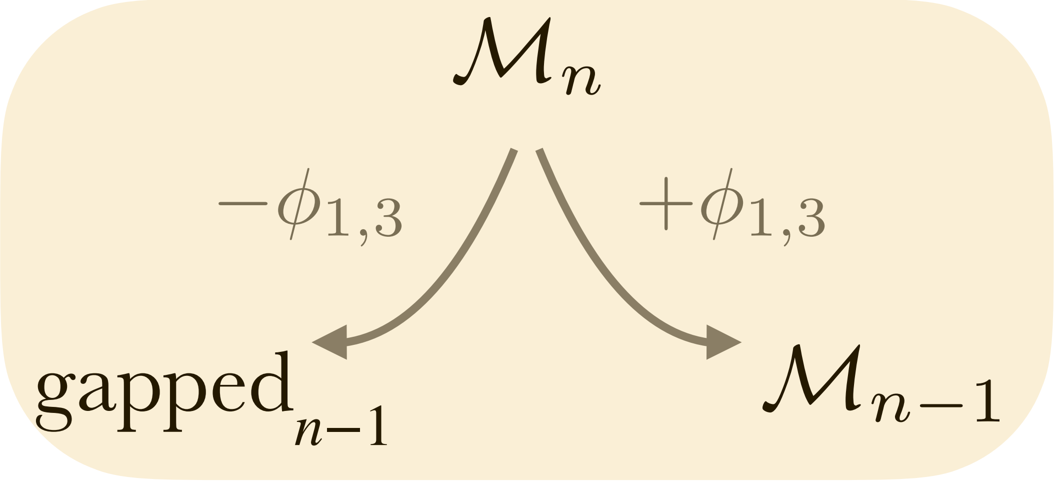



- $\mathscr{M}_n \pm \phi_{1,3}$ $\quad \mathscr{L}_{r,1} \to \mathscr{A}_n$ fusion category



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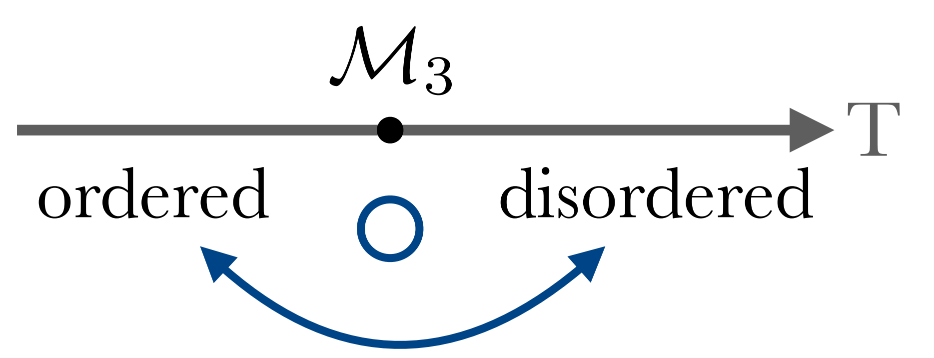


$\mathcal{N}\;\varepsilon = -\varepsilon\;\mathcal{N}$ $\rightarrow \langle\varepsilon\varepsilon\varepsilon\rangle_{\mathscr{M}_3} = 0$

$\mathcal{N}\;\sigma = \mu \overset{\eta}{-\!-}\;\mathcal{N}$ $\sigma$ (local) $\leftrightarrow$ $\mu$ (twisted)

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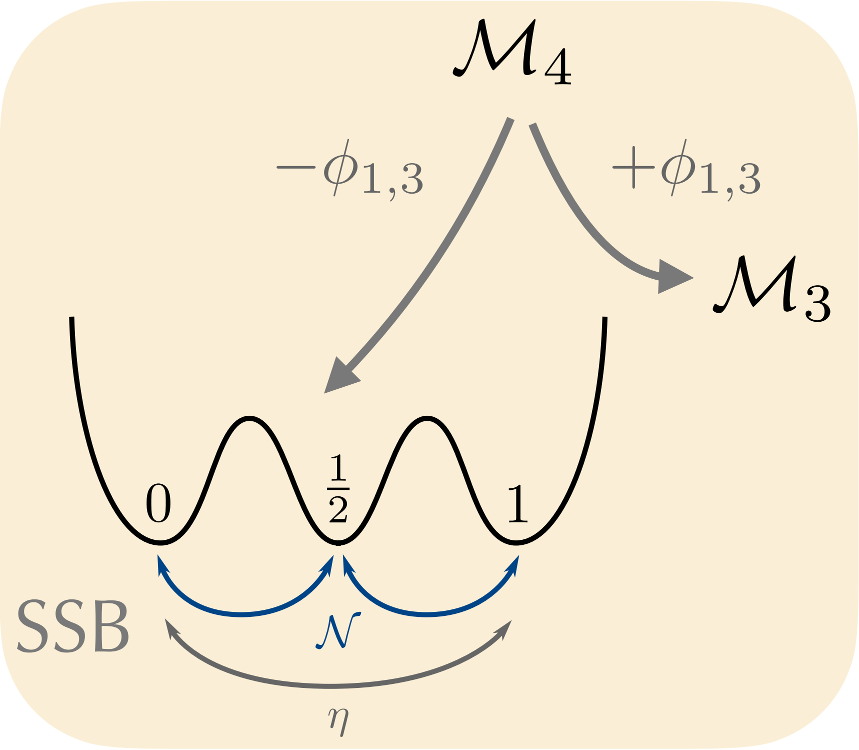


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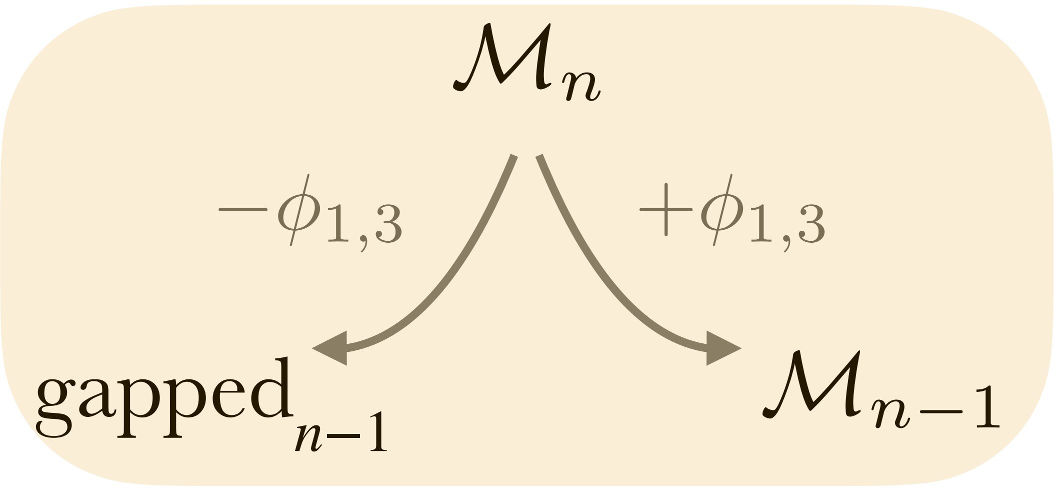
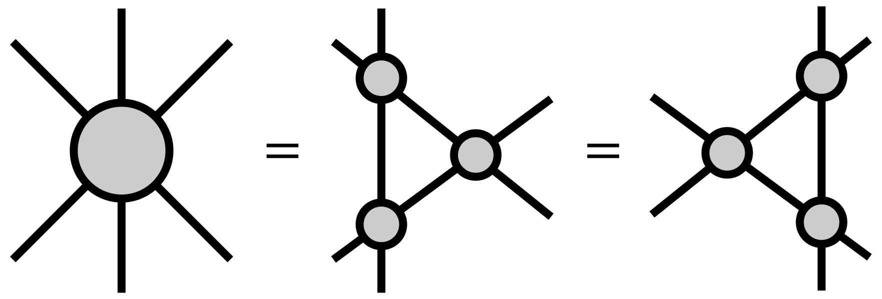



- $\phi_{1,3}$, $\phi_{2,1}$, $\phi_{1,2}$ deformations are **integrable**. Exact S-matrix



# Scattering Amplitudes and Modified Crossing

# Scattering amplitudes in 1+1d

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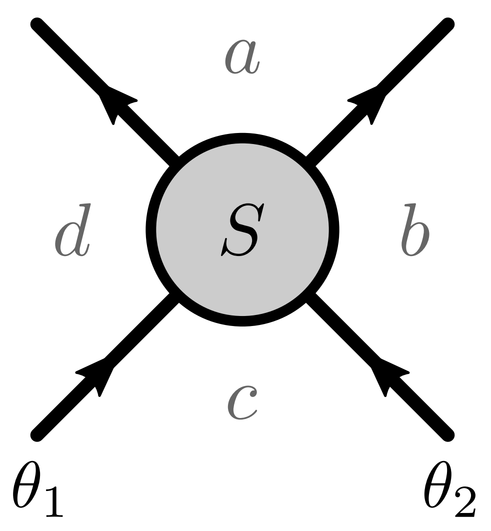
$$S^{ab}_{dc}(\theta) =$$




Massive kinks interpolating between neighbouring vacua

$\theta = \theta_1 - \theta_2$

$s = 4m^2 \cosh^2(\theta/2)$

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# Scattering amplitudes in 1+1d

(standard) **A X I O M S**

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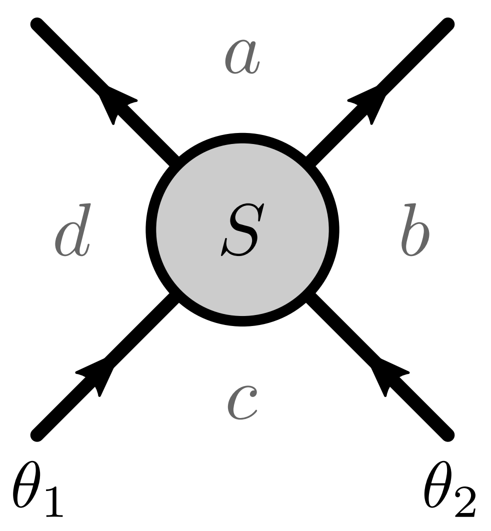


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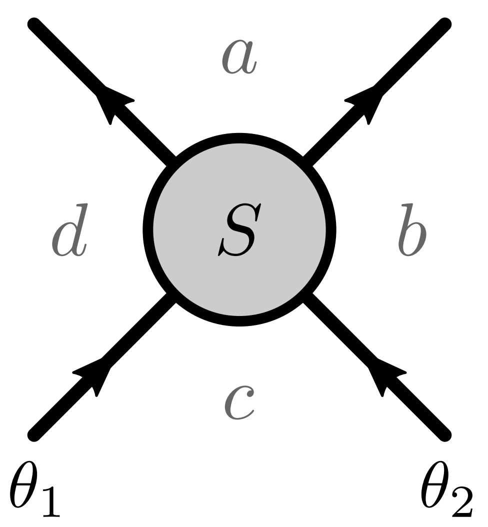
$S_{dc}^{ab}(\theta) =$ 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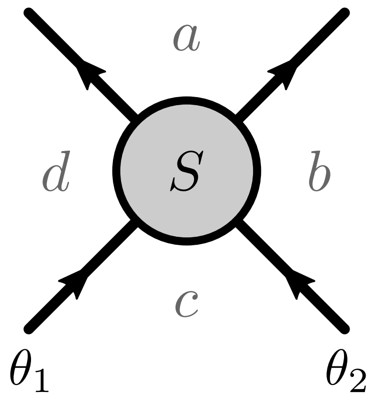
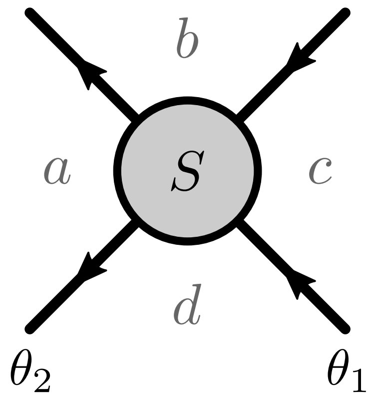
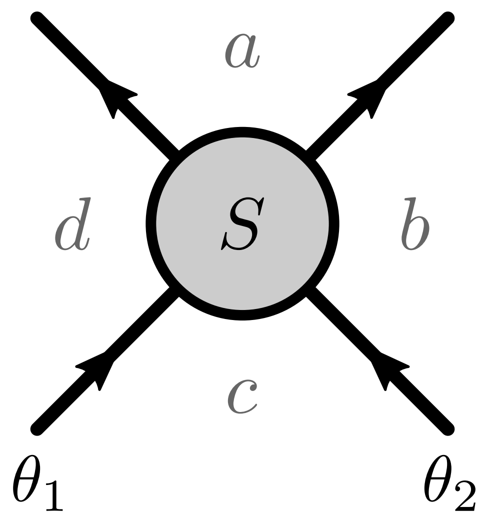
$S_{dc}^{ab}(\theta) =$ 

$=$ 

$= S_{ad}^{bc}(i\pi - \theta)$

$S_{dc}^{ab}(\theta) =$ 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# Scattering amplitudes in 1+1d

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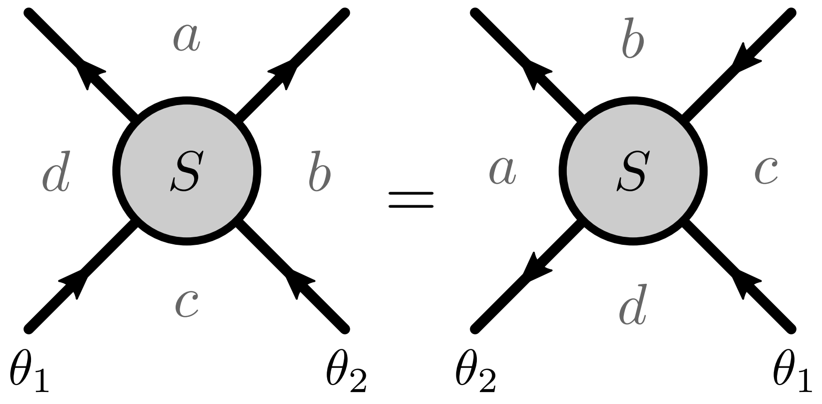
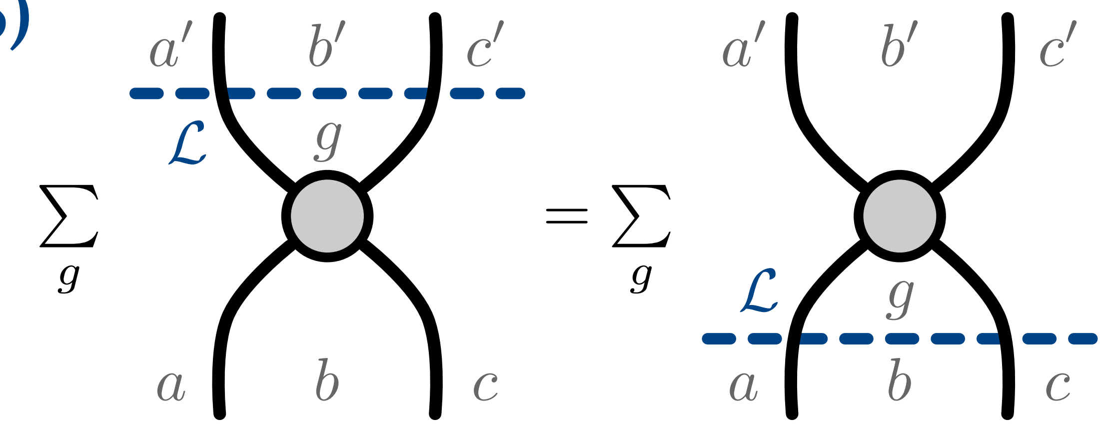



$= S^{bc}_{ad}(i\pi - \theta)$

**Symmetries (S)**

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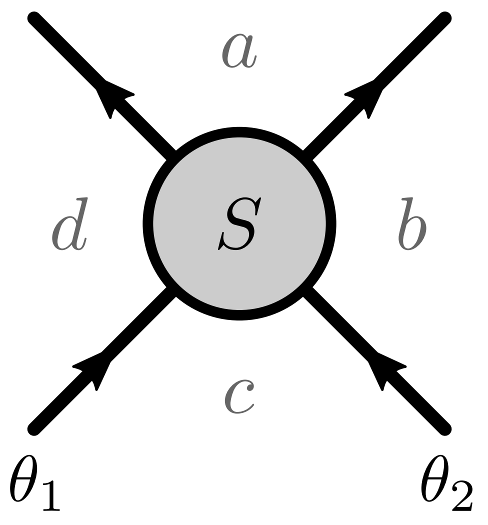


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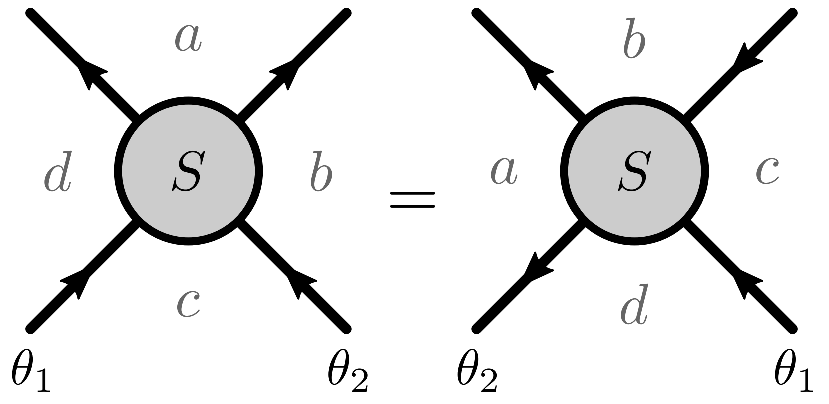


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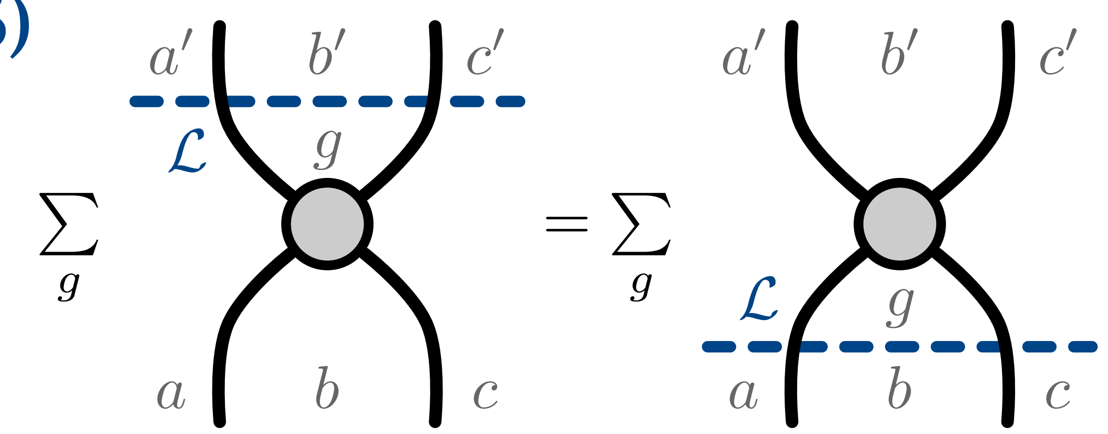
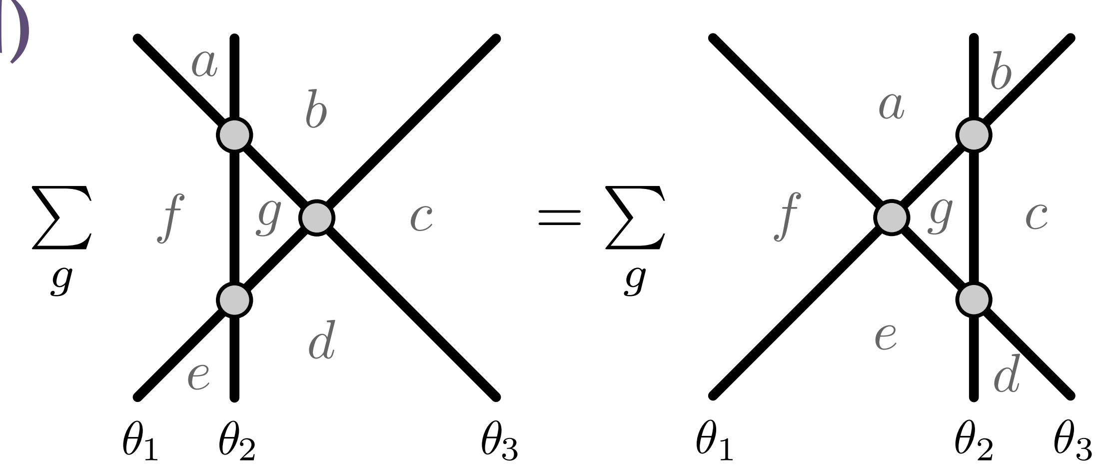



***Integrability (I)**

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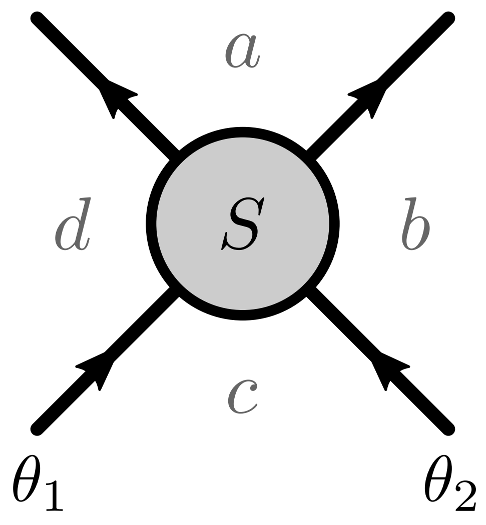


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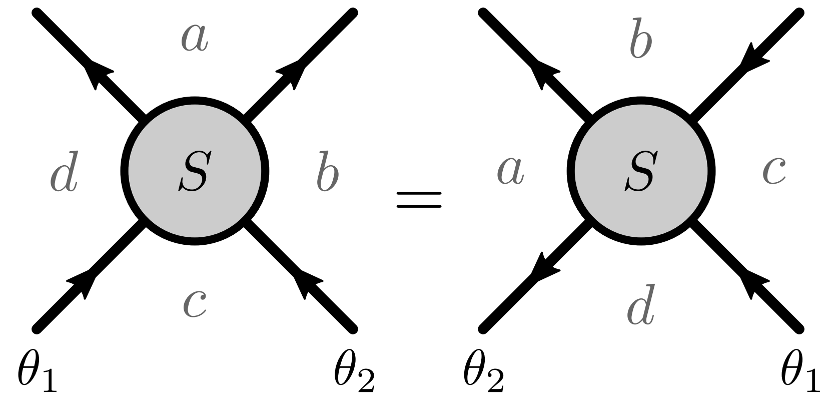


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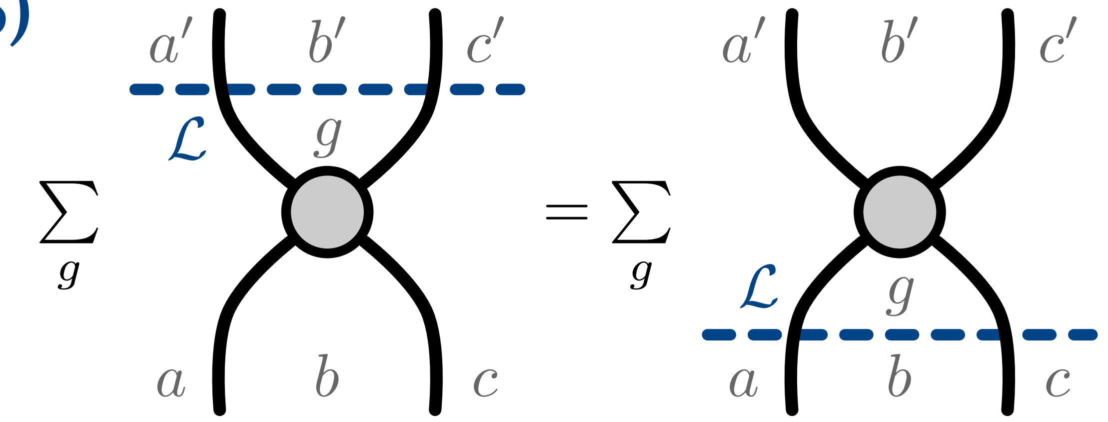


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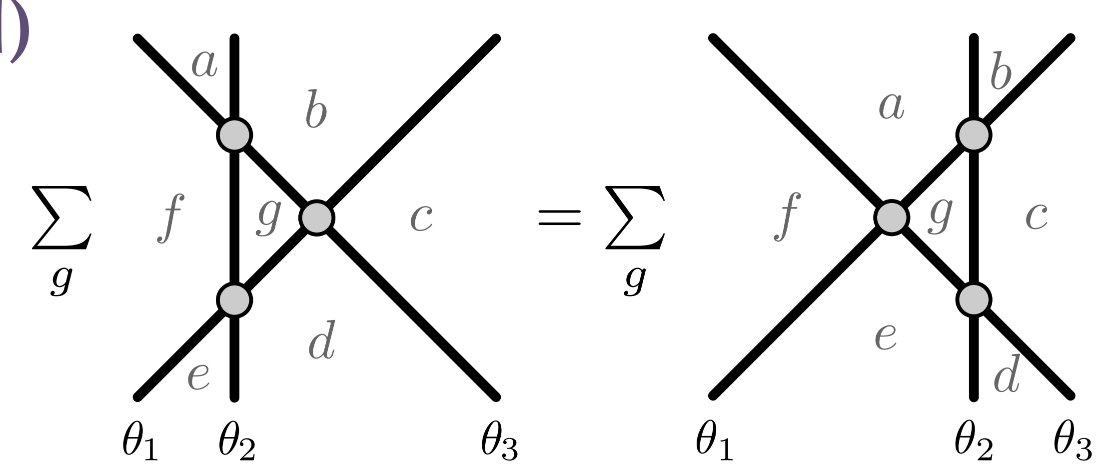


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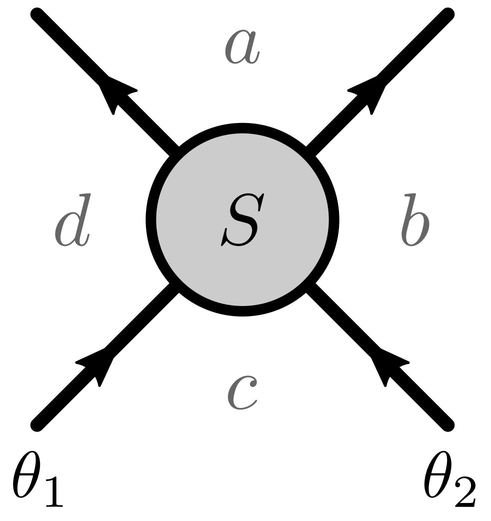


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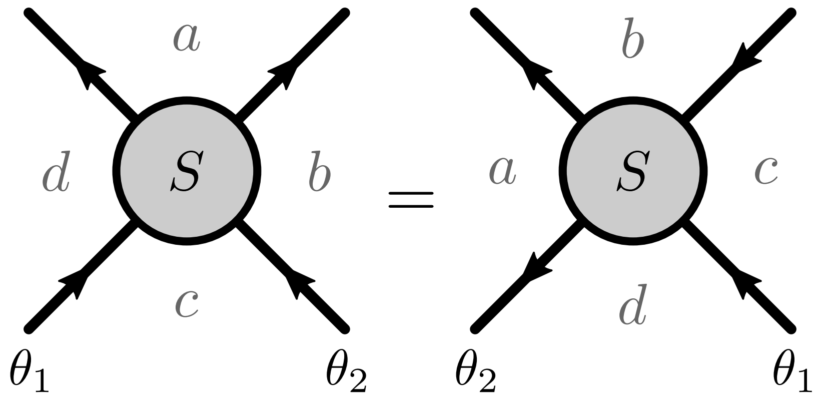


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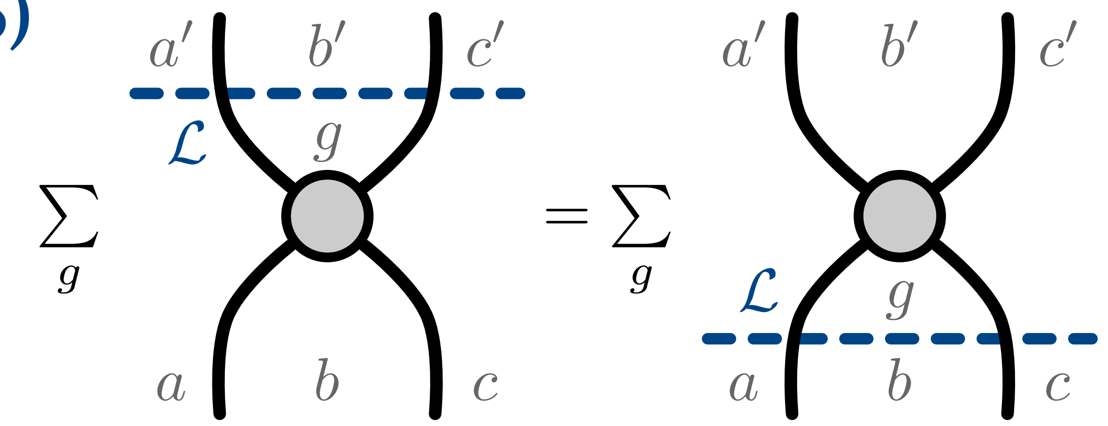


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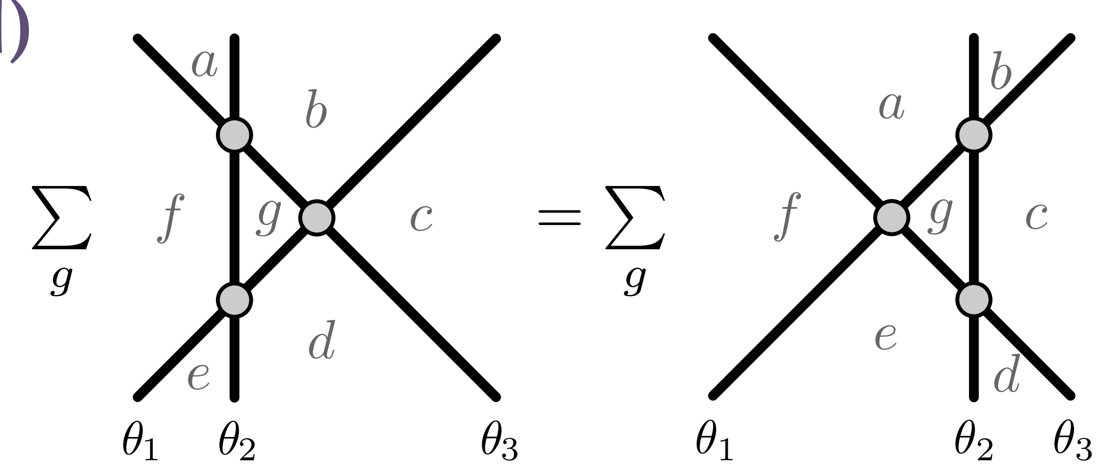


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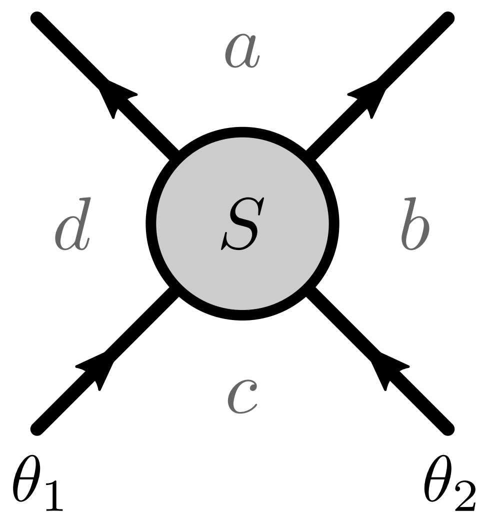


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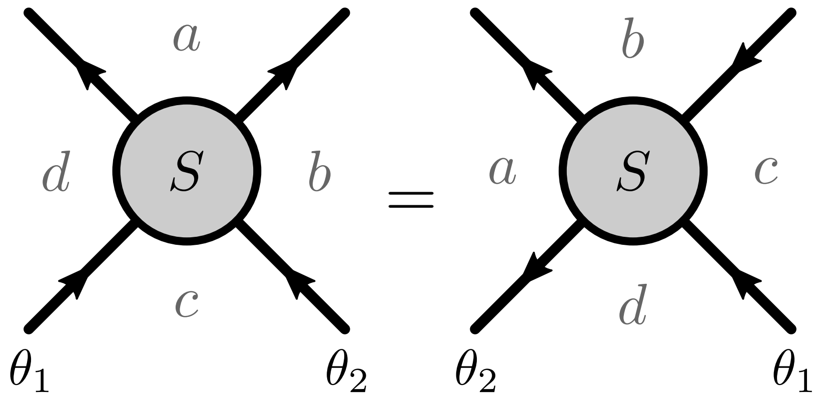
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$



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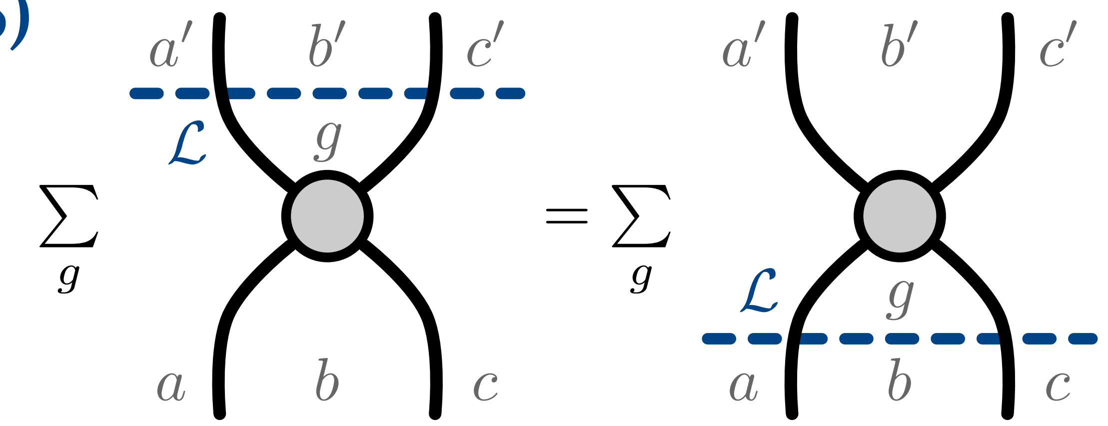



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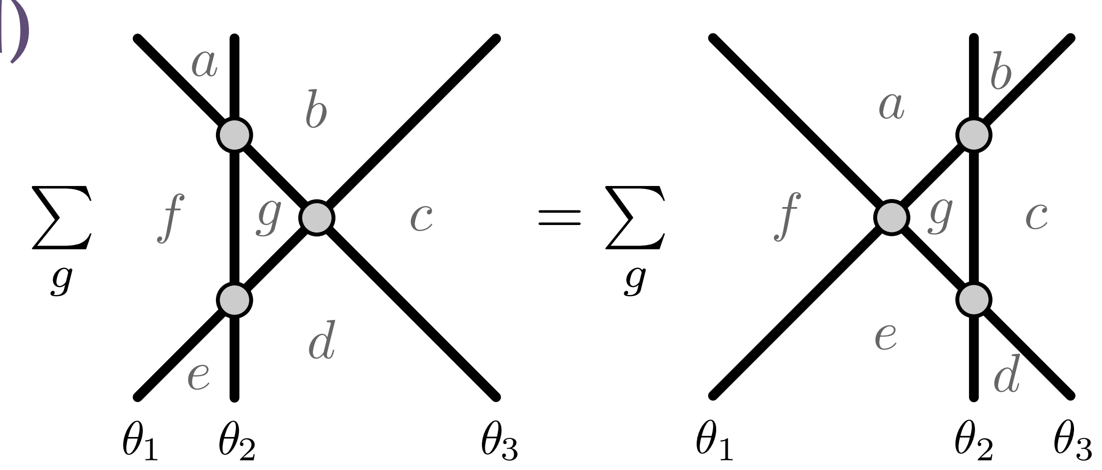



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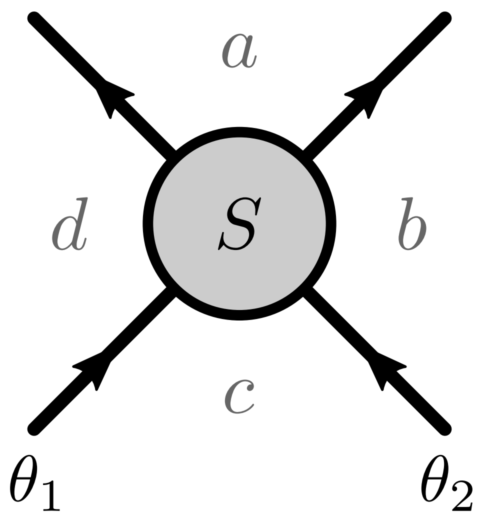
$S^{ab}_{dc}(\theta) =$ 



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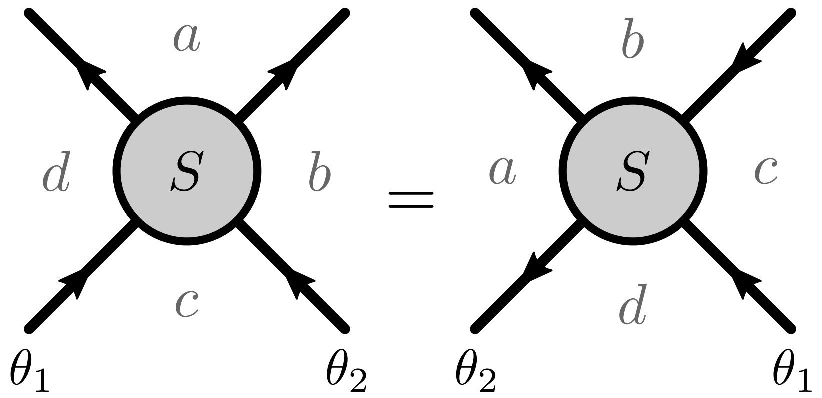
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$



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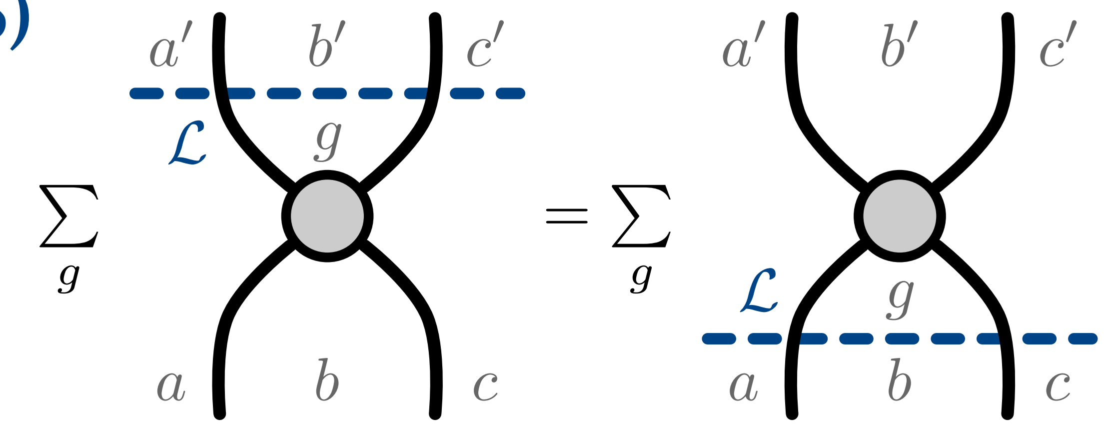



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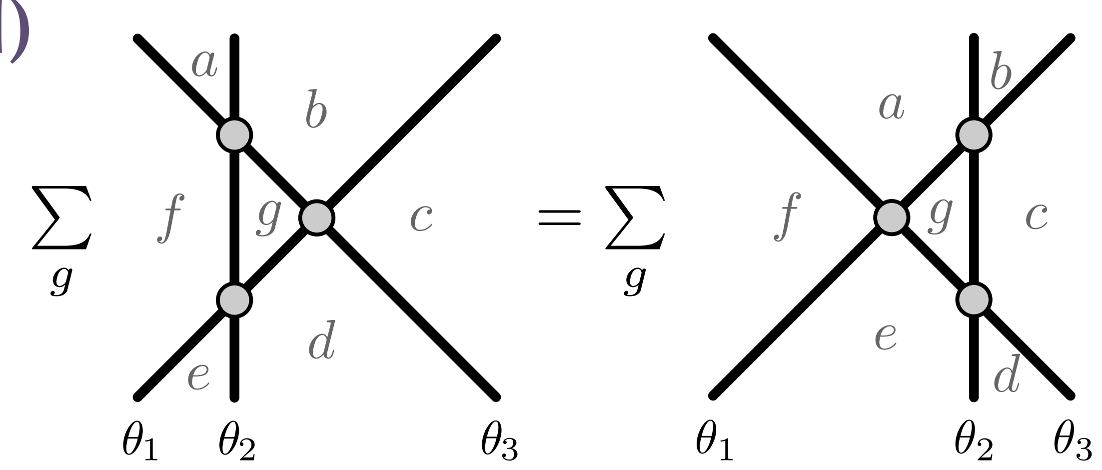



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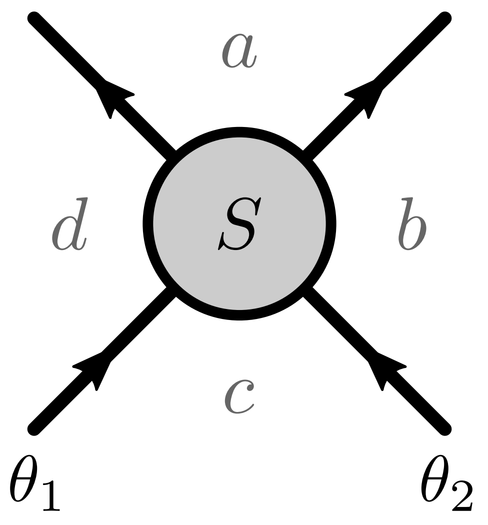
$S^{ab}_{dc}(\theta) =$ 



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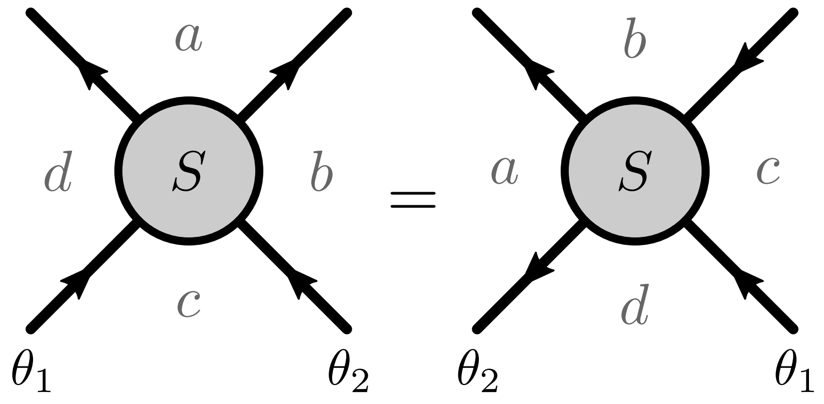
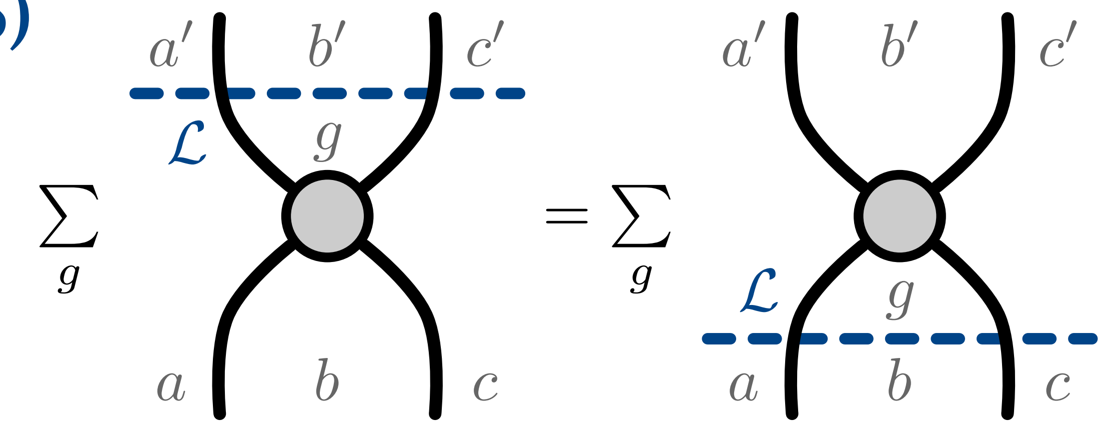
$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$

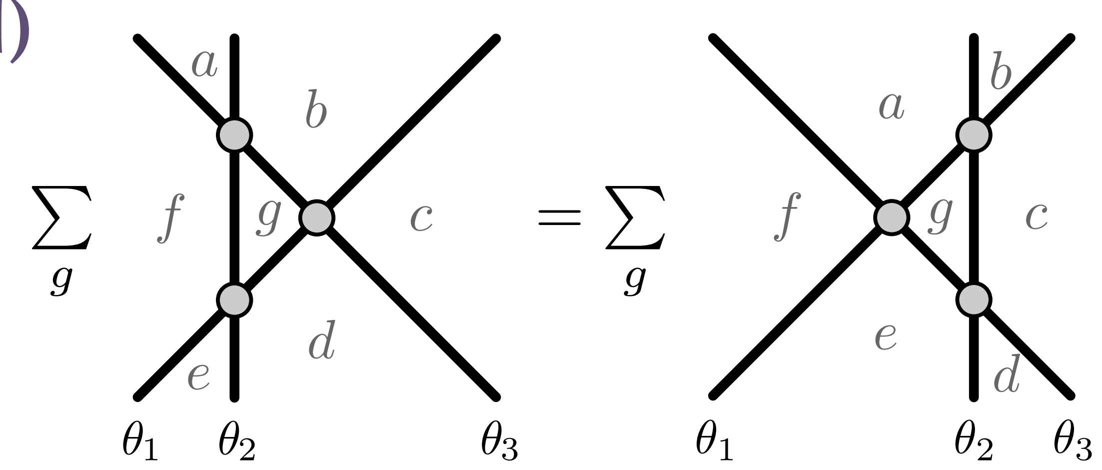


**Symmetries (S)**



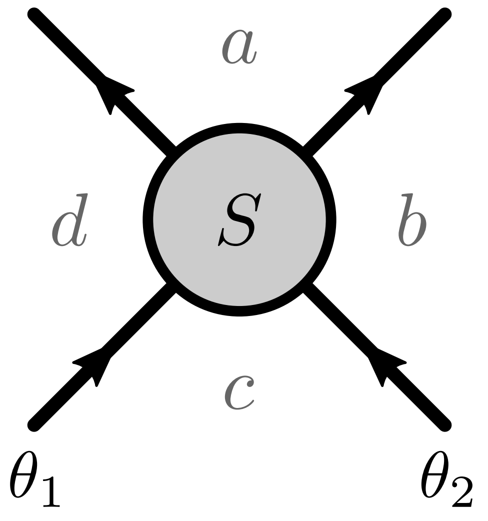


***Integrability (I)**





Factorized scattering, no particle production

$S^{ab}_{dc}(\theta) =$ 



Massive kinks interpolating between neighbouring vacua

$\theta = \theta_1 - \theta_2$
$s = 4m^2\cosh^2(\theta/2)$
$t = 4m^2 - s$

- Example: $\mathscr{M}_n - \phi_{1,3}$ RG flow

**U+C+I** give: [Zamolodchikov '89] $\quad a = 1, \ldots, n-1 \quad \mathrm{d}_a = \dfrac{\sin \pi a/n}{\sin \pi/n}$

$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta)\, \cancel{\left(\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}}} \left[\sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\, \sinh\left(\frac{\theta}{n}\right)\delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

note oscillations

→ does not commute w/ non-invertible syms!

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# Scattering amplitudes in 1+1d

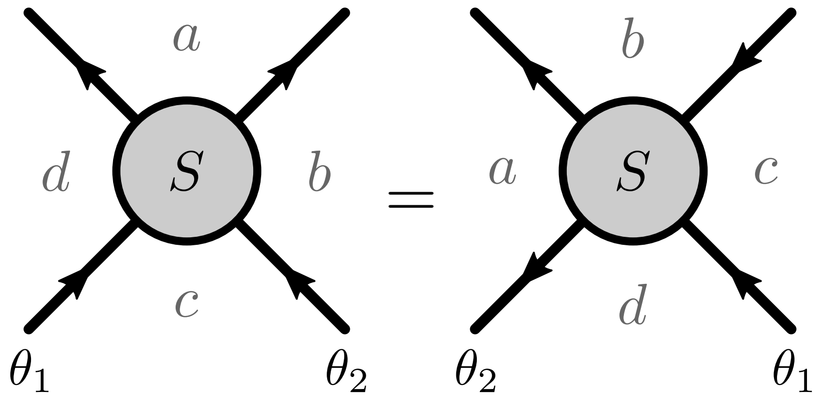
## (standard) AXIOMS

**Unitarity (U)**

$$|S^{ab}_{dc}(\theta \geq 0)|^2 \leq 1$$

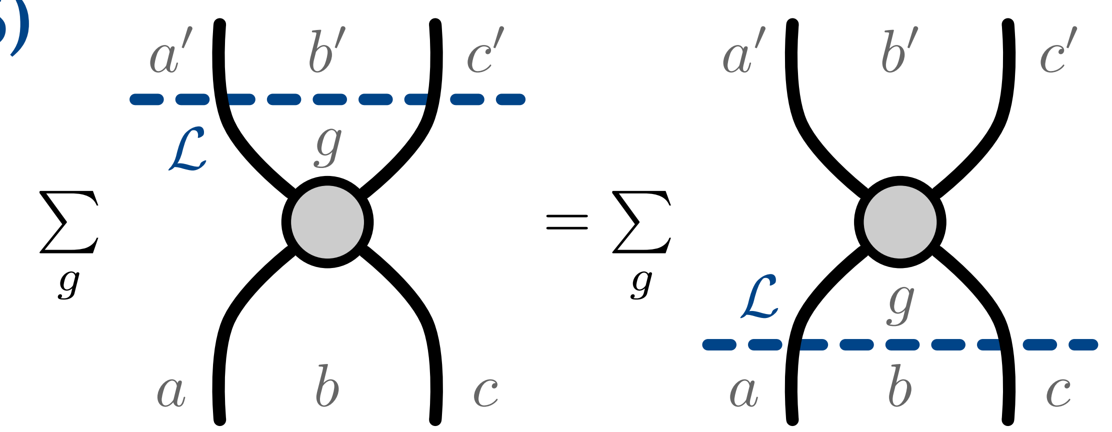
**Crossing (C)**

$S^{ab}_{dc}(\theta) =$




$= S^{bc}_{ad}(i\pi - \theta)$

**Symmetries (S)**

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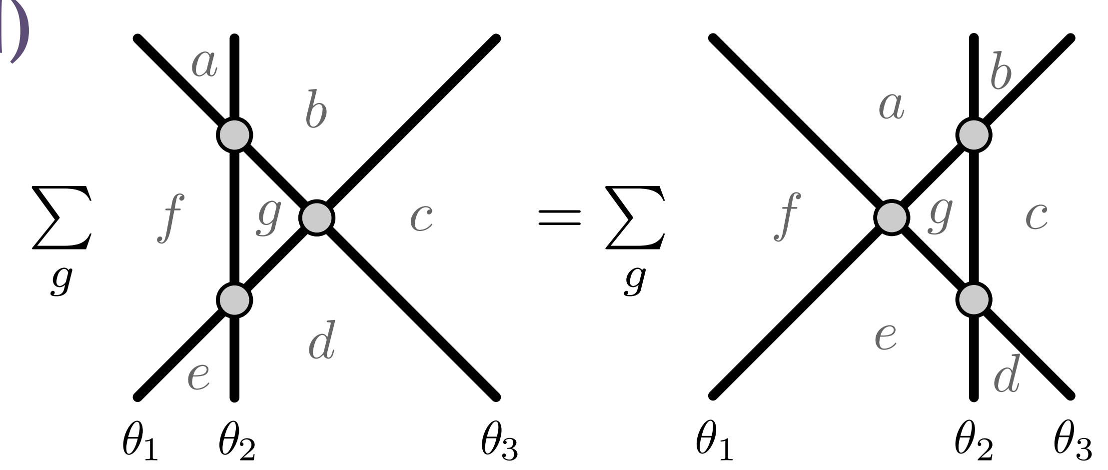


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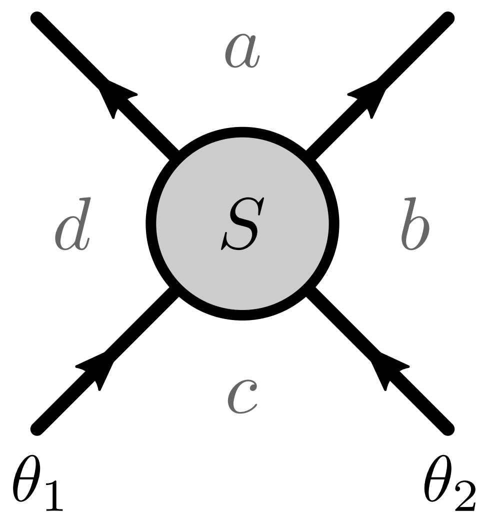


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$\Longrightarrow$ **Modified crossing (mC)**

[Copetti, LC, Komatsu '24]

$$\boxed{S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)}$$

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- New crossing rules can be understood from proper normalization of $|\text{in}\rangle$ and $|\text{out}\rangle$ states, taking into account topological degrees of freedom.

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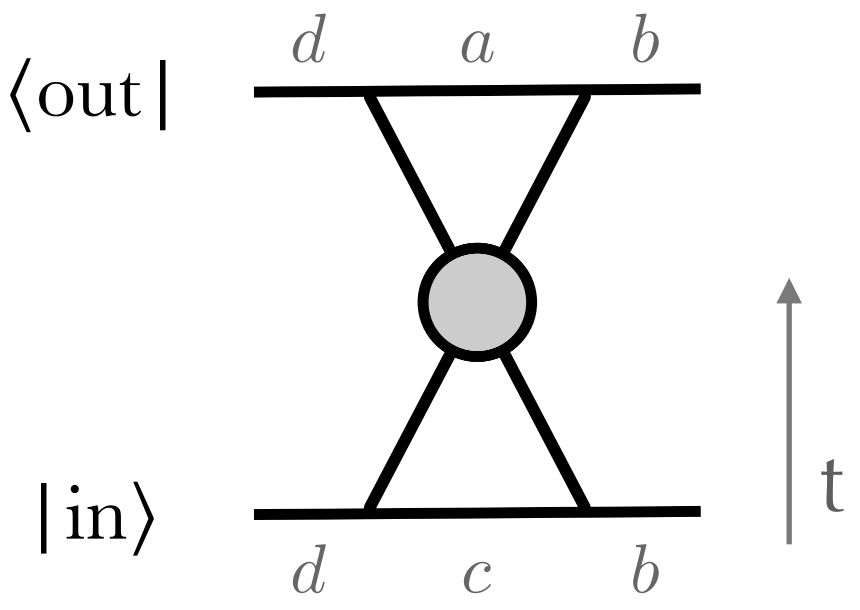


$$\langle \mathrm{out} | \mathrm{in} \rangle = S^{ab}_{dc}(\theta)\, \underbrace{(2\pi)^2\, 2\sqrt{s}\sqrt{s-4m^2}\, \delta^2(p_1+p_2-p_1'-p_2')}_{\text{‘}\,\delta^2(\,\cdot\,)\,\text{’}}$$

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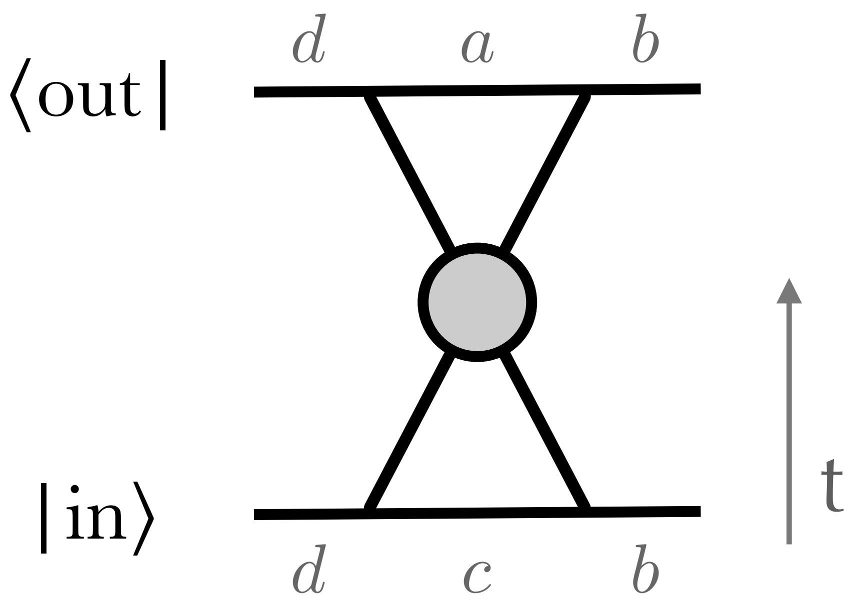


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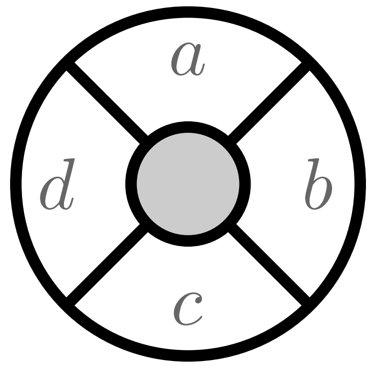


$\Big|_{\mathrm{analyt.\ cont.}}$

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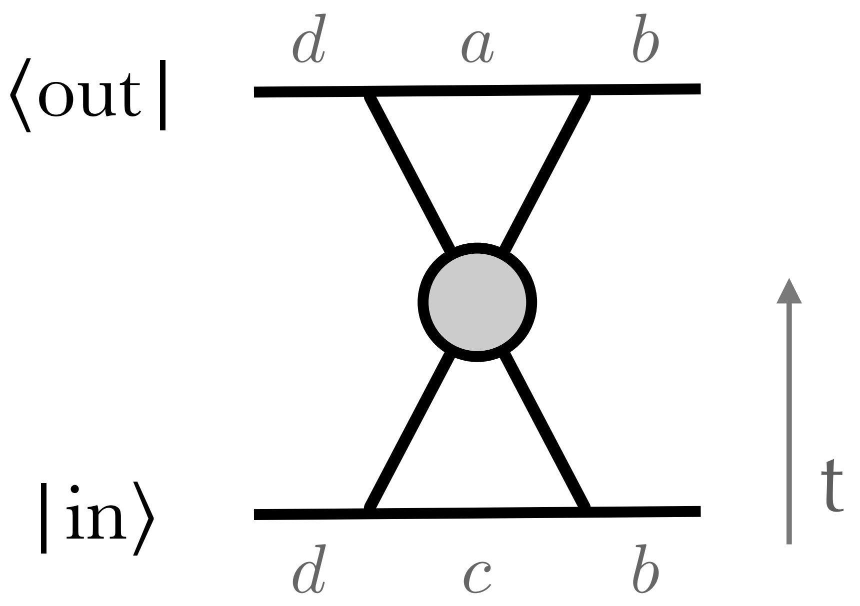


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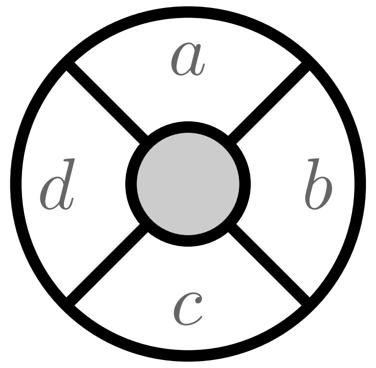
$S^{ab}_{dc}(\theta) \propto$ 

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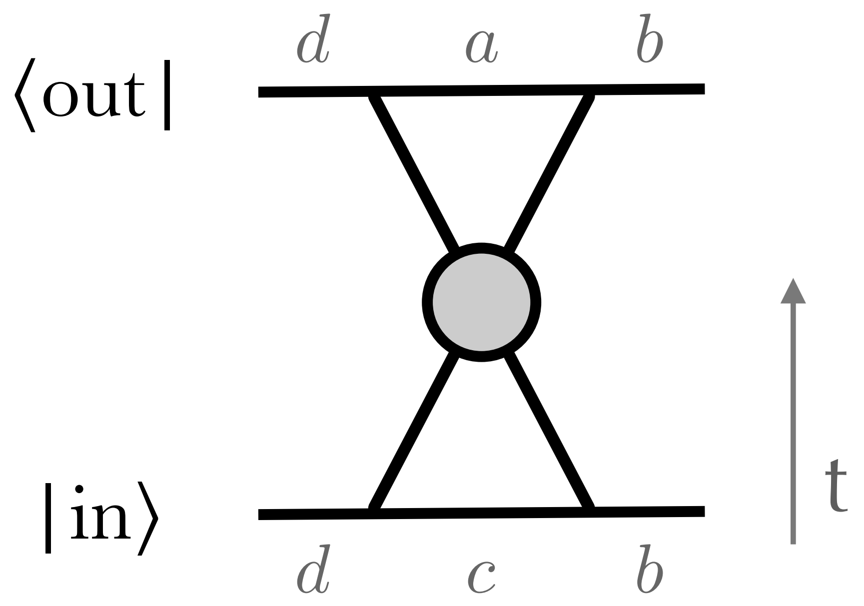


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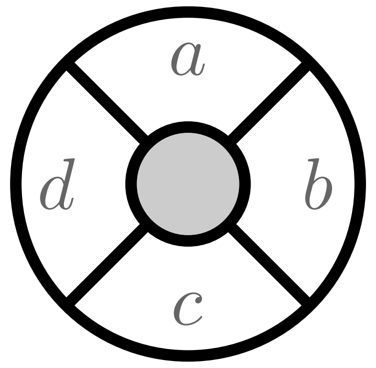


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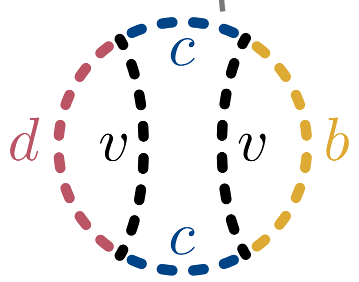
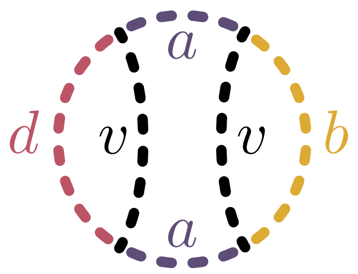
s-channel

$\mathrm{d}_v\sqrt{\mathrm{d}_b \mathrm{d}_d}$

$\langle \mathrm{in}|\mathrm{in}\rangle_s = $  $\times\,\delta^2(\,\cdot\,)$


$\langle \mathrm{out}|\mathrm{out}\rangle_s = $  $\times\,\delta^2(\,\cdot\,)$

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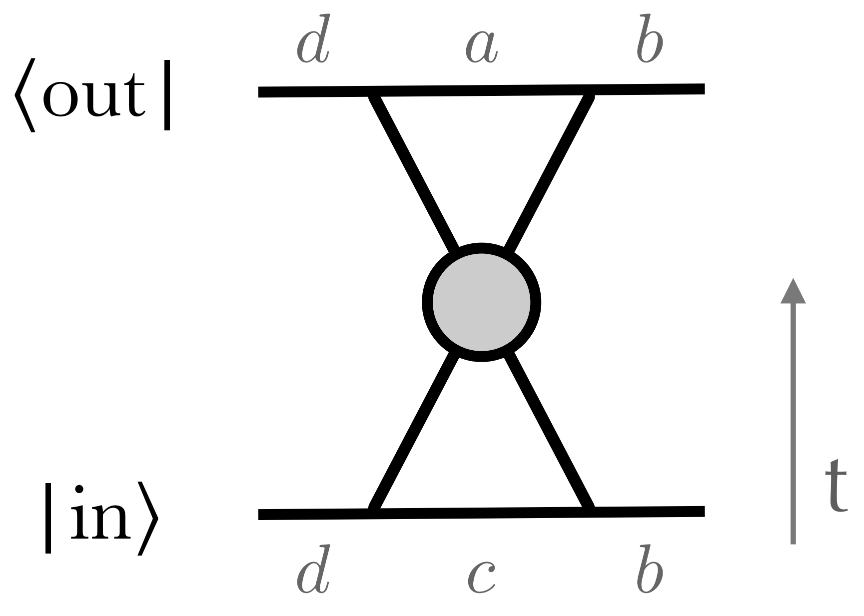
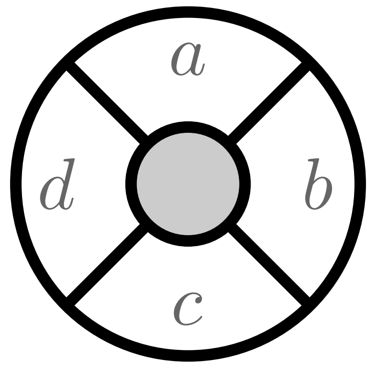



$$\langle \mathrm{out}|\mathrm{in}\rangle = S_{dc}^{ab}(\theta)\ \underbrace{(2\pi)^2\, 2\sqrt{s}\sqrt{s-4m^2}\,\delta^2(p_1+p_2-p_1'-p_2')}_{\text{‘ }\delta^2(\,\cdot\,)\text{ ’}}$$

$S_{dc}^{ab}(\theta) \propto$  $\Big|_{\text{analyt. cont.}}$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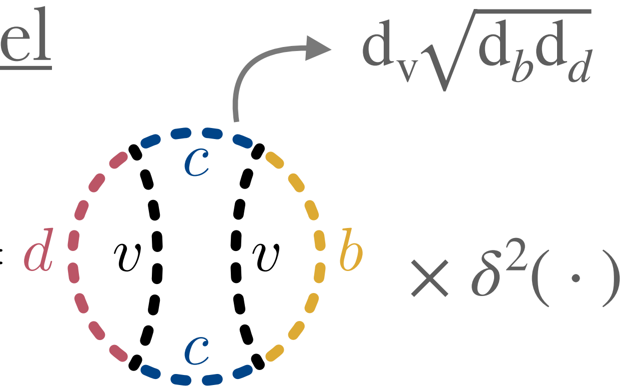
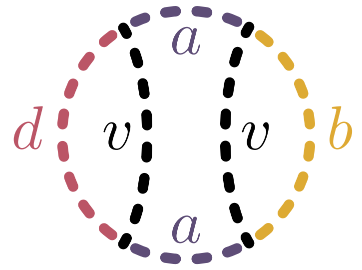
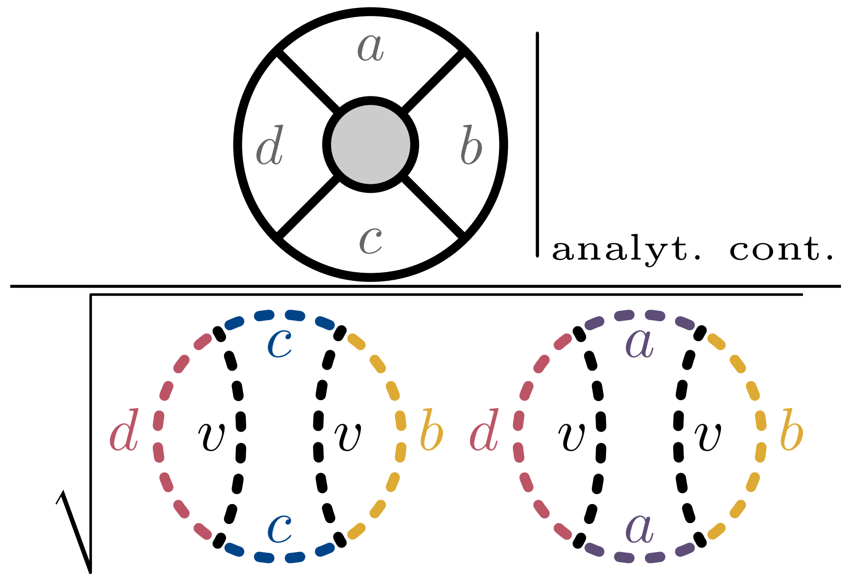
$\langle \mathrm{in}|\mathrm{in}\rangle_s =$  $\times\, \delta^2(\,\cdot\,)$


$\langle \mathrm{out}|\mathrm{out}\rangle_s =$  $\times\, \delta^2(\,\cdot\,)$


$S_{dc}^{ab}(\theta) =$ 

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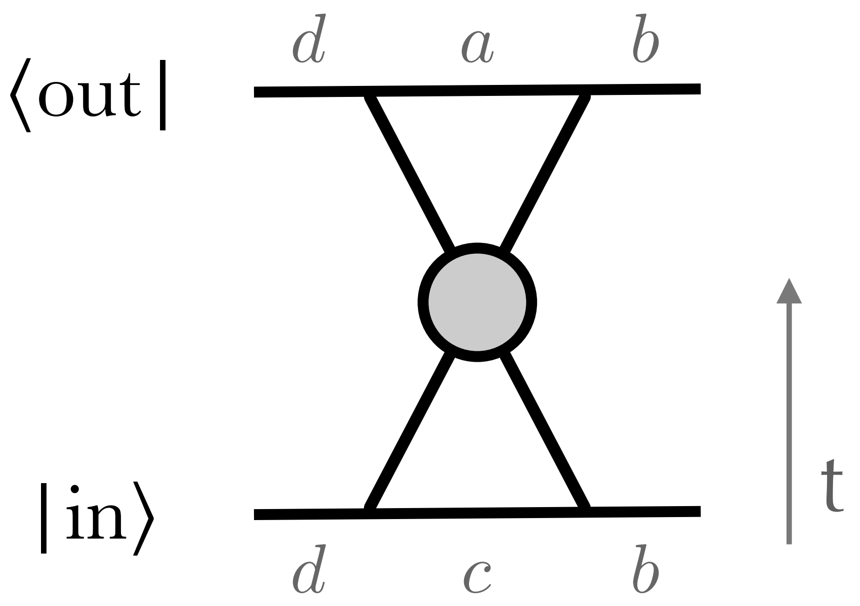


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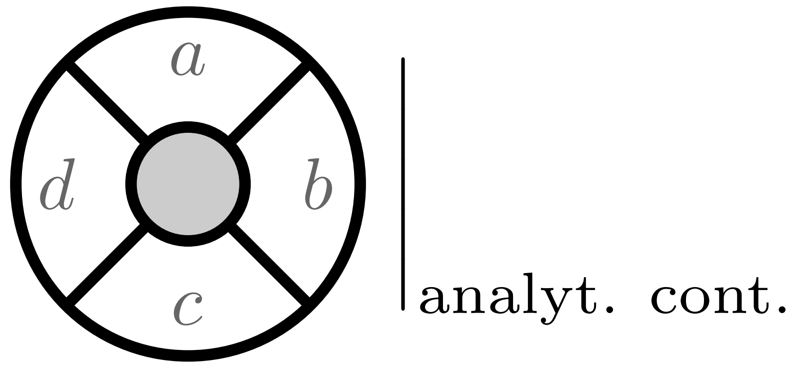
$S^{ab}_{dc}(\theta) \propto$ 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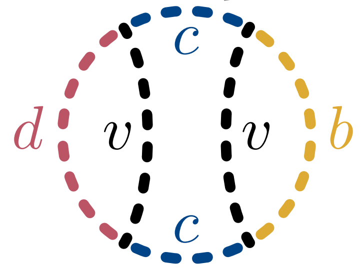
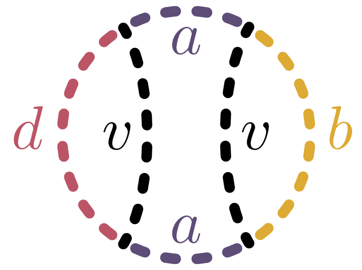
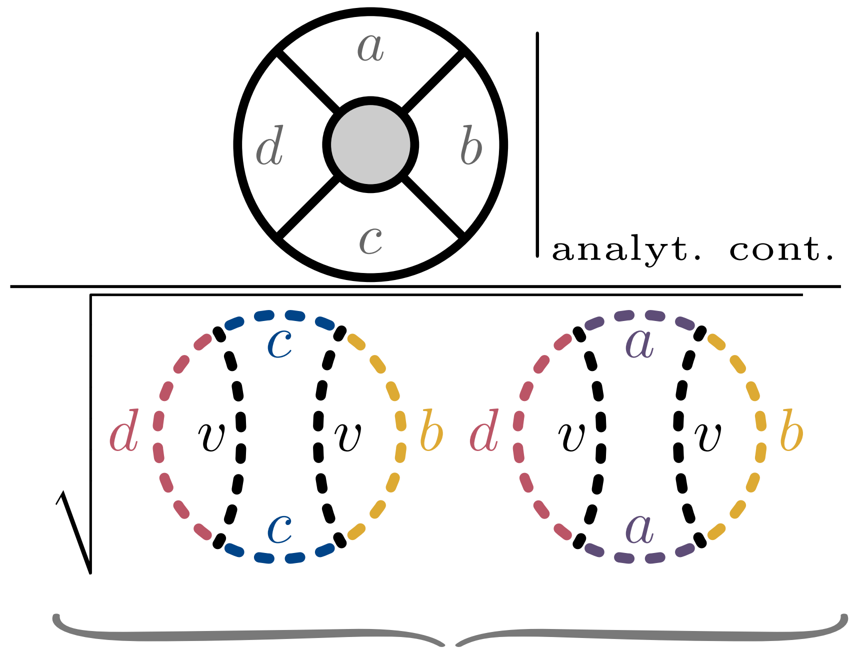
$\langle \text{in}|\text{in}\rangle_s =$  $\times\, \delta^2(\,\cdot\,)$


$\langle \text{out}|\text{out}\rangle_s =$  $\times\, \delta^2(\,\cdot\,)$


$S^{ab}_{dc}(\theta) =$ 


different in t-channel

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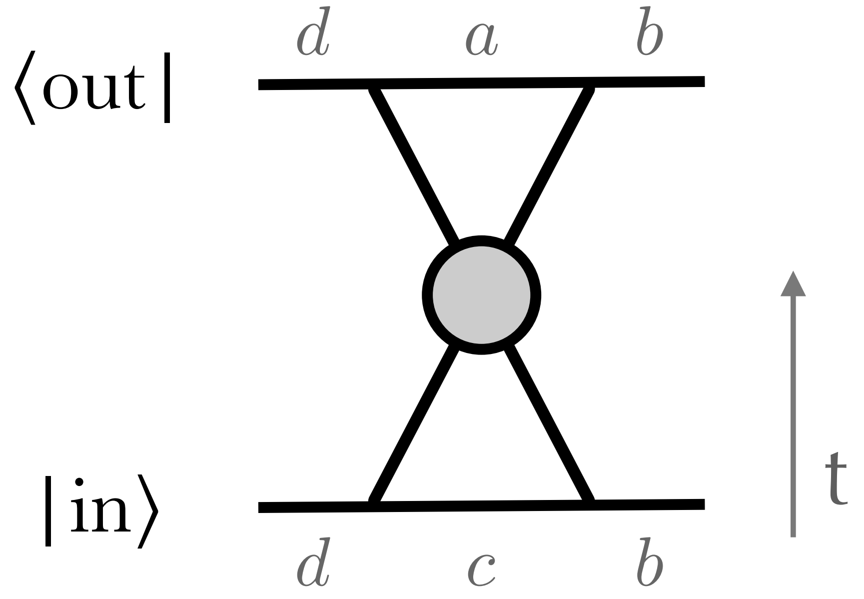


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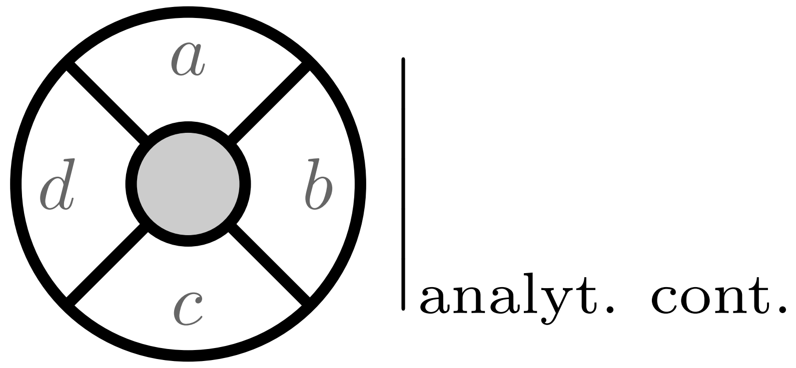
$S^{ab}_{dc}(\theta) \propto$ 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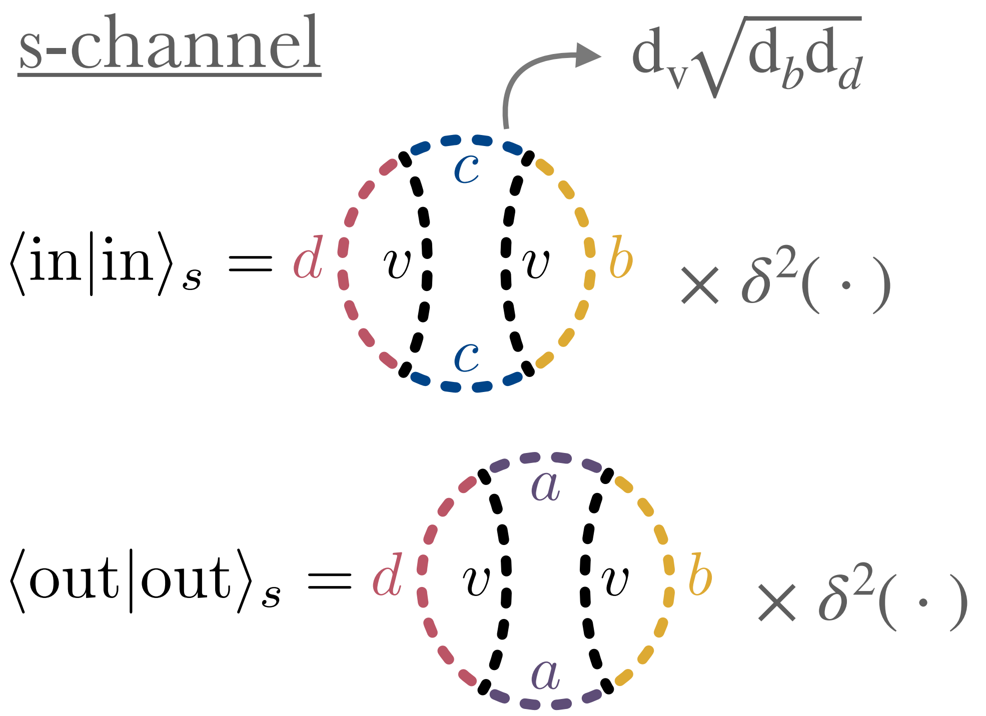


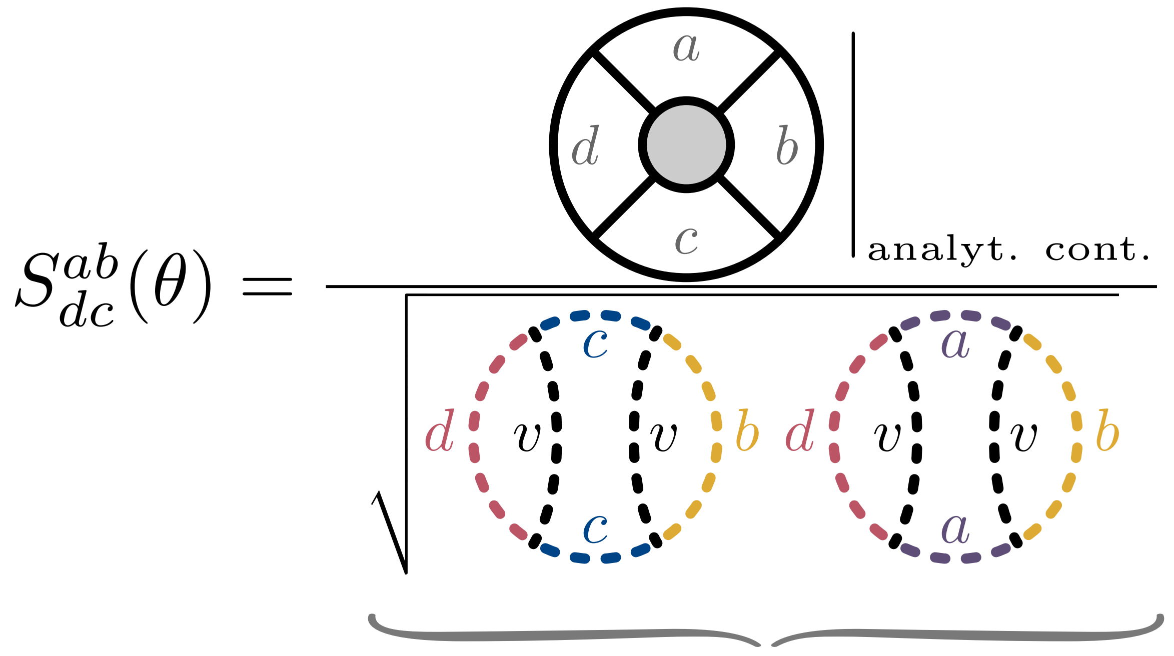


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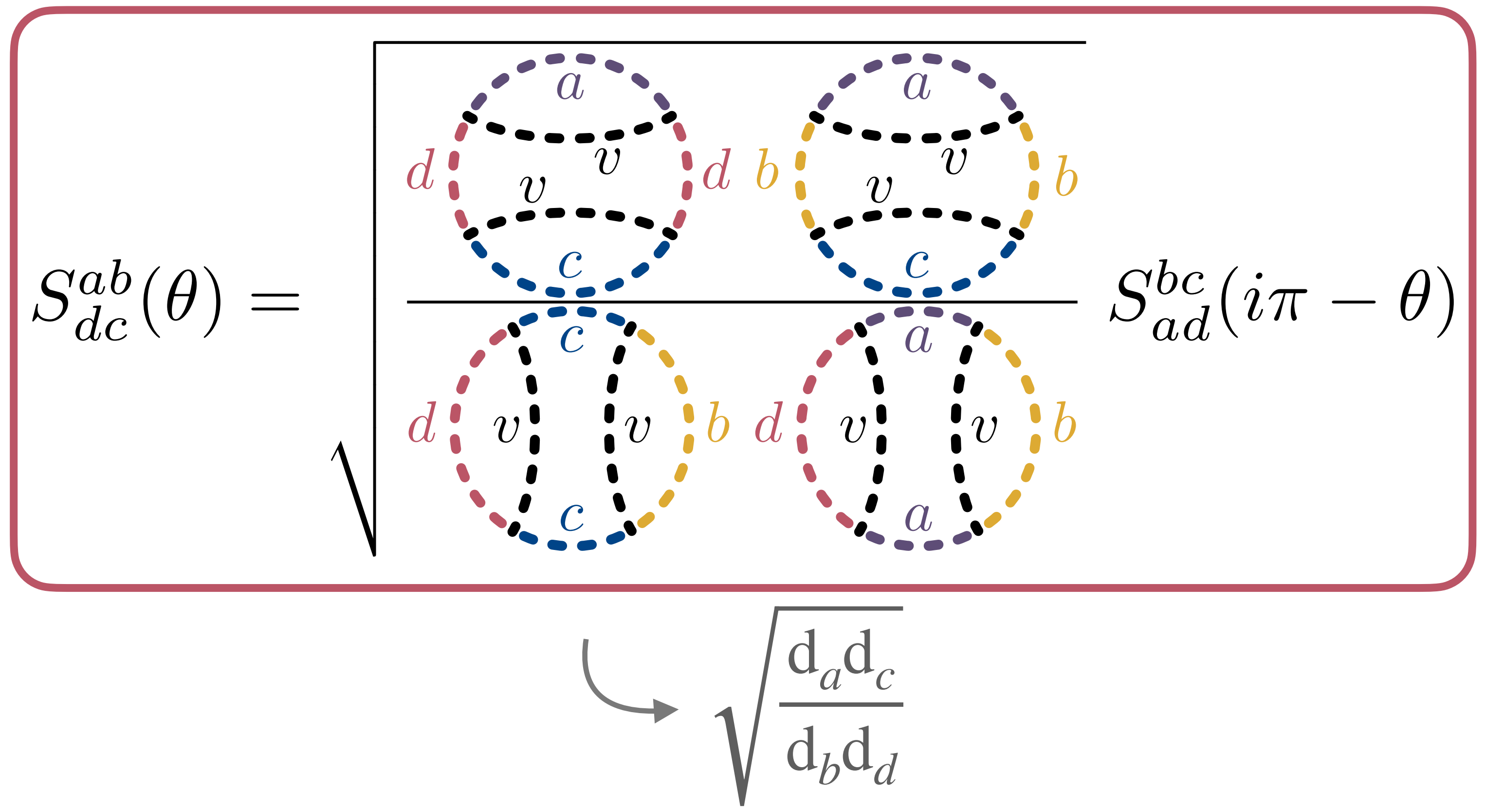

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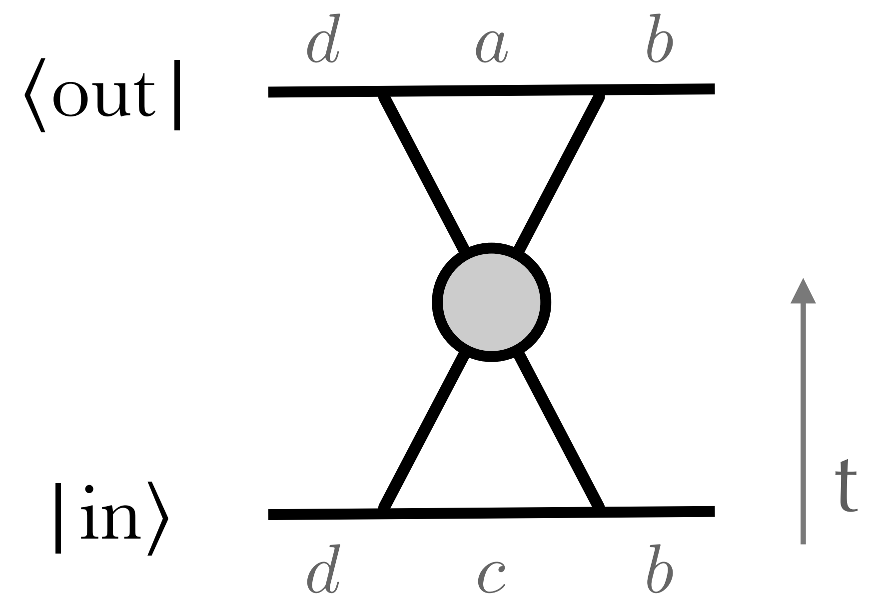


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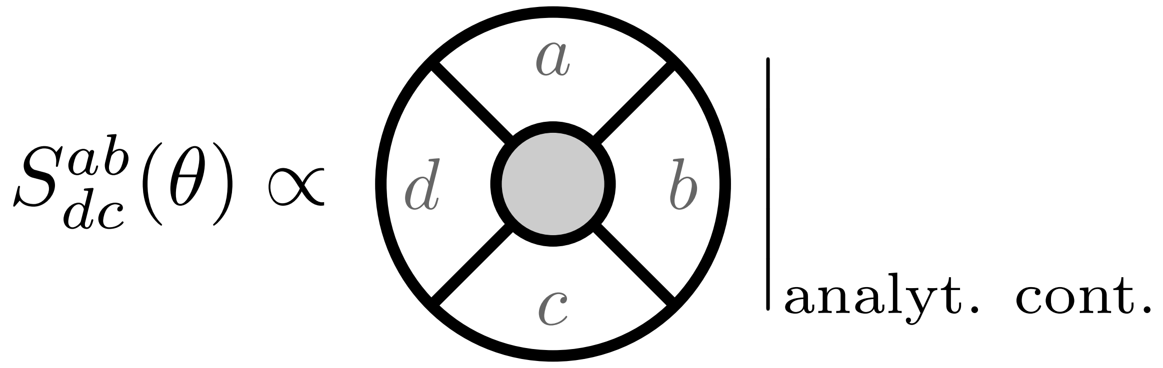


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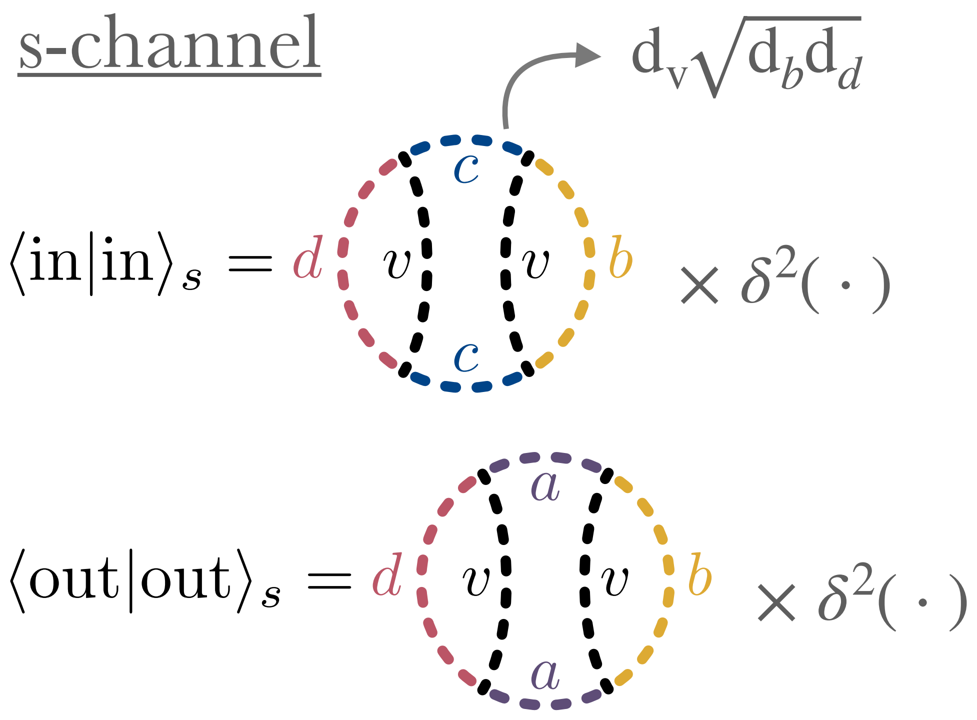


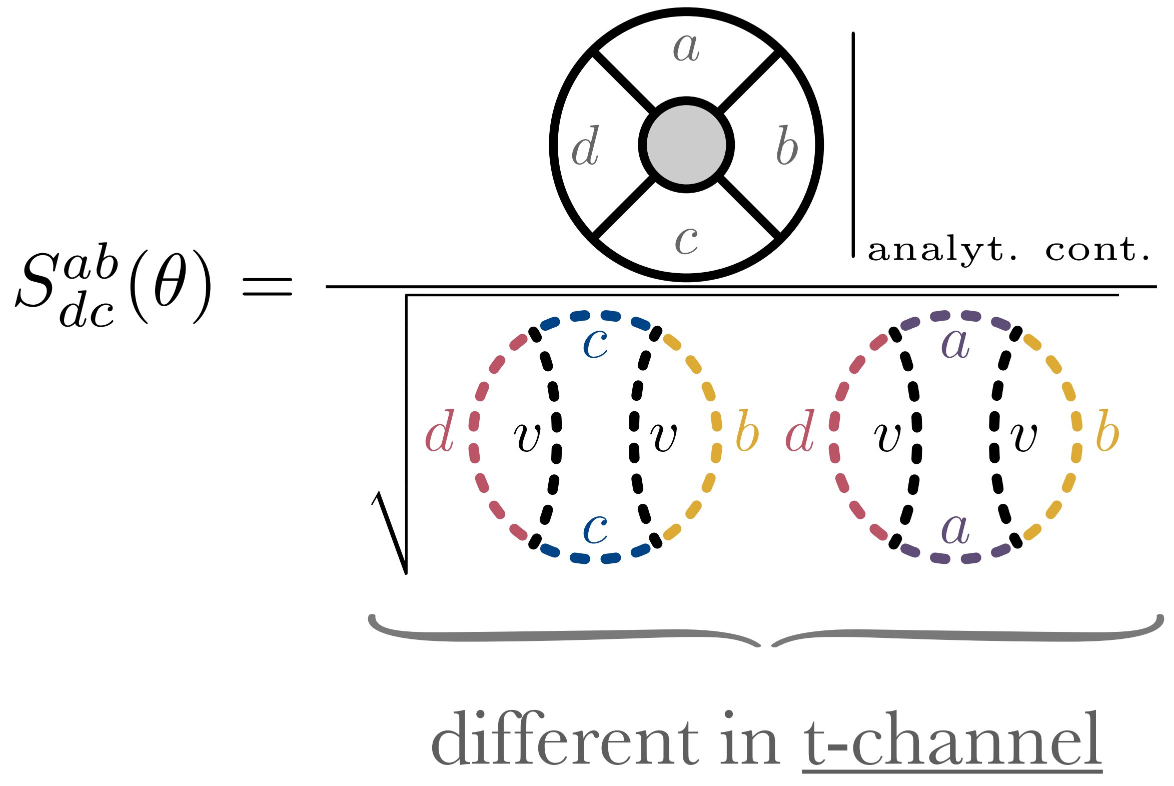


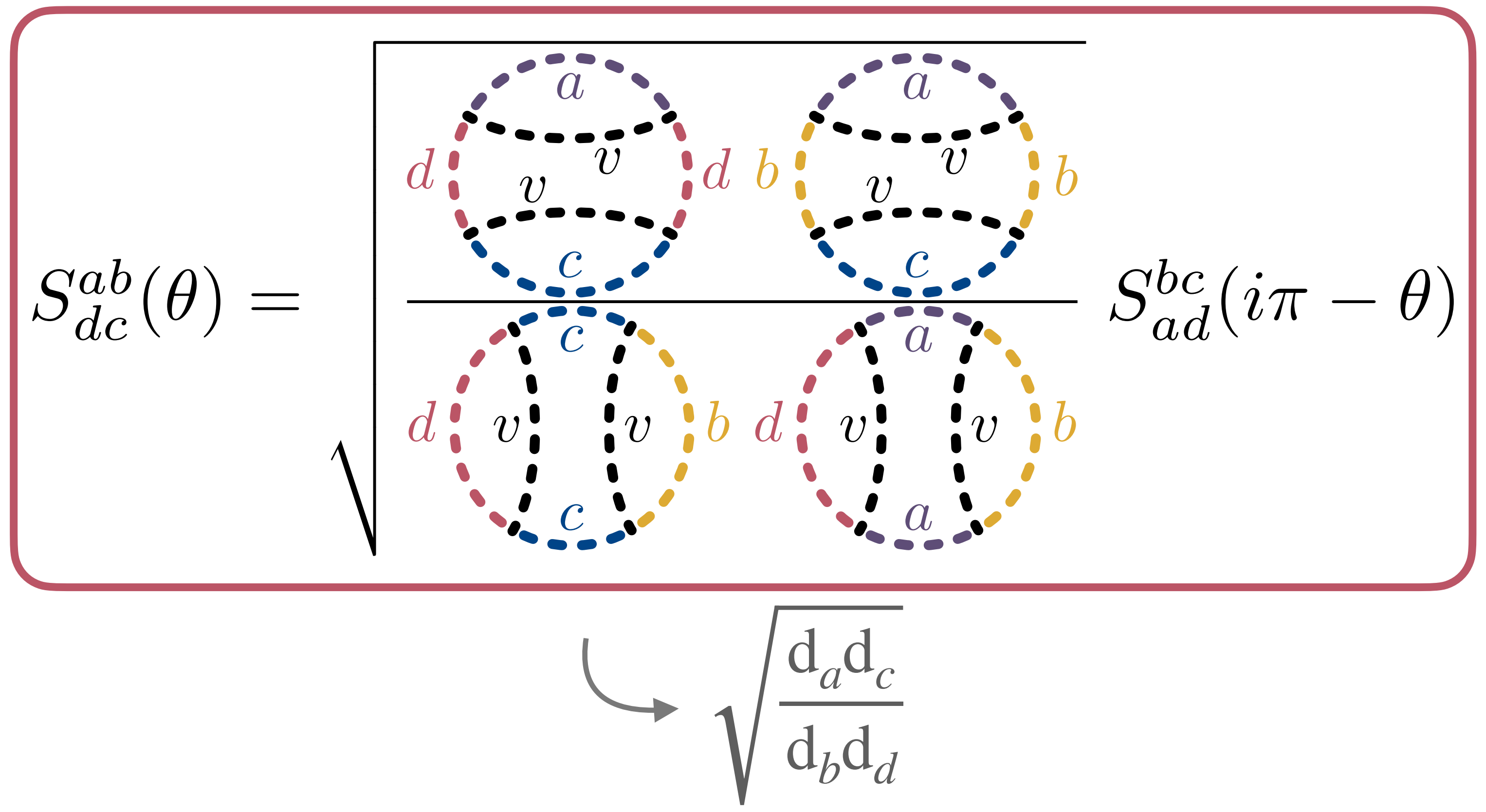


- Modification from: non-trivial IR TQFT + scattering non-local objects *("dressing" with symmetry line $v$)*.

# S-matrix Bootstrap &
# Non-invertible Symmetries

# S-matrix Bootstrap

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Find the space of consistent $2 \to 2$ kink scattering amplitudes $S_{dc}^{ab}(s)$ with a given fusion category

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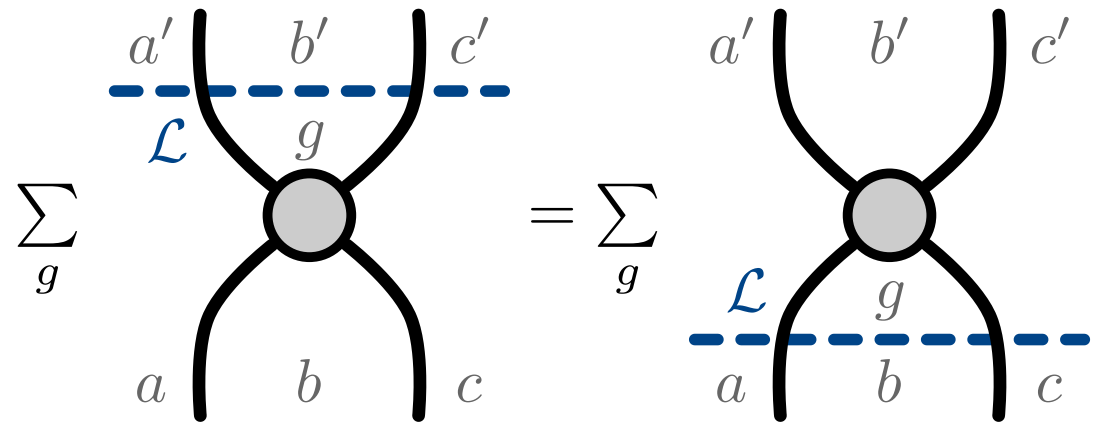

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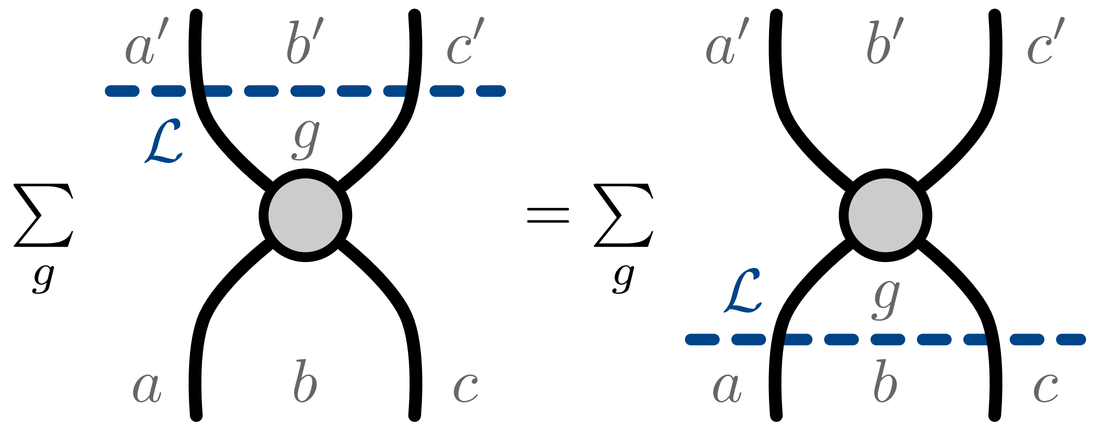


$$S^{ab}_{dc}(s) = \sum_{\chi} A_{\chi}(s)(P_{\chi})^{ab}_{dc}$$

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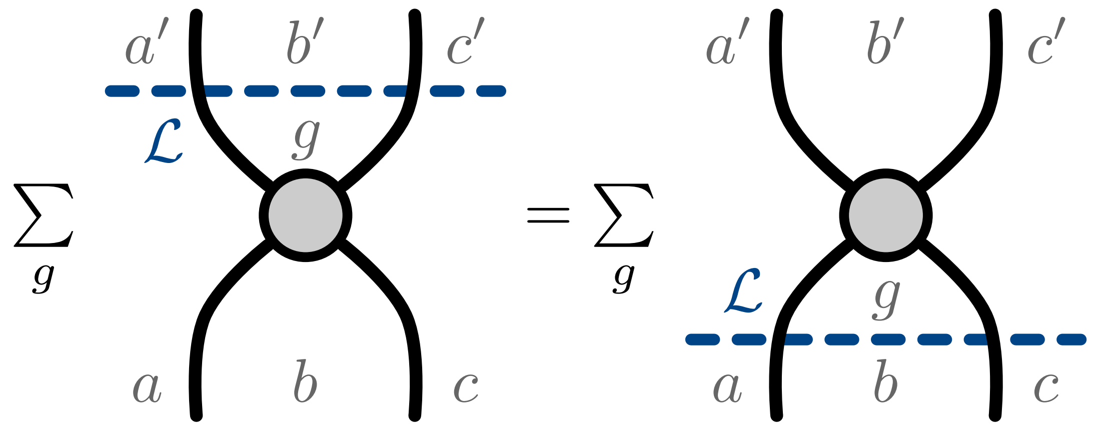
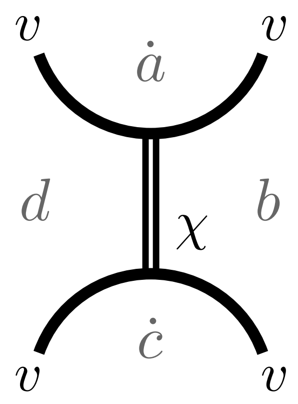
*SSB all syms → regular representation*




$$S^{ab}_{dc}(s) = \sum_{\chi} A_\chi(s)(P_\chi)^{ab}_{dc}$$

$(P_\chi)^{ab}_{dc} = $  $= \sqrt{\mathrm{d}_a\mathrm{d}_c}\,\mathrm{d}_\chi \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$



[see also Aasen, Fendley, Mong '20]

*e.g.* $\mathscr{A}_n$ *category in* $\mathscr{M}_n - \phi_{1,3}$

$$(P_0)^{ab}_{dc} = \frac{1}{\mathrm{d}_v}\sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\delta_{bd} \quad (P_1)^{ab}_{dc} = \delta_{ac} - \frac{1}{\mathrm{d}_v}\sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\delta_{bd}$$

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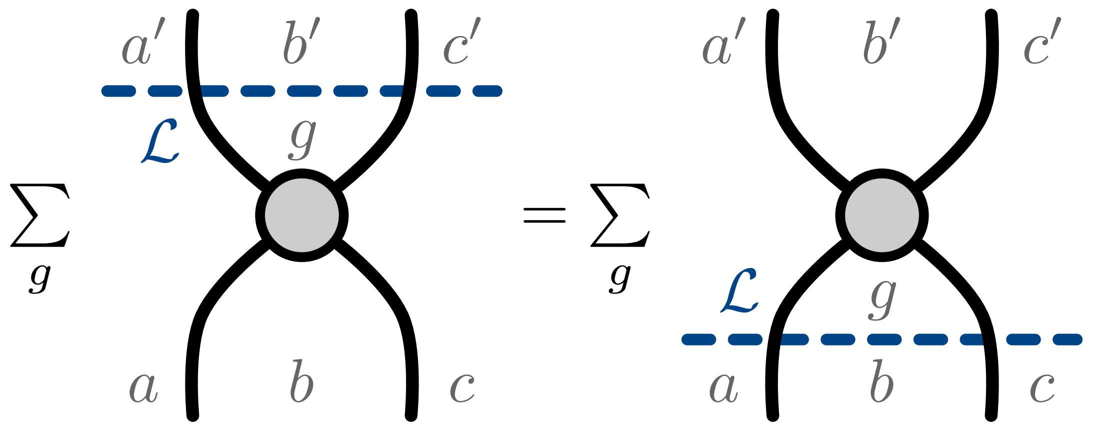


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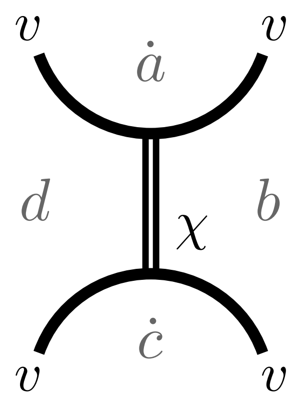
$(P_\chi)_{dc}^{ab} = d$  $b = \sqrt{\mathrm{d}_a \mathrm{d}_c}\, \mathrm{d}_\chi \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$



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*e.g.* $\mathscr{A}_n$ *category in* $\mathscr{M}_n - \phi_{1,3}$

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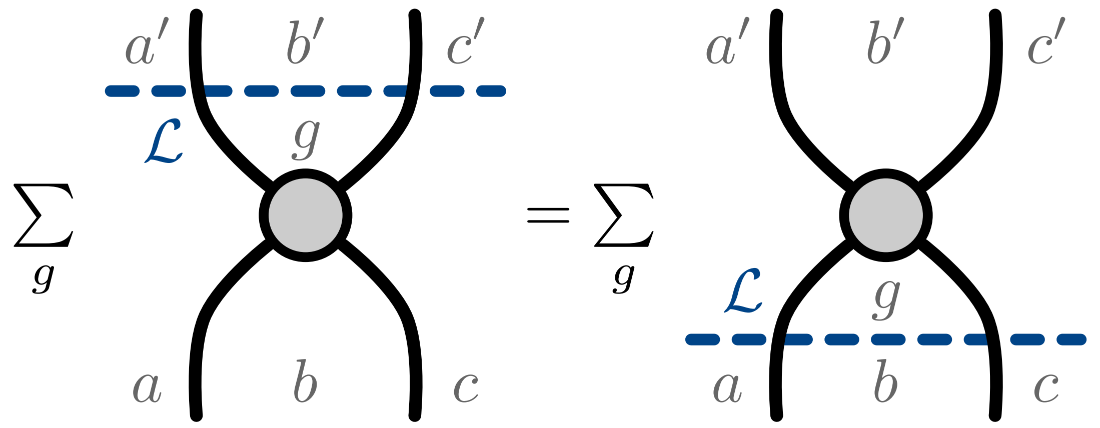


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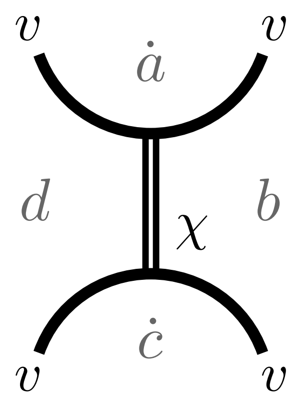
$(P_\chi)^{ab}_{dc} = d$ 

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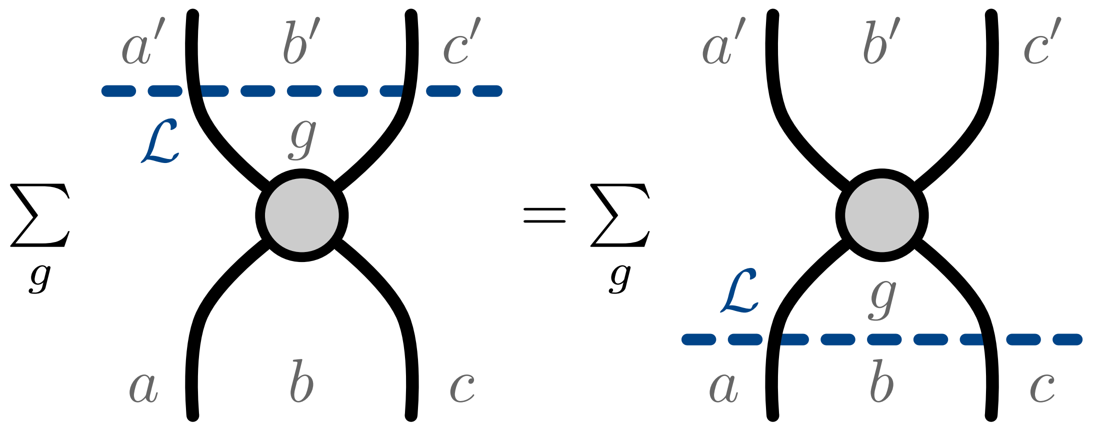


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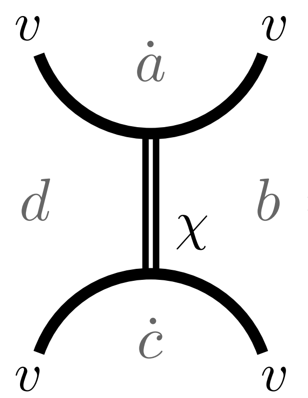
$(P_\chi)^{ab}_{dc} = d$  $b = \sqrt{\mathrm{d}_a \mathrm{d}_c}\,\mathrm{d}_\chi \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$



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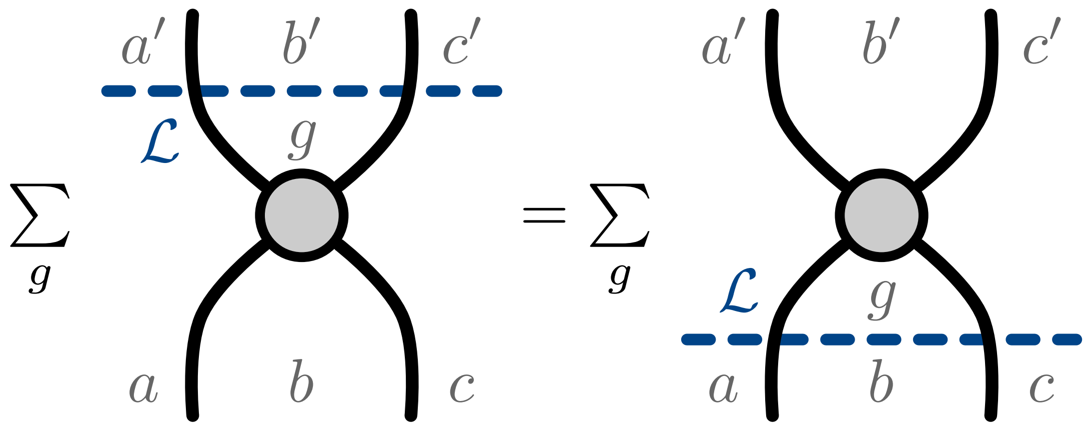


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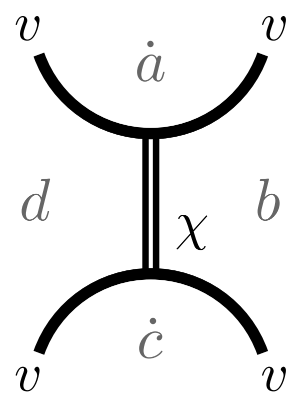
$(P_{\chi})^{ab}_{dc} = d$  $b = \sqrt{\mathrm{d}_a \mathrm{d}_c}\, \mathrm{d}_{\chi} \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$



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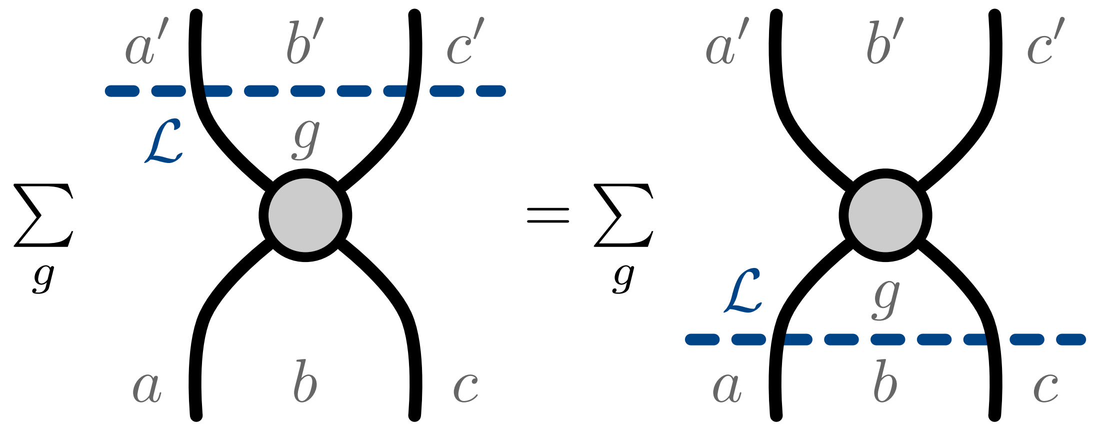


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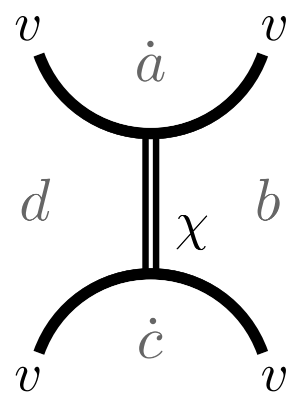
$(P_\chi)^{ab}_{dc} = d$  $b = \sqrt{\mathrm{d}_a \mathrm{d}_c}\, \mathrm{d}_\chi \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$



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1. Write ansatz that trivializes **A+mC+S**
2. Impose **U** numerically for $s_j \geq 4m^2$
3. Bound parameters, i.e. max functionals

*e.g.* $\mathscr{F}[A_\chi] = \sum_\chi n_\chi A_\chi(s_*)$

*or dual* $\mathscr{F}[A_\chi] \leq \mathscr{F}_d[K_\chi] = \int \sum_\chi |K_\chi(s)|$

# Examples: $\mathscr{A}_n$ and Fibonacci

$\mathcal{A}_n$

Fibonacci

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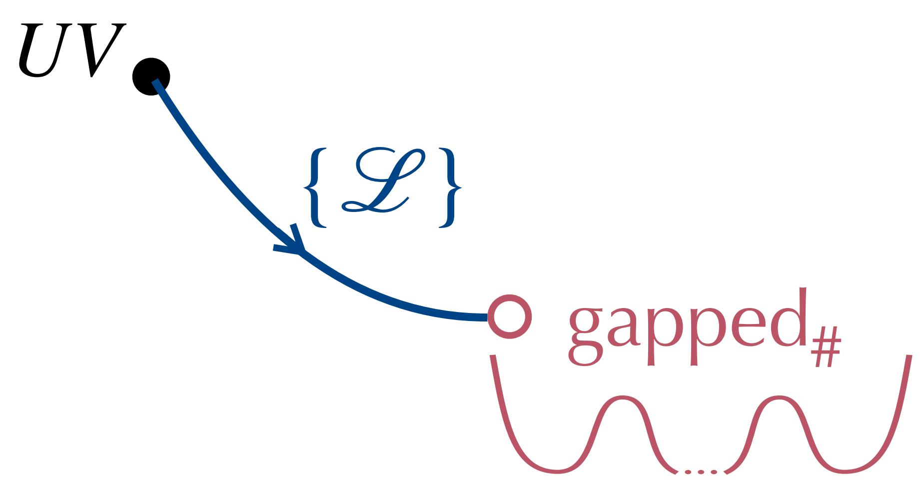


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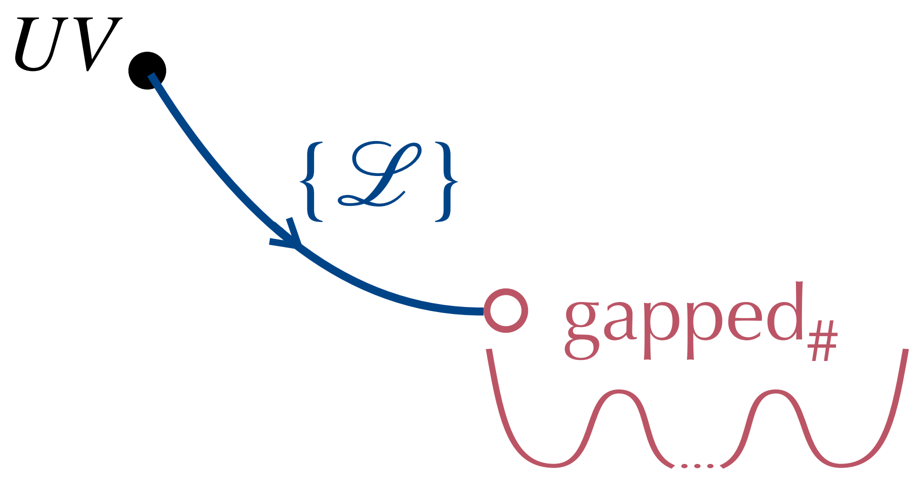


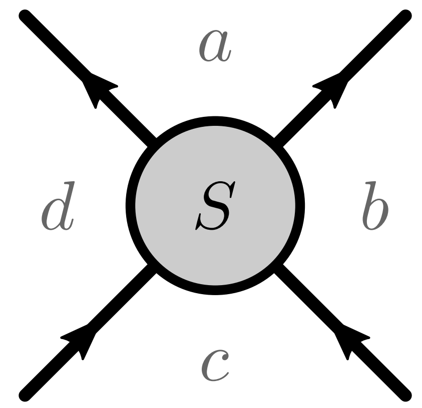


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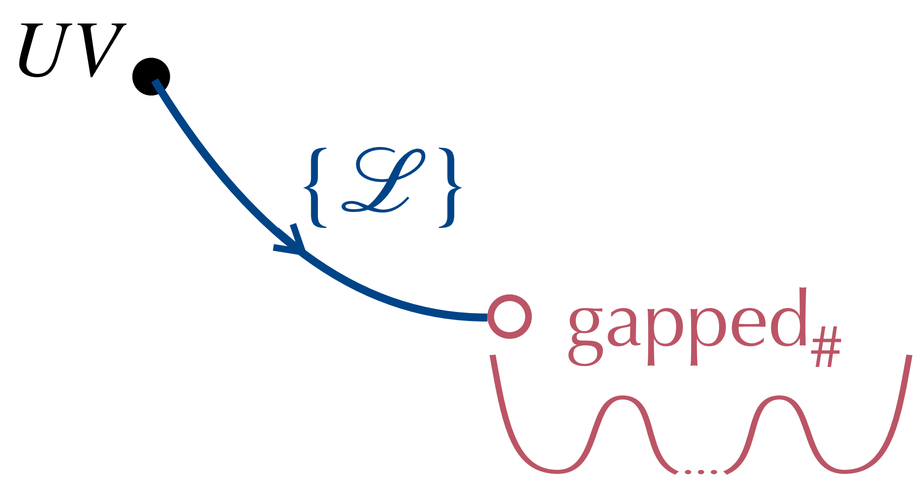


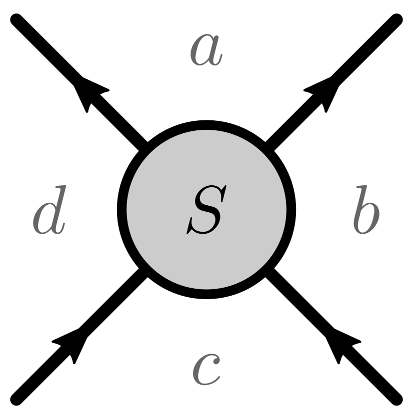


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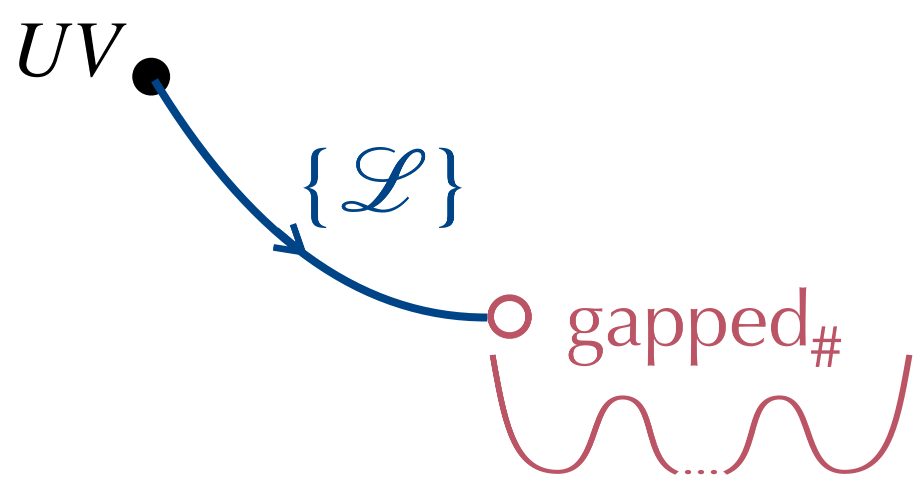


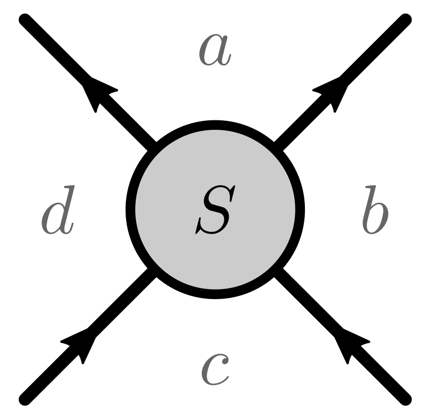


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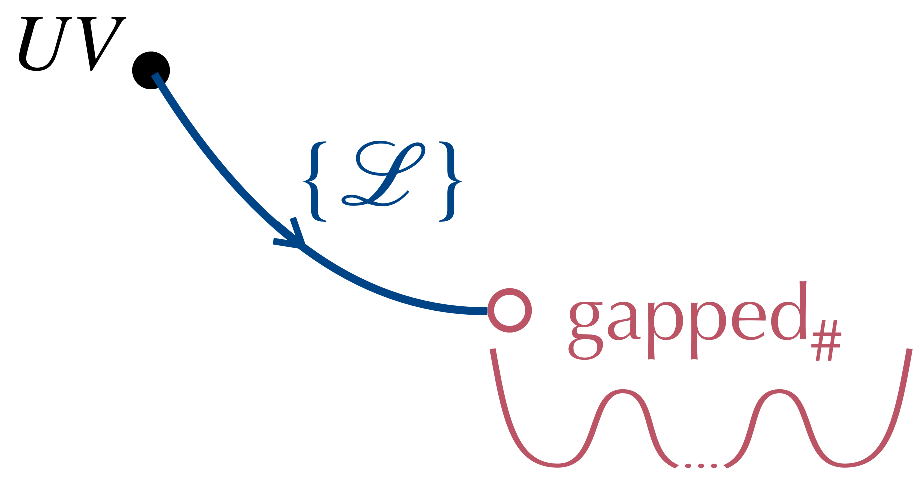


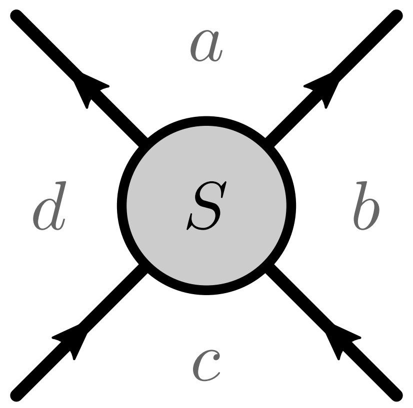


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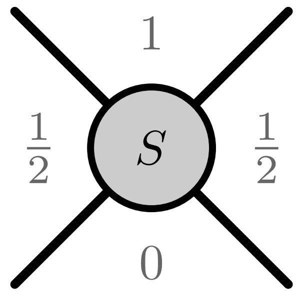


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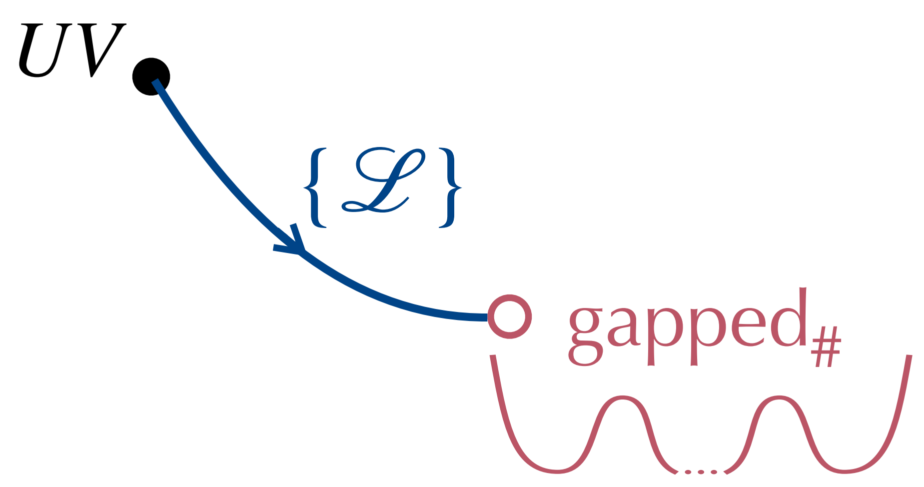


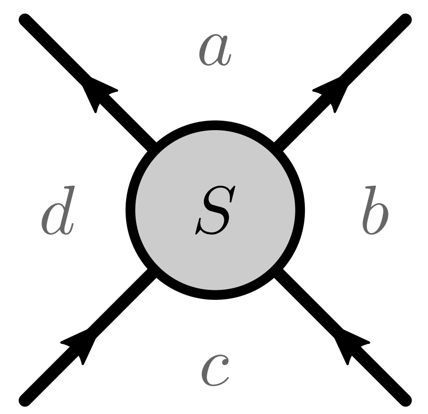


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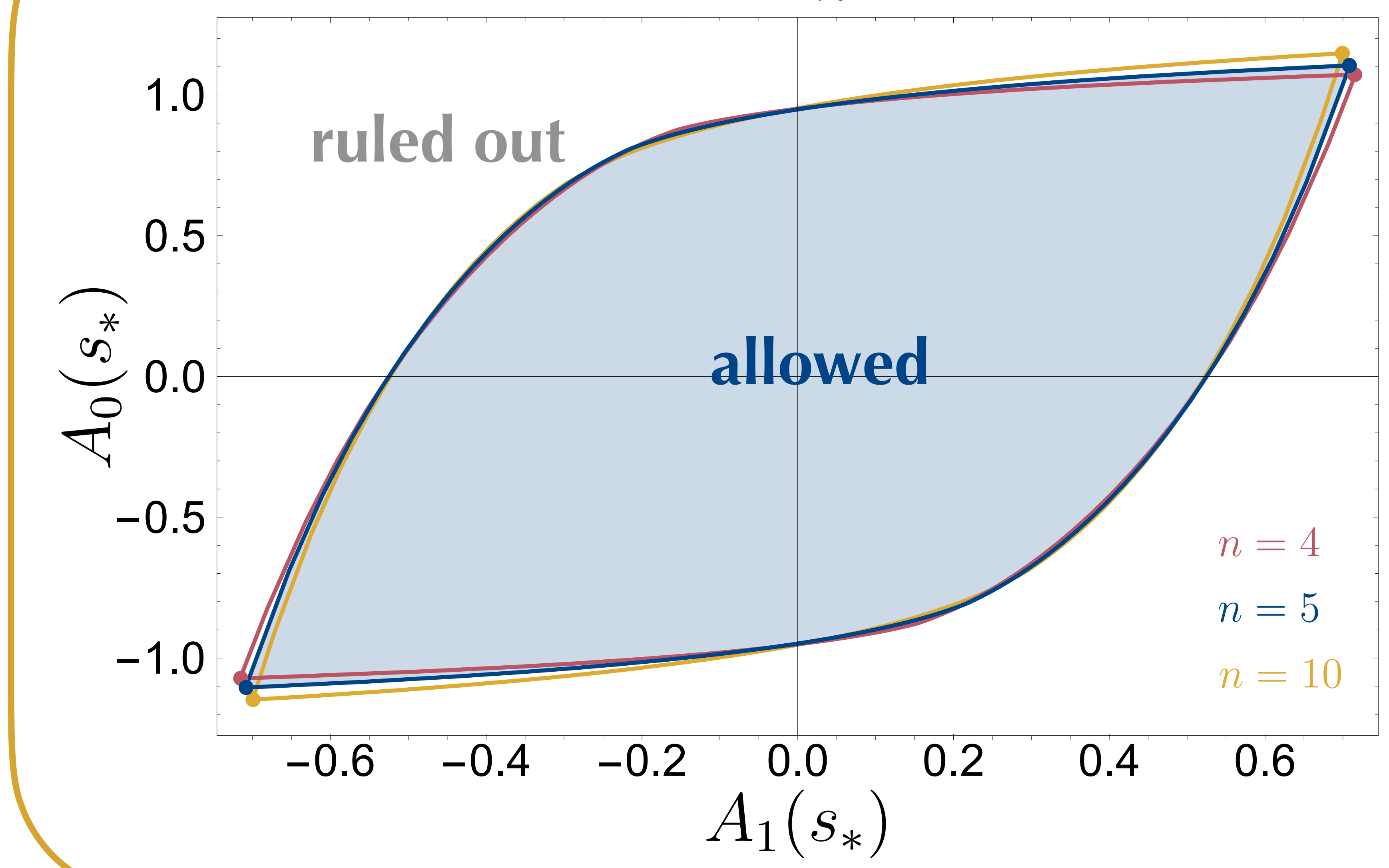


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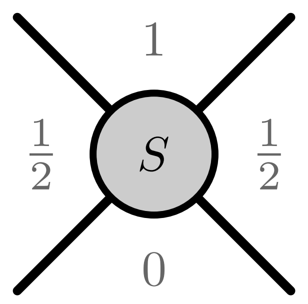


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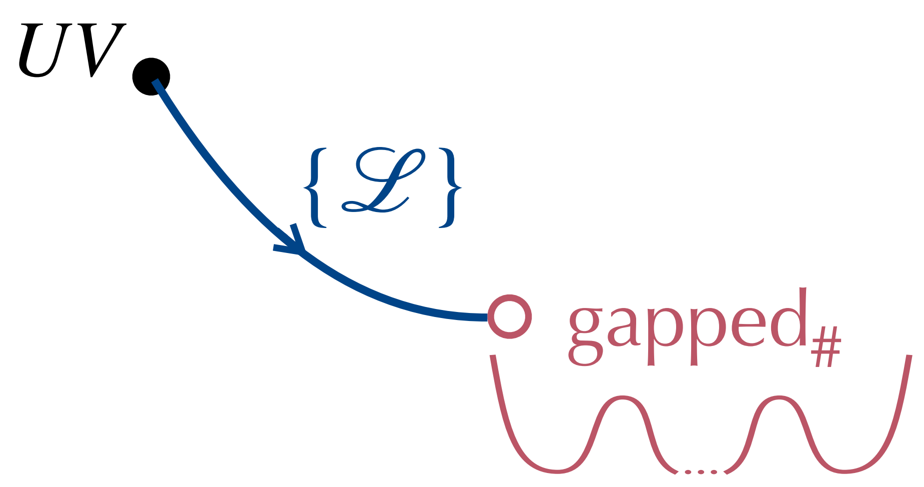


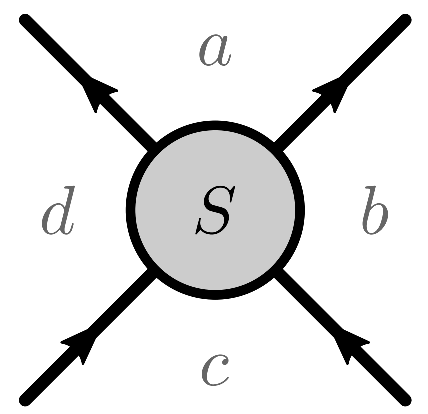


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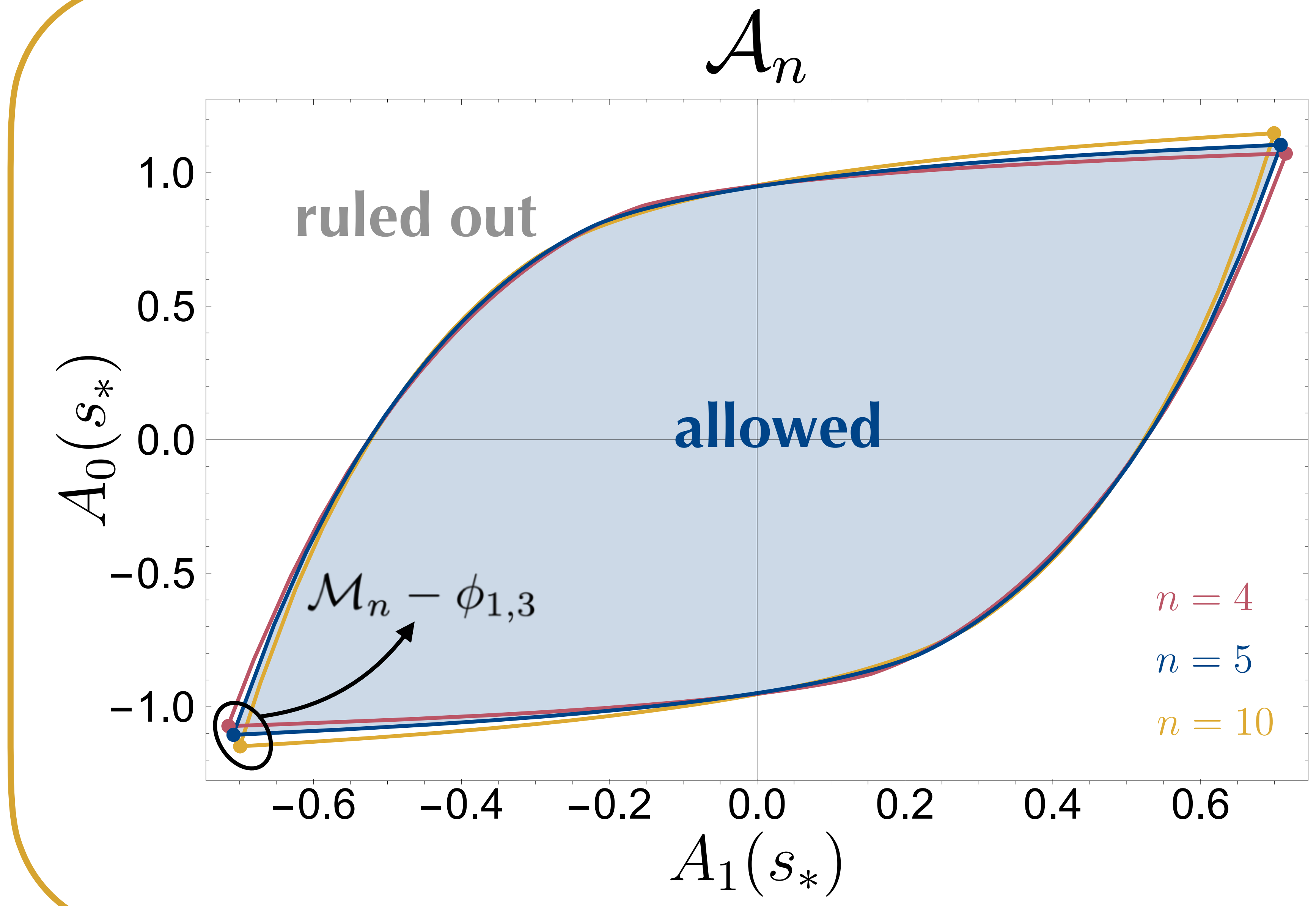


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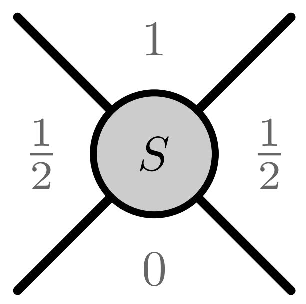


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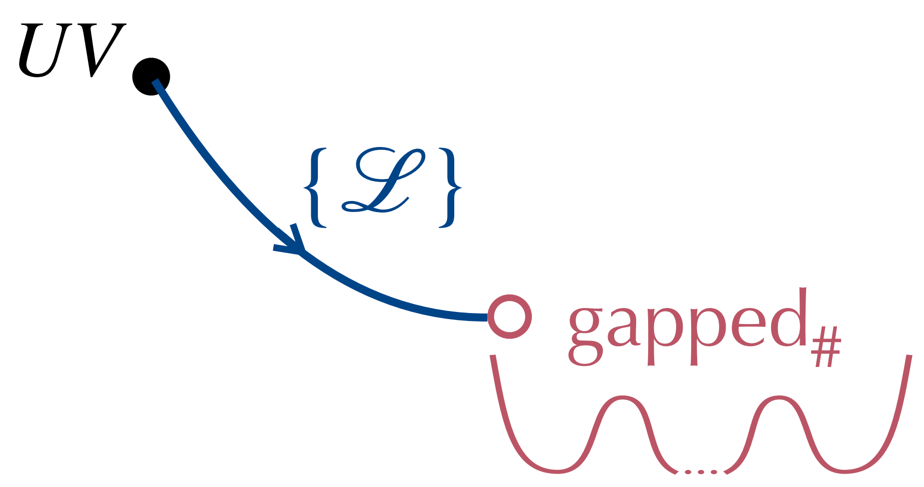


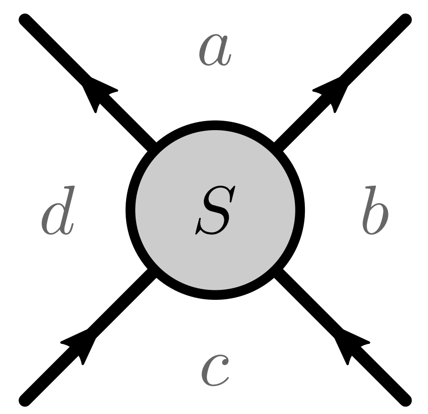


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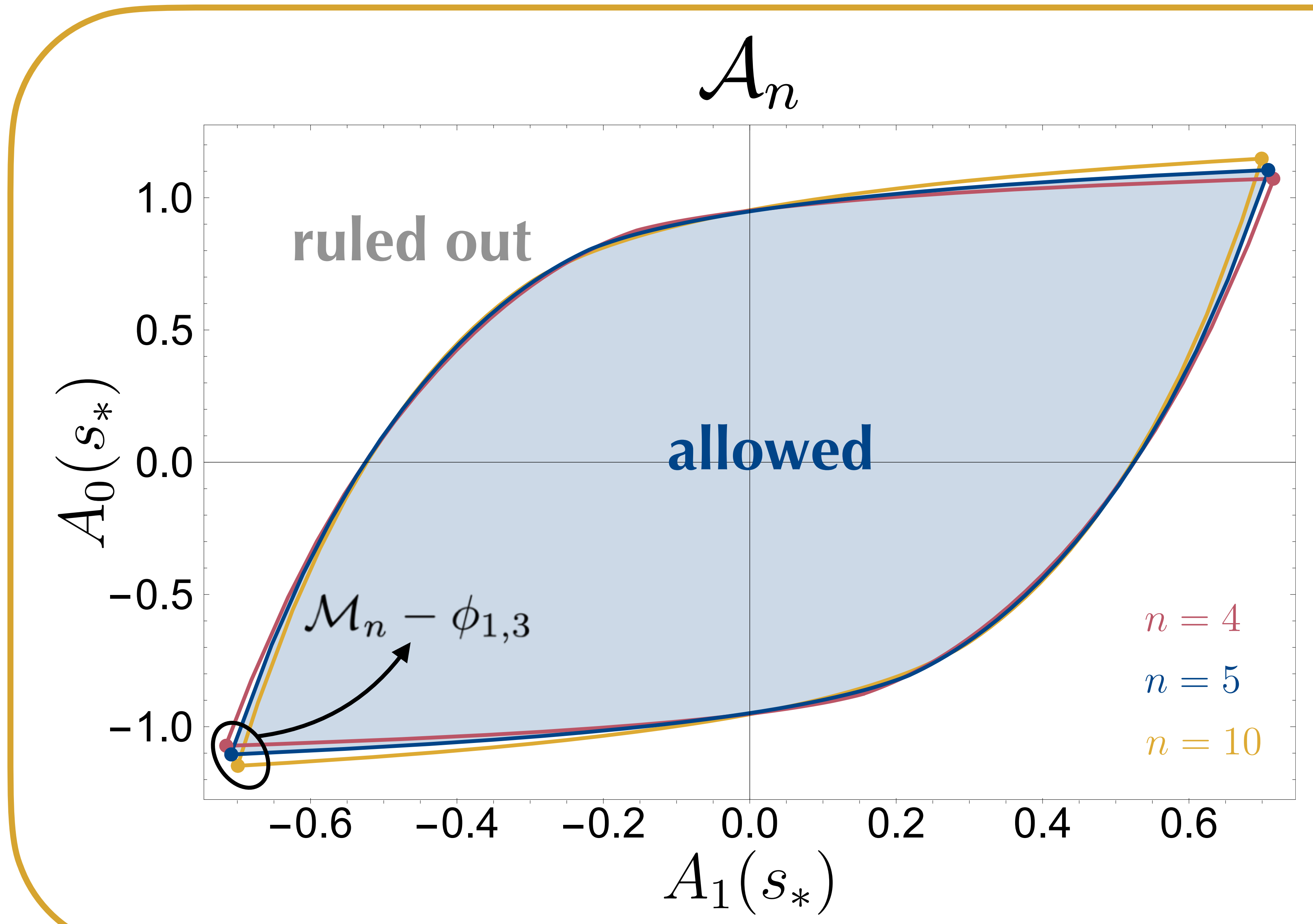


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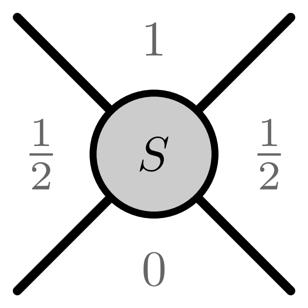


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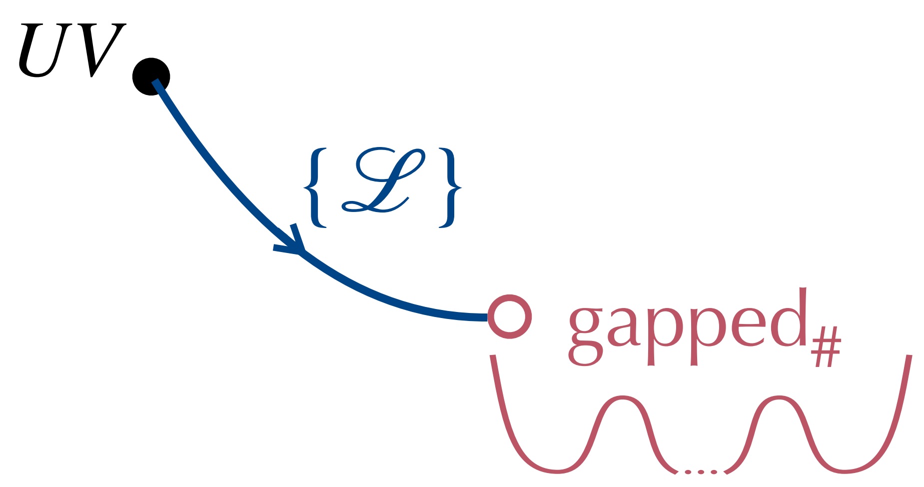


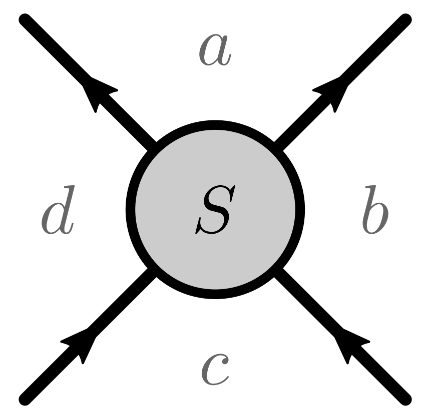


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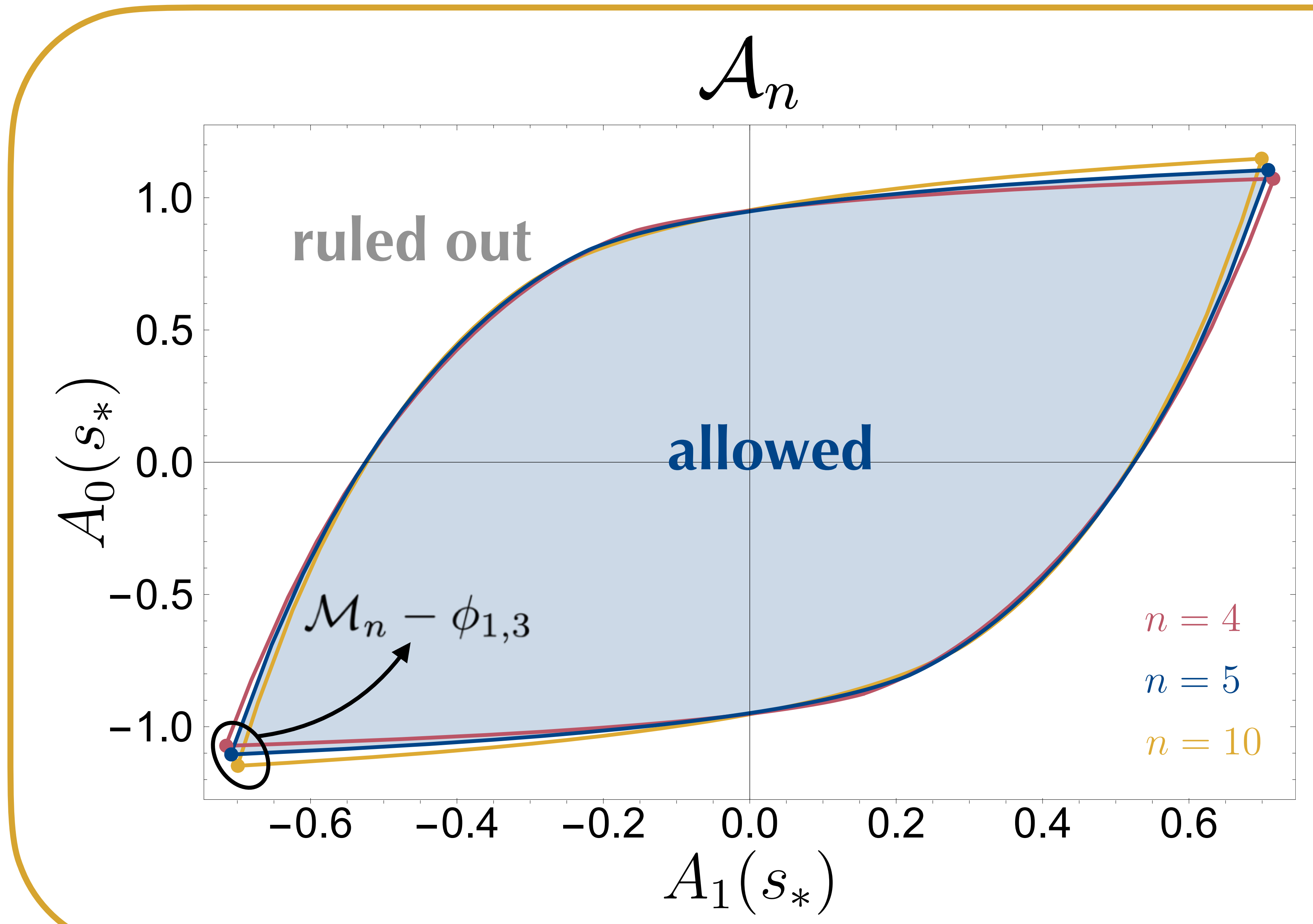


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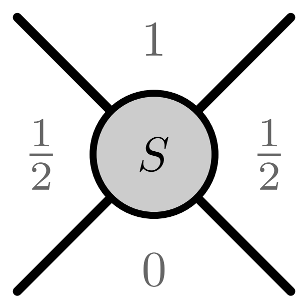


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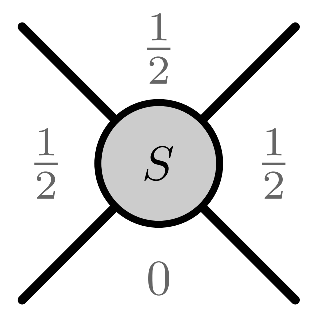


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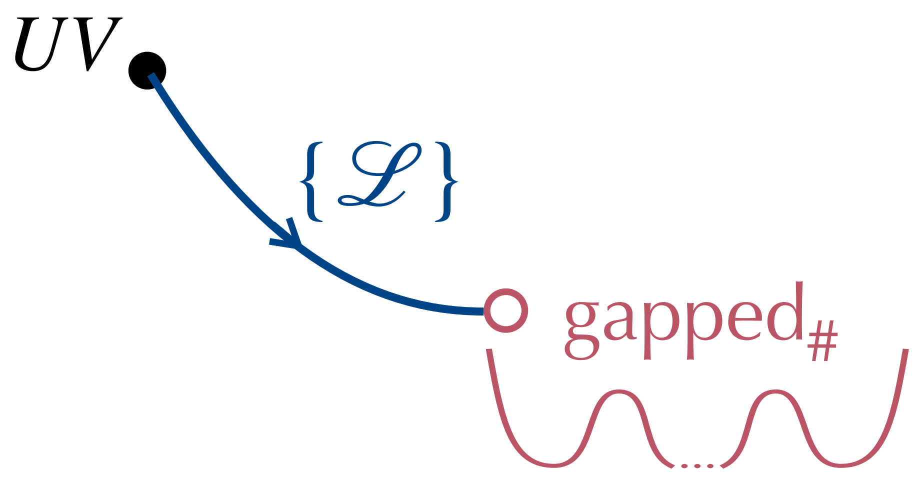


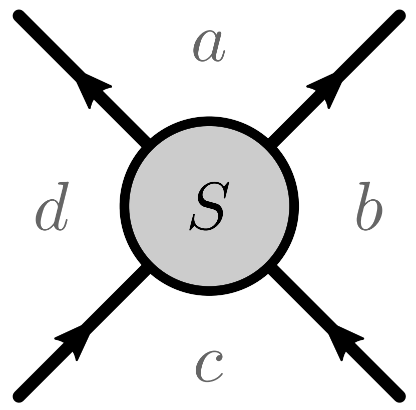


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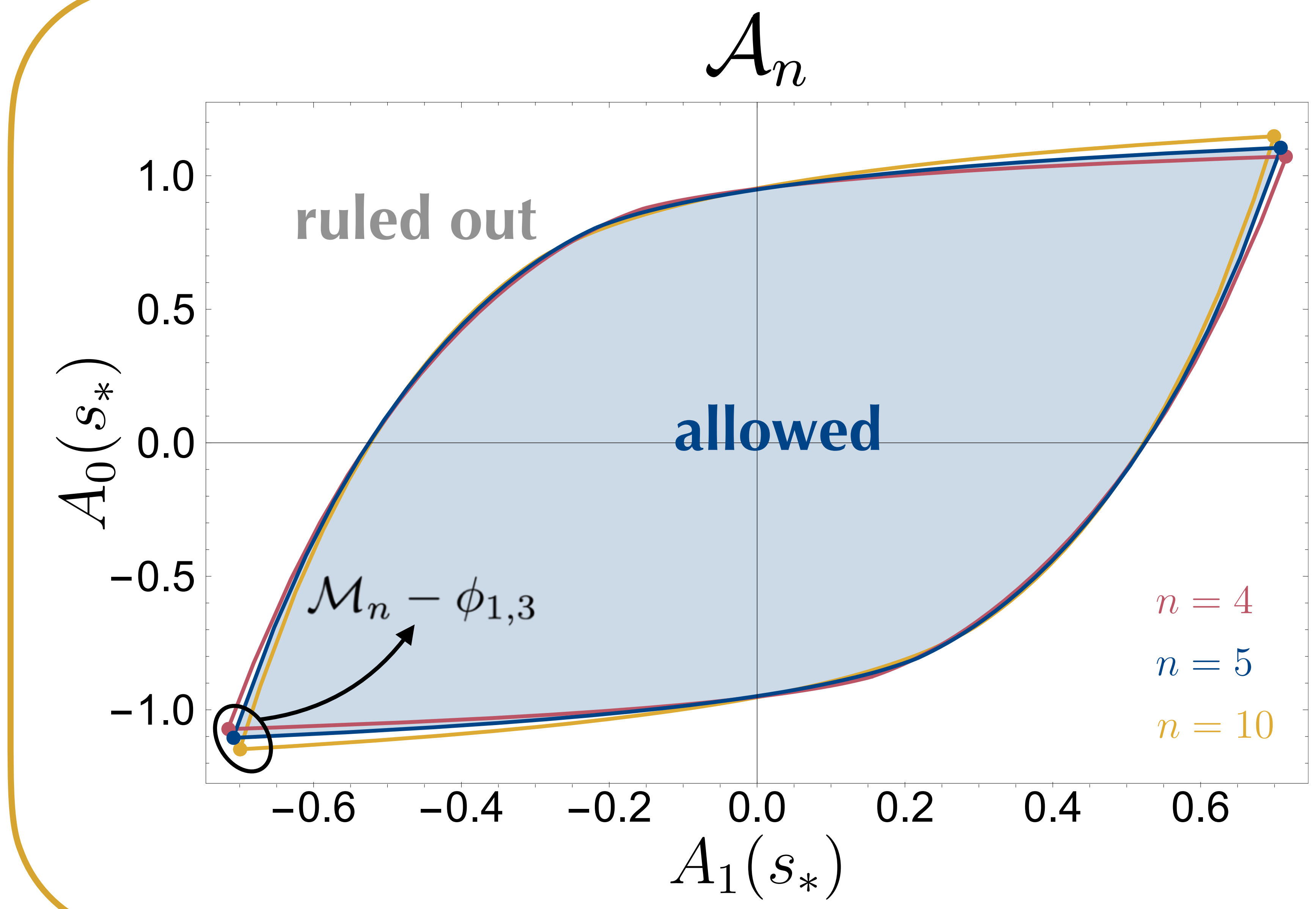


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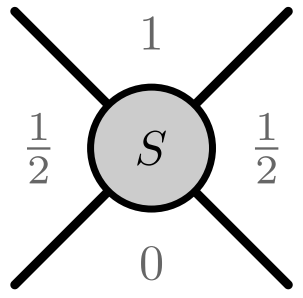


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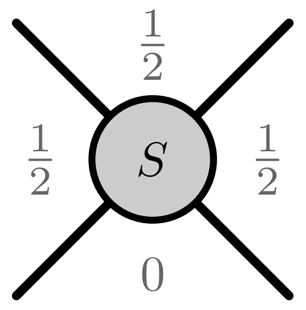


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# Examples: $\mathscr{A}_n$ and Fibonacci

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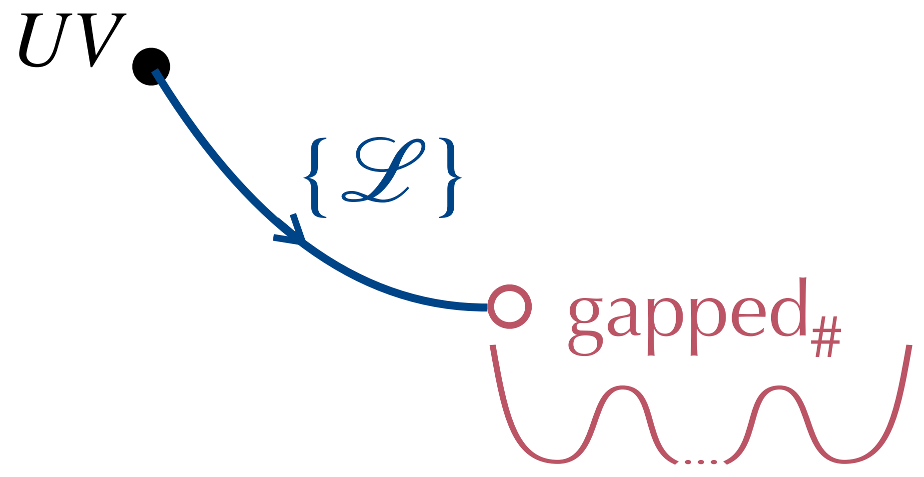


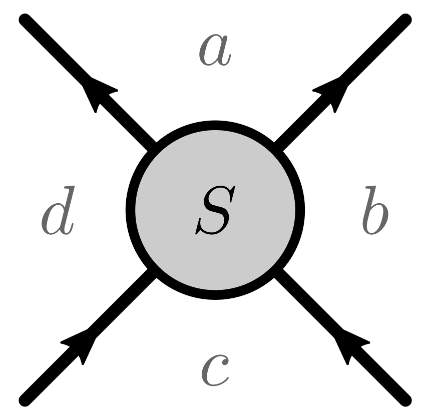


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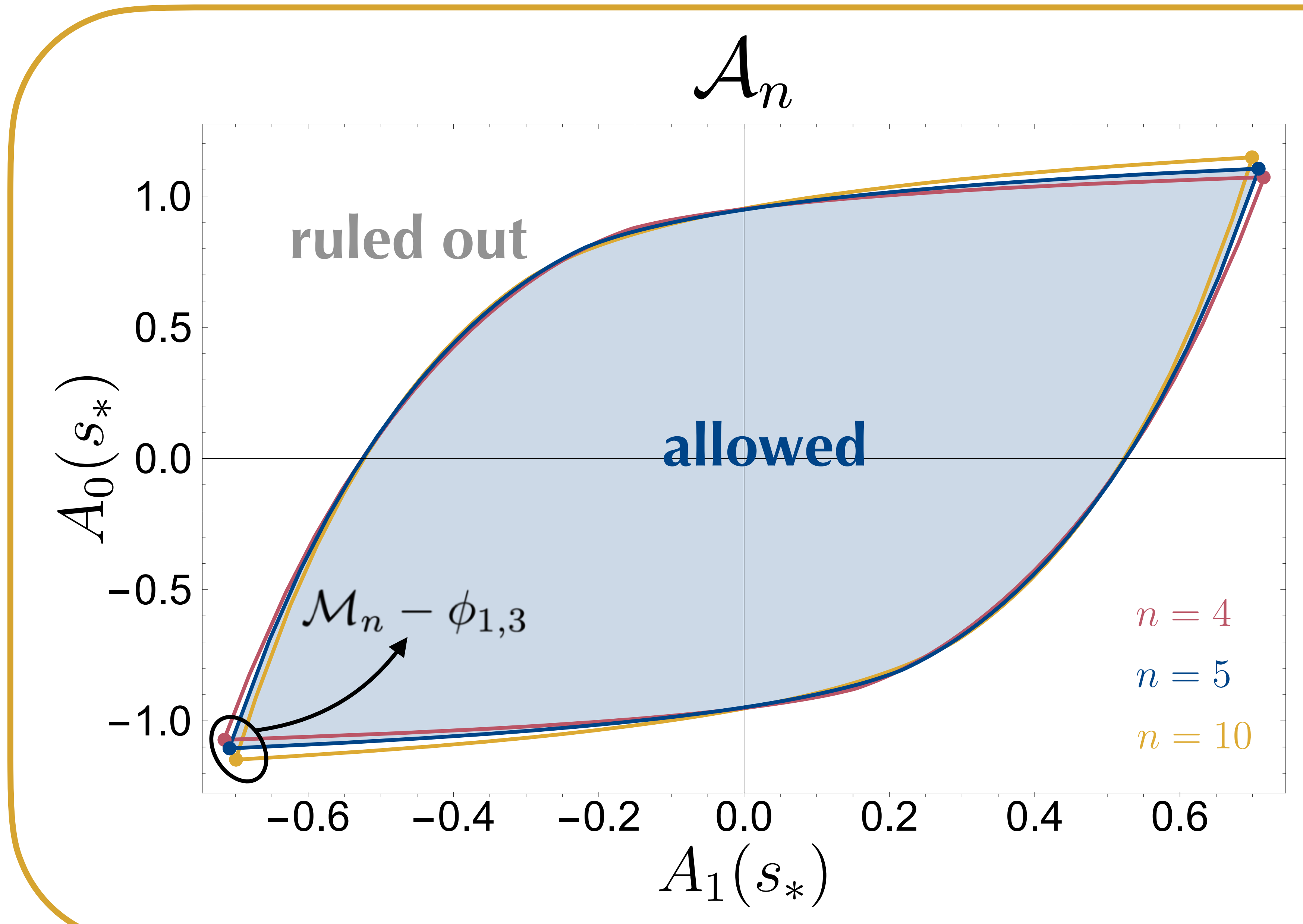


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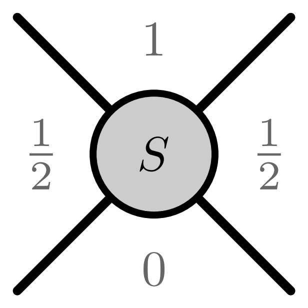


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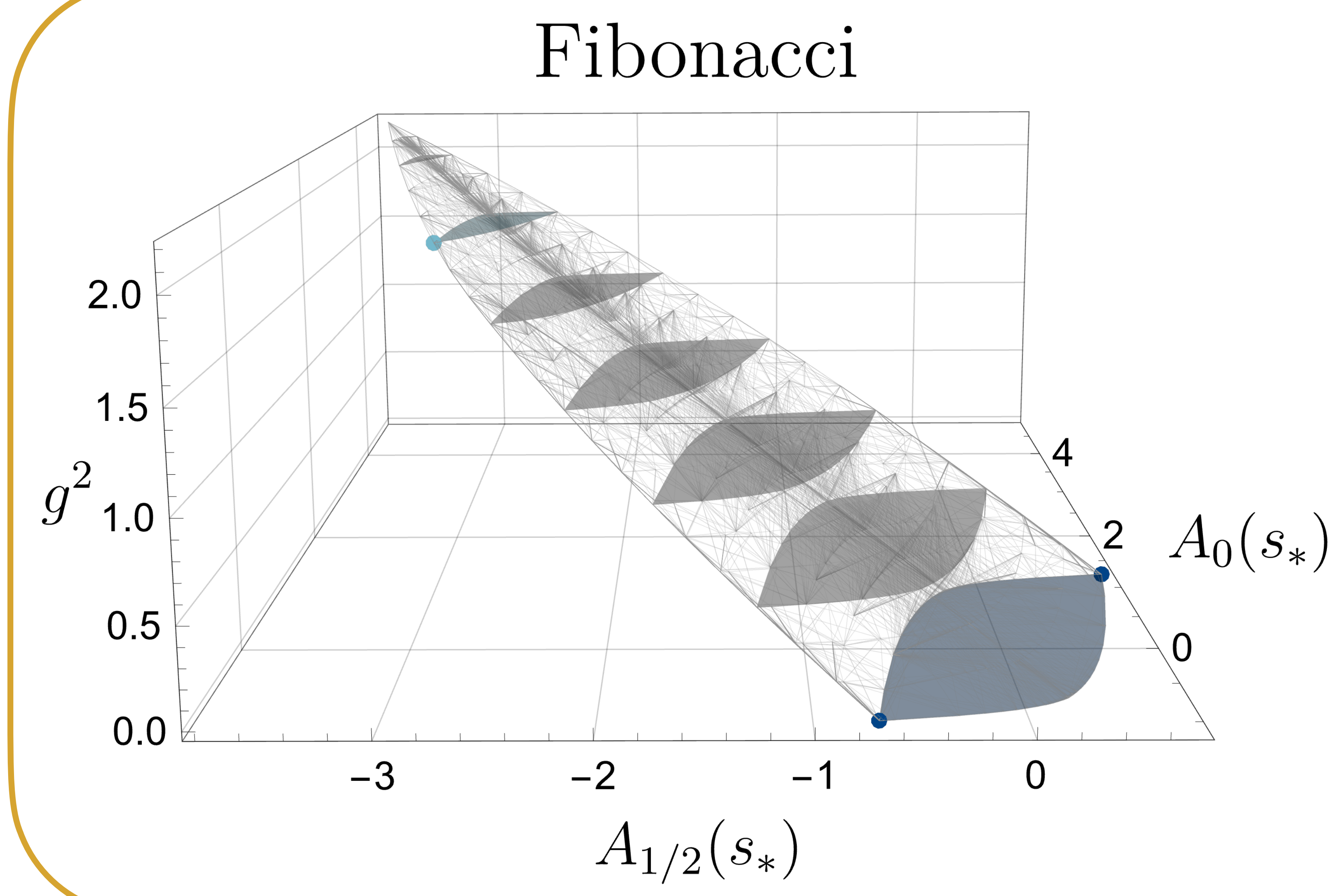


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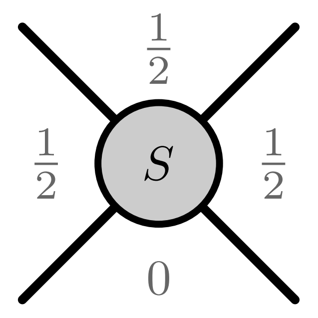


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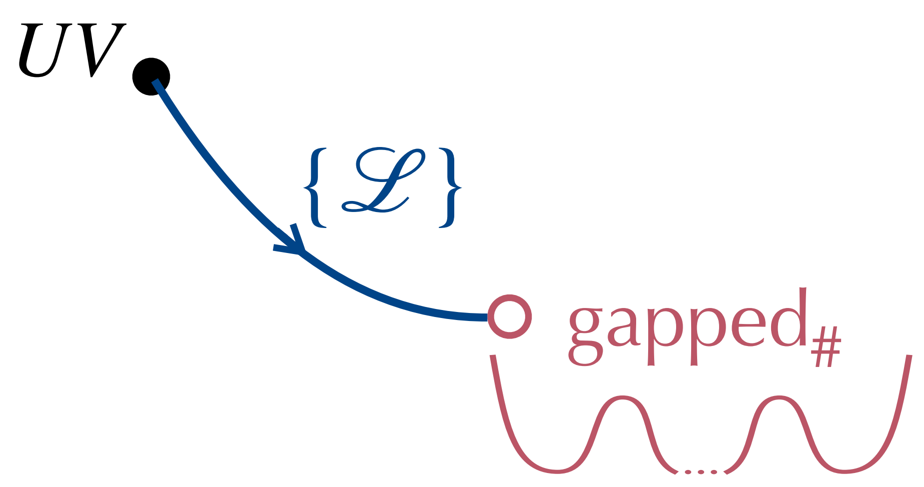


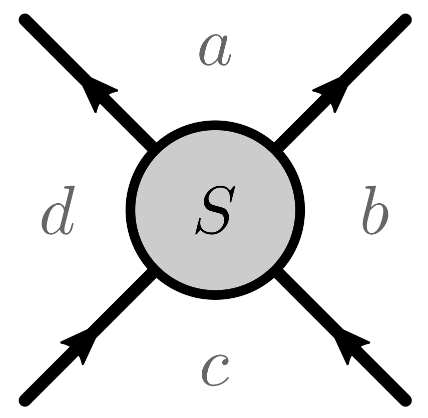


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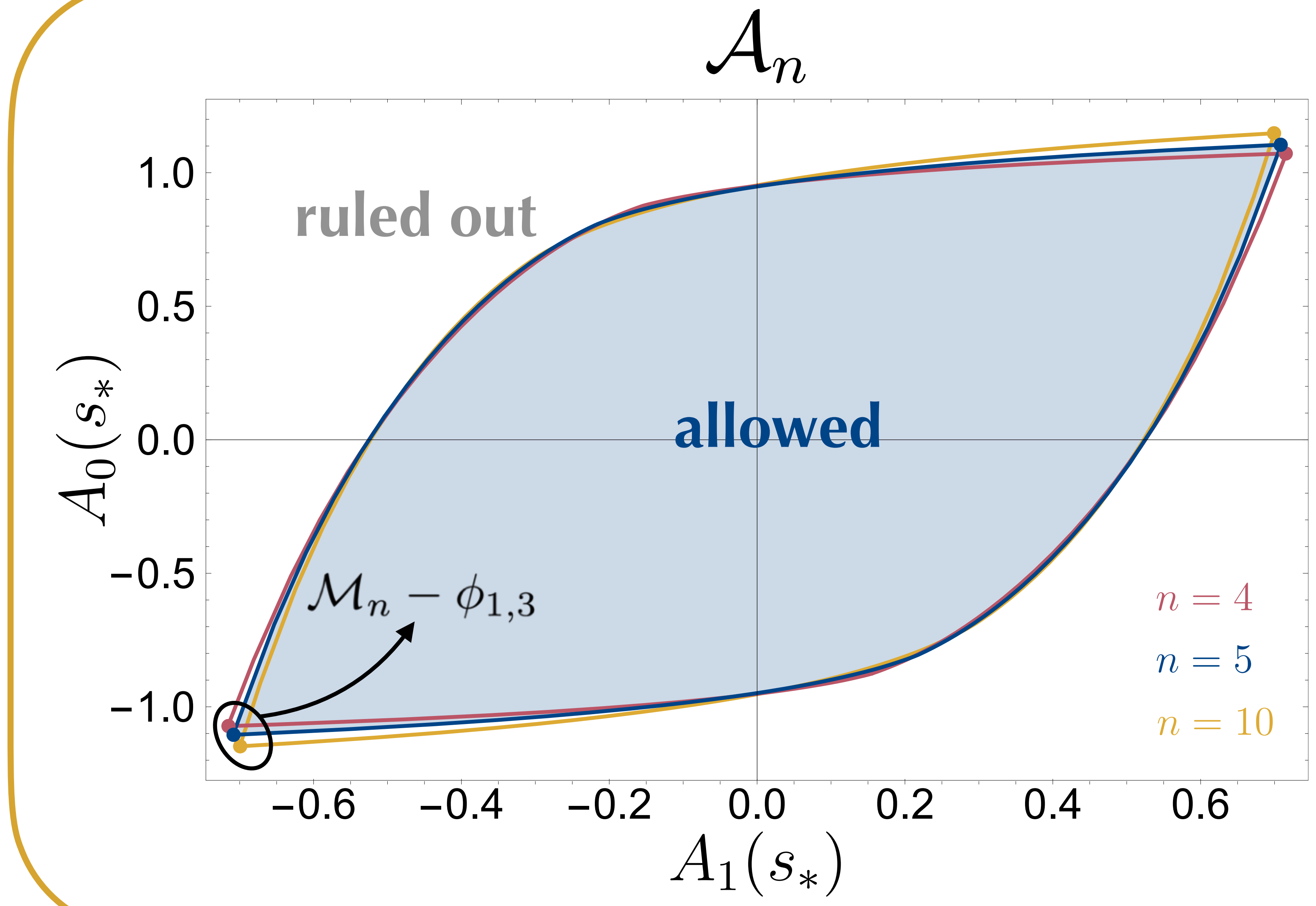


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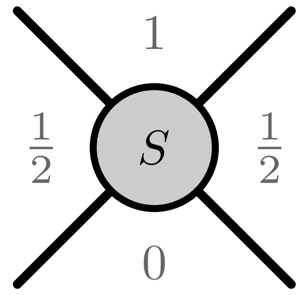


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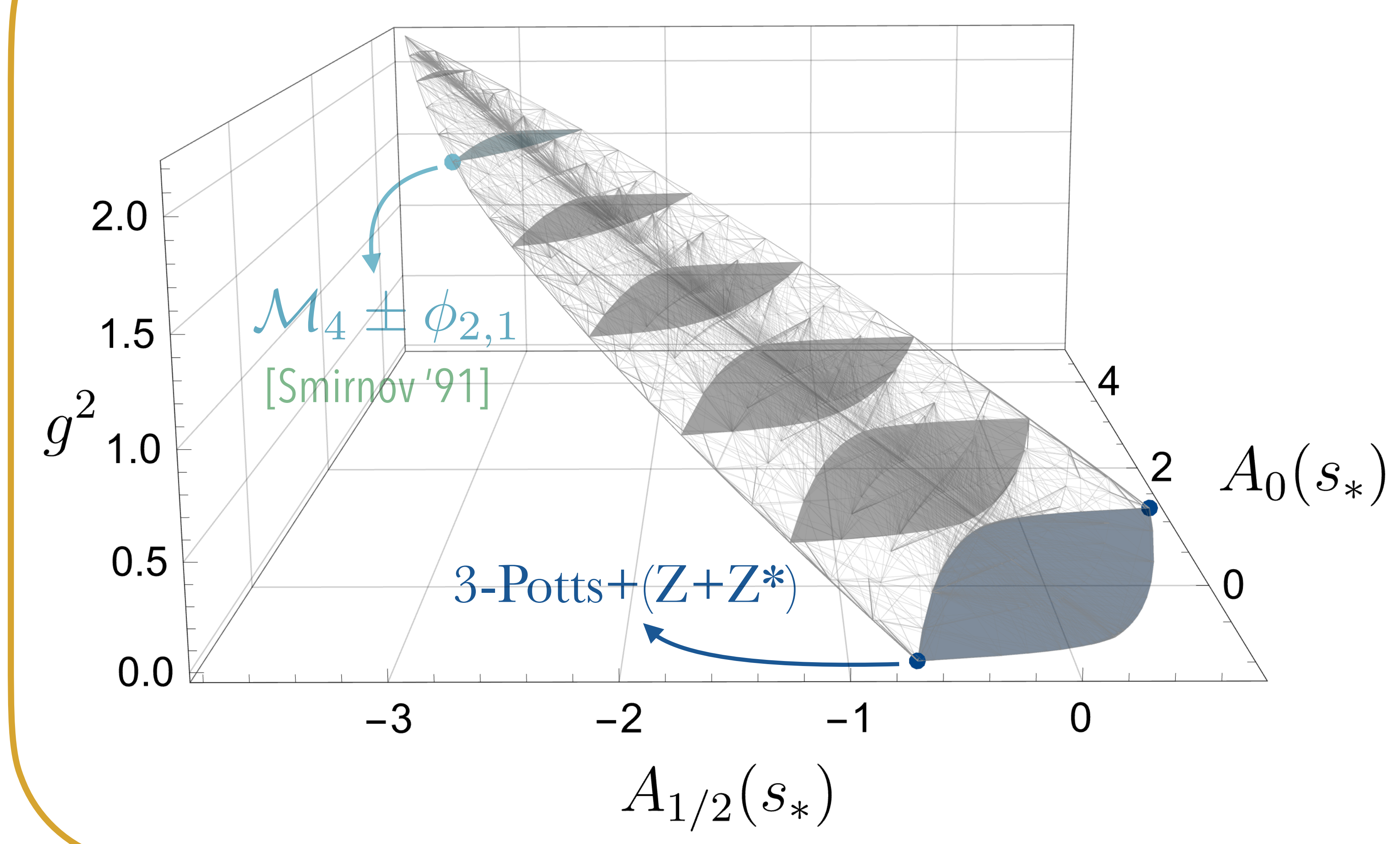


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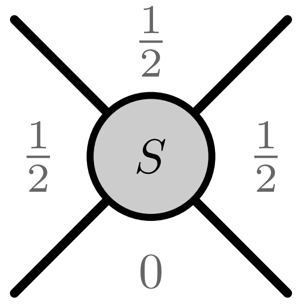


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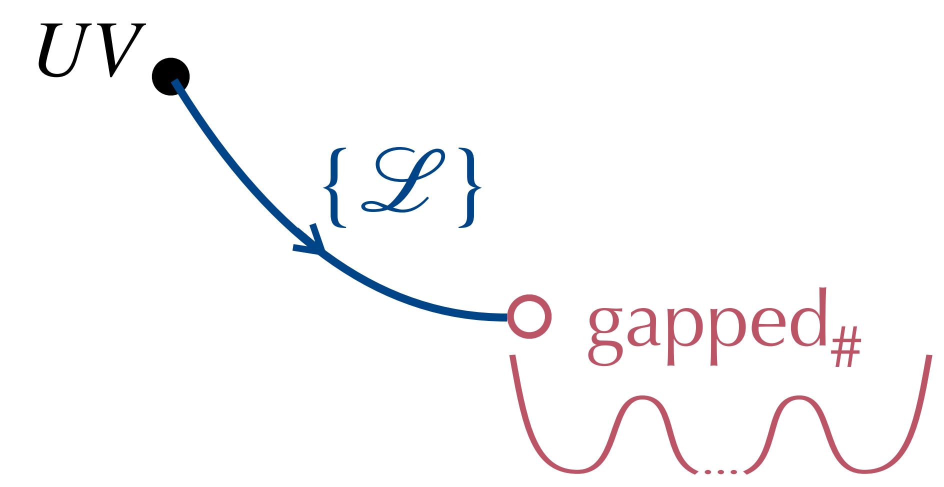


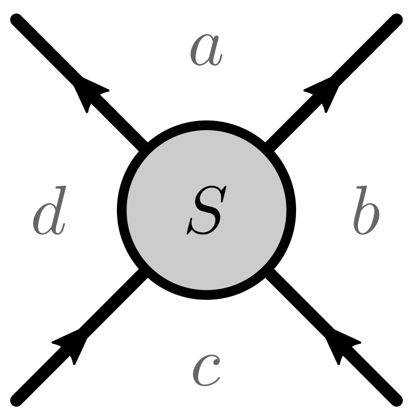


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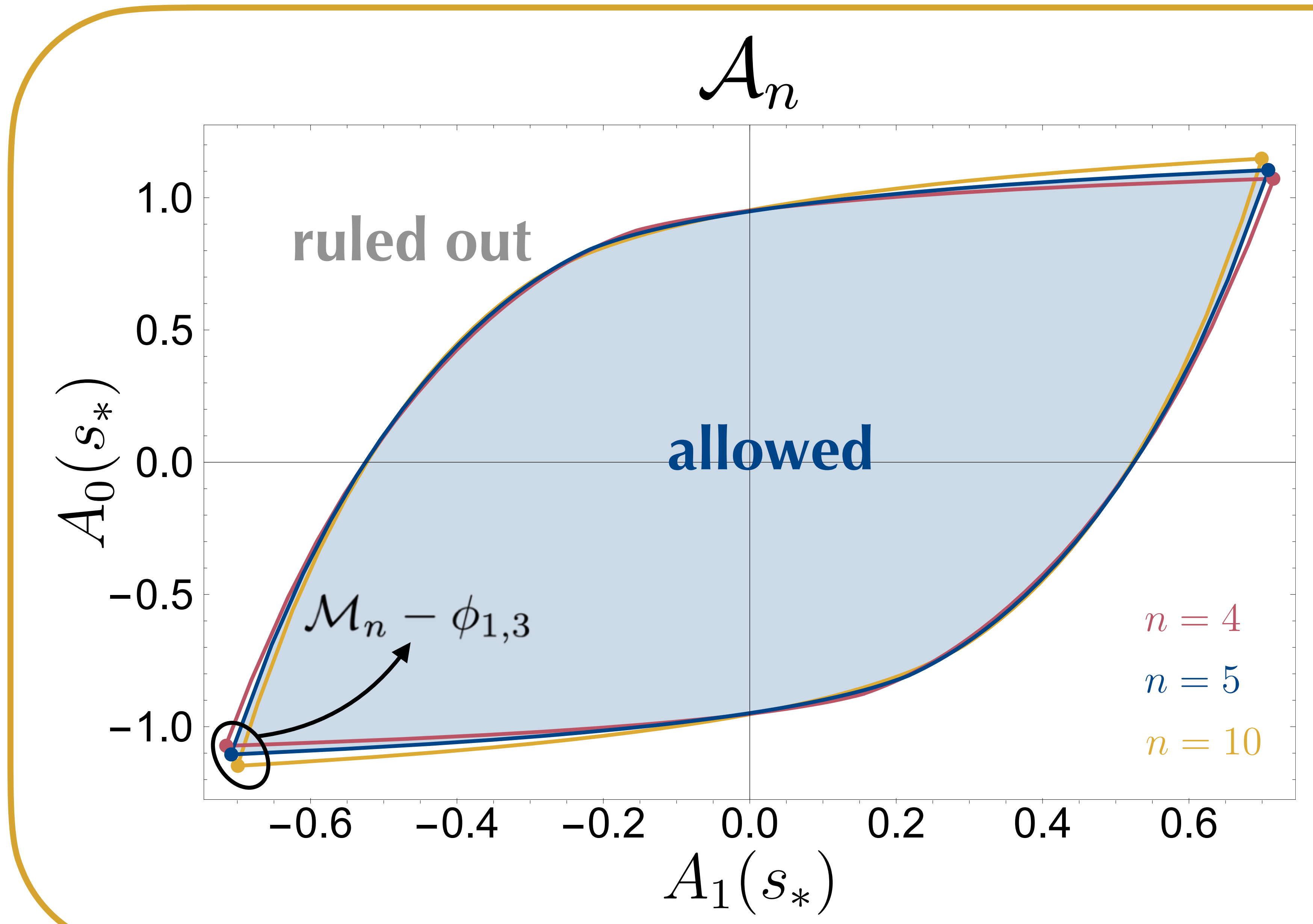


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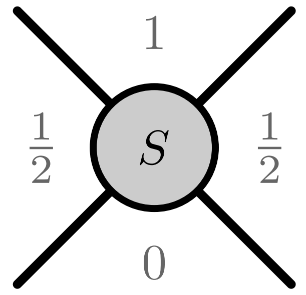


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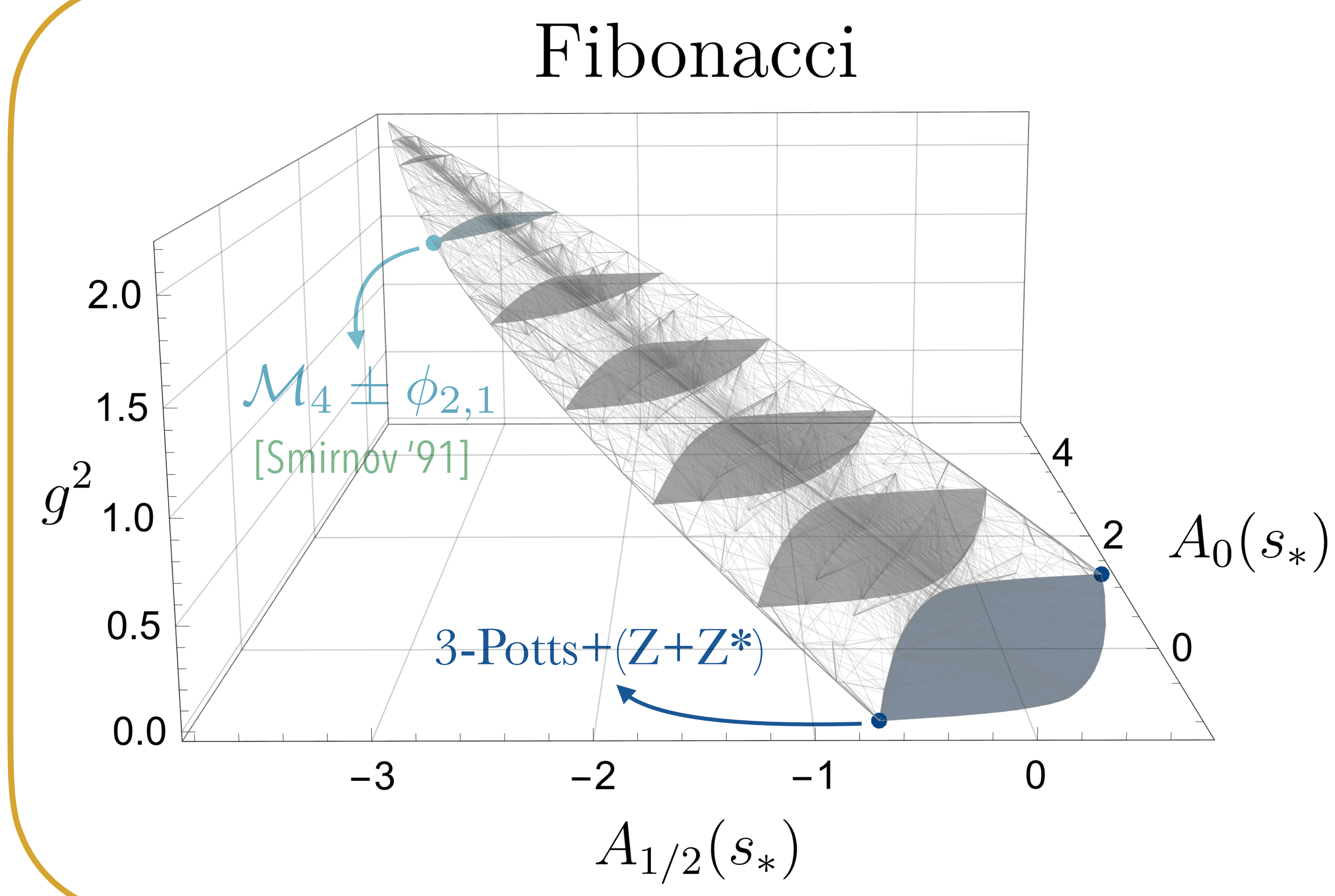


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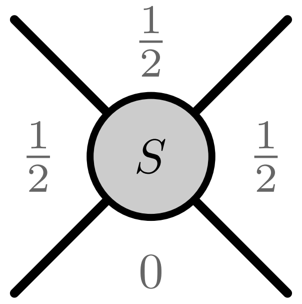


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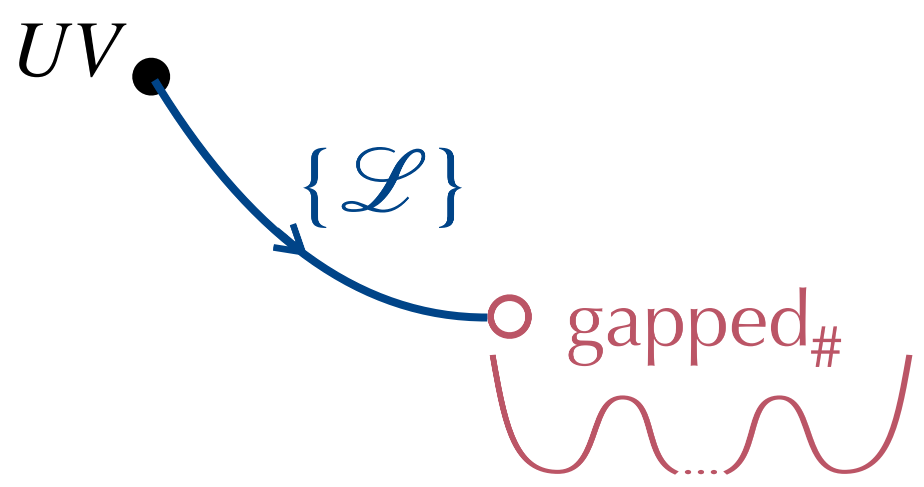


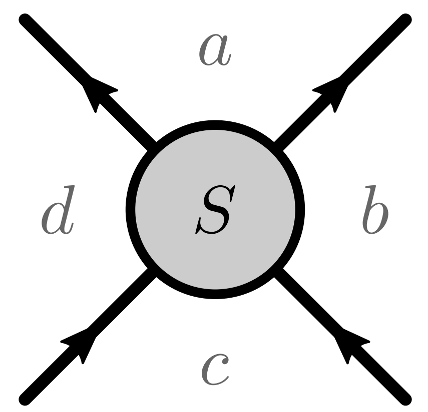


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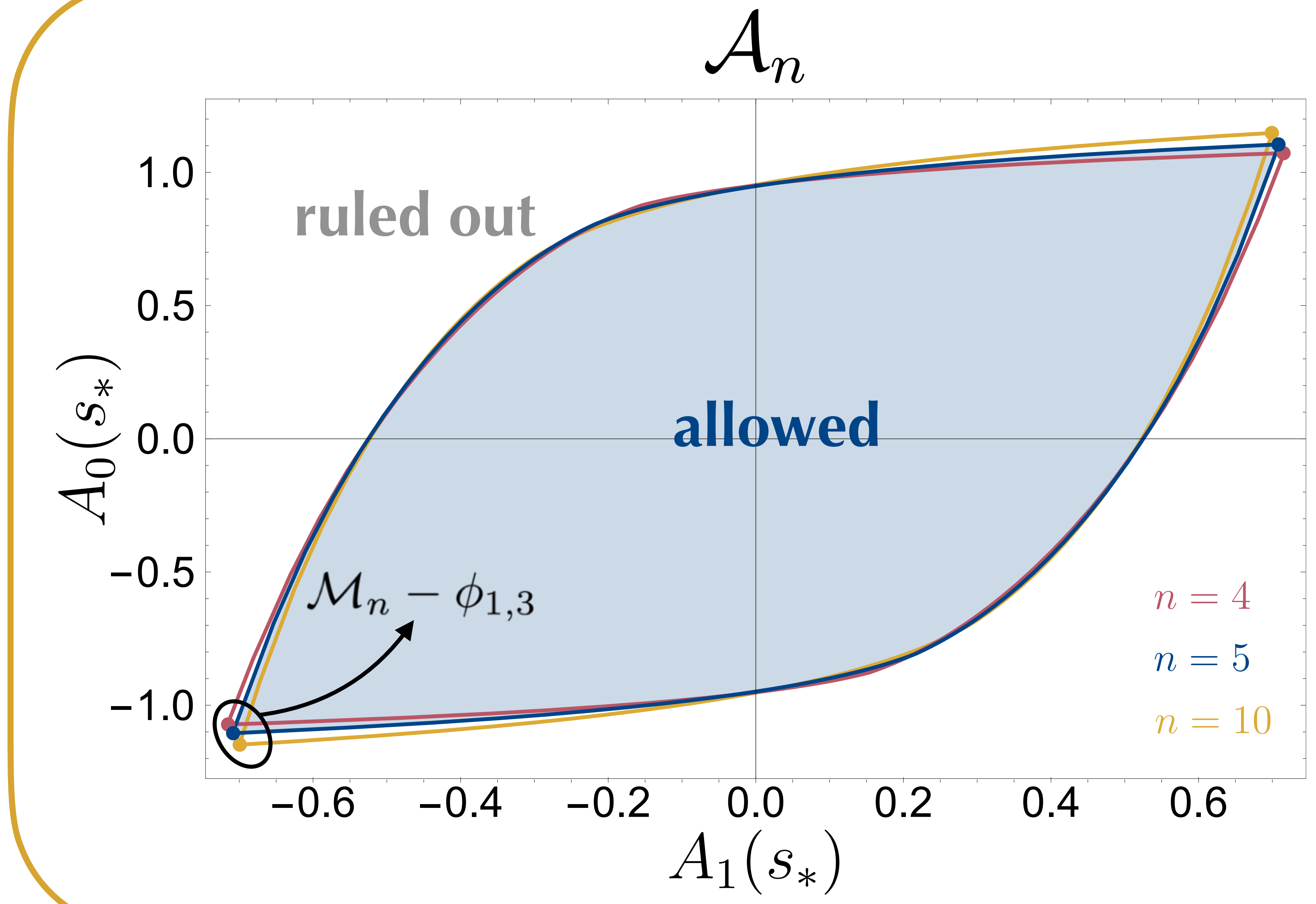


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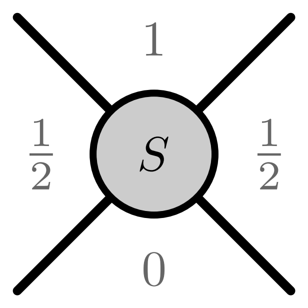


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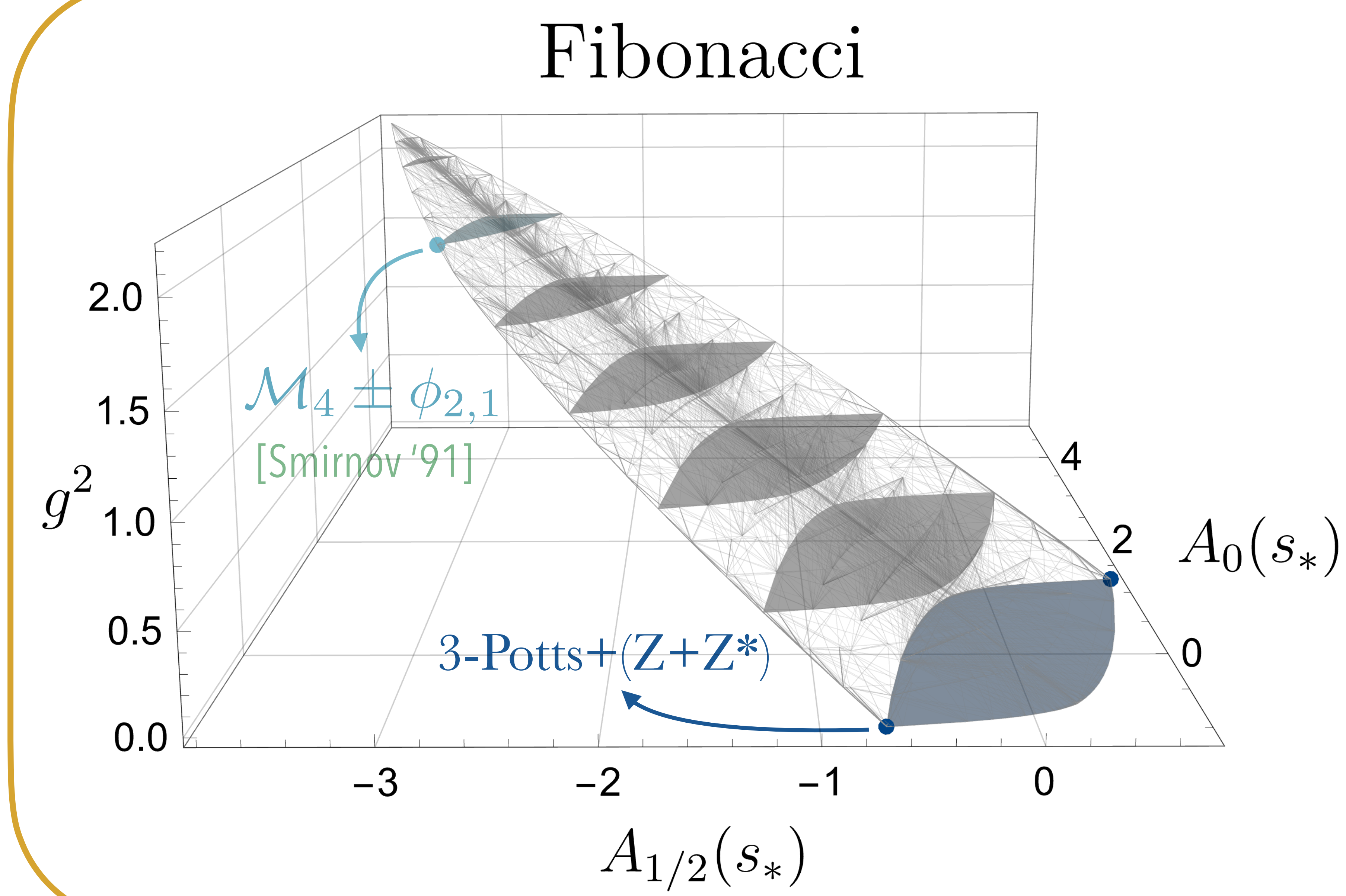


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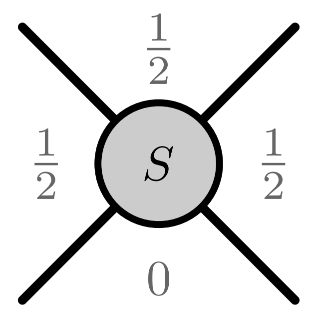


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- 6 topological lines: $\{\underbrace{1, \alpha, \bar{\alpha}}_{\mathrm{d}_{\alpha_i} = 1}, \underbrace{\rho, \alpha\rho, \bar{\alpha}\rho}_{\mathrm{d}_{\rho_i} = \frac{1}{2}(3+\sqrt{13}) = \zeta}\}$. Fusion rules: $\rho \times \bar{\alpha} = \alpha\rho\,, \quad \rho \times \rho = 1 + \rho + \alpha\rho + \bar{\alpha}\rho$

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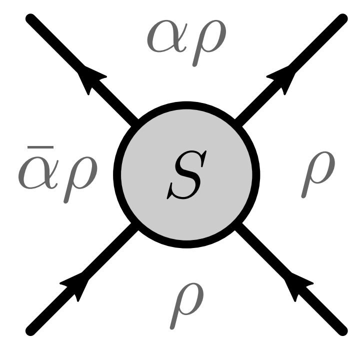

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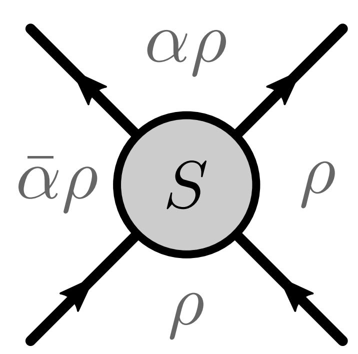


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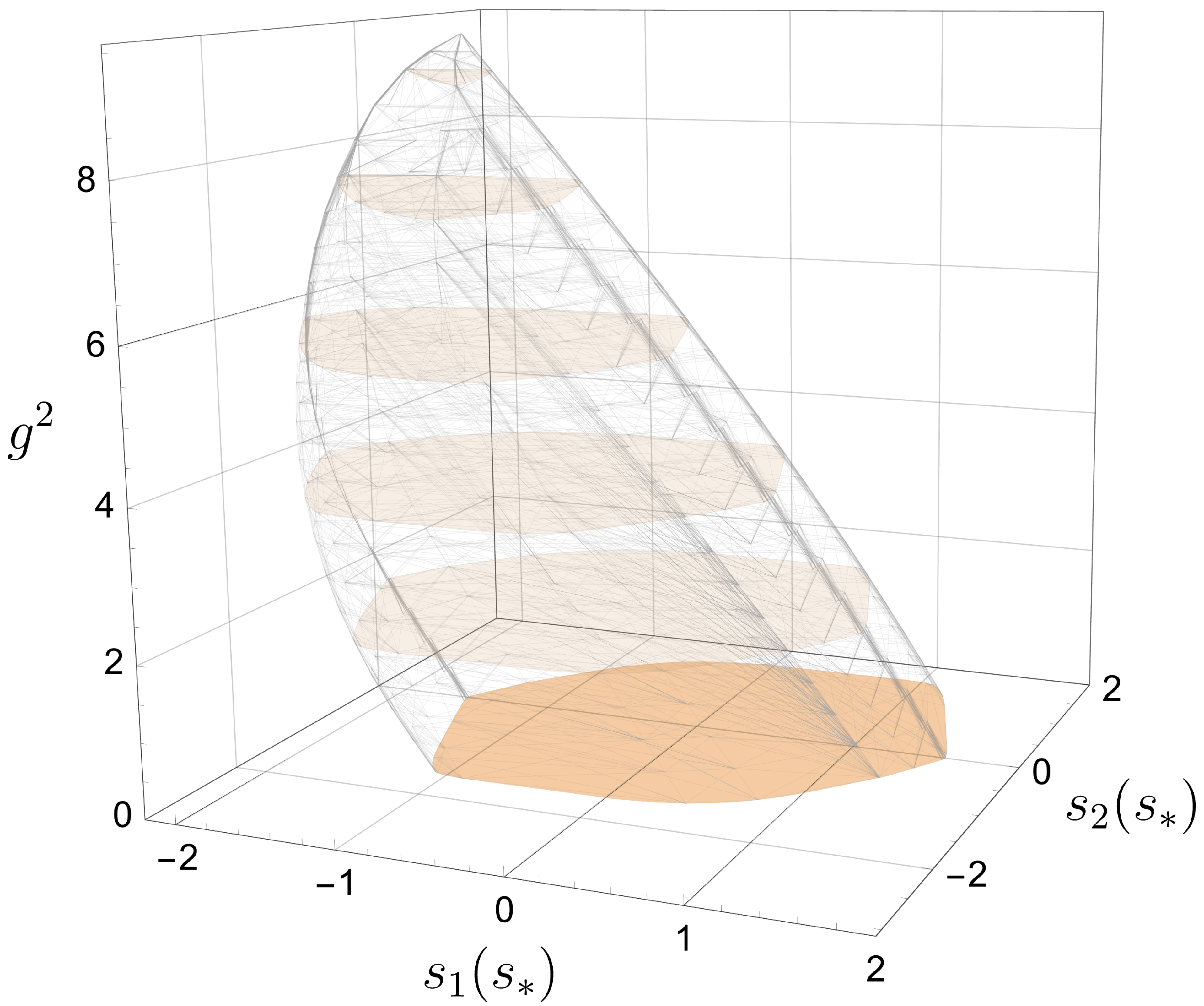


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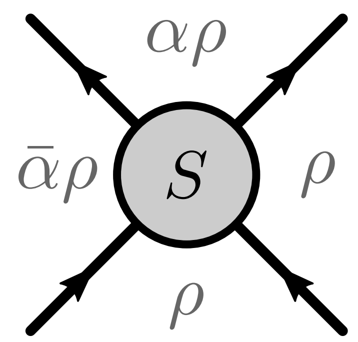


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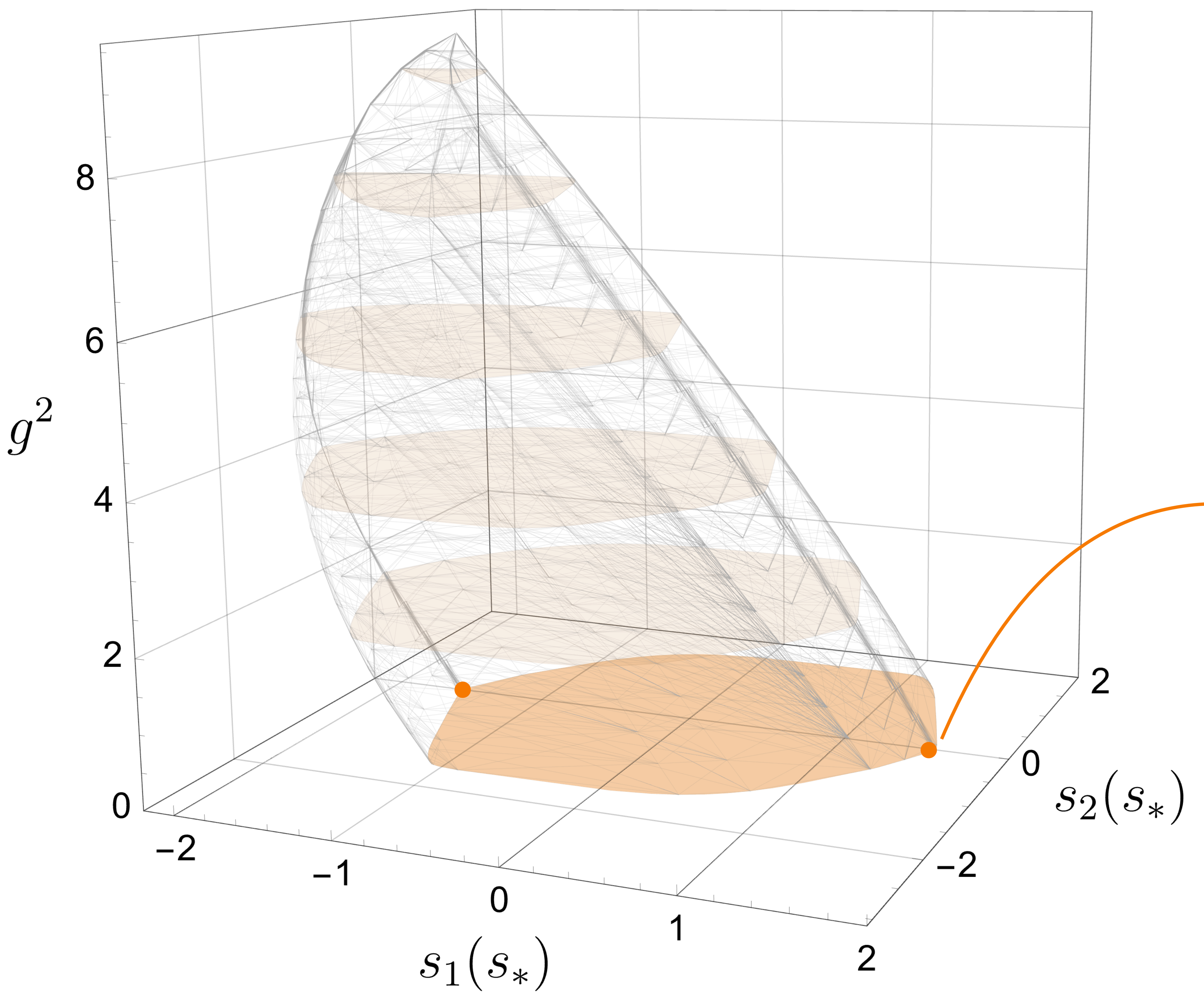


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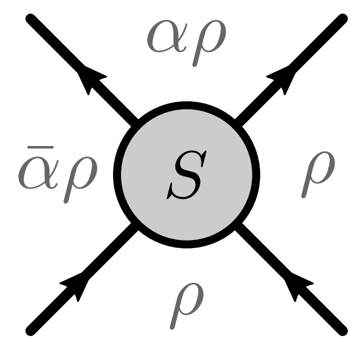


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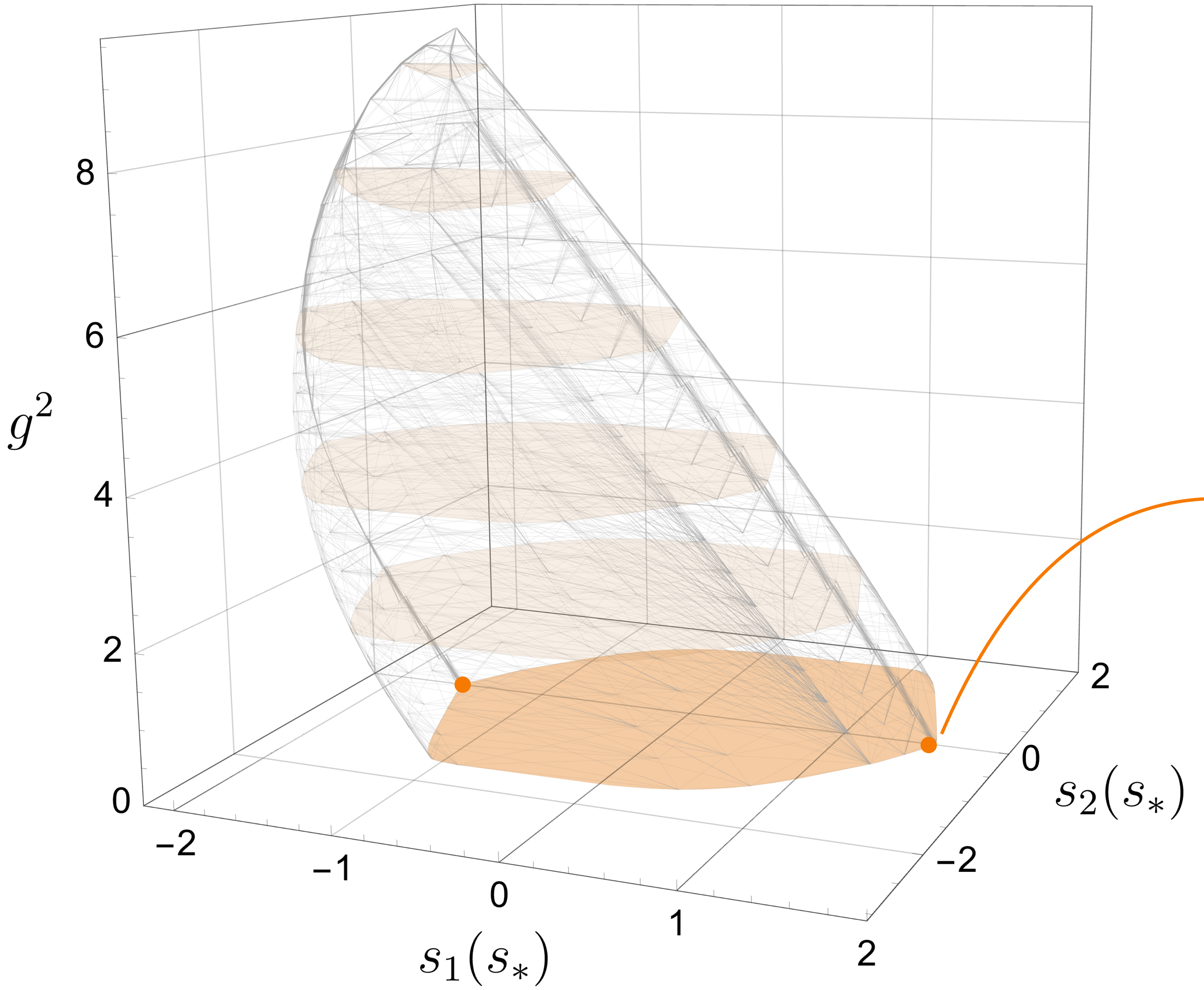


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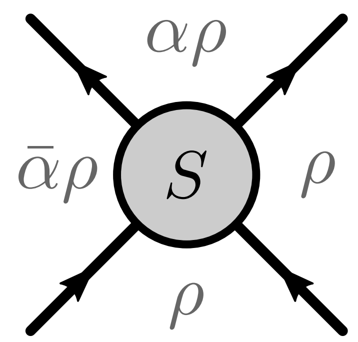


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Analytic solution: $$\frac{A_\rho(\theta)}{A_1(\theta)}=\frac{\alpha+i\tanh(\mu\,\theta/\pi)}{\alpha-i\tanh(\mu\,\theta/\pi)}$$

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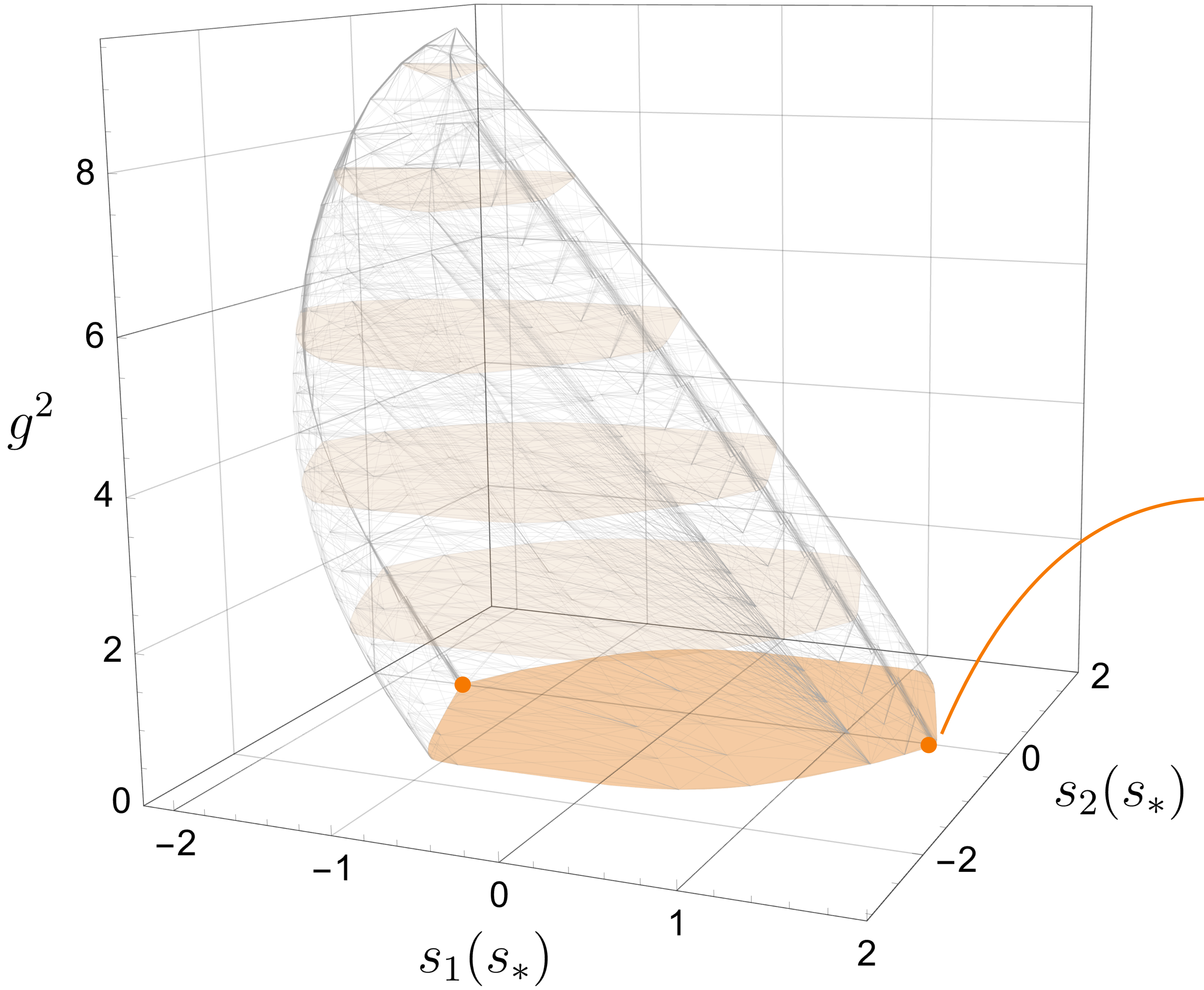
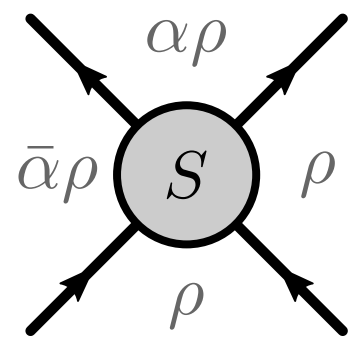
- 6 topological lines: $\{\underbrace{1,\alpha,\bar\alpha}_{d_{\alpha_i}=1},\underbrace{\rho,\alpha\rho,\bar\alpha\rho}_{d_{\rho_i}=\frac{1}{2}(3+\sqrt{13})=\zeta}\}$. Fusion rules: $\rho\times\bar\alpha=\alpha\rho\,,\quad \rho\times\rho=1+\rho+\alpha\rho+\bar\alpha\rho$
- No known field theory realization! *Hints for CFTs with $c\sim 2, 3/2$*

[Vanhove, Lootens, Van Damme, Wolf, Osborne, Haegeman, Verstraete '21; Huang, Lin, Ohmori, Tachikawa,Tezuka '21; Corcoran, de Leeuw '24; Bottini, Schäfer- Nameki '24; Albert, Honda, Kaidi, Zheng '25]

Haagerup $\mathcal{H}_3$ [LC, to appear]




- 6 vacua; 4 fusion channels $A_1(s)$, $A_\rho(s)$, $A_{\alpha\rho}(s)$, $A_{\bar\alpha\rho}(s)$
- Many kinks $K_{a,b}$ (15), including breather $K_{\rho,\rho}$




$$A_\rho(s)\sim\frac{g^2}{s-m^2}\,,$$

$\{s_1,s_2\}$: crossing sym. combinations of $A_\chi(s)$

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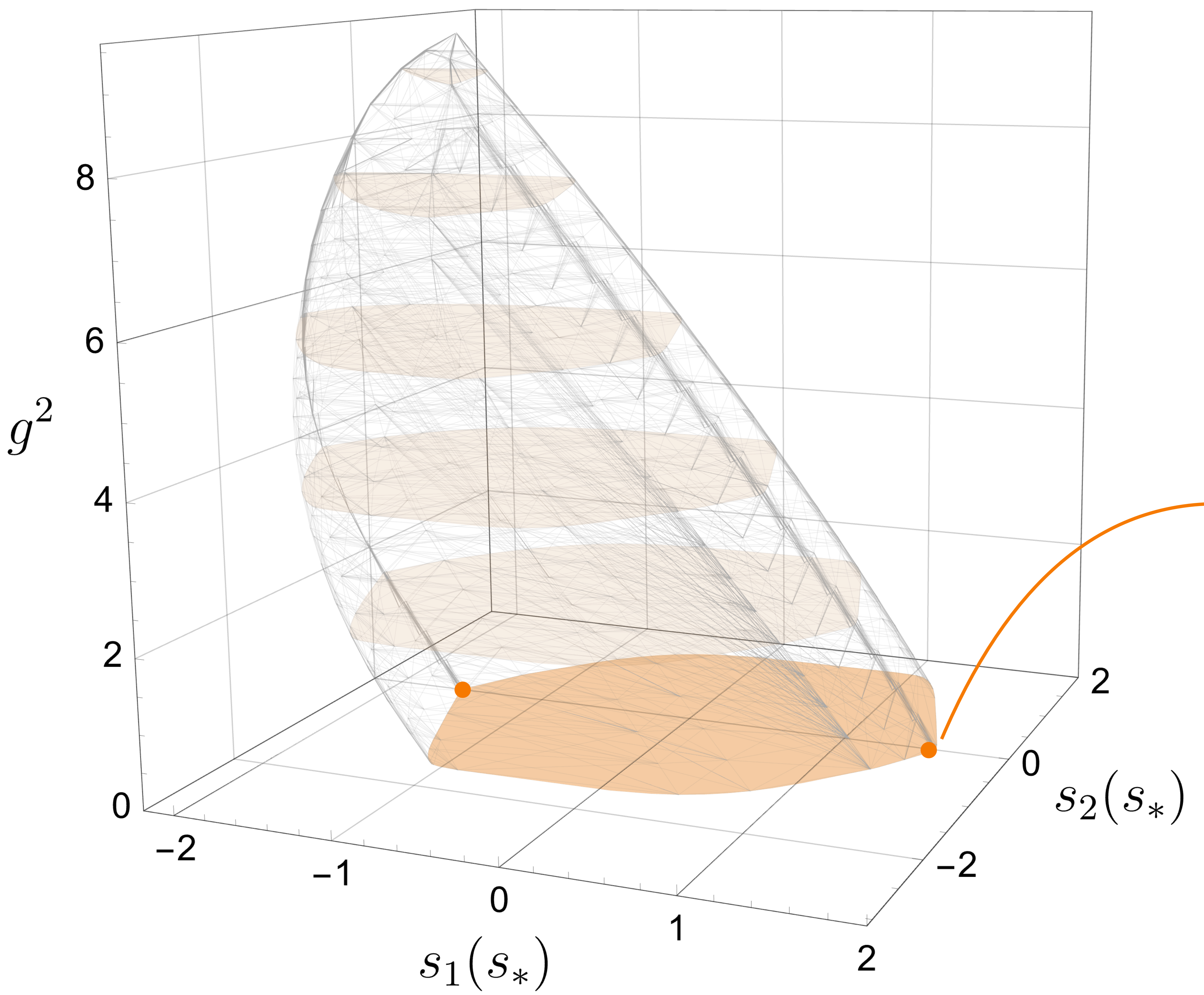


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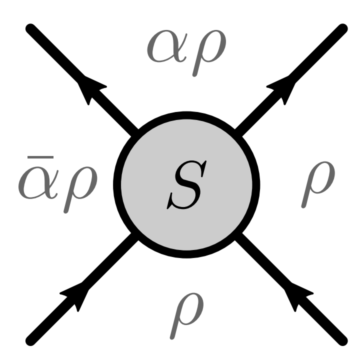


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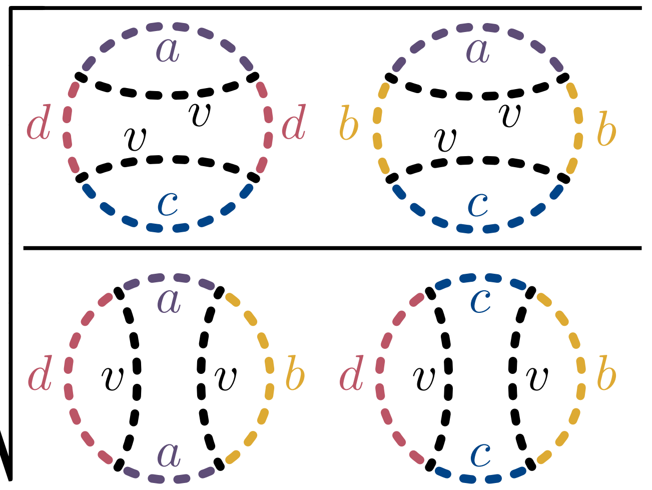
- Conjecture: no integrable solutions in this space

Extremal amplitudes do not satisfy YBE

# Final Remarks

# Summary

- Non-invertible symmetries and anomalies in 1+1d gapped RG flows lead to **modified crossing**.

$S^{ab}_{dc}(s) = \sqrt{\quad}$  $S^{bc}_{ad}(t)$

- Generic for when IR described by non-trivial TQFT
- Modification from corrections to norms of $|\text{in}\rangle, |\text{out}\rangle$ states

- **S-matrix Bootstrap** Categorical symmetry $\mathscr{C}$ can be used to explore the space of consistent $\mathscr{C}$-sym QFTs.
  - $\mathscr{A}_n$ and Fibonacci: known integrable models appear at vertices
  - Haagerup $\mathscr{H}_3$: new models expected, no sign of integrability

- Other symmetry breaking patterns: *Non-regular representation. Module category* $\mathscr{M}$, $d_a \to g_a$ *relative Euler terms,*
$F \to \varphi^*$ *dual boundary F-symbol*

# Future directions

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# Thank you!