

Self-Dual Gravity from Twistor String Compactifications

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Table of Contents

- 1 Motivation
- 2 Self-Dual Gravity
- 3 Compactifying the Twistor String
- 4 Twisted Holography

Motivation

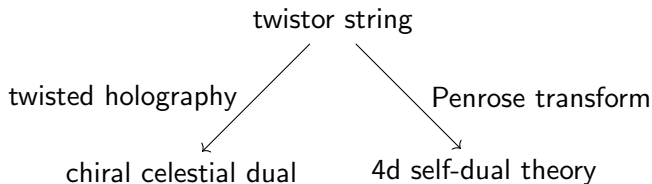
Goal: To construct holographic dualities between gravitational theories in four-dimensional asymptotically flat space-times and conformal field theories in the celestial sphere.

We face many challenges in attempting to construct dualities of this type:

- ▶ Which bulk gravitational theories have celestial duals?
- ▶ Is there a dimensionless quantity in the bulk that can be used as an expansion parameter?
- ▶ Is celestial CFT local?

It is therefore useful to have toy examples for which these questions can be answered.

A family of examples are provided by top-down dualities between four-dimensional **self-dual** theories and two-dimensional **chiral** CFTs.



So far this has been applied to a 4d integrable σ -model on Burns space [Costello et al. 22, 23], and the self-dual sector of gauge theory [RB et al. 24].

In this talk I will apply this construction to **self-dual gravity**.

What is Self-Dual Gravity?

Let g be a metric on the 4d manifold \mathcal{M} . The **self-dual vacuum Einstein equations** (without cosmological constant) state that

$$C = *C, \quad \text{Ric} = 0$$

for C the Weyl tensor and Ric the Ricci tensor.

Remark

In Lorentzian signature $*^2 = -1$, so metric must be complexified. Alternatively can work with real metrics in Euclidean or split $(++--)$ signature.

Can understand self-dual gravity (sdgr) as a chiral weak coupling limit of Einstein gravity in the first order formalism.

Writing

$$e^{\dot{\alpha}\alpha} \in \Omega^1(\mathcal{M}), \quad \Gamma^\alpha{}_\beta \in \Omega^1(\mathcal{M})$$

for the vierbeins and left-handed spin connection, the Palatini action for Einstein gravity is

$$S_{\text{Palatini}}[e, \Gamma] = \frac{1}{2} \int_{\mathcal{M}} \epsilon_{\dot{\alpha}\beta} e^{\dot{\alpha}\alpha} \wedge e^{\dot{\beta}\beta} \wedge (d\Gamma_{\alpha\beta} + \kappa \Gamma_{\alpha\gamma} \wedge \Gamma^\gamma{}_\beta).$$

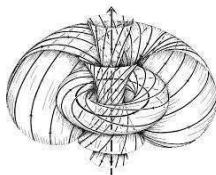
Sending $\kappa \rightarrow 0$ recovers sdgr.

Twistor Theory

A powerful tool for studying sdgr is Penrose's **twistor theory**. His **non-linear graviton** identifies:

- four-dimensional Riemannian manifolds \mathcal{M} with a self-dual vacuum metric.
- Certain complex three-folds \mathcal{Z} which fibre holomorphically over \mathbb{CP}^1 with holomorphic symplectic fibres.

\mathcal{Z} is the **twistor space** of \mathcal{M} . [Penrose, 76]



Picture credit: [Atiyah et al. 17]

The twistor space of \mathbb{R}^4 , denoted \mathbb{PT} , is the total space of the rank two holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{CP}^1 \quad \text{with co-ordinates } v^{\dot{0}}, v^{\dot{1}}, z.$$

By the Penrose transform, self-dual vacuum fluctuations of \mathbb{R}^4 can be interpreted as complex structure deformations of \mathbb{PT} .

In the **Kodaira-Spencer formalism**, to deform the complex structure we shift the $\bar{\partial}$ operator by a **Beltrami differential**

$$\bar{\partial} \mapsto \bar{\partial} + u, \quad u = u^z \partial_z + u^{\dot{\alpha}} \partial_{v^{\dot{\alpha}}} \in \Omega^{0,1}(\mathbb{PT}, T_{\mathbb{PT}}^{1,0}).$$

Consistency of the Cauchy-Riemann equations demands

$$\bar{\partial}u + \frac{1}{2}[u, u] = 0.$$

A generic complex structure deformation of \mathbb{PT} corresponds to a self-dual fluctuation of the **conformal structure** of \mathbb{R}^4 .

Self-dual vacuum Einstein fluctuations correspond to Beltrami differentials of a restricted form:

$$\bar{\partial} \mapsto \bar{\partial} + \epsilon^{\dot{\beta}\dot{\alpha}} \partial_{v\dot{\alpha}} h \partial_{v\dot{\beta}}$$

$h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ is the **Hamiltonian**. This deformation is integrable if

$$\bar{\partial} h + \frac{1}{2} \epsilon^{\dot{\beta}\dot{\alpha}} \partial_{v\dot{\alpha}} h \wedge \partial_{v\dot{\beta}} h = 0.$$

It's possible to introduce an action for h on twistor space which descends to sdgr on space-time. **[Mason, Wolf, 07]**

Twistor String

We will see that sdgr is an inevitable output of the **twistor string**, i.e., the B-model topological string on twistor space. **[Witten, 03]**

The closed string sector of the B-model describes deformations of the target W as a Calabi-Yau manifold.

Its dynamical fields include a Beltrami differential, u , which must be divergence free

$$\operatorname{div} u \propto \partial(u \lrcorner \Sigma) = 0,$$

where Σ is the holomorphic top form.

But also other fields, e.g., a divergence free bi-vector and a $(0, 2)$ -form gerbe. Altogether we have a poly-vector field.

There is a catch: twistor space is not Calabi-Yau.

To overcome this Witten suggests using a super-Calabi-Yau, e.g.,

$$\mathbb{P}\mathbb{T}^{\mathcal{N}=1} = \Pi(\mathcal{O}(1) \oplus \mathcal{O}(3)) \rightarrow \mathbb{P}\mathbb{T}.$$

Here Π indicates Grassmann parity reversal, so the fibres are fermionic.

The closed string sector includes the spectrum of self-dual (super-)conformal gravity, since it allows arbitrary complex structure deformations of twistor space **[Witten, Berokvits, 04]**.

But in fact the theory has many more degrees of freedom. This is easiest to see by using the following well known duality of the B-model:

$$\text{B-model on } V \leftrightarrow \text{B-model on } \text{IV}^V$$

Therefore the twistor string can be rewritten on the bosonic Calabi-Yau

$$\mathcal{O}(-1) \oplus \mathcal{O}(-3) \rightarrow \mathbb{P}^1.$$

Since the fibres are non-compact, there are infinite towers of closed string fields.

Resolution

In order to get an honest 4d gravitational theory we should **compactify** the fibres to get a Calabi-Yau five-fold $X \rightarrow \mathbb{P}^1$.

Compactification

We will now see that a generic compactification generates sdgr on space-time.

There are two natural choices for the fibres of X , either T^4 or $K3$. We will concentrate on the former. Since $\mathbb{P}\mathbb{T}$ is not Calabi-Yau, we cannot take X to be the trivial T^4 bundle.

Instead we should take a non-trivial T^4 fibration $\pi_Y : Y \rightarrow \mathbb{C}\mathbb{P}^1$ with canonical bundle $K_Y \cong \pi_Y^* \mathcal{O}(2)$, and form the fibre product

$$X = Y \times_{\mathbb{C}\mathbb{P}^1} \mathbb{P}\mathbb{T}.$$

There are many ways of engineering a suitable Y , e.g., fibre products of Weierstrass models.

Now let's compactify the B-model on X to twistor space.

Locally X looks like $\mathbb{C}^3 \times T^4$. The B-model on this background has been studied in [Costello, Paquette, 20; Fernández et al. 24].

There the authors argue that the compactified theory is described by poly-vector fields valued in the cohomology

$$H^{0,\bullet}(\wedge^\bullet T_{T^4}^{1,0}) = \mathbb{C}[\xi_1, \xi_2, \eta^1, \eta^2].$$

Here ξ_a, η^b are fermionic variables representing holomorphic vector fields and $(0, 1)$ -cohomology representatives on the T^4 fibres.

These are essentially KK-modes.

For the non-trivial fibration $X \rightarrow \mathbb{P}^1$ this local picture changes because the complex structure on the fibre varies over the base.

Shifting $z \mapsto z + \delta z$ on the base changes the complex structure of the fibre by

$$\bar{\partial}(\partial_z) = \kappa(z) = \kappa^a{}_b(z) \eta^b \xi_a \in H^{0,1}(X_z; T_{X_z}^{1,0}).$$

This is the **Kodaira-Spencer map**.

Consider the B-model on X with dynamical poly-vector s , which we expand as

$$s = u + u^a \xi_a + u_b \eta^b + u^a{}_b \eta^b \xi_a + \dots$$

Here the components $u, u^a, u_b, u^a{}_b, \dots$ are poly-vector fields on $\mathbb{P}\mathbb{T}$. Acting on vector field with a ∂_z component generates an extra term

$$\bar{\partial}_X(u^z \partial_z) = \bar{\partial}_{\mathbb{P}\mathbb{T}}(u^z \partial_z) + u^z \kappa^a{}_b(z) \eta^b \otimes \xi_a.$$

This enters the equations of motion for $u^a{}_b$

$$u^z \kappa^a{}_b(z) + \bar{\partial}_u u^a{}_b + [u^a, u_b] = 0.$$

Away from zeros of the Kodaira-Spencer map, this can be used to eliminate the $u^z \partial_z$ component of the Beltrami differential on $\mathbb{P}\mathbb{T}$.

The divergence free condition implies that

$$D_z u^z + \partial_{v^{\dot{\alpha}}} u^{\dot{\alpha}} = \partial_{v^{\dot{\alpha}}} u^{\dot{\alpha}} = 0,$$

so that $u^{\dot{\alpha}}$ can be written in terms of a Hamiltonian $h \in \Omega^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(2))$

$$u^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{v^{\dot{\beta}}} h.$$

This is the twistor description of a self-dual Einstein graviton.

We are seeing that the non-trivial fibration $X \rightarrow \mathbb{P}\mathbb{T}$ spontaneously breaks the local Weyl invariance of self-dual conformal gravity. **Any compactification of the twistor string will break Weyl invariance in this way.**

The theory also has many extra fields. These can be cut down by using type I strings and/or compactifying on a K3.

Twisted Holography

In order to apply twisted holography to the above set up we wrap N_5 D5 branes on the locus $Y_0 = \{v^0 = v^1 = 0\} \subset X$, and N_1 D1 branes on the distinguished holomorphic locus $\mathbb{CP}^1 \subset Y_0$.

The backreacted geometry only depend on N_1, N_5 through $N = N_1 N_5$.

As above, we will view this a modification of twisted holography on $\mathbb{C}^3 \times T^4$, which proposes a duality between **[Costello, Paquette, 20; Fernández et al. 24]**

B-model on
the super-conifold \leftrightarrow chiral de Rham complex
on $(T^4)^N/S_N$ as $N \rightarrow \infty$.

See also **[Witten, 91; Malikov et al. 99; Kapustin 05; ...]**

Remark

This is the twist of the more familiar $\text{AdS}_3/\text{CFT}_2$ duality between type IIB supergravity on $\text{AdS}_3 \times S^3 \times T^4$ and the large N limit of the D1-D5 system.

This physical duality has been well studied, and much machinery carries over from the $\mathcal{N} = (4, 4)$ symmetric orbifold.

Let's briefly remind ourselves of the states of the chiral de Rham complex on $(T^4)^N/S_N$ [Dijkgraaf et al. 96; de Boer 98].

The state space can be assembled from twist n sectors corresponding to cyclic permutations of length n . These are dual to single particle states in the bulk.

The twist n sector is generated by chiral primaries for a small $\mathcal{N} = 4$ super-conformal algebra in the untwisted sector.

One of these, $\sigma_n^+(z)$, corresponds to modes of the poly-vector u in the bulk:

chiral algebra state	bulk mode	Penrose transform
$\sigma_n^+(z_0)$	$\bar{\delta}_{z=z_0}(v^{\dot{0}})^n \partial / \partial v^{\dot{1}} + \dots$	Einstein graviton
$G_{-1/2}^{\dot{1}} \tilde{G}_{-1/2}^{\dot{1}} \sigma_n^+(z_0)$	$\bar{\delta}_{z=z_0}(v^{\dot{0}})^{n-1} \partial / \partial z + \dots$	conformal graviton

Here $G^{\dot{\alpha}}$, $\tilde{G}^{\dot{\beta}}$ are super-conformal generators, and dotted indices are R-symmetry/right-handed spinor indices.

Evidence for self-dual gravity:

- If the torus moduli begin to vary, the operator corresponding to the Kodaira-Spencer map is switched on. Half of the super-conformal generators, $G^{\dot{\alpha}}$, are no longer BRST closed, so the conformal gravitons are eliminated from the spectrum.
- Up to an overall normalization

$$w_m^p(z) \propto (J_0^{\dot{1}1})^{p-1-m} \sigma_{2p-3}^+(z).$$

The $2 \rightarrow 1$ OPE of these operators is known [**Lunin, Mathur, 00; 02**]. The genus zero contribution to their OPE coincides with $\mathcal{L}w_{1+\infty} +$ corrections from backreaction.

Summary

We have seen:

- For the closed string sector of the B-model to describe a four-dimensional theory of gravity it should be compactified.
- Doing so spontaneously breaks the Weyl gauge symmetry of conformal gravity. Conformal gravitons become massive and can be integrated out.
- We can find the expected $\mathcal{L}w_{1+\infty}$ symmetry appear in the dual D1-D5 system.

Much more to understand: backreaction, loop corrections, K3 compactifications, . . .

Thank you for listening.

More Detail on the Dual

The twist n sector of the chiral de Rham complex is described by a $\beta\gamma - bc$ system into $(T^4)^n$, subject to cyclic boundary conditions

$$\begin{aligned}\partial_z \gamma_i^a(e^{2\pi i} z) &= \partial_z \gamma_{i+1}^a(z), & \beta_{a,i}(e^{2\pi i} z) &= \beta_{a,i+1}(z), \\ c_i^a(e^{2\pi i} z) &= c_{i+1}^a(z), & b_{a,i}(e^{2\pi i} z) &= b_{a,i+1}(z).\end{aligned}$$

We can arrange the fermionic fields into a R-symmetry vector using the holomorphic symplectic structure

$$\psi_{a,i}^{\dot{\alpha}} = (\epsilon_{ab} c_i^b, b_{a,i}).$$

The theory is also accompanied by the zero-modes of the right moving sector, which survive the half-twist $\tilde{c}_i^a, \tilde{b}_{a,i}$.

In terms of these fields, the small $\mathcal{N} = 4$ super-conformal algebra is generated by:

$$T = \sum_{i=1}^n (\beta_{a,i} \partial_z \gamma_i^a + \frac{1}{2} \epsilon^{ba} \epsilon_{\dot{\alpha}\dot{\beta}} \psi_{a,i}^{\dot{\alpha}} \partial_z \psi_{b,i}^{\dot{\beta}})$$

$$G^{\dot{\alpha}} = \sum_{i=1}^n \psi_{a,i}^{\dot{\alpha}} \partial_z \gamma_i^a$$

$$\tilde{G}^{\dot{\alpha}} = \sum_{i=1}^n \epsilon^{ba} \beta_{a,i} \psi_{b,i}^{\dot{\alpha}}$$

$$J^{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^n \frac{1}{2} \epsilon^{ba} \psi_{a,i}^{\dot{\alpha}} \psi_{b,i}^{\dot{\beta}}$$

We have

$$\sigma_1^+(z) = J^{\dot{0}\dot{0}}(z).$$

The operator switched on when the fibres vary over the base is

$$\sum_{i=1}^n \lambda^a_b(z) \beta_{a,i}(z) \tilde{c}_i^b.$$

This modifies the BRST differential, which now acts by

$$QG^{\dot{\alpha}}(z) = \sum_{i=1}^n \kappa^a_b(z) \psi^{\dot{\alpha}}_{a,i}(z) \tilde{c}_i^b$$

where $\kappa^a_b(z) = \partial_z \lambda^a_b(z)$ is the Kodaira-Spencer map.