A Holographic Model for Soft Photons and Gravitons in 4D

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Soft factorization

A scattering amplitude in gauge theory factorizes into a universal soft factor and a non-universal hard factor. Consider an amplitude involving m soft photons and n hard particles

$$A_{m+n} = S_m \widetilde{A}_n.$$

$\widetilde{A}_n$ is the hard factor. The soft factor S_m receives contribution from the m real soft photons (leading soft photon theorem) and virtual soft photons in the internal loops.

In four spacetime dimensions,

$$S_m = \mathcal{J}_{a_1}(x_1)\cdots\mathcal{J}_{a_m}(x_m)\exp\left[-\frac{\alpha}{2\pi e^2}\int d^2x\,\mathcal{J}^a(x)\mathcal{J}_a(x)\right], \qquad \alpha = \ln\frac{\Lambda}{\mu}, \qquad \mathcal{J}_a(x) = -\frac{e^2}{4\pi}\sum_k Q_k \frac{\hat{p}_k\cdot\epsilon_a(x)}{\hat{p}_k\cdot\hat{q}(x)}$$

We work in coordinates where the transverse directions are flat

$$\hat{q}^\mu(x) = \frac{1}{2}(1+x^a x_a, 2x^a, 1-x^a x_a), \qquad \epsilon_a(x) = \partial_a\hat{q}(x), \qquad a = 1,2.$$

There is also an infinite phase coming from virtual soft photons that we ignore for now.

Soft photon and Goldstone operators

The gauge field at the asymptotic future (+) and past (−) admits the following mode expansion

$$A_\mu^\pm(X) = e \int \frac{d^3q}{(2\pi)^3(2q^0)} \left[\epsilon_\mu^a(q)\mathcal{O}_a^\pm(q)e^{iq\cdot X} + \text{h.c.}\right]$$

with the canonical commutator

$$[\mathcal{O}_a^\pm(q), \mathcal{O}_b^{\pm\dagger}(q')] = \delta_{ab}(2\pi)^3(2q^0)\delta^3(q-q')$$

The photon operator has the soft expansion

$$\mathcal{O}_a^\pm(\omega, x) = -\frac{4\pi}{e}\left(\frac{N_a^\pm(x)}{\omega \pm i\varepsilon} - 2\pi i C_a^\pm(x)\delta(\omega)\right) + \cdots, \qquad N_a(x) = \partial_a N(x), \qquad C_a(x) = \partial_a C(x)$$

where $N_a^\pm$ is the soft photon operator, and $C_a^\pm$ is the superphaserotation Goldstone operator,

$$[C^\pm(x), N^\pm(x')] = -ie^2 G(x-x'), \qquad G(x-x') = \frac{1}{4\pi}\ln[(x-x')^2].$$

Soft effective action

The dynamics of the soft photon and Goldstone modes are governed by the *soft effective action* [Kapec, Mitra '21]

$$S_0 = \frac{i\alpha}{2\pi e^2}\int d^2x\, N_a(x)N^a(x) - \frac{1}{e^2}\int d^2x\, C^a(x)N_a(x), \qquad N_a(x) = N_a^+(x) - N_a^-(x)$$

which can be derived from the Maxwell action [He, Mitra, Sivaramakrishnan, Zurek '24]

$$S = -\frac{1}{2e^2}\int_{M_4} F\wedge *F + \frac{1}{e^2}\int_{\mathcal{I}^+} A\wedge *F - \frac{1}{e^2}\int_{\mathcal{I}^-} A\wedge *F$$

by isolating the contributions of the soft and Goldstone modes.

The boundary terms implement Neumann boundary conditions that allow radiation flux through null infinities.

The antipodal matching condition $C_a^+ = C_a^- \equiv C_a$ has been put in by hand.

Generalized soft effective action

We propose two simple but important modifications to the story presented in the literature.

- N and C are invariant under spacetime translations, but not under Lorentz transformations. The soft contribution to the Lorentz charge is

$$\frac{1}{e^2}\int d^2x\,\mathcal{L}_Y C_a^{\pm}(x)N^{\pm a}(x).$$

  The perturbative Lorentz-invariant vacuum is assumed to be C eigenstate with eigenvalue $C_a = 0$ in the literature. We take it to be the N eigenstate with eigenvalue $N_a(x) = 0$.

- Instead of imposing matching condition, we distinguish the future and past modes C^+ and C^-, which yields the *generalized soft effective action*

$$S = \frac{i\alpha}{2\pi e^2}\int d^2x\,N_a(x)N^a(x) - \frac{1}{e^2}\int d^2x\,C^{+a}(x)N_a^+(x) + \frac{1}{e^2}\int d^2x\,C^{-a}(x)N_a^-(x)$$

Generalized soft effective action

$$S = \frac{i\alpha}{2\pi e^2}\int d^2x\, N_a(x)N^a(x) - \frac{1}{e^2}\int d^2x\, C^{+a}(x)N_a^+(x) + \frac{1}{e^2}\int d^2x\, C^{-a}(x)N_a^-(x)$$

For an amplitude involving m soft photons and n hard particles, the soft factor can be computed from the generalized soft effective action by the path integral

$$\begin{aligned}&\langle 0,+|\, N_{a_1}\cdots N_{a_m} O_1\cdots O_n\,|0,-\rangle\,|_{\text{soft}}\\ &= \int [dC^+][dC^-][dN^+][dN^-]N_{a_1}\cdots N_{a_m}\exp\left[iS + \frac{i}{e^2}\int d^2x(C^{+a}\mathcal{J}_a^+ + C^{-a}\mathcal{J}_a^-)\right]\end{aligned}$$

where terms involving the soft factors $\mathcal{J}_a^{\pm} = -\frac{e^2}{4\pi}\sum_{k\in {\text{out} \atop \text{in}}} Q_k \frac{\hat{p}_k\cdot\epsilon_a}{\hat{p}_k\cdot\hat{q}}$ are the net contribution of the hard operators O_i to the soft sector [Kapec, Mitra '21].

We show that this generalized soft effective action is capable of re-deriving the bulk 4D infrared physics in an extremely economic manner.

Antipodal matching condition

- We can rewrite the generalized action in terms of the difference $N_a = N_a^+ - N_a^-$ and the average $N_a^{\text{avg}} = \frac{1}{2}(N_a^+ + N_a^-)$,

$$S = \frac{i\alpha}{2\pi e^2}\int d^2x\, N_a N^a - \frac{1}{2e^2}\int d^2x\,(C^{+a} + C^{-a})N_a - \frac{1}{e^2}\int d^2x\,(C^{+a} - C^{-a})N_a^{\text{avg}}$$

- The action is linear in N_a^{avg}, so it is not a dynamical mode.

- Also, in scattering problems, the soft graviton operator is always inserted in the form of the difference

$$\lim_{\omega\to 0}[\omega\mathcal{O}_a(\omega\hat{q}(x))] = -\frac{4\pi}{e}[N_a^+(x) - N_a^-(x)]$$

- Therefore, we can simply integrate out N_a^{avg} to obtain a delta function imposing the constraint

$$C_a^+(x) = C_a^-(x)$$

which is precisely the antipodal matching condition.

Soft theorem and IR divergence

- The relevant path integral of the soft sector is

$$
\begin{aligned}
&\langle 0,+|\, N_{a_1}\cdots N_{a_m} O_1 \cdots O_n \,|0,-\rangle\,|_{\mathrm{soft}} \\
&= \int [dC^+][dC^-][dN^+][dN^-] N_{a_1}\cdots N_{a_m} \exp\left[iS + \frac{i}{e^2}\int d^2x (C^{+a}\mathcal{J}_a^+ + C^{-a}\mathcal{J}_a^-)\right]
\end{aligned}
$$

- After integrating N_a^{avg} out, the path integral involving

$$
\langle 0,+|\, N_{a_1}\cdots N_{a_m} O_1 \cdots O_n \,|0,-\rangle\,|_{\mathrm{soft}} = \int [dC][dN] N_{a_1}\cdots N_{a_m} \exp\left[-\frac{\alpha}{2\pi e^2}\int d^2x\, N_a N^a - \frac{i}{e^2}\int d^2x\, C_a (N^a - J^a)\right].
$$

- Integrating C out, we obtain a delta function imposing $N_a = J_a$. Evaluating the N integral, we obtain

$$
\langle 0,+|\, N_{a_1}\cdots N_{a_m} O_1 \cdots O_n \,|0,-\rangle\,|_{\mathrm{soft}} = \mathcal{J}_{a_1}\cdots \mathcal{J}_{a_m} \exp\left[-\frac{\alpha}{2\pi e^2}\int d^2x\, \mathcal{J}_a \mathcal{J}^a\right].
$$

  The first factor $\mathcal{J}_{a_1}\cdots\mathcal{J}_{a_m}$ correspond to the soft theorem.
  The exponential reflects IR divergence: as we remove the IR cutoff $\mu \to 0$, $\alpha \to +\infty$ and the exponential goes to 0.

Faddeev-Kulish dressed states

- The presence of IR divergence implies that Fock states are not the correct asymptotic states. Rather, the correct asymptotic states are Fock states dressed with a cloud of soft photons.

- Dressed states are inserted by the operators $\widetilde{O}_k \equiv e^{R_k^\pm} O_k$, where

$$R_k^\pm = eQ_k \int_{\text{soft}} \frac{d^3q}{(2\pi)^3(2q^0)} \frac{\hat{p}_k \cdot \epsilon}{\hat{p}_k \cdot q} (\mathcal{O}_a^\pm(q) - \mathcal{O}_a^{\pm\dagger}(q))$$
$$= \frac{i}{4\pi} Q_k \int d^2x \left(C^{\pm a}(x) + \frac{1}{2} N^{\pm a}(x) \right) \frac{\hat{p}_k \cdot \varepsilon_a(x)}{\hat{p}_k \cdot \hat{q}(x)},$$
$$\sum_{k=1}^{n} R_k^{\eta_k} = -\frac{i}{e^2} \int d^2x \left(C^a \mathcal{J}_a^+ + C^{-a} \mathcal{J}_a^- + \frac{1}{2} N^{+a} \mathcal{J}_a^+ - \frac{1}{2} N^{-a} \mathcal{J}_a^- \right).$$

- One finds that the dressings cancel IR divergence: the path integral evaluates to a finite number!

$$\langle 0, +| \, \widetilde{O}_1 \cdots \widetilde{O}_n \, |0, -\rangle \, |_{\text{soft}} = \int [dC^+][dC^-][dN^+][dN^-] \exp\Big(\sum_{k=1}^{n} R_k^{\eta_k} \Big) e^{iS + \frac{i}{e^2} \int d^2x (C^{+a} \mathcal{J}_a^+ + C^{-a} \mathcal{J}_a^-)} = 1$$

General IR-finite dressed states

- It is well known that Faddeev-Kulish states are not the only states that cancel IR divergence. We can also see this from our generalized soft effective action.

- Consider a general dressing such that

$$\sum_{k=1}^{n} R_k^{\eta_k} = -\frac{i}{e^2}\int d^2x \left(C^a \mathcal{P}_a^+ + C^{-a}\mathcal{P}_a^- + N^{+a}\mathcal{Q}_a^+ + N^{-a}\mathcal{Q}_a^-\right)$$

It turns out that the only necessary condition for IR-finiteness, i.e. $\langle 0,+|\, \widetilde{O}_1 \cdots \widetilde{O}_n \,|0,-\rangle\,|_{\text{soft}} = 1$, is

$$\mathcal{P}_a^+(x) + \mathcal{P}_a^-(x) = \mathcal{J}_a^+(x) + \mathcal{J}_a^-(x).$$

- There is absolutely no constraint on $\mathcal{Q}_a^\pm$, which means there is an infinite set of IR-finite dressed states.

- Observe that the crucial ingredient is C^a, which is why we choose N eigenstate as the perturbative vacuum. If it is a C eigenstate, then this part of the dressing becomes trivial.

Gravity

- The generalized soft effective action has exactly the same form in gravitational theories as the one in gauge theory,

$$S = \frac{i\alpha}{2\pi\kappa^2}\int d^2x\, N_{ab}(x)N^{ab}(x) - \frac{1}{\kappa^2}\int d^2x\left(C^{+ab}(x)N^+_{ab}(x) - C^{-ab}(x)N^-_{ab}(x)\right), \qquad \kappa^2 = 32\pi G_N$$

  where $N^{\pm}_{ab}(x) = \partial_a\partial_b N^{\pm}(x)$ is the soft graviton and $C_{ab}(x) = \partial_a\partial_b C(x)$ is the supertranslation Goldstone, and $N_{ab} = N^+_{ab} - N^-_{ab}$.

- The path integral relevant for a scattering process involving m soft gravitons and n hard particles is

$$\begin{aligned}&\langle 0,+|\, N_{a_1b_1}\cdots N_{a_mb_m}O_1\cdots O_n\,|0,-\rangle\,|_{\text{soft}}\\ &= \int [dC^+][dC^-][dN^+][dN^-]N_{a_1b_1}\cdots N_{a_mb_m}\exp\left[iS + \frac{i}{\kappa^2}\int d^2x(C^{+ab}\mathcal{J}^+_{ab} + C^{-ab}\mathcal{J}^-_{ab})\right]\end{aligned}$$

  where $\mathcal{J}^+_{ab}$ and $\mathcal{J}^-_{ab}$ are the soft graviton factors for out and in particles respectively.

- One can derive the antipodal matching condition, soft theorem, IR divergence, and Faddeev-Kulish states in exactly the same way as in gauge theory.

Schwinger-Keldysh formalism

- It is now well understood that IR divergences in gauge/gravity theories are associated with vaccuum transitions in scattering amplitudes.

- Quantum systems that undergo transitions are said to be out-of-equilibrium, and the Schwinger-Keldysh formalism is a framework for describing such systems.

- Correlators in such systems are described by the Schwinger-Keldysh action

$$S_{SK} = S[q_+] - S[q_-] + I_{FV}[q_+, q_-]$$

  where $S[q]$ is the action of the system, and I_{FV} is the Feynman-Vernon influence functional that describes the interaction of the open system with the environment.

- The generalized soft effective action has a natural interpretation as a Schwinger-Keldysh action if we make the identification

$$S[q_\pm] = -\frac{1}{e^2}\int d^2x\, C^{\pm a} N_a^\pm, \qquad I_{FV}[q_+, q_-] = \frac{i\alpha}{2\pi e^2}\int d^2x (N_a^+ - N_a^-)^2.$$

Summary

- We generalize the soft effective action by distinguishing future and past Goldstone modes instead of imposing matching condition by hand.
- In addition, we choose the N eigenstate with eigenvalue $\partial_a N = 0$ as the Lorentz-invariant perturbative vacuum, instead of the C eigenstate assumed in the literature.
- This generalized soft effective action exhibits complicated 4D infrared physics such as the antipodal matching condition, soft theorems, IR divergences, and IR-finite dressed states in a remarkably simple way.
- The generalized action has a natural and interesting interpretation as a Schwinger-Keldysh action that describes an open system coupled to an environment.

Thank you!