Exceptional Structures and The Standard Model

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Outline

My goal is not to say something new, but rather describe several ideas that I find important

- $\mathrm{Spin}(10) \supset \mathrm{Spin}(8)$ is octonionic

- The minimal dimension of an internal KK manifold M with G_{SM} as the group of isometries is seven

  But odd-dimensional M can never lead to chiral fermions, which gives eight as the minimal dimension

  In both dimensions exceptional holonomy is possible, so potentially relevant

- $G_{\mathrm{SM}} \subset \mathrm{Spin}(10)$ is the stabiliser of a certain geometric structure, which is easiest to describe using octonions

  Choice of a unit imaginary octonion encodes the choice of this structure

Dubois-Violette, Todorov, Furey, Hudges, Boyle

Spin(8) and octonions

Spin(8) is octonionic, as implied by triality

V, S_+, S_- Smallest dimension irreps of Spin(8), all $\mathbb{R}^8$ as vector spaces

$$v|\psi_+\rangle = |\psi_-\rangle \qquad v \in V, \psi_\pm \in S_\pm$$

This is the octonionic product $\mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$

More concretely, Cl(8) is generated by

$$\Gamma_x = \begin{pmatrix} 0 & L_{\bar{x}} \\ L_x & 0 \end{pmatrix} \qquad x \in \mathbb{O}$$

L_x operator of left multiplication by an octonion x. Very concretely, 8x8 matrices

Spin(8) spinors are real and $S_\pm \sim \mathbb{O}$

Spin(10) and octonions

Any Spin group containing Spin(8) admits an octonionic description

Cl(10) is generated by the following 32 x 32 matrices

$$\Gamma_x = \begin{pmatrix} 0 & 0 & 0 & L_{\bar{x}} \\ 0 & 0 & L_x & 0 \\ 0 & L_{\bar{x}} & 0 & 0 \\ L_x & 0 & 0 & 0 \end{pmatrix} \quad \Gamma_9 = \begin{pmatrix} 0 & 0 & \mathbb{I}_8 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_8 \\ \mathbb{I}_8 & 0 & 0 & 0 \\ 0 & -\mathbb{I}_8 & 0 & 0 \end{pmatrix} \quad \Gamma_{10} = \begin{pmatrix} 0 & 0 & -i\mathbb{I}_8 & 0 \\ 0 & 0 & 0 & -i\mathbb{I}_8 \\ i\mathbb{I}_8 & 0 & 0 & 0 \\ 0 & i\mathbb{I}_8 & 0 & 0 \end{pmatrix}$$

Γ_x, Γ_9 are real, so Spin(9) spinors are real $\quad S^{\mathrm{Spin}(9)} \sim \mathbb{O}^2$

Γ_{10} is imaginary, so Spin(10) spinors are complex $\quad S_{\pm}^{\mathrm{Spin}(10)} \sim \mathbb{O}_{\mathbb{C}}^2$

Octonionic description of the Lie algebra

$$\mathfrak{spin}(10)\Big|_{S_+} \ni \begin{pmatrix} A + ia & -L_{\bar{x}} + iL_{\bar{y}} \\ L_x + iL_y & A' - ia \end{pmatrix} \qquad A, A' \in \mathfrak{spin}(8), \quad a \in \mathbb{R}, \quad x, y \in \mathbb{O}$$

This model is from R. Bryant, but also from “Spinors and Calibrations” by R. Harvey

These facts, together with the experimental relevance of Spin(10), explain why octonions are relevant for SM fermions

SM gauge group as group of isometries

In Kaluza-Klein compactification, the gauge group arises as the group of isometries of the internal space M

Witten '81 - The minimal dimension of M with $G_{\rm SM} = {\rm SU}(3) \times {\rm SU}(2) \times {\rm U}(1)$ is seven $M_7 = \mathbb{CP}^2 \times \mathbb{CP}^1 \times S^1$

This was an exciting observation because 4+7=11

And 11 is the maximal dimension where SUGRA exists

But was also realised that compactification on M_7 cannot gives rise to chiral fermions

The minimal even dimension of M with $G_{\rm SM}$ as the group of isometries is eight $\dim(M_8 \times M_{3,1}) = 12$

In this dimension and with signature (11,1) there is a very attractive option for the spinor Lagrangian

$$\int_{M_8 \times M_{3,1}} \langle \hat\Psi, \not{D}\Psi \rangle$$

$\Psi \in S_+^{{\rm Spin}(11,1)} \sim \mathbb{C}^{32}$ chiral spinor

$\hat\Psi \in S_-^{{\rm Spin}(11,1)}$ complex conjugate spinor of opposite chirality

Note that no mass term is possible! $\langle \hat\Psi, \Psi \rangle$ - does not exist

Weyl spinors are necessarily massless in this dimension!

Dimensional reduction

Under dimensional reduction $\not{D} = \not{D}_8 \otimes \mathbb{I}_{3,1} + \mathbb{I}_8 \otimes \not{D}_{3,1}$

$$\Psi = \psi^8_+ \otimes \psi^{3,1}_+ + \psi^8_- \otimes \psi^{3,1}_-$$

$$\hat{\Psi} = \hat{\psi}^8_+ \otimes \hat{\psi}^{3,1}_+ + \hat{\psi}^8_- \otimes \hat{\psi}^{3,1}_-$$

S^8_+ $S^{3,1}_-$ S^8_- $S^{3,1}_+$

$$\langle \hat{\Psi}, \not{D}\Psi \rangle = \langle \hat{\psi}^8_+ \otimes \hat{\psi}^{3,1}_+ + \hat{\psi}^8_- \otimes \hat{\psi}^{3,1}_-, \not{D}_8 \psi^8_+ \otimes \psi^{3,1}_+ + \not{D}_8 \psi^8_- \otimes \psi^{3,1}_- + \psi^8_+ \otimes \not{D}_{3,1}\psi^{3,1}_+ + \psi^8_- \otimes \not{D}_{3,1}\psi^{3,1}_- \rangle$$

$$= \quad \langle \hat{\psi}^8_+, \not{D}_8 \psi^8_- \rangle \langle \hat{\psi}+^{3,1}, \psi^{3,1}_- \rangle + \langle \hat{\psi}^8_+, \psi^8_+ \rangle \langle \hat{\psi}+^{3,1}, \not{D}_{3,1}\psi^{3,1}_+ \rangle + \langle \hat{\psi}^8_-, \not{D}_8 \psi^8_+ \rangle \langle \hat{\psi}-^{3,1}, \psi^{3,1}_+ \rangle + \langle \hat{\psi}^8_-, \psi^8_- \rangle \langle \hat{\psi}-^{3,1}, \not{D}_{3,1}\psi^{3,1}_- \rangle$$

Mass terms for 4D spinors

Kinetic terms for 4D spinors

Dimensional reduction continued

The non-zero eigenvalues of the internal Dirac operator are large (small internal space)

Only Dirac zero modes (harmonic spinors) are relevant at 4D low energies

If the spectrum of harmonic spinors on the internal space was chiral, would get chiral 4D massless fermions

Unfortunately, have the following no-go: Non-trivial isometry -> Harmonic spinors are non-chiral

Atiyah-Hirzebruch '70 : Any compact spin manifold admitting a non-trivial action of a compact connected Lie group has vanishing $\hat{A}$ -genus. Hence the dimensions of the spaces of positive and negative chirality harmonic spinors are the same.

Even stronger statement if scalar curvature positive: No harmonic spinors at all (Lichnerowicz)

But these obstructions disappear if one switches on a fundamental U(1) gauge field

Examples: harmonic spinors exist and are of a single chirality on $\mathbb{CP}^1, \mathbb{CP}^2$ with U(1) gauge field turned on.

So, it is possible that the KK scenario can still lead to interesting developments

Related to the subject of this workshop because geometry in eight dimensions is deeply related to octonions

In particular, special holonomy (Kahler, hyper-Kahler) is relevant, and exceptional holonomy Spin(7) potentially relevant

Geometric structures stabilised by G_{SM}

There are several basic observations:

- Choosing a unit imaginary octonion $\mathbf{u} \in \mathbb{O}$ endows $\mathbb{O}$ with a complex structure $J = L_{\mathbf{u}}$

  There is another complex structure, which is also important and relevant $J' = R_{\mathbf{u}}$

- This accomplishes two things: $\mathbb{O} \sim \mathbb{C}^4$

  But also have a preferred copy of the complex plane $\mathbb{C} = \mathrm{Span}(1, \mathbf{u})$ so that $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$

  This can be understood nicely by introducing $I = JJ' : I^2 = \mathbb{I}$

  Then $\mathbb{C} \subset \mathbb{O}$ is the eigenspace of eigenvalue +1 of I

- Given that spinors of Spin(8), Spin(9), Spin(10) are octonions $S_{\pm}^{\mathrm{Spin}(8)} = \mathbb{O}$ $S^{\mathrm{Spin}(9)} = \mathbb{O}^2$ $S_{\pm}^{\mathrm{Spin}(10)} = \mathbb{O}_{\mathbb{C}}^2$

  A very natural question to ask is what is the subgroup of these groups that commutes with the chosen $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$

  (and thus with both complex structures on the space of spinors)

Stabilisers of J, J'

$\mathrm{Spin}(8) \supset G_J = \mathrm{SU}(4)$

$\mathrm{Spin}(9) \supset G_J = \mathrm{SU}(4) \times \mathrm{SU}(2)/\mathbb{Z}_2$

$\mathrm{Spin}(10) \supset G_J = G_{\mathrm{PS}} = \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$

$\mathrm{Spin}(8) \supset G_{J,J'} = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(3))$

From KK: 1912.11282

$\mathrm{Spin}(9) \supset G_{J,J'} = G_{\mathrm{SM}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\mathbb{Z}_6.$

$\mathrm{Spin}(10) \supset G_{J,J'} = G_{\mathrm{LR}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\mathbb{Z}_6$

From Boyle: 2006.16265

This seems clearly relevant to the question of how Spin(10) breaks to G_{SM}

But also clear that can only be a part of the answer, because need to break the symmetry further

Pure spinors

Another set of basic observations: There are several different types of orbits of Spin(10) in $S_{\pm}^{\mathrm{Spin}(10)}$

Pure spinor - stabilised by $G_\psi = \mathrm{SU}(5)$

Real unit spinor - stabilised by $G_\psi = \mathrm{Spin}(7)$

General impure spinor - stabilised by $G_\psi = \mathrm{SU}(4)$

Pure spinor of Spin(10) defines a complex structure on $\mathbb{R}^{10}$

Example of a pure spinor in octonionic description $\psi = \begin{pmatrix} 1 + i\mathbf{u} \\ 0 \end{pmatrix}$ $\mathbf{u}$ -unit imaginary octonion

Theorem: (KK 2209.05088) G_{SM} is the subgroup of Spin(10) that stabilises a pure spinor ψ and projectively stabilises another pure spinor ψ' with ψ, ψ' such that $\psi + \psi'$ is again a pure spinor.

Concretely, can take $\psi = \begin{pmatrix} 1 + i\mathbf{u} \\ 0 \end{pmatrix}$ $\psi' = \begin{pmatrix} 0 \\ 1 + i\mathbf{u} \end{pmatrix}$

Symmetry breaking by copies of Higgs in the spinor representation

Summary and Outlook

The relevance of Spin(10) to SM is the standard GUT fact

Spin(10) is simplest to describe using octonions

Some natural geometric structures are easiest to describe in this language

Commutants of such structures are very close or coincide with SM gauge group

But these ideas are still kinematical

What is missing is a convincing dynamical principle that incorporates these ideas

It is possible that the way to make progress is to marry these kinematical ideas with 12 -> 4 KK dimensional reduction idea

Thank you!