STANDARD MODEL SYMMETRIES

AND THE

NESTED EMBEDDINGS

OF

$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$

N. FUREY

EXCEPTIONAL STRUCTURES + SM

U. EDINBURGH 2026.07.13

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BASIC RESULT ALREADY ONLINE

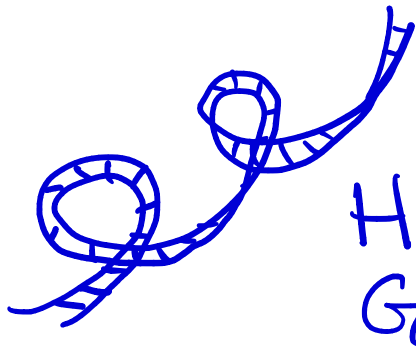
A SUPERALGEBRA WITHIN...

arXiv: 2505.07923

~ NEW VERSION ~

IDEA : NATURE'S EFFICIENCY

NATURE : ACGT $\Rightarrow$



HUMAN GENOME

HOW? SEQUENCES.

US: $a_i \in \mathbb{A}$ $\Rightarrow$ $\sum_i a_1(a_2(a_3(a_4)))$

SM IRREPS

HOW? SEQUENCES.

$\mathbb{A}$ is 16 $\mathbb{R}$ DIM $\Rightarrow$ $\mathrm{End}_{\mathbb{R}}(\mathbb{A}) = M_{16}(\mathbb{R}) \simeq Cl(0,8)$

WHY $C\ell(0,8)$?

(1) SIZE

ROUGHLY OFF-SHELL UNCONSTRAINED DOF OF SM GAUGE BOSONS, 3 GEN FERMIONS, HIGGS, $3 \times \nu_R$ $(2^{44}\mathbb{R})$

(2) BOTT PERIODICITY.

$$C\ell(s,t) \simeq C\ell(s_0,t_0) \otimes C\ell(0,8) \otimes C\ell(0,8) \otimes \cdots$$

$$s_0, t_0 \in \{0,\ldots,7\}$$

$$\boxed{C\ell(0,8) \longleftrightarrow \text{SPACE OF 1-PARTICLE STATES}}$$

Bott Periodic Fock Space

$$
\begin{aligned}
F_{s_0,t_0} := \; & Cl(s_0,t_0) && |0\rangle \\
\oplus \; & Cl(s_0,t_0)\otimes Cl(0,8) && |p_i\rangle \\
\oplus \; & Cl(s_0,t_0)\otimes Cl(0,8)\otimes Cl(0,8) && |p_j\rangle|p_k\rangle \\
\oplus \; & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots && \vdots
\end{aligned}
$$

$$F := \sum_{s_0,t_0=0}^{7} F_{s_0,t_0} \longleftarrow \{Cl(s,t)\}$$

To Do: (Anti-)Symmetrization, etc.

PROBLEM:

$Cl(0,8) \rightsquigarrow$ PARTICLE IRREPS?

CLUE:

$\mathcal{H}_{16}(\mathbb{C})$

$p^\ell_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	$b_R^\dagger$ $b_R^\dagger$	$(\nu_e, e)_L^\dagger$ $(\nu_\mu, \mu)_L^\dagger$	$e_R^\dagger$ $\mu_R^\dagger$	$\nu_{eR}^\dagger$ $\nu_{\mu R}^\dagger$
b_R b_R	$p^q_\mu\sigma^\mu$ $G^j_\mu\Lambda_j\sigma^\mu$ $B_\mu\sigma^\mu$	$(u, d)_L$ $(c, s)_L$	u_R c_R	d_R s_R
$(\nu_e, e)_L$ $(\nu_\mu, \mu)_L$	$(u, d)_L^\dagger$ $(c, s)_L^\dagger$	$p^a_\mu\sigma^\mu$ $W^m_{L\mu}\tau_m\sigma^\mu$	(h) (h)	$(\nu_\tau, \tau)_L$ $(\nu_\tau, \tau)_L$
e_R μ_R	$u_R^\dagger$ $c_R^\dagger$	$(h^\dagger)$ $(h^\dagger)$	$p^{\bar{a}}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	τ_R τ_R
ν_{eR} $\nu_{\mu R}$	$d_R^\dagger$ $s_R^\dagger$	$(\nu_\tau, \tau)_L^\dagger$ $(\nu_\tau, \tau)_L^\dagger$	$\tau_R^\dagger$ $\tau_R^\dagger$	$p^{\bar{a}}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$

MISSING:
$(t, b)_L$ t_R

EXTRA:
b_R' t_R' n h'

$\Downarrow$

HEAVIEST QUARKS COMPOSITE?

A SUPERALGEBRA WITHIN...
arXiv: 2505.07923

GENERATIONS: 3 PRINTS...
JHEP 2014

CLUE:

PEIRCE DECOMP?

$\mathcal{H}_{16}(\mathbb{C})$

$p^\ell_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	$b_R^\dagger$ $b_R^\dagger$	$(\nu_e,\, e)_L^\dagger$ $(\nu_\mu,\, \mu)_L^\dagger$	$e_R^\dagger$ $\mu_R^\dagger$	$\nu_{eR}^\dagger$ $\nu_{\mu R}^\dagger$
b_R b_R	$p^q_\mu\sigma^\mu$ $G^j_\mu\Lambda_j\sigma^\mu$ $B_\mu\sigma^\mu$	$(u,\, d)_L$ $(c,\, s)_L$	u_R c_R	d_R s_R
$(\nu_e,\, e)_L$ $(\nu_\mu,\, \mu)_L$	$(u,\, d)_L^\dagger$ $(c,\, s)_L^\dagger$	$p^a_\mu\sigma^\mu$ $W^m_{L\mu}\tau_m\sigma^\mu$	(h) (h)	$(\nu_\tau,\, \tau)_L$ $(\nu_\tau,\, \tau)_L$
e_R μ_R	$u_R^\dagger$ $c_R^\dagger$	$(h^\dagger)$ $(h^\dagger)$	$p^{\bar a}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	τ_R τ_R
ν_{eR} $\nu_{\mu R}$	$d_R^\dagger$ $s_R^\dagger$	$(\nu_\tau,\, \tau)_L^\dagger$ $(\nu_\tau,\, \tau)_L^\dagger$	$\tau_R^\dagger$ $\tau_R^\dagger$	$p^{\bar a}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$

A SUPERALGEBRA WITHIN... GENERATIONS: 3 PRINTS...

arXiv: 2505.07923 JHEP 2014

CLUE:

$\mathcal{H}_{16}(\mathbb{C})$

$p^\ell_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	$b^\dagger_R$ $b^\dagger_R$	$(\nu_e,\, e)^\dagger_L$ $(\nu_\mu,\, \mu)^\dagger_L$	$e^\dagger_R$ $\mu^\dagger_R$	$\nu^\dagger_{eR}$ $\nu^\dagger_{\mu R}$	
b_R b_R	$p^q_\mu\sigma^\mu$ $G^j_\mu\Lambda_j\sigma^\mu$ $B_\mu\sigma^\mu$	$(u,\, d)_L$ $(c,\, s)_L$	u_R c_R	d_R s_R	$\mathbb{O}$
$(\nu_e,\, e)_L$ $(\nu_\mu,\, \mu)_L$	$(u,\, d)^\dagger_L$ $(c,\, s)^\dagger_L$	$p^a_\mu\sigma^\mu$ $W^m_{L\mu}\tau_m\sigma^\mu$	(h) (h)	$(\nu_\tau,\, \tau)_L$ $(\nu_\tau,\, \tau)_L$	$\mathbb{H}$
e_R μ_R	$u^\dagger_R$ $c^\dagger_R$	$(h^\dagger)$ $(h^\dagger)$	$p^{\bar{a}}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	τ_R τ_R	$\mathbb{C}$
ν_{eR} $\nu_{\mu R}$	$d^\dagger_R$ $s^\dagger_R$	$(\nu_\tau,\, \tau)^\dagger_L$ $(\nu_\tau,\, \tau)^\dagger_L$	$\tau^\dagger_R$ $\tau^\dagger_R$	$p^{\bar{a}}_\mu\sigma^\mu$ $B_\mu\sigma^\mu$	$\mathbb{R}$

A SUPERALGEBRA WITHIN... arXiv: 2505.07923

GENERATIONS: 3 PRINTS... JHEP 2014

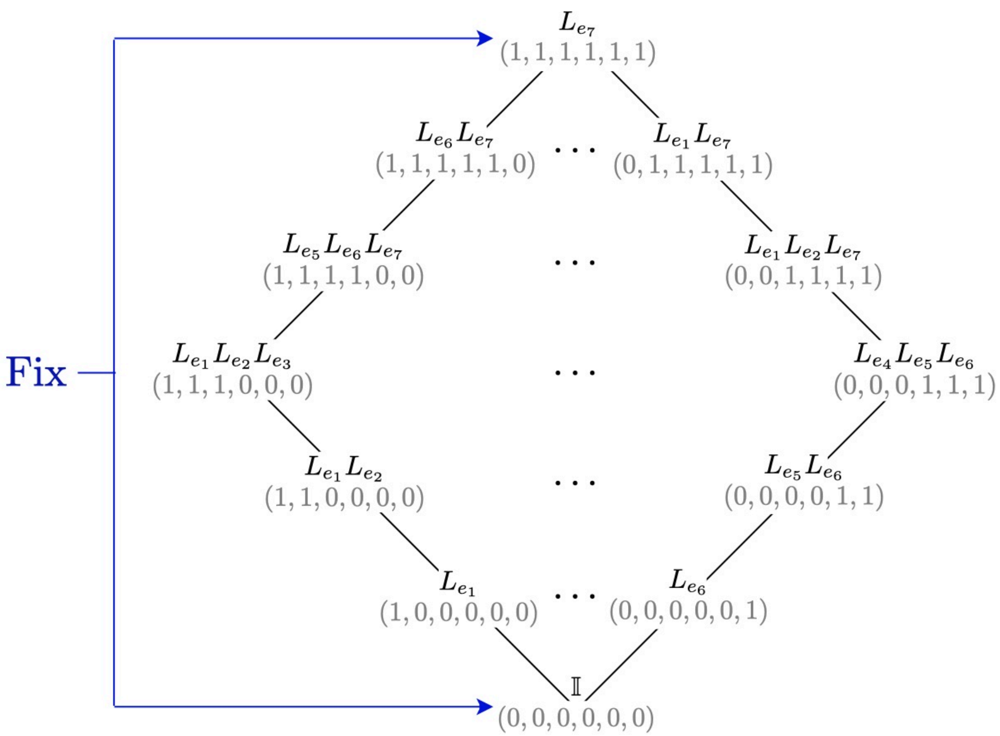
STEPPING STONE

$$\underline{\underline{A}} = \mathbb{D} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$$

$$15\ \mathbb{R}$$

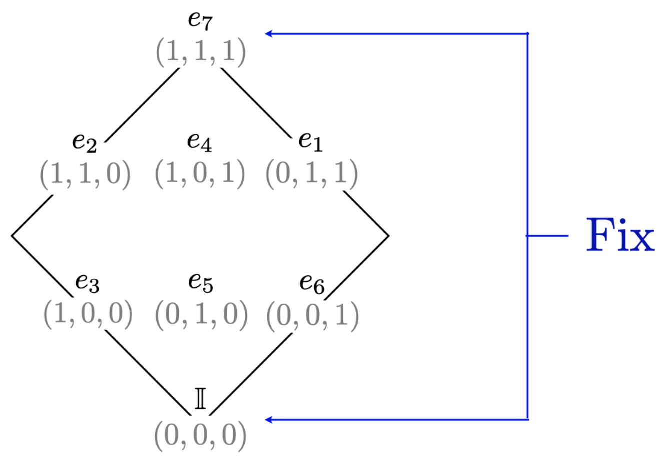
Find $\mathfrak{g}_{SM} = su(3) \oplus su(2) \oplus u(1)$

(1)



$\mathrm{End}_{\mathbb{R}}(\mathbb{O})$

$\mathbb{Z}_2^6$-graded



$\mathbb{O}$

$\mathbb{Z}_2^3$-graded

Günaydin + Gürsey JMP 1973

FIND $\mathfrak{g}_{SM} = su(3) \oplus su(2) \oplus u(1)$

(2) $\mathrm{Tr}(J_{\mathbb{O}}\, \ell_{\mathbb{O}}) = \mathrm{Tr}(J_{\mathbb{H}}\, \ell_{\mathbb{H}}) = \mathrm{Tr}(J_{\mathbb{C}}\, \ell_{\mathbb{C}})$

$$\Rightarrow \ell \in su(3)_c \oplus su(2)_L \oplus u(1)_Y$$

$$\left.\begin{aligned} Y &= \frac{1}{3} P_{\mathbb{O}_2} + \frac{1}{2} P_{\mathbb{H}} + P_{\mathbb{C}} \\ Q &= \frac{1}{3} P_{\mathbb{O}_2} + P_{\mathbb{H}_2} + P_{\mathbb{C}} \end{aligned}\right\} \text{SIMPLE}$$

EACH TERM $\sim \frac{1}{n} \mathbb{1}_{n \times n}$ MAX. MIXED STATE

EMBED $\underbrace{\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}}_{15\,\mathbb{R}}$ IN $\underbrace{\mathbb{V}}_{16\,\mathbb{R}}$

$$\mathbb{V} = \mathbb{O} \oplus \mathbb{O}, \quad \mathbb{S}, \quad \mathbb{C} \otimes \mathbb{O} = \mathbb{O} + i\,\mathbb{O}, \quad \ldots$$

PLEASE SEE MODELS OF MANY AUTHORS

EMBED $\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ IN $\mathbb{V}$

$15\,\mathbb{R}$ $\qquad$ $16\,\mathbb{R}$

$$\mathbb{V} = \underbrace{\mathbb{O} \oplus \mathbb{O}}_{e_i = 1},\ \underbrace{\mathbb{S}}_{e_i = e_8},\ \mathbb{C} \otimes \mathbb{O} = \mathbb{O} + \underbrace{i\,\mathbb{O}}_{e_i = i},\ \ldots$$

$$\mathbb{V} = e_i\mathbb{O} \oplus \underbrace{e_j\mathbb{H} \oplus \underbrace{e_k\mathbb{C} \oplus \underbrace{e_\ell\mathbb{R} \oplus \underbrace{\mathbb{R}}_{\mathbb{R}}}_{\mathbb{C}}}_{\mathbb{H}}}_{\mathbb{O}}$$

EMBED $\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ IN $\mathbb{V}$

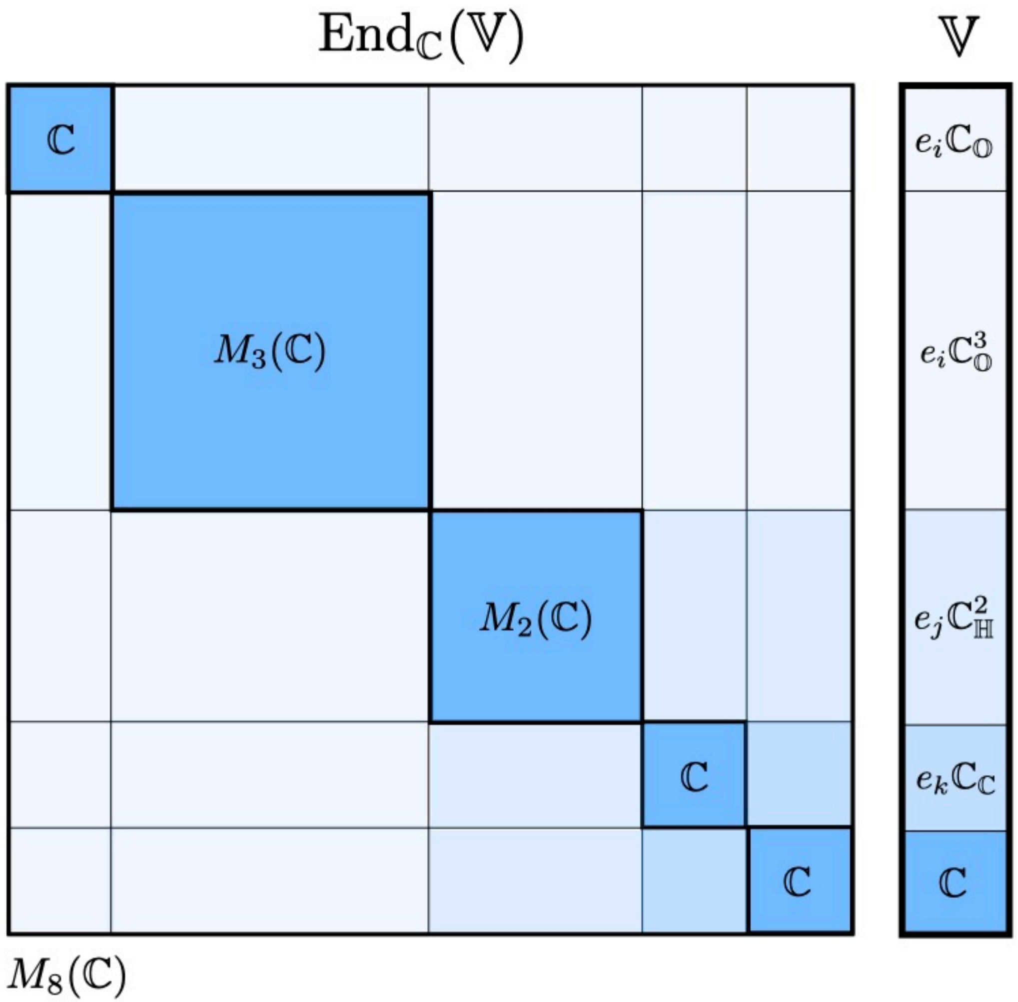
15 $\mathbb{R}$ 16 $\mathbb{R}$

$\mathbb{V} = \mathbb{O} \oplus \mathbb{O}$, $\mathbb{S}$, $\mathbb{C} \otimes \mathbb{O} = \mathbb{O} + i\,\mathbb{O}$, ...

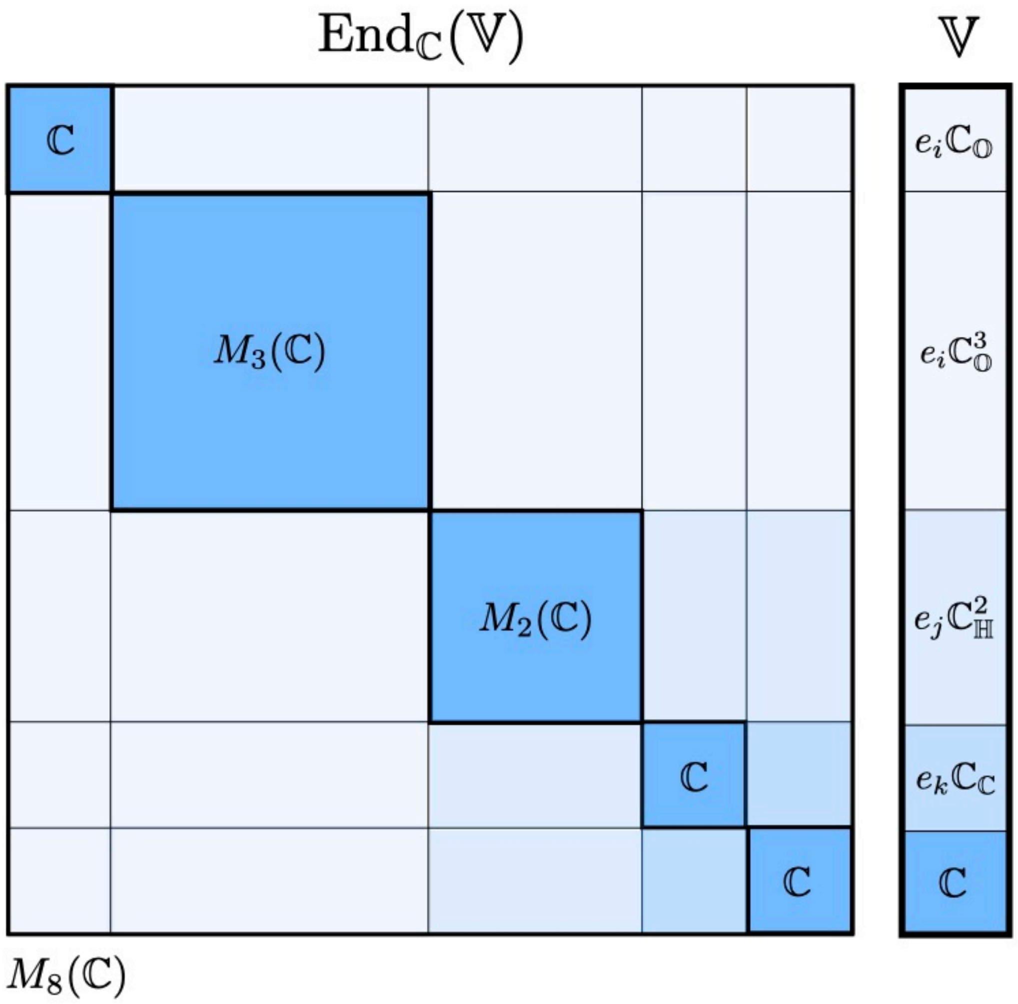
$e_i = 1$ $e_i = e_8$ $e_i = i$

$$\mathbb{V} = e_i\mathbb{O} \oplus \underbrace{e_j\mathbb{H} \oplus \underbrace{e_k\mathbb{C} \oplus \underbrace{e_\ell\mathbb{R} \oplus \underbrace{\mathbb{R}}_{\mathbb{R}}}_{\mathbb{C}}}_{\mathbb{H}}}_{\mathbb{O}}$$

$\mathbb{V} = \mathbb{S} \Rightarrow$ ENTIRELY CAYLEY-DICKSON TOWER


$\mathrm{End}_{\mathbb{C}}(\mathbb{V})$
$\mathbb{V}$
$\mathbb{C}$
$M_3(\mathbb{C})$
$M_2(\mathbb{C})$
$\mathbb{C}$
$\mathbb{C}$
$M_8(\mathbb{C})$
$e_i\mathbb{C}_{\mathbb{O}}$
$e_i\mathbb{C}^3_{\mathbb{O}}$
$e_j\mathbb{C}^2_{\mathbb{H}}$
$e_k\mathbb{C}_{\mathbb{C}}$
$\mathbb{C}$

ON-SHELL COUNTING



MISSING:
$(t, b)_L \quad t_R$

EXTRA:
$b'_R \quad t'_R \quad \nu h'$

$\Downarrow$

HEAVIEST QUARKS COMPOSITE?

SAME INTERNAL IRREPS AS A SUPERALGEBRA WITHIN... arXiv: 2505.07923

Cousin:

$$e_8 = \mathrm{tri}(\mathbb{O}_1) \oplus \mathrm{tri}(\mathbb{O}_2) \oplus 3 \cdot \mathbb{O}_1 \otimes \mathbb{O}_2$$

$$\mathbb{O}_1 \otimes \mathbb{O}_2 \mapsto \mathbb{O}_1 \otimes (e_i \mathbb{H} \oplus e_j \mathbb{C} \oplus e_k \mathbb{R} \oplus \mathbb{R})$$

$\Rightarrow$ Similar particle content to SuperAlg. within...
arXiv: 2505.07923