EXCEPTIONAL JORDAN ALGEBRA, QUANTUM MECHANICS , SPACE-TIME SUPERSYMMETRY AND THE STANDARD MODEL

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Broad outline of the topics I will try to cover in my talk:

- Jordan formulation of quantum mechanics.
- Octonionic quantum mechanics and the exceptional Jordan algebra.
- Exceptional Lie algebras and the exceptional Jordan algebra. Magic square of Freudenthal, Rozenfeld and Tits and symmetries of elementary particles.
- Generalized space-times coordinatized by Jordan algebras and space-time supersymmetry.
- Magical supergravity theories
- Octonionic magical supergravity(OMS) versus maximal $N = 8$ supergravity and the exceptional Jordan algebra.
- Minimal unitary representation of $E_{7(-25)}$ and the composite scenario to accommodate the physics of the standard model
- Minimal unitary representation of $E_{7(-25)}$ and bosonic supersymmetry in ten dimensions.
- Quantum degeneracies of BPS black holes of 5d OMS and singular modular forms of $E_{7(-25)}$
- Quantum completion of octonionic magical supergravity theory and other open problems

JORDAN FORMULATION OF QUANTUM MECHANICS

- In the years 1932-33 Pascual Jordan proposed a novel formulation of the quantum mechanics that came to be called Jordan formulation. His main motivation was to generalize the quantum mechanical formalism such that the then observed beta decay phenomena can be explained within this generalized framework! He argued that the commutator of Hermitian operators corresponding to observables that act on an Hilbert space does not preserve their hermiticity:

$$H_1^\dagger = H_1 \quad , \quad H_2^\dagger = H_2 \longrightarrow \quad [H_1, H_2]^\dagger \neq [H_1, H_2]$$

Jordan proposed using the symmetric anti-commutator product among Hermitian operators under which they remain Hermitian. Under the symmetric product $H_1 \circ H_2 = 1/2(H_1 H_2 + H_2 H_1)$ Hermitian operators acting on the physical Hilbert space satisfy the identities:

$$H_1 \circ H_2 = H_2 \circ H_1$$

$$H_1 \circ (H_2 \circ H_1^2) = (H_1 \circ H_2) \circ H_1^2$$

which are taken as defining identities of Jordan algebras.
Jordan hoped that there would be a rich family of algebras satisfying the above identities and which can not be realized in terms of linear operators acting on a vector (Hilbert) space with the product taken as 1/2 the anticommutator.

Dirac formulation of quantum mechanics over an Hilbert space

Pure states $\Longleftrightarrow$ rays $|\psi\rangle q$ with $\bar{q}q = 1$

Jordan formulation

Pure states $\Longleftrightarrow$ $|\psi\rangle\langle\psi| = P_\psi = P_\psi^2$

Propositional calculus $\Longleftrightarrow$ Projective geometry

Axioms of quantum mechanics $\Longleftrightarrow$ Axioms of projective geometry

Pure states: $Tr(P_\psi) = 1$ $\Longleftrightarrow$ Points in projective geometry

Superpositions of $|\psi\rangle$ and $|\xi\rangle$ $\Longleftrightarrow$ Line connecting the points P_ψ and P_ξ

- Jordan , von Neumann and Wigner (JvNW) (1934) gave a complete classification of all finite dimensional simple Jordan algebras and showed that with one exception all finite dimensional simple Jordan algebras are special i.e. they can be realized as linear operators acting on a vector space with the product being 1/2 the anticommutator.
- The complete list simple Euclidean Jordan algebras are as follows:
  i) Dirac gamma matrices $\Gamma(d)$ in d Euclidean dimensions.
  ii) Jordan algebras $J_n^{\mathbb{R}}$ $J_n^{\mathbb{C}}$ $J_n^{\mathbb{H}}$ generated by $n \times n$ " Hermitian" matrices over reals $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$
  iii) Exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over the division algebra of octonions $\mathbb{O}$, which can not be realized in terms of linear operators acting on some vector space.
- The proposal of Pascual Jordan to generalize the algebraic framework of quantum mechanics led to a single novel algebraic structure, namely $J_3^{\mathbb{O}}$ in the finite dimensional case. Let alone explaining the beta decay phenomena it was not at all obvious at the time if all the axioms of quantum mechanics could be satisfied in the exceptional case due to its intrinsic non-associativity.
- During the subsequent decades the exceptional Jordan algebra $J_3^{\mathbb{O}}$ had a distinguished career in mathematics. After the work of JvNW mathematicians established deep connections between the exceptional Jordan algebra and the other exceptional groups F_4, E_6, E_7, E_8.

Freudenthal-Rozenfeld-Tits Magic square:

	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$\mathbb{H}$	$USp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

MG and Gürsey (1971):
The compact magic square contains basically all the internal symmetry groups of hadronic world that were uncovered by the physicists in the 1950s and 1960s. It begged the question whether the exceptional groups could also be relevant for the physics of elementary particles . In particular could one understand the observed internal symmetries by extending the underlying field of quantum mechanics from complex numbers to octonions!?
Our work led to the so-called algebraic confinement scheme in which quarks are represented by transverse octonionic fields with $SU(3)$ automorphisms identified as the color $SU(3)_C$. The states in color singlet sector are described by an ordinary complex Hilbert space. This proposal gave a nice mathematical model of the original suggestion of Gell-Mann that color quarks operate in a **fictitious** Hilbert space and only the color singlet sector is observable (Gell-Mann 1972). However this scheme did not incorporate dynamics. Shortly afterwards quantum chromodynamics was established as the correct theory of strong interactions in which the quarks are **confined**.

- Exceptional Jordan algebra as the basis of charge space including the color degrees of quarks. MG 1973.
- E_6 GUT (Gürsey, Ramond & Sikivie (1975))
  E_8 GUT (Bars & MG (1980) with family unification.
- $E_8 \times E_8$ appeared as gauge symmetry of the heterotic string (Gross, Harvey, Martinec and Rohm 1984)
- **Formulation of Octonionic Quantum Mechanics over the exceptional Jordan algebra satisfying the axioms of quantum mechanics as formulated by von Neumann.** MG, Piron , Ruegg (1978)
  Quantum Mechanics with projection operators belonging to the exceptional Jordan algebra $J_3^{\mathbb{O}}$. Corresponding projective geometry is the octonionic Moufang plane. The result of successive, compatible experiments do not depend on the order in which they are performed, in spite of the non-associativity of underlying octonions.
  The quantum mechanics described by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ describes two octonionic degrees of freedom and its projective geometry is non-Desarguian. It was hoped that there might exist infinite dimensional exceptional Jordan algebras that describe an extension of the octonionic quantum mechanics.
  However these hopes were dashed by the remarkable results of Zelmanov who showed that there are NO INFINITE DIMENSIONAL EXCEPTIONAL JORDAN ALGEBRAS (Zelmanov 1979-1983). (These results were referred to as "Russian revolution in Jordan algebras" by McCrimmon.)

QUADRATIC JORDAN FORMULATION OF QM

Hilbert Space Formulation	Jordan Formulation	Quadratic Jordan Formulation
$\mid \alpha >$	$\mid \alpha >< \alpha \mid \;=\; P_\alpha$	P_α
$H \mid \alpha >$	$H \circ P_\alpha$	$\Pi_H P_\alpha = \{H P_\alpha H\}$
$< \alpha \mid H \mid \beta >$	?	?
$< \alpha \mid H \mid \alpha >$	$Tr\; H \circ P_\alpha$	$Tr\; \Pi_{P_\alpha} H \;=\; Tr\; \{P_\alpha H P_\alpha\}$
$\mid< \alpha \mid H \mid \beta >\mid^2$	$Tr\; P_\alpha \circ \{H P_\beta H\}$	$Tr\; \Pi_{P_\alpha} \Pi_H P_\beta$
	$=\; Tr\; \{H P_\alpha H\} \circ P_\beta$	$=\; Tr\; \Pi_{P_\beta} \Pi_H P_\alpha$
$[H_1, H_2]$	?	$4(\Pi_{H_1} \circ \Pi_{H_2} - \Pi_{H_1 \circ H_2})$

Quadratic Jordan formulation of QM involves only the Jordan triple product:
$\{ABC\} \equiv (A \circ B) \circ C + A \circ (B \circ C) - (A \circ C) \circ B$ with $A \circ B = \frac{1}{2}(AB + BA)$

$$\Pi_H A \equiv \{HAH\}$$

Quadratic Jordan formulation extends to the octonionic quantum mechanics as well as to formulation of quantum mechanics over finite fields and to Jordan superalgebras, including the exceptional one. However one loses the probability interpretation if the trace form is not a real number.

(MG 1978,1991)

Space-time Supersymmetry

Early days of spacetime supersymmetry following the seminal work of Wess and Zumino (1974) two of the most important problems were:
1. How to formulate a local gauge theory of spacetime supersymmetry that necessarily requires gravity ?
2. Is there an exceptional superalgebra whose local gauge theory would lead to a unified theory of all interactions including gravity ?
My first paper on supersymmetry was an attempt to answer the second question in which the concept of generalized spacetimes coordinatized by Jordan algebras was introduced. MG (1975)
Minkowski spacetime can be coordinatized by Hermitian 2×2 matrices:
$x = \sigma_\mu x^\mu$ where $\sigma_\nu = (1_2, \vec{\sigma})$ can be considered as elements of the Jordan algebra $J_2^{\mathbb{C}}$ with the Jordan product taken as $1/2$ the anticommutator.
Automorphism group of $J_2^{\mathbb{C}} = SU(2) \rightarrow$ rotation group
Invariance group of the norm form of $J_2^{\mathbb{C}}$:
$N(x) = Det(x) = \eta_{\mu\nu} x^\mu x^\nu = SL(2, C) \rightarrow$ Lorentz group =reduced structure group
Linear fractional group of $J_2^{\mathbb{C}} = SU(2,2) \rightarrow$ Conformal group
Twistor theory is based on such a coordinatization of Minkowski space-time in $d = 4$.

Symmetry groups of generalized space-times coordinatized by the simple Jordan algebras of degree three

J	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$Rot(J)$	$SO(3)$	$SU(3)$	$USp(6)$	F_4
$Lor(J)$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
$Conf(J)$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$

Table: Simple EUCLIDEAN Jordan algebras of degree 3 and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. $J_3^{\mathbb{A}}$ denotes a Jordan algebra of 3×3 hermitian matrices over $\mathbb{A}$. These spacetimes are all causal.

J	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}_s}$	$J_3^{\mathbb{H}_s}$	$J_3^{\mathbb{O}_s}$
$Rot(J)$	$SO(3)$	$SL(3,R)$	$Sp(6,R)$	$F_{4(4)}$
$Lor(J)$	$SL(3,\mathbb{R})$	$SL(3,R)\times SL(3,R)$	$SL(6,R)$	$E_{6(6)}$
$Conf(J)$	$Sp(6,\mathbb{R})$	$SL(6,R)$	$SO(6,6)$	$E_{7(7)}$

Table: Simple SPLIT Jordan algebras of degree 3 and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$ represent the split forms of composition algebras.

Tits Construction of the Magic Square

	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$\mathbb{H}$	$USp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	$F(4)$	E_6	E_7	E_8

Magic Square

$$L = Aut(\mathbb{A}) \oplus \mathbb{A}_0 \otimes J_{3_0}^{\mathbb{A}'} \oplus Aut(J_3^{\mathbb{A}'})$$

$$E_8 = G_2 \oplus (7 \otimes 26) \oplus F_4$$

Goal was to construct Lie superalgebras by replacing the real parameters multiplying the spinorial components of the Jordan algebra by Grassmann parameters in the Tits construction. MG 1975

Implicitly I was working with Jordan superalgebras which were defined and classified by Viktor Kac later. Kac 1977

UNIFIED CONSTRUCTION OF LIE AND LIE-SUPER ALGEBRAS OVER TRIPLE SYSTEMS

Bars , MG (1978)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$OSp(1/2)$	$SU(2/1)$	$OSp(4/2)$	$F(4)$
$\mathbb{C}$	$SU(1/2)$	$SU(1/2) \times SU(2/1)$	$SU(4/2)$	?
$\mathbb{H}$	$OSp(4/2)$	$SU(4/2)$	$OSp(4/8)$	?
$\mathbb{O}$	$F(4)$	?	?	?

Super Magic Square

Extension of the Kantor's construction of Lie algebras over triple systems to unified construction of Lie and Lie superalgebras. Taking the triple system over Grassmann variables leads to Lie superalgebras. (triple systems $\leftrightarrow$ three algebras)
$G(3) \supset G_2 \times SU(2)$
$F(4) \supset SO(7) \times SU(2)$
Construction of $G(3)$ and $F(4)$ using octonions Sudbery (1983)
NO SUPERALGEBRAS CORRESPONDING TO THE EXCEPTIONAL LIE ALGEBRAS OF THE E-SERIES OVER THE FIELD OF REAL AND COMPLEX NUMBERS.
Classification of simple Lie superalgebras Kac (1977)
Much later mathematicians have shown that super analogs of the exceptional Lie algebras of the E-series exist over the fields of characteristic 3 and 5 !!

Appearance of exceptional groups in maximal supergravity:

Cremmer and Julia (1978)

d	$\#vector\ \ fields$	$U\,duality\ \ group$	$scalar\ \ manifold$
$d=5$	27	$E_{6(6)}$	$\frac{E_{6(6)}}{USp(8)}$
$d=4$	28	$E_{7(7)}$	$\frac{E_{7(7)}}{SU(8)}$
$d=3$	$-$	$E_{8(8)}$	$\frac{E_{8(8)}}{SO(16)}$

Table: Global symmetries of maximal supergravity in $5, 4$ and 3 dimensions. Note that $E_{6(6)}$ and $E_{7(7)}$ are the Lorentz and conformal groups of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$.

$N=2$ MAXWELL-EINSTEIN SUPERGRAVITY THEORIES (MESGT) IN FIVE DIMENSIONS

MG, Sierra and Townsend (1983)

- Bosonic part of the $5D$ $N=2$ MESGT Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} &= -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) + \\ &\quad + \frac{e^{-1}}{6\sqrt{6}}\, C_{IJK}\, \varepsilon^{\mu\nu\rho\sigma\lambda} F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda \end{aligned}$$

  coupling of (n_V-1) vector multiplets $(A^a_\mu, \lambda^{ai}, \varphi^a)$ to $N=2$ supergravity $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$ $(I,J,K=1,\ldots,n_V,$ i=1,2 , $x,a=1,\ldots,(n_V-1)$ $)$

- $5D$, $N=2$ MESGT is uniquely determined by the constant symmetric tensor C_{IJK}.

- $5D$ MESGTs with symmetric scalar manifolds G/H such that G is a symmetry of the Lagrangian $\Longleftrightarrow$ C_{IJK} is given by the norm (determinant) $\mathcal{N}_3$ of a Euclidean Jordan algebra J of degree 3.

$$\mathcal{N}_3(J) = C_{IJK}h^I h^J h^K$$

  Euclidean $J :\Longleftrightarrow X^2 + Y^2 = 0 \Longrightarrow X = Y = 0 \;\forall X,\, Y \in J$

- Scalar manifold in $d=5$ is $\mathcal{M}_5 = \frac{Lorentz(J)}{Rotation(J)}$

- Scalar manifolds of 5d MESGTs defined by a Jordan algebra J of degree three are $\mathcal{M}_5 = Lor(J)/Rot(J)$.
- Unified $N=2$ Maxwell-Einstein Supergravity theories in 5d $\Leftrightarrow$ all the vectors fields including the graviphoton transform in an irreducible representation of a simple U-duality group of the action.
- There exist only four unified MESGTs in $d=5$ with symmetric target spaces . They are defined by the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ and describe the coupling of 5, 8, 14 and 26 vector multiplets to supergravity. Their symmetries in 5,4 and 3 dimensions give the groups of the Magic Square of Freudenthal, Rozenfeld and Tits $\Longrightarrow$ *Magical supergravity theories* (GST 1983)
- Scalar manifolds of five dimensional magical sugras are the irreducible symmetric spaces

$$\begin{array}{ccccccc} J & = & J_3^{\mathbb{R}} & J_3^{\mathbb{C}} & J_3^{\mathbb{H}} & J_3^{\mathbb{O}} \\ \mathcal{M}_5 & = & SL(3,\mathbb{R})/SO(3) & SL(3,\mathbb{C})/SU(3) & SU^*(6)/USp(6) & E_{6(-26)}/F_4 \end{array}$$

- The generic Jordan family defined by non-simple Jordan algebras $J = \Gamma_{(1,n-1)} \oplus \mathbb{R}$ has the scalar manifold $\left(SO(n-1,1) \times SO(1,1)\right)/SO(n-1)$

J	$\mathcal{M}_5 =$ Lor (J) / Rot (J)	$\mathcal{M}_4 =$ Conf (J) / $\widetilde{\text{Lor}}\,(J) \times U(1)$	$\mathcal{M}_3 =$ QConf$(\mathcal{F}(J))/\widetilde{\text{Conf}}\,(J) \times \mathrm{SU}(2)$
$J_3^{\mathbb{R}}$	$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$	$\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$	$\mathrm{F}_{4(4)}/\mathrm{USp}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{C}}$	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$	$\mathrm{SU}(3,3)/\mathrm{S}\,(\mathrm{U}(3) \times \mathrm{U}(3))$	$\mathrm{E}_{6(2)}/\mathrm{SU}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{H}}$	$\mathrm{SU}^*(6)/\mathrm{USp}(6)$	$\mathrm{SO}^*(12)/\mathrm{U}(6)$	$\mathrm{E}_{7(-5)}/\mathrm{SO}(12) \times \mathrm{SU}(2)$
$J_3^{\mathbb{O}}$	$\mathrm{E}_{6(-26)}/\mathrm{F}_4$	$\mathrm{E}_{7(-25)}/\mathrm{E}_6 \times \mathrm{U}(1)$	$\mathrm{E}_{8(-24)}/\mathrm{E}_7 \times \mathrm{SU}(2)$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$\frac{\mathrm{SO}(n-1,1)\times\mathrm{SO}(1,1)}{\mathrm{SO}(n-1)}$	$\frac{\mathrm{SO}(n,2)\times\mathrm{SU}(1,1)}{\mathrm{SO}(n)\times\mathrm{SO}(2)\times\mathrm{U}(1)}$	$\frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2)\times\mathrm{SO}(4)}$

Table: Scalar manifolds $\mathcal{M}_d$ of $N=2$ MESGT's defined by Jordan algebras J of degree 3 in $d=3,4,5$ dimensions. $\widetilde{\text{Lor}}\,(J)$ and $\widetilde{\text{Conf}}\,(J)$ denote the compact real forms of the Lorentz group $\text{Lor}\,(J)$ and conformal group $\text{Conf}\,(J)$ of a Jordan algebra J. ($\text{QConf}\,(\mathcal{F}\,(J))$ denotes the quasiconformal group associated with J.

In addition there exist three infinite families of unified MESGT's in $5d$. They are defined by Lorentzian Jordan algebras generated by $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ that are Hermitian with respect to the Lorentzian metric η (MG, Zagermann 2003)

$$(A\eta)^\dagger = A\eta \qquad \eta = diag(+,+,..,+,-)$$

Note the theories defined by Euclidean Jordan algebras $J_3^{\mathbb{C}}, J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ are equivalent to the theories defined by Lorentzian Jordan algebras $J_{(3,1)}^{\mathbb{R}}, J_{(3,1)}^{\mathbb{C}}$ and $J_{(3,1)}^{\mathbb{H}}$. Except for these special cases the the scalar manifolds of these Lorentzian theories are neither symmetric nor homogeneous.

J	D	$\mathrm{Aut}(J)$	No. of vector fields	No. of scalars
$J^{\mathbb{R}}_{(1,N)}$	$\frac{1}{2}(N+1)(N+2)$	$SO(N,1)$	$\frac{1}{2}N(N+3)$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,N)}$	$(N+1)^2$	$SU(N,1)$	$N(N+2)$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,N)}$	$(N+1)(2N+1)$	$USp(2N,2)$	$N(2N+3)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	27	$F_{4(-20)}$	26	25

Table: List of the simple Lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,N)}$. The columns show, respectively, their dimensions D, their automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$, the number of vector fields $(n+1)=(D-1)$ and the number of scalars $n=(D-2)$ in the corresponding MESGTs. Their C-tensors are given by the structure constants of the traceless elements of the Lorentzian Jordan algebra. Note the theories defined by Euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$ are equivalent to the theories defined by Lorentzian Jordan algebras $J^{\mathbb{R}}_{(3,1)}$, $J^{\mathbb{C}}_{(3,1)}$ and $J^{\mathbb{H}}_{(3,1)}$. Except for these special cases the the scalar manifolds of these Lorentzian theories are neither symmetric nor homogeneous.

- Under dimensional reduction of 5d magical MESGTs to four dimensions :
$$\mathcal{M}_5 = \frac{Lor(J)}{Rot(J)} \Rightarrow \mathcal{M}_4 = \frac{Conf(J)}{\widetilde{Lor}(J) \times U(1)}$$
where $\widetilde{Lor}(J)$ is the compact real form of the Lorentz group of J.
- In 5D: Vector fields $A^{\mu}_{I} \Leftrightarrow$ Elements of Jordan algebra J
- In 4D: $F^{A}_{\mu\nu} \oplus \tilde{F}^{A}_{\mu\nu} \Leftrightarrow$ Freudenthal triple system (FTS) $\mathcal{F}(J)$:

$$\mathcal{F}(J) \ni X = \begin{vmatrix} \mathbb{R} & \mathbf{J} \\ \tilde{\mathbf{J}} & \mathbb{R} \end{vmatrix} \Leftrightarrow \begin{vmatrix} F^{0}_{\mu\nu} & F^{I}_{\mu\nu} \\ \tilde{F}^{I}_{\mu\nu} & \tilde{F}^{0}_{\mu\nu} \end{vmatrix}$$

- Scalar manifolds of 4d magical sugras are the symmetric spaces :

$$\begin{array}{cccccc} J & = & J_3^{\mathbb{R}} & J_3^{\mathbb{C}} & J_3^{\mathbb{H}} & J_3^{\mathbb{O}} \\ \mathcal{M}_4 & = & \frac{Sp(6,\mathbb{R})}{U(3)} & \frac{SU(3,3)}{S(U(3)\times U(3))} & \frac{SO^*(12)}{U(6)} & \frac{E_{7(-25)}}{E_6 \times U(1)} \end{array}$$

- $N = 2$ MESGTs reduce to $N = 4$ supersymmetric sigma models coupled to gravity in $d = 3$. The target manifolds of magical supergravity theories in $d = 3$ are the exceptional quaternionic symmetric spaces:

$$\frac{F_{4(4)}}{Usp(6) \times USp(2)}, \frac{E_{6(2)}}{SU(6) \times SU(2)}, \frac{E_{7(5)}}{SO(12) \times SU(2)}, \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

U-duality Orbits and Jordan Algebras:

- Study of U-duality Orbits of Extremal , Spherically Symmetric Stationary Black Hole Solutions of Supergravity Theories with Symmetric Target Spaces in $d = 5$ and $d = 4$ using the underlying Jordan algebras and Freudenthal triple systems
  MG and Ferrara, 1997
- This work led to the proposal that $4d$ U-duality groups act as spectrum generating conformal groups of the underlying Jordan algebras that define the corresponding $5d$ supergravity theories. $Conf[J]$ leaves invariant light-like separations with respect to a cubic distance function $\mathcal{N}_3(J_1 - J_2)$ and admits a 3-grading with respect to their Lorentz subgroups

  $$Conf[J] = K_J \oplus Lor(J) \times \mathcal{D} \oplus T_J$$

  $Lor(J)$ is the $5D$ U-duality group that leaves the cubic norm invariant.
- Question: Can the $3D$ U-duality groups act as spectrum generating conformal symmetries of corresponding $4D$ supergravity theories ? (MG, Koepsell, Nicolai 1997)
  Problem: $E_{8(8)}$ and $E_{8(-24)}$ appear as $3d$ U-duality groups. No conformal realization for any real forms of E_8, G_2 and $F_4 \Leftrightarrow$ No 3-grading with respect to a subgroup of maximal rank.
- However, all simple Lie algebras admit a 5-grading with respect to a subalgebra of maximal rank

  $$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

  such that the grade ± 2 subspaces are one-dimensional.

  $$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K$$

  $A, B, C ... = 1, ...2N$ and $(K, \Delta, \tilde{K})$ form an $sl(2)$ subalgebra.
  $U_A \in \mathfrak{g}^{+1}$ and $A \in \mathcal{F}$ where $\mathcal{F}$ is a Freudenthal triple system.

▶ QUASICONFORMAL REALIZATION OF $E_{8(8)}$ MG, Koepsell, Nicolai, 2000

$$
\begin{aligned}
E_{8(8)} &= \mathbf{1}_{-2} \oplus \mathbf{56}_{-1} \oplus E_{7(7)} + SO(1,1) \oplus \mathbf{56}_{+1} \oplus \mathbf{1}_{+2} \\
\mathfrak{g} &= \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K
\end{aligned}
$$

over a space $\mathcal{T}$ coordinatized by the elements X of the exceptional FTS $\mathcal{F}(J_3^{\mathbb{O}_S})$ plus an extra singlet variable x: $\mathbf{56}_{+1} \oplus \mathbf{1}_{+2} \Leftrightarrow (X, x) \in \mathcal{T}$:

$$
\begin{aligned}
K(X) = 0, \quad & U_A(X) = A, \qquad S_{AB}(X) = (A, B, X), \\
K(x) = 2, \quad & U_A(x) = \langle A, X \rangle, \quad S_{AB}(x) = 2\langle A, B \rangle x, \\
\tilde{U}_A(X) &= \frac{1}{2}(X, A, X) - Ax \\
\tilde{U}_A(x) &= -\frac{1}{6}\langle (X, X, X), A \rangle + \langle X, A \rangle x \\
\tilde{K}(X) &= -\frac{1}{6}(X, X, X) + Xx \\
\tilde{K}(x) &= \frac{1}{6}\langle (X, X, X), X \rangle + 2x^2
\end{aligned}
$$

Freudenthal triple product $\Leftrightarrow (X, Y, Z)$
Skew-symmetric invariant form $\Leftrightarrow \langle X, Y \rangle = -\langle Y, X \rangle$
Quartic invariant of $E_{7(7)} \Leftrightarrow \langle (X, X, X), X \rangle$
$A, B, .. \in \mathcal{F}(J_3^{\mathbb{O}_S})$

- Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$?
- Define a quartic norm of $\mathcal{X} = (X, x) \in \mathcal{T}$ as $\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2$
  $\mathcal{Q}_4(X)$ is the quartic norm of the underlying Freudenthal system and $X \in \mathcal{F}$.
- Define a quartic "distance" function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

  $$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}))$$

  $\delta(\mathcal{X}, \mathcal{Y})$ is the "symplectic" difference of $\mathcal{X}$ and $\mathcal{Y}$:

  $$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y \rangle) = -\delta(\mathcal{Y}, \mathcal{X})$$

- Light-like separations $d(\mathcal{X}, \mathcal{Y}) = 0$ are left invariant under quasiconformal group action.
  $\longrightarrow$ *Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.*
- $E_{8(8)}$ is the invariance group of a quartic light-cone in 57 dimensions!
- Quasiconformal realizations extend to all Lie algebras and Lie superalgebras.

- The space-times defined by simple Jordan algebras of degree 3 $J_3^{\mathbb{A}}$ correspond to extensions of Minkowski space-times in the critical dimensions $d = 3, 4, 6, 10$ by a dilatonic (ρ) *and* commuting spinorial coordinates (ξ^a).

$$
\begin{aligned}
J_3^{\mathbb{R}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0,1,2 \quad \alpha = 1,2 \\
J_3^{\mathbb{C}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0,1,2,3 \quad \alpha = 1,2,3,4 \\
J_3^{\mathbb{H}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0,\ldots,5 \quad \alpha = 1,\ldots,8 \\
J_3^{\mathbb{C}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0,\ldots,9 \quad \alpha = 1,\ldots,16
\end{aligned}
$$

- The commuting spinors ξ are represented by a 2×1 matrix over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The cubic norm of a "vector" with coordinates $X^I = (\rho, x_m, \xi^\alpha)$ is given by

$$
\mathcal{V}(\rho, x_m, \xi^\alpha) = C_{IJK} X^I X^J X^K = \sqrt{2}\rho x_m x_n \eta^{mn} + x^m \bar{\xi}\gamma_m \xi
$$

  Their Lorentz groups have the Lorentz groups Minkowskian spacetimes in $d = 3, 4, 6, 10$ as subgroups:

$$
\begin{aligned}
J = \mathbb{R} \oplus J_2^{\mathbb{R}} &: \mathrm{SO}(1,1) \times \mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbb{R}) \\
J = \mathbb{R} \oplus J_2^{\mathbb{C}} &: \mathrm{SO}(1,1) \times \mathrm{SO}(3,1) \subset \mathrm{SL}(3,\mathbb{C}) \\
J = \mathbb{R} \oplus J_2^{\mathbb{H}} &: \mathrm{SO}(1,1) \times \mathrm{SO}(5,1) \subset \mathrm{SU}^*(6) \\
J = \mathbb{R} \oplus J_2^{\mathbb{O}} &: \mathrm{SO}(1,1) \times \mathrm{SO}(9,1) \subset \mathrm{E}_{6(-26)}
\end{aligned}
$$

- Remarkable fact: the adjoint identity satisfied by C_{IJK} tensor $\Longleftrightarrow$ Fierz identities required for the existence of supersymmetric Yang-Mills theory in the critical dimensions $d = 3, 4, 6, 10$ Sierra (1987).

EXCEPTIONAL $N = 2$ versus MAXIMAL $N = 8$ SUPERGRAVITY:

- The exceptional $N = 2$ supergravity is defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over real octonions $\mathbb{O}$. Its global invariance group in $5D$ is $E_{6(-26)}$ with maximal compact subgroup F_4.
- The C-tensor C_{IJK} of $N = 8$ supergravity in five dimensions can be identified with the symmetric tensor given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ defined over split octonions $\mathbb{O}_s$. Its global invariance group in $5D$ is $E_{6(6)}$ with maximal compact subgroup $USp(8)$.
- In $D = 4$ and $D = 3$ the exceptional supergravity has $E_{7(-25)}$ and $E_{8(-24)}$ as its U-duality group while the maximal $N = 8$ supergravity has $E_{7(7)}$ and $E_{8(8)}$, respectively.
- One can couple 28 hypermultiplets to exceptional $N = 2$ supergravity in $d = 4$ parametrizing the coset space $\frac{E_{8(-24)}}{E_7 \times SU(2)}$. The resulting theory has FHSV model as a subtheory $SO(10,2) \times SU(1,1) \times SO(12,4) \subset E_{7(-25)} \times E_{8(-24)}$ In three dimensions this exceptional theory with exceptional matter has the moduli space $\frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$ and descends from an anomaly free theory in $d = 6$.
  MG (Paris , 2006)
- The magical supergravity theories including the exceptional one coupled to hypermultiplets in $d = 6$ was constructed by MG, Samtleben and Sezgin (2010).

- Neither the maximal supergravity nor the exceptional supergravity can accommodate the standard model of particle physics directly since possible gaugings of both theories can not accommodate the standard model gauge group $SU(2) \times U(1) \times SU(3)$.
- In the early days of maximal supergravity Murray Gell-Mann argued that to make contact with observation some of the observed particles in Nature may have to be composites of the fields of maximal supergravity.
  Cremmer and Julia suggested that the composite $SU(8)$ local gauge symmetry of maximal supergravity might become dynamical at the quantum level (1979).
  Ellis, Gaillard, Maiani and Zumino extended this proposal and suggested the possibility that bound states of maximal supergravity might lead to a grand unified theory based on $SU(5)$ which requires the breaking of the family unifying GUT group $SU(8)$. Furthermore the bound states might fall into unitary representations of U-duality group $E_{7(7)}$ which are all infinite dimensional.
- Unitary representations of the U-duality groups of extended supergravity theories ($N \geq 4$) using oscillators transforming in the same representation as the vector field strengths and their magnetic duals in four dimensions were given by MG and Saclioglu (1982).
- Gunaydin, Sierra, Townsend proposed a composite scenario for the exceptional supergravity in which the composite local symmetry $E_6 \times U(1)$ becomes dynamical (1983). The exceptional group E_6 had earlier been proposed as a grand unified gauge group by Gürsey, Ramond and Sikivie.
- Furthermore in contrast to the U-duality group $E_{7(7)}$ of maximal supergravity the U-duality group $E_{7(-25)}$ of the exceptional supergravity admits unitary lowest weight representations which are intrinsically chiral. (MG1984)

The minimal unitary realization of the conformal group $E_{7(-25)}$ of the exceptional Jordan algebra $J_3^{\mathbb{O}}$

MG& Pavlyk (2005) and MG (2025)

- As a conformal group of $J_3^{\mathbb{O}}$ the Lie algebra of $\mathrm{E}_{7(-25)}$ admits the natural 3-graded decomposition :

$$E_{7(-25)} = \overline{27} \oplus E_{6(-26)} \oplus SO(1,1) \oplus 27$$

  $(2\bar{7})$ and 27 denoting translations and special conformal transformations.

- However the minimal unitary representation is obtained by quantization of geometric quasiconformal realization of $\mathrm{E}_{7(-25)}$ with the natural 5-graded decomposition :

$$\begin{array}{ccccccccccc} E_{7(-25)} & = & \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2} \\ & & \mathbf{M} & \oplus & \mathbf{16}+\bar{\mathbf{16}} & \oplus & (SO(10,2)\oplus\Delta) & \oplus & \mathbf{16}+\bar{\mathbf{16}} & \oplus & \mathbf{K} \end{array}$$

  The minrep is realized over the tensor product of the Fock space of 16 bosonic oscillators $Z^{ab} = -Z^{ba}$, $Z^{78} = -Z^{87}$ $(a, b = 1, ..6)$ and the Hilbert space of conformal quantum mechanics with coordinate and momentum (x, p).
  16+1=17 is the dimension of the minrep of $E_{7(-25)}$.

$$[\tilde{Z}_{ab}, Z^{cd}] = \frac{1}{2}(\delta_a^c\delta_b^d - \delta_b^c\delta_a^d) \ , \ \ [\tilde{Z}_{78}, Z^{78}] = 1/2$$

- Grade -2 generator is simply $M = \frac{1}{2}x^2$ and the grade -1 generators are given by bilinears :
  $M^{ab} = xZ^{ab} \quad , \quad M^{78} = xZ^{78} \quad , \ a, b = 1, .., 6$
  $\tilde{M}_{ab} = x\tilde{Z}_{ab} \quad , \quad \tilde{M}_{78} = x\tilde{Z}_{78}$
  They transform in spinor representation 16 and $\bar{16}$ of $SO(10)$ and together they form a Majorana-Weyl spinor of $SO(10,2)$.

  The grade $+2$ generator is given by $K = \frac{p^2}{2} + \frac{2}{x^2}\left(C_2 + \frac{99}{16}\right)$ where C_2 is the Casimir of grade zero subalgebra $\mathfrak{so}(10,2)$.
  The grade $+1$ generators are given by $K^{ab} = i[M^{ab}, K] \quad , \quad \tilde{K}_{ab} = i[\tilde{M}_{ab}, K]$
  $K^{78} = i[M^{78}, K] \quad , \quad \tilde{K}_{78} = i[\tilde{M}_{78}, K]$
- The grade ± 2 generators K, M and Δ generate the $SU(1,1)$ symmetry of conformal quantum mechanics with the Casimir C_2 playing the role of coupling constant.
  Casimir of $E_{7(-25)} = -14$
- Generators S_{AB} of $SO(10,2)$ are realized as bilinears of 16 bosonic oscillators

$$[S_{A,B}, S_{C,D}] = -i\left(\eta_{AC}S_{B,D} + \eta_{BD}S_{A,C} - \eta_{BC}S_{A,D} - \eta_{AD}S_{B,C}\right)$$

  where $\eta_{AB} = \mathrm{Diag}(1, ..., 1, -1, -1)$ ($A, B, \cdots = 1, 2, \cdots, 12$). Lie algebra of $SO(10,2)$ admits a compact three grading given by the generator $H = S_{11,12}$

$$\mathfrak{so}(10,2) = \mathfrak{s}^{-1} \oplus \mathfrak{so}(10) + \mathfrak{u}(1)_H \oplus \mathfrak{s}^{+1}$$

- The Fock space of 16 bosonic oscillators decompose into infinitely many unitary lowest weight representations of $SO(10,2)$.

-
- $SO(10,2)$ as a conformal group in ten dimensional Minkowskian spacetime admits a noncompact three grading with respect to the dilatation generator $D = S_{1,12}$ such that grade -1 and +1 generators are momentum P_μ and special conformal generators K_μ:

$$\mathfrak{so}(10,2) = K_\mu \oplus (S_{\mu,\nu} + D) \oplus P_\mu$$

where $P_\mu = S_{\mu,1} - S_{\mu,12} \quad , \quad K_\mu = S_{\mu,1} + S_{\mu,12} \quad , \mu, \nu, .. = 2, 3, ..., 11.$
- The Poincare mass operator P^2 as well as K^2 vanish identically as *operators in the above realization*:

$$P^2 = P_\mu P_\nu \eta^{\mu\nu} = 0 \quad , \quad K^2 = K_\mu K_\nu \eta^{\mu\nu} = 0$$

Hence the resulting unitary irreps of $SO(10,2)$ describe conformally massless fields $\Phi_{-(4+n/2)}(x_\mu; (0,0,0,0,n)_D)$ transforming in the representation $(0,0,0,0,n)_D$ of the Lorentz group $SO(9,1)$.

They can be represented as symmetric tensors $\Psi_{\alpha\beta...\gamma}(x_\mu)$ in spinor indices $\alpha, \beta, .. = 1, 2, .., 16$ subject to certain constraints under projection to irreps of $SO(9,1)$:

$$\Phi_{-(4+n/2)}(x_\mu; (0,0,0,0,n)) \Longleftrightarrow \underbrace{\Psi_{\alpha\beta...\gamma}}_{n}(x_\mu)$$

The field in the $(0,0,0,0,1)$ representation is simply the Weyl spinor Ψ_α in ten dimensions. The field with the Dynkin label $(0,0,0,0,2)$ is a symmetric tensor $\Psi_{\alpha\beta}$ subject to the constraint $\sigma_\mu^{\alpha\beta}\Psi_{\alpha\beta} = 0$ where $\sigma_\mu(\mu, \nu = 0, 1, 2, .., 9)$ are the symmetric ten dimensional sigma matrices.

$SU(1,1)_K$ subgroup of $E_{7(-25)}$ and conformal quantum mechanics

- The operator $L_0 = \frac{1}{2}(K+M)$ is the compact generator of $SU(1,1)_K$ generated by Δ, M and K and determines its compact three grading

$$[L_0, L_\pm] = \pm L_\pm \quad , \quad [L_-, L_+] = 2L_0$$

where $L_- = \frac{1}{2}(-\Delta + M - K)$ and $L_+ = \frac{1}{2}(\Delta + M - K)$.

$$L_0 = \frac{1}{4}\left(x^2 + p^2\right) + \frac{1}{2x^2}\mathcal{G}$$

where $\mathcal{G} = 2(C_2(SO(10,2)) + \frac{99}{16})$. $V(x) = \frac{\mathcal{G}}{x^2}$

$$L_+ = \frac{1}{4}(x - ip)^2 - \frac{1}{2\,x^2}\mathcal{G} \quad , \quad L_- = \frac{1}{4}(x + ip)^2 - \frac{1}{2\,x^2}\mathcal{G}$$

For a given eigenvalue g of $\mathcal{G}$ there exists a wave-function $\psi^{\alpha_g}_{l_{\alpha_g}}$ in the Hilbert space of the conformal oscillator that is the lowest weight vector of a unitary irreducible representation of $SU(1,1)_K$:

$$\psi^{\alpha_g}_{l_{\alpha_g}}(x) = C_0\, x^{\alpha_g} e^{-x^2/2}$$

where $\alpha_g = \frac{1}{2} \pm \sqrt{2g + \frac{1}{4}}$. This wave-function satisfies

$$L_- \psi^{\alpha_g}_{l_{\alpha_g}}(x) = 0 \quad , \quad L_0 \psi^{\alpha_g}_{l_{\alpha_g}}(x) = l_{\alpha_g} \psi^{\alpha_g}_{l_{\alpha_g}}(x)$$

where $l_{\alpha_g} = \frac{\alpha_g}{2} + \frac{1}{4} = \frac{1}{2} \pm \frac{1}{2}\sqrt{2g + \frac{1}{4}}$.

Compact three-grading of $E_{7(-25)}$ with respect to the compact $E_6 \times U(1)$ subgroup

- $$E_{7(-25)} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} = 27_{-1} \oplus E_6 + U(1)_R \oplus \bar{27}_{+1}$$

  $R = S_{11,12} + \frac{1}{2}(K + M) = S_{11,12} + L_0$.
  The generators R^I_+ and R^-_I in grade ± 1 subspaces transform under the $SO(10)$ subgroup as

  $$R^I_+ = S^+_i \oplus Q^\alpha_+ \oplus L_+ = 10 + 16 + 1$$
  $$R^-_I = S^-_i \oplus Q^-_{\dot\alpha} \oplus L_- = 10 + \bar{16} + 1$$

  with α and $\dot\alpha$ denoting the spinor indices of $SO(10)$ and its conjugate, respectively.

- S^-_i and $Q^-_{\dot\alpha}$ annihilate the Fock vacuum $|0\rangle$ and the state $\psi_2^{7/2}(x)$ is annihilated by L_- .
  Hence the tensor product state $|\psi_2^{7/2}(x)\,;0\rangle = \psi_2^{7/2}(x) \otimes |0\rangle$ is the lowest weight vector of the minimal unitary representation of $E_{7(-25)}$. It satisfies

  $$\begin{aligned} R|\psi_2^{7/2}(x)\,;0\rangle &= (L_0 + S_{11,12})|\psi_2^{7/2}(x)\,;0\rangle = 6|\psi_2^{7/2}(x)\,;0\rangle \\ R^-_I|\psi_2^{7/2}(x)\,;0\rangle &= 0 \quad , I = 1, 2, \cdots, 27 \end{aligned}$$

  The lowest weight vector $|\psi_2^{7/2}(x)\,;0\rangle$ is a singlet of the compact subgroup E_6 .

Decomposition of the minrep of $E_{7(-25)}$ with respect to the $SO(10,2) \times SU(1,1)_K$ subgroup and massless conformal fields in ten dimensional Minkowskian space-time

- The lowest weight vector $|\psi_2^{7/2}(x);0\rangle$ of the minrep of $E_{7(-25)}$ satisfies

$$R_I^- |\psi_2^{7/2}(x);0\rangle = 0 \quad , \quad I = 1,2,...,27$$

  and is a singlet of the maximal compact subgroup E_6 with the $U(1)_R$ charge 6. Acting on $|\psi_2^{7/2}(x);0\rangle$ with the 27 grade $+1$ operators repeatedly we generate an infinite number of states that form the basis of the minrep of $E_{7(-25)}$:

$$\left(|\psi_2^{7/2}(x);0\rangle \oplus R_+^I |\psi_2^{7/2}(x);0\rangle \oplus R_+^I R_+^J |\psi_2^{7/2}(x);0\rangle \oplus \cdots \right) \Leftrightarrow \text{Minrep of } E_{7(-25)}$$

- In Dynkin labels the K-type decomposition of the minrep of $E_{7(-25)}$ with respect to its maximal compact subgroup $E_6 \times U(1)_R$ is given by

$$\text{Minrep of } E_{7(-25)} \equiv \sum_{n=0}^{\infty} \bigoplus (n,0,0,0,0,0)_{(n+6)}$$

  where the subscript indicates the $U(1)$ R-charge. First few levels the irreps of E_6 have dimensions

$$\begin{aligned} (1,0,0,0,0,0) &= 27 \\ (2,0,0,0,0,0) &= \overline{351}' \\ (3,0,0,0,0,0) &= 3003 \\ (4,0,0,0,0,0) &= 19305' \end{aligned}$$

- We shall denote the states that transform in the representation of $SO(10)$ with Dynkin labels $(m_1, m_2, m_3, m_4, m_5)$ and $U(1)_H$ charge E as $|\Psi_E(m_1, m_2, m_3, m_4, m_5)\rangle$ and its tensor product with conformal wave function $\psi^{\alpha_g}_{l_{\alpha_g}}(x)$ with $U(1)_{L_0}$ charge l_{α_g} as

$$|\psi^{\alpha_g}_{l_{\alpha_g}}(x); \Psi_E(m_1, m_2, m_3, m_4, m_5)\rangle$$

In this notation the lowest weight vector $|\psi^{7/2}_2(x)\,;0\rangle$ takes the form $|\psi^{7/2}_2(x); \Psi_4((0,0,0,0,0))\rangle$
Recall that $R^l_+ = S^+_i \oplus Q^\alpha_+ \oplus L_+ = 10 + 16 + 1$

- Acting on the state $|\psi^{7/2}_2(x)\,;0\rangle$ with the raising operators S^+_i repeatedly one generates the basis of the minimal unitary representation of $SO(10,2)$ that describes a massless scalar field in ten dimensional Minkowskian spacetime.
- On the other hand by acting on the lowest weight vector $|\psi^{7/2}_2(x); 0)\rangle$ with the generators Q^α_+ we generate a new lowest weight irrep of $SO(10,2) \times SU(1,1)_K$

$$S^-_i\, Q^+_\alpha |\psi^{7/2}_2(x); 0\rangle = 0 \quad , \quad L_-\, Q^+_\alpha |\psi^{7/2}_2(x); 0\rangle = 0$$

that transform in the spinor representation $16 = (0,0,0,0,1)$ of $SO(10)$ with $E = 9/2$ and has $U(1)_{L_0}$ charge $l_{\alpha_g} = 5/2$. By repeated applications of the generators Q^+_α one generates an infinite tower of lowest K-types of $SO(10,2) \times SU(1,1)$.

- If we denote the Hilbert spaces of the UIRs of $SU(1,1)_K$ and $SO(10,2)$ with lowest weight irreps $\psi^{\alpha=7/2+p}_{l_\alpha=2+p/2}(x)$ and $|\Psi_{(4+p/2)}(0,0,0,0,p)\rangle$ as $\mathcal{H}(\psi^{\alpha=7/2+p}_{l_\alpha=2+p/2}(x)$ and $\mathcal{H}(\Psi_{(4+p/2)}(0,0,0,0,p)\rangle)$, respectively, then we can write down the decomposition of the Hilbert space of the minrep of $E_{7(-25)}$ as

$$\mathcal{H}_{minrep}(E_{7(-25)}) = \sum_{p=0}^{\infty} \bigoplus \left(\mathcal{H}(\Psi_{(4+p/2)}(0,0,0,0,p) \bigotimes \mathcal{H}(\psi^{\alpha=7/2+p}_{l_\alpha=2+p/2}(x)) \right)$$

- The results above show that the spectrum of the minimal unitary representation of $E_{7(-25)}$ decomposes into infinitely many irreps of $SO(10,2) \times SU(1,1)_K$ and one can interpret the minrep as an infinite multiplet with *bosonic supersymmetry* generators transforming in spinor representation 16 of $SO(10)$ and its conjugate. Each irrep of $SO(10,2)$ inside the minrep of $E_{7(-25)}$ comes with infinite multiplicity which form an irrep of the " bosonic R-symmetry group" $SU(1,1)_K$. This provides an example of bosonic symmetry algebra whose unitary spectrum exhibits spacetime supersymmetry
- The possibility that a bosonic symmetry algebra may exhibit spacetime supersymmetry in its spectrum was first mentioned to me by Edward Witten in connection with hyperbolic Lie algebra E_{10} (Cambridge, 1985).
- Whether this can be realized within the framework of attempts to relate the hyperbolic Lie algebra E_{10} to M-theory is still an open problem. For recent reviews on the subject and further references see the latest papers by Kleinschmidt and Nicolai.
- The results obtained for $E_{7(-25)}$ can be truncated to noncompact groups $SO^*(12), SU(3,3)$ and $Sp(6,\mathbb{R})$ which are the conformal groups of simple Euclidean Jordan algebras of degree three over the quaternions, complex numbers and real numbers.

Minimal unitary representation of the global symmetry group $E_{7(-25)}$ of OMS and the composite scenario.

- When $E_6 \times U(1)$ is restricted to the $SO(10) \times U(1) \times U(1)$ subgroup inside the minrep of $E_{7(-25)}$ the representation $16 = (0,0,0,0,1)$ of $SO(10)$ appears with infinite multiplicity with different $U(1)_H \times U(1)_{L_0}$ charges.

$$E_6 \times U(1) \to SO(10) \times U(1)_H \times U(1)_{L_0} :$$

$$Minrep \to \sum_{q=0}^{\infty} \bigoplus (0,0,0,0,1)^{(L_0=q+5/2)}_{(H=9/2)} + \mathrm{e}xcitations$$

  Assuming $SO(10)$ composite local symmetry becomes dynamical one has an effective $SO(10)$ GUT with infinite families of quarks and leptons in 16 0f $SO(10)$. The spectrum is intrinsically chiral! No conjugate families in $\overline{16}$.

- However one has to break $N=2$ supersymmetry at high energies in order to have truly chiral families of fermions. Then we have an infinite family of fermions in 16 of GUT group $SO(10)$. To make contact with the phenomenology of the standard model one would then have to find a mechanism to make the fermions beyond three families of quarks and leptons very heavy while breaking $SO(10)$ down to the standard model gauge group $SU(3) \times SU(2) \times U(1)$.
- Clearly composite scenarios are very speculative. Composite gauge fields becoming dynamical at the quantum level was established in two dimensions. It is not known how to extend those results to four dimensions.
-

The question whether the spectrum falls into unitary representations of the U-duality group at the quantum level is independent of the question whether the composite gauge fields become dynamical. At the quantum level we expect the continuous U-duality groups to be broken down to their discrete arithmetic subgroups. If one can show that the quantum degeneracies of the spectrum are given by the Fourier coefficients of modular forms corresponding to some unitary representations of the U-duality group this would represent some evidence! For the octonionic magical supergravity we do have such an evidence.

Exceptional Jordan Algebra over the Integral Octonions and Exceptional Modular Forms

- The real octonions with integer coefficients $\mathbb{O}(\mathbb{Z})$ form a ring. However, as was shown by Coxeter, the ring $\mathbb{O}(\mathbb{Z})$ is not maximal. There is a maximal order $\mathcal{R}$ which has $\mathbb{O}(\mathbb{Z})$ as a subring generated by $\mathbb{O}(\mathbb{Z})$ and four additional octonions with all half-integer coefficients.
- The trace form $Tr(x) = x + \bar{x}$ and the norm $\mathcal{N}(x) = x\,\bar{x}$ take integer values in $\mathcal{R}$ hence its elements were called integral Cayley numbers (octonions) by Coxeter.
- Elkies and Gross define a $\mathbb{Z}$ lattice JL inside $J_3^{\mathbb{O}}$ by considering 3×3 Hermitian matrices over integral octonions $\mathcal{R}$:

$$J(a,b,c;x,y,z) = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \equiv J_3^{\mathbb{O}}(\mathcal{R}) \Leftrightarrow JL \quad , \quad a,b,c \in \mathbb{Z} \, , \; x,y,z \in \mathcal{R}$$

$$\mathcal{N}(J) = \det(J) = \frac{1}{6}(J,J,J)$$

- The cubic norm of elements in JL take on integral values and its invariance group is the discrete arithmetic subgroup $E_{6(-26)}(\mathbb{Z})$ of $E_{6(-26)}$.

The elements E belonging to the exceptional cone $\mathcal{C}$ with $\mathcal{N}(E) = 1$ are called *POLARIZATIONS*. Given a polarization E one can define a LINEAR form that maps the elements of $J_3^{\mathbb{O}}$ into $\mathbb{R}$:

$$\begin{aligned} T_E : \quad J_3^{\mathbb{O}} & \Longrightarrow \mathbb{R} \\ T_E(J) &= \frac{1}{2}(E, E, J) \qquad , \forall\, J \in J_3^{\mathbb{O}} \end{aligned}$$

as well as a QUADRATIC form

$$\begin{aligned} R_E : \quad J_3^{\mathbb{O}} & \Longrightarrow \mathbb{R} \\ R_E(J) &= \frac{1}{2}(E, J, J) \qquad , \forall\, J \in J_3^{\mathbb{O}} \end{aligned}$$

which is associated with the bilinear form

$$(A, B)_E = R_E(A + B) - R_E(A) - R_E(B) = (E, A, B)$$

The bilinear form $(A, B)_E$ has signature $(1, 26)$.
One can also define another bilinear form $\langle A, B \rangle_E$, where

$$\langle A, B \rangle_E = T_E(A) T_E(B) - (A, B)_E$$

and has signature (27,0). The linear form T_E as well as the bilinear forms are invariant under the subgroup of the reduced structure group $E_{6(-26)}$ of $J_3^{\mathbb{O}}$ that leaves the polarization E invariant.

- A remarkable result (Benedict Gross 1996): while the continuous $E_{6(-26)}$ acts transitively on positive polarizations with determinant one over the reals $\mathbb{R}$, its arithmetic subgroup $E_{6(-26)}(\mathbb{Z})$ does NOT act transitively on the polarizations $E > 0$ in JL with determinant one.
- There exist precisely two orbits under the action of $E_{6(-26)}(\mathbb{Z})$ represented by the identity polarization $I = J(1,1,1,0,0,0)$ and the "indecomposable" polarization

$$E = J(2,2,2;\beta,\beta,\beta) = \begin{pmatrix} 2 & \beta & \bar{\beta} \\ \bar{\beta} & 2 & \beta \\ \beta & \bar{\beta} & 2 \end{pmatrix}$$

where $\beta = \frac{1}{2}(-1 + j_1 + j_2 + j_3 + j_4 + j_5 + j_6 + j_7)$
$Tr(\beta) = \beta + \bar{\beta} = -1 \quad , \quad \mathcal{N}(\beta) = \beta\bar{\beta} = 2 \quad , \quad \beta^2 + \beta + 2 = 0$.
- There exist THREE distinct rank one elements $A \in JL$ with respect to the identity polarization I with $T_I(A) = 1$.
- There are NO rank one elements $A \in JL$ with $T_E(A) = 1$.

Integral Jordan Roots:

Given a polarization E Jordan roots $S \in \mathcal{C}$ have rank one and satisfy

$$T_E(S) = 2 = 1/2(E,E,S) \quad , \quad (S,S) = (E,S,S) = 0$$

$$\langle S,S\rangle = (T(S))^2 = 4 \quad . \quad S \circ_E S = 2S$$

A root triple is defined as 3 mutually orthogonal Jordan roots S_1, S_2 and S_3 that satisfy:

$$2E = S_1 + S_2 + S_3 \quad , \quad \langle S_i, S_j\rangle = 0 \qquad i \neq j \quad , \quad S_i \circ S_j = 0 \qquad i \neq j \ ,$$

where $i,j = 1,2,3$.

Elkies and Gross:

- The linear form T_E maps the lattice JL defined by $J_3^{\mathbb{O}}(\mathcal{R})$ into $\mathbb{Z}$.
- Rank(A)=1 elements satisfy $T_E(A) \geq 2$ in the polarization E.
- The number $\mathfrak{N}(n)$ of rank 1 elements A such that $T_E(A) = n$ with $n \in \mathbb{N}$ is given by

$$\mathfrak{N}(n) = \sum_{T_E(A)=n;\ rank(A)=1} \left(\sum_{d|c(A)} d^3 \right) = \frac{3 \cdot 7 \cdot 13}{691} \left(\sigma_{11}(n) - \tau(n)\right) \ .$$

$\sigma_{11}(n) = \sum_{d|n} d^{11}$, where $d|n$ indicates the sum over positive divisors of n including 1 and $c(A)$ denote the largest integer such that $A/c(A) \in JL$.

- The Ramanujan $\tau(n)$ function is given by the $q-$series of the 24^{th} power of the Dedekind eta function

$$\Delta = \eta(\tau)^{24} = q \prod_{m \geq 1} (1 - q^m)^{24} = \sum_{n \geq 1} \tau(n) q^n, \quad q := e^{2\pi i \tau} \ .$$

- The rank 1 elements with $T_E(A) = 2$ correspond to integral Jordan roots and their number is $\mathfrak{N}(2) = 819$.
- The proof of the formulas for the multiplicity $\mathfrak{N}(n)$ of rank 1 elements uses the theory of modular forms of weight 12 on the upper half-plane and on the exceptional tube domain.

The space of modular forms of weight 12 for $SL(2,\mathbb{Z})$ is two dimensional spanned by Δ and the Eisenstein series E_{12}. The unique holomorphic modular $f(q)$ form of weight 12 for $SL(2,\mathbb{Z})$ whose Fourier series begins as $f(q) = 1 + 0q + (\#)q^2 + \cdots$ is given by

$$\begin{aligned} f(q) &= \frac{65520}{691}(E_{12} - \Delta) = 1 + \frac{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}{691} \sum_{n \geq 1} (\sigma_{11}(n) - \tau(n)) \\ &= 1 + 240 \sum_{n \geq 1} \mathfrak{N}(n) = 1 + 240 \cdot 819 q^2 + \ldots\ldots \end{aligned}$$

The modular form $f(q)$ is the theta function of the Leech lattice $\Lambda \subset \mathbb{R}^{24}$ which is an even unimodular lattice with minimal norm > 2. (no root vectors)

Elkies and Gross obtain the modular form $f(q)$ by restriction of a singular modular form over the exceptional tube domain studied by Kim.

The unique *exceptional tube domain* $\mathcal{D}$ of complex dimension 27 is simply the "upper half-plane" of the exceptional Jordan algebra $J_3^{\mathbb{O}}$ spanned by elements of the form $Z = (X + iY)$ where $X, Y \in J_3^{\mathbb{O}}$ and $Y > 0$.

The conformal group $E_{7(-25)}$ of $J_3^{\mathbb{O}}$ acts holomorphically on the exceptional tube domain $\mathcal{D}$. Since the conformal group $Conf(J)$ includes translations T_J by the elements of $J_3^{\mathbb{O}}$ the Fourier coefficients of the modular forms of the arithmetic subgroup $E_{7(-25)}(\mathbb{Z})$ of $E_{7(-25)}$ are expected to describe the degeneracies of quantum extremal black holes of the octonionic magical supergravity theory whose charges are labelled by the elements of the exceptional Jordan algebra.

Modular Forms of $E_{7(-25)}$ and Quantum Degeneracies of Extremal Black Holes of the $5d$ Octonionic Magical Supergravity

- The continuous U-duality group G of any supergravity theory gets broken down to its discrete arithmetic subgroup $G(\mathbb{Z})$ by the stringy corrections. Hence the quantum completion of the octonionic magical supergravity is expected to break its continuous U-duality group $E_{6(-26)}$ down to its arithmetic subgroup $E_{6(-26)}(\mathbb{Z})$. The arithmetic subgroup $E_{6(-26)}(\mathbb{Z})$ was first studied by Benedict Gross.
- The bare charge lattice of black holes in $5d$ are lattices with integer valued coordinates. However the lattice JL defined by the exceptional Jordan algebra over the Coxeter order of integral octonions involve integer as well as half-integer coefficients. Hence *the lattice JL can not be identified with bare charges but must be identified with dressed charges*
- h^I that determines the polarization depends on the scalar fields. If the bare charges are q_I, then $h^I q_I$ corresponds to the central charge and the physical (dressed) graviphoton is given by the linear combination $h^I A_{I\mu}$. In a given vacuum given by the VEV $\langle h^I \rangle$ the physical graviphoton is $\langle h^I \rangle A_{I\mu}$. If we choose the identity polarization $\langle h^I \rangle J_I = I_3$ then the graviphoton coincides with the bare graviphoton.
- Under the action of the continuous $E_{6(-26)}$, the identity polarization can be mapped into any other polarization. However under the action of the arithmetic subgroup $E_{6(-26)}(\mathbb{Z})$ we have two distinct orbits namely the polarizations in the orbit of the identity polarization I_3 and the polarizations that are in the orbit of the indecomposable polarization E. Hence at the quantum level we have two distinct vacua that are not connected by the arithmetic subgroup of the continuous U-duality group. We refer. to the vacuum corresponding to the identity polarization as the **perturbative vacuum** and the one defined by the indecomposable polarization E as the **non-perturbative vacuum**.

Rank 1 Black Holes

- Black holes of supergravity described by rank 1 elements of $J_3^{\mathbb{O}}$ have vanishing entropy (area) and their orbit under the continuous U-duality group is

$$\frac{E_{6(-26)}}{SO(9,1)\circledS T^{16}} \quad , \quad dim = 17 = dim \ of \ minrep$$

- In addition to rank 1 condition, they are uniquely labelled by the trace form $T(A)$. For quantum black holes described by rank 1 elements A of the exceptional Jordan algebra over the integral octonions, $T(A)$ takes on integral values. We interpret the number of rank 1 elements with a given value of $T(A)$ as the quantum degeneracy of critical light-like (small) black holes with the unique quantum number $T(A)$.
- The number of distinct rank one elements A with $T(A) = n$ $(n \in \mathbb{Z})$ in the indecomposable polarization E was given by $\mathfrak{N}(n)$. For $n = p$ prime it simplifies to

$$\mathfrak{N}(p) = \frac{3 \cdot 7 \cdot 13}{691} \left(p^{11} - \tau(p) + 1\right).$$

- For $p = 2$ corresponding to Jordan roots, we have $\mathfrak{N}(2) = 819$. The group ${}^3D_4(2)$ acts transitively on the 819 Jordan roots and the stabilizer of a given Jordan root is the subgroup $2_+^{1+8} \cdot L_2(8)$. The finite group ${}^3D_4(2)$ is also expected to act transitively on the set of $A \geq 0$ in JL with rank(A)=1 and $T(A) = 3$ and the stabilizer is a maximal subgroup of ${}^3D_4(2)$ isomorphic to $L_2(2) \times L_2(8)$. Elkies and Gross

Degeneracies of critical light-like black holes with $T(A) = n$ with $n \geq 2$ are given by the Fourier coefficients of the singular modular form of degree 4 over the exceptional domain $\mathcal{D}$ which has the Fourier expansion

$$E_4(Z) = 1 + 240 \sum_{\substack{T \geq 0,\ T \in L, \\ \mathrm{rank}(T)=1}} \sigma_3(c(T)) e^{2\pi i \mathrm{Tr}(T \circ_I Z)} \quad , \quad Z \in \mathcal{D}$$

where $\sigma_3(n) = \sum_{d|n} d^3$, where $d|n$ indicates the sum over positive divisors of n including 1.

If we restrict $Z = \tau E^{\#}$ where τ is the complex coordinate in the upper half-plane $\mathcal{H}$, then we obtain the modular form of weight 12 of $SL(2,\mathbb{Z})$. All rank one elements A in the polarization E have $T_E(A) > 1$.

The resulting modular form of weight twelve of $SL(2,\mathbb{Z})$ turns out to be the theta function of the Leech lattice. Its Fourier coefficients give the quantum degeneracy of critical light-like BPS black holes with $T_E(A) = n$ with $n > 1$

$$\mathfrak{N}(n) = \frac{3 \cdot 7 \cdot 13}{691} \left(\sigma_{11}(n) - \tau(n) \right) \ .$$

Rank 2 Black Holes

- Rank 2 black holes are called light-like and are characterized by the conditions that the underlying element T of the Jordan algebra has rank two

$$\mathcal{N}(T) = 0 \quad , \quad T^{\#} \neq 0 \; .$$

Black holes whose charges lie on rank two elements T have vanishing entropy.

- Under the action of continuous U-duality group $E_{6(-26)}$ the rank 2 elements can be brought to the form

$$S_{ij} = \lambda(P_i + P_j) \qquad (i \neq j)$$

or to the form $A_{ij} = \lambda(P_i - P_j)$. $P_i(i = 1, 2, 3)$ are the irreducible idempotents. For $\lambda > 0$, S_{ij} lie in the exceptional cone and describe BPS black holes. Their orbits S_{ij} are given by the coset space

$$\frac{E_{6(-26)}}{SO(9) \circledS T^{16}} \quad , \quad dim = 17 + 9 = 26 = dim \ of \ deformed \ minrep$$

Degeneracies of quantum BPS black holes of rank 2 are given by a modular form of weight 8 of $E_{7(-25)}(\mathbb{Z})$. By squaring the exceptional singular modular form of weight 4, one gets an exceptional singular modular form of weight 8. In the formulation of Krieg it takes the form

$$E_8(Z) = E_4(Z)^2 = \sum_{T \in L\,,\, T \geq 0} \alpha(T)\, e^{2\pi i \mathrm{Tr}(T \circ_J Z)} \quad , \quad Z \in \mathcal{D},$$

where

$$\alpha(T) = \begin{cases} 1 & \text{if } T = 0 \\ 480 \cdot \sigma_7(c(T)) & \text{if } \mathrm{rank}(T) = 1 \\ 240 \cdot 480 \cdot \displaystyle\sum_{d \in \mathbb{N}, d | c(T)} d^7 \sigma_3(c(T^\# / d^2)) & \text{if } \mathrm{rank}(T) = 2 \\ 0 & \text{if } \mathrm{rank}(T) = 3 \end{cases} .$$

The Fourier coefficients $\alpha(T)$ above are all rational integers. Coefficients of rank 2 elements T with $T^\# \neq 0$ count the degeneracies of rank 2 black holes.
Choosing $Z = \tau E^\#$, the modular form $E_8(Z)$ reduces to modular form of weight 24 for $SL(2, \mathbb{Z})$.
$E_4(Z)$ and $E_8(Z) = (E_4(Z))^2$ are the only two singular modular forms on the exceptional domain.

Rank 3 Black Holes

For rank three elements T, the entropy (area) of the large black hole is given by the squareroot of $\det(T)$.

- By squaring the singular modular form E_4 one obtains a modular form whose Fourier coefficients give the degeneracies of rank two black holes. Hence the Fourier coefficients of the cube of $E_4(Z)$ (and higher powers) are expected to count the degeneracies of rank three black holes with non-vanishing entropy.

$$E_4(Z) = 1 + 240 \sum_{\substack{T \geq 0,\ T \in L, \\ \mathrm{rank}(T)=1}} \sigma_3(c(T)) e^{2\pi i \mathrm{Tr}(T \circ_J Z)} \quad , \quad Z \in \mathcal{D}$$

However how to disentangle the degeneracies of higher powers of E_4 into terms corresponding to rank one , rank two and rank three is a challenging open problem.
The singular modular form describing the quantum degeneracies of rank one BPS black holes is known to be related to the minimal unitary representation of $E_{7(-25)}$. Similarly the modular form $E_8 = E_4^2$ is related to the next to minimal unitary representation. This suggests that the full spectrum at the quantum level must correspond to a unitary representation of $E_{7(-25)}$.

Quantum Completion of Octonionic Magical Supergravity

- Proposal for a composite scenario based on a dynamical E_6 GUT with a family $U(1)$ group GST 1983
- Proposal that the theory we are looking for must truncate both to maximal supergravity and octonionic magical supergravity MG May 1984
- Anomaly cancellation work of Green and Schwarz that ushered in the so-called first string revolution. Summer 1984.
- Does there exist an "exceptional" CY manifold that leads to the octonionic magical supergravity from Type II superstring whose toroidal compactification leads to the maximal supergravity GST 1986.
- Does there exist a CY manifold whose moduli space of Hodge structures is given by the symmetric space $E_{7(-25)}/E_6 \times U(1)$? Benedict Gross 1994.
- M-theory proposal of Witten 1996
- Anomaly considerations led to the suggestion that the CY manifold hunted for must be self-mirror with Hodge numbers $h^{11} = h^{21} = 27$ such that M/Superstring theory compactified over it leads to octonionic magical supergravity coupled to 28 hypermultiplets with the target manifold $E_{8(-24)}/E_7 \times SU(2)$. The resulting theory would contain the FHSV model as a subtheory. MG 2006.

- Bianchi and Ferrara argued that octonionic magic supergravity theory may admit a string interpretation closely related to the Enriques model derivation of FHSV model. Their proposed Calabi-Yau manifold is of Borcea-Voisin type (2007).
- However Borcea-Voisin CY manifolds are ruled out since their moduli spaces of Hodge structures are direct product manifolds as in the FHSV model and hence can not include the exceptional symmetric space $E_{7(-25)}/E_6 \times U(1)$.
  MG & Kidambi 2022
- In the space of toric CY threefolds, there is a sharp peak in the number of threefolds with Hodge numbers $h^{11} = h^{21} = 27$ with 910,113 Calabi-Yau threefolds. Deep physical and mathematical meaning of this fact is not known!
- Whether the octonionic magical supergravity coupled to exceptional matter can be obtained by compactification of a CY threefold requires showing that the intersection numbers C_{IJK} coincide with the C-tensor given by the cubic norm of the exceptional Jordan algebra.
- Calculation of the intersection numbers for CY manifolds with such large Hodge numbers within a reasonably finite amount of time is beyond the reach of currently available computer programs , some including AI and machine learning!
- Consensus in math literature seems to be that the answer to the question posed by Benedict Gross is in the negative.
- Cecotti argued that the magical supergravities are not in the swampland. They include the octonionic magical supergravity whose quantum completion is not known.
- Where does the quantum completion of the octonionic magical supergravity lie?
- Could the octonionic magical supergravity describe the low energy effective theory of a new phase of M-theory?

Some open problems and future directions

- Extension of the above results on the minimal and next to minimal unitary representations of $E_{7(-25)}$ and modular forms of weight 4 and 8 of $E_{7(-25)}(\mathbb{Z})$ to higher modular forms and their relations to large BPS black holes.
- extension of the $5d$ results to the quasiconformal extensions of $4d$ U-duality groups as spectrum generating symmetries
- Modular forms of $E_{8(-24)}$ and their relations to quantum black holes of $4d$ octonionic magical supergravity and their connections to the minrep and next to minrep of $E_{8(-24)}$.
- Symmetry algebra of the octonionic magical supergravity reduced to one dimension is a non-split real form $E_{10(nonsplit)}$ of the hyperbolic Lie algebra E_{10} which has the Lie algebra $E_{8(-24)}$ as a subalgebra. The symmetry algebra of octonionic magical supergravity coupled to exceptional matter parametrized by the coset space $E_{8(-24)}/E_7 \times SU(2)$ would have the symmetry algebra $E_{10(nonsplit)} \times E_{10(nonsplit)}$. This again suggests the possibility that there is another phase of M-theory whose strongly coupled phase is related to the octonionic magical supergravity similar to the connection between split $E_{10(10)}$ and maximal supergravity.