The Standard Model Gauge Group from the Exceptional Jordan Algebra

$$
\begin{tikzcd}
 & \mathfrak{h}_3(\mathbb{O}) & \\
\mathfrak{h}_3(\mathbb{C}) \arrow[ur] & & \mathfrak{h}_2(\mathbb{O}) \arrow[ul] \\
 & \mathfrak{h}_2(\mathbb{C}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

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The gauge group of the Standard Model of particle physics is

$$\begin{aligned} \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)) &:= \left\{ x \in \mathrm{SU}(5) : x = \left(\begin{array}{ccccc} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array} \right) \right\} \\ &\cong \left(\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)\right)/\mathbb{Z}_6 \end{aligned}$$

Its representation on one generation of fermions and antifermions comes from restricting the obvious representation of $\mathrm{SU}(5)$ on $\Lambda\mathbb{C}^5$ to $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3))$.

Is there something special about this?

Jordan, von Neumann and Wigner proved:

Theorem. Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, **where $n \leq 3$**.
- The "spin factor" $\mathbb{R} \oplus \mathbb{R}^n$, with

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x}).$$

Building on work of Todorov and Dubois-Violette, Paul Schwahn and I showed:

Theorem 1. Suppose A, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that

$$A \cong h_2(\mathbb{O}) \qquad B \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B \cong \mathfrak{h}_2(\mathbb{C})$$

Then

$$\mathrm{Stab}_0(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

where Stab_0 means the *identity component* of the stabilizer.

So, the Standard Model gauge group consists of symmetries of an octonionic qutrit that restrict to act on a complex qutrit and a compatible octonionic qubit.

$$\begin{tikzcd} & \mathfrak{h}_3(\mathbb{O}) & \\ \color{blue}{\mathfrak{h}_3(\mathbb{C})} \arrow[ur] & & \color{red}{\mathfrak{h}_2(\mathbb{O})} \arrow[ul] \\ & \color{magenta}{\mathfrak{h}_2(\mathbb{C})} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(More precisely it's the identity component of this symmetry group: there's another component that acts as antiunitary transformations.)

For example, since

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R},\ x, y, z \in \mathbb{O} \right\}$$

we can take

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R},\ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

$$B = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R},\ x, y, z \in \mathbb{C} \right\} \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R},\ z \in \mathbb{C} \right\} \cong \mathfrak{h}_2(\mathbb{C})$$

Todorov and Dubois-Violette showed that *in this example*

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

So they showed that the theorem holds *in this special case*.

I'll sketch an argument for this.

Schwahn and I showed that this case implies the general case.

We can build the octonions as

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$$

with multiplication defined using only $\mathrm{SU}(3)$-invariant operations: the inner product and a funny cross product on $\mathbb{C}^3$. Thus $\mathrm{SU}(3)$ acts as automorphisms of $\mathbb{O}$ preserving $\mathbb{C} \subset \mathbb{O}$.

Conversely, if we start with $\mathbb{O}$ and choose any square root of -1, $\mathbb{O}$ gets a complex structure, and splits as a copy of $\mathbb{C}$ and a 3d complex vector space as above.

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In physics, this suggests how there are 3 colors of quark ($\mathbb{C}^3$) for each lepton ($\mathbb{C}$), transforming under the fundamental and trival representations of $\mathrm{SU}(3)$, respectively.

Now, $\mathrm{SU}(3) \times \mathrm{SU}(3)$ acts to give all the Jordan algebra automorphisms of $\mathfrak{h}_3(\mathbb{O})$ preserving $\mathfrak{h}_3(\mathbb{C})$:

- The left $\mathrm{SU}(3)$ acts to *mix up the matrix entries*, conjugating matrices in $\mathfrak{h}_3(\mathbb{O})$ by $\mathrm{SU}(3)$ elements.
- The right $\mathrm{SU}(3)$ acts *separately on each matrix entry* as octonion automorphisms that preserve $\mathbb{C} \subset \mathbb{O}$.

Thus, the whole right copy of $\mathrm{SU}(3)$ preserves

$$\left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \; \alpha, \beta \in \mathbb{R}, \; z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

This is the strong force gauge group.

Of the left copy, only $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \subset \mathrm{SU}(3)$ preserves this block. This is the electroweak force gauge group.

Thus, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{SU}(3)$ acts on $\mathfrak{h}_3(\mathbb{O})$ preserving both $\mathfrak{h}_3(\mathbb{C})$ and $\mathfrak{h}_2(\mathbb{O})$. But there's a $\mathbb{Z}_3$ that acts trivially, and

$$\big(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \times \mathrm{SU}(3)\big)/\mathbb{Z}_3 \;\cong\; \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

This is the idea of Theorem 1 in this special case.

In proving Theorem 1, Paul Schwahn and I noticed:

Theorem 2. Suppose X, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ with

$$X \cong h_2(\mathbb{C}) \qquad B \cong \mathfrak{h}_3(\mathbb{C}) \qquad X \subset B$$

Then

$$\mathrm{Stab}_0(X) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

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(More precisely it's the identity component.)