Projective geometry, Jordan algebras, and the Standard Model gauge group

(joint with John Baez)

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Reminder: Euclidean Jordan algebras

A *Jordan algebra* is a non-associative algebra $(J, \circ)$ which satisfies

1. commutativity: $x \circ y = y \circ x$ for all $x, y \in J$,
2. the Jordan identity: $(x \circ x) \circ (x \circ y) = x \circ ((x \circ x) \circ y)$ for all $x, y \in J$.

In particular, the k-th power x^k is well-defined for any $x \in J$, $k \in \mathbb{N}$.
A Jordan algebra over $\mathbb{R}$ is called *Euclidean* if

3. $x_1^2 + \ldots + x_n^2 = 0$ implies $x_1 = \ldots = x_n = 0$ for any $x_1, \ldots, x_n \in J$.

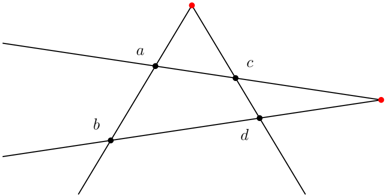
There is a canonically defined *trace form* $\mathrm{tr} : J \to \mathbb{R}$.
The *simple* Euclidean Jordan algebras are:

- the classical ones: $\mathfrak{h}_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $n \geq 3$,
- the exceptional one: $\mathfrak{h}_3(\mathbb{O})$,
- and the spin factors: $\mathbb{R} \times \mathbb{R}^n$, $n \geq 0$. Note $\mathfrak{h}_2(\mathbb{K}) \cong \mathbb{R} \times \mathbb{R}^{\dim \mathbb{K}+1}$.

Projective spaces, axiomatically

A *projective space* is a pair (P, L) of sets ("points" / "lines") and an incidence relation ("$p \in P$ lies on $\ell \in L$") such that:

- Any two distinct points $p, q \in P$ lie on exactly one line $\ell_{p,q} \in L$.
- Veblen's axiom: If $a, b, c, d \in P$ are distinct points and the lines $\ell_{a,b}$ and $\ell_{c,d}$ meet, then so do the lines $\ell_{a,c}$ and $\ell_{b,d}$.



- Any line has at least 3 points on it.

If any two lines meet, and there are at least two lines, (P, L) is called a *projective plane*.

Some concrete projective spaces

If V is a vector space over a division ring $\mathbb{K}$, then its *projectivization* $\mathbb{P}(V)$ is a projective space:

- Points of $\mathbb{P}(V)$ = one-dimensional subspaces of V.
- Lines of $\mathbb{P}(V)$ = two-dimensional subspaces of V.
- p lies on ℓ if $p \subset \ell$.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \geq 0$ we denote $\mathbb{KP}^n := \mathbb{P}(\mathbb{K}^{n+1})$. These are projective planes if $n = 2$.
For $\mathbb{K} = \mathbb{O}$ this doesn't work because $\mathbb{O}$ is non-associative, i.e. not a division ring!

Jordan algebras $\rightsquigarrow$ Projective spaces

Observation

Every simple Euclidean Jordan algebra $(J, \circ)$ gives rise to a projective space in the following way.

- Points: $P = \{p \in J \mid p^2 = p,\ \operatorname{tr} p = 1\}$.
- Lines: $L = \{\ell \in J \mid \ell^2 = \ell,\ \operatorname{tr} \ell = 2\}$.
- Incidence: p lies on ℓ if $p \circ \ell = p$.

The automorphism group $\operatorname{Aut}(J, \circ)$ acts by "isometric" collineations, and transitively on both P and L.

In particular:

- $\mathfrak{h}_n(\mathbb{K}) \rightsquigarrow \mathbb{KP}^{n-1}$, with automorphism groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$.
- $\mathfrak{h}_3(\mathbb{O}) \rightsquigarrow \mathbb{OP}^2$ (we take this as definition of $\mathbb{OP}^2$!), with automorphism group F_4.

The main theorems

Theorem 1

Let X, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that $X \cong \mathfrak{h}_2(\mathbb{C})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $X \subset B$. Then

$$\mathrm{Stab}(X) \cap \mathrm{Stab}(B)_0 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)).$$

Theorem 2

Let A, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that $A \cong \mathfrak{h}_2(\mathbb{O})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$. Then

$$\mathrm{Stab}(A) \cap \mathrm{Stab}(B)_0 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)).$$

This means that $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ can be realized as the intersection of the maximal connected subgroups $\mathrm{Stab}(A) \cong \mathrm{Spin}(9)$ and $\mathrm{Stab}(B)_0 \cong (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ of F_4.

The main theorems

John's talk: These statements are true for the "standard" subalgebras of $\mathfrak{h}_3(\mathbb{O})$. To show them in general, it suffices to establish transitivity of F_4 on these configurations:

Proposition

- Let X, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that $X \cong \mathfrak{h}_2(\mathbb{C})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $X \subset B$. Then there exists an element $g \in \mathrm{F}_4$ such that $g \cdot X = \mathfrak{h}_2(\mathbb{C})$, $g \cdot B = \mathfrak{h}_3(\mathbb{C})$.
- Let A, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that $A \cong \mathfrak{h}_2(\mathbb{O})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$. Then there exists an element $g \in \mathrm{F}_4$ such that $g \cdot A = \mathfrak{h}_2(\mathbb{O})$, $g \cdot B = \mathfrak{h}_3(\mathbb{C})$.

So let us study how those subalgebras of $\mathfrak{h}_3(\mathbb{O})$ behave under the action of F_4.

Peirce subspaces

- For every idempotent $e \in J$ ($e^2 = e$), there is a *Peirce decomposition* $J = E_0(e) \oplus E_{1/2}(e) \oplus E_1(e)$, where $E_\lambda(e) := \{x \in J \mid e \circ x = \lambda x\}$. Further, $E_0(e)$ and $E_1(e)$ are subalgebras.
- If $\ell \in J$ is a line, then $E_1(\ell)$ is spanned by the set of points on ℓ.

Lemma

If ℓ is a line in $\mathfrak{h}_3(\mathbb{K})$, then $E_1(\ell) \cong \mathfrak{h}_2(\mathbb{K})$. Conversely, every subalgebra isomorphic to $\mathfrak{h}_2(\mathbb{K})$ is of this form.

Morally: $\{\text{Lines of } \mathbb{KP}^2\} = \{\mathbb{KP}^1\text{s in } \mathbb{KP}^2\}$.

Corollary

F_4 acts transitively on $\mathfrak{h}_2(\mathbb{K})$-subalgebras of $\mathfrak{h}_3(\mathbb{K})$.

The action of F_4 on $\mathbb{OP}^2$

To summarize:

- F_4 acts transitively on the points of $\mathbb{OP}^2$, with stabilizer $\mathrm{Spin}(9)$. Thus $\mathbb{OP}^2 \cong \mathrm{F}_4/\,\mathrm{Spin}(9)$.
- F_4 acts transitively on lines, i.e. on $\mathfrak{h}_2(\mathbb{O})$-subalgebras of $\mathfrak{h}_3(\mathbb{O})$.

A major part of our work is the following:

Lemma

F_4 acts transitively on the set of $\mathfrak{h}_3(\mathbb{C})$-subalgebras of $\mathfrak{h}_3(\mathbb{O})$.

Morally: F_4 acts transitively on the set of $\mathbb{CP}^2$s in $\mathbb{OP}^2$.

Proof of Theorem 2

- Let A, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$, where $A \cong \mathfrak{h}_2(\mathbb{O})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$.
- It suffices to show that F_4 acts transitively on such pairs (A, B).
- First, move B to the standard $\mathfrak{h}_3(\mathbb{C}) \subset \mathfrak{h}_3(\mathbb{O})$ using the previous lemma. From here on, assume $B = \mathfrak{h}_3(\mathbb{C})$.
- $\mathrm{Stab}(\mathfrak{h}_3(\mathbb{C}))_0 \cong (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ contains an $\mathrm{SU}(3)$-subgroup acting by automorphisms on $\mathfrak{h}_3(\mathbb{C})$.
- Therefore, by transitivity on lines, we can move $A \cap \mathfrak{h}_3(\mathbb{C})$ to the standard $\mathfrak{h}_2(\mathbb{C})$ using an element of $\mathrm{Stab}_0(\mathfrak{h}_3(\mathbb{C}))$.

Proof of Theorem 2

We have managed to move B to $\mathfrak{h}_3(\mathbb{C})$ and $A \cap B$ to $\mathfrak{h}_2(\mathbb{C})$. Now it remains to see that this has to move A to the standard $\mathfrak{h}_2(\mathbb{O})$. For this purpose:

Lemma (Uniqueness lemma)

Let $\mathbb{K}, \mathbb{L}$ be division algebras. Every subalgebra $X \subset \mathfrak{h}_3(\mathbb{K})$ with $X \cong \mathfrak{h}_2(\mathbb{L})$ is contained in a unique subalgebra isomorphic to $\mathfrak{h}_2(\mathbb{K})$. Namely, $E_1(\ell)$ where ℓ is the unit of X.

With this, we have

$$\mathrm{Stab}(A) \cap \mathrm{Stab}(B)_0 \cong \mathrm{Stab}(\mathfrak{h}_2(\mathbb{O})) \cap \mathrm{Stab}(\mathfrak{h}_3(\mathbb{C}))_0 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)).$$

This concludes the proof of Theorem 2.

Proof of Theorem 1

- Let X, B be subalgebras of $\mathfrak{h}_3(\mathbb{O})$, where $X \cong \mathfrak{h}_2(\mathbb{C})$, $B \cong \mathfrak{h}_3(\mathbb{C})$, and $X \subset B$.
- By the uniqueness lemma, there is a unique subalgebra $A \subset \mathfrak{h}_3(\mathbb{O})$ with $X \subset A$ and $A \cong \mathfrak{h}_2(\mathbb{O})$.
- Now one shows that $X = A \cap B$.
- Since X uniquely determines A, and conversely A, B uniquely determine X, one has

$$\mathrm{Stab}(X) \cap \mathrm{Stab}(B)_0 = \mathrm{Stab}(A) \cap \mathrm{Stab}(B)_0$$

and we can apply Theorem 2, which gives

$$\mathrm{Stab}(X) \cap \mathrm{Stab}(B)_0 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)).$$

Thank you!

Figure: The plane of Sir George Cayley (the "father of aviation"), 1853.

(Getty Images)