Standard Model from Exceptional Jordan Triple Systems

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Jordan triple system

Jordan triple

Vector space V over $\mathbb{C}$

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Triple product $\{\cdot\,,\cdot\,,\cdot\}: V \times V \times V \to V$ $\longrightarrow$ $\square: V \times V \to End(V)$

$\mathrm{a}\square\mathrm{b} = \{\mathrm{a}, \mathrm{b}\,,\cdot\}$

- $\{\cdot\,,\cdot\,,\cdot\} \;\to\;$ $\mathbb{C}$-linear in first and last arguments and anti-linear in the second
- $\{\mathrm{a}\,,\mathrm{b}\,,\mathrm{c}\} = \{\mathrm{c}, \mathrm{b}, \mathrm{a}\}$
- $[a\square b, u\square v] = (a\square b\; u)\square v - u\square(b\square a\; v)$

Example: $V = M_{\mathrm{p,q}}(\mathbb{C})$ and $\{a, b, c\} = \frac{1}{2}(ab^{\dagger}c + cb^{\dagger}a)$

If $p \neq q$, not a (unital) Jordan algebra

Classification of Jordan algebras

Formally real simple Jordan algebras

Vector space over $\mathbb{R}$	Jordan product
$\mathfrak{h}_{\mathrm{n}}(\mathbb{R})$	$a \circ b = \frac{1}{2}(ab + ba)$
$\mathfrak{h}_{\mathrm{n}}(\mathbb{C})$	$a \circ b = \frac{1}{2}(ab + ba)$
$\mathfrak{h}_{\mathrm{n}}(\mathbb{H})$	$a \circ b = \frac{1}{2}(ab + ba)$
$\mathbb{R}^n \bigoplus \mathbb{R}$	$(x, t) \circ (x', t') = (\mathrm{t}x' + t'x, x \cdot x' + tt')$
$\mathfrak{h}_3(\mathbb{O})$	$a \circ b = \frac{1}{2}(ab + ba)$

Classification of Jordan triple systems

Vector space over $\mathbb{C}$	Jordan product	Compact Hermitian symmetric space
$\mathrm{M_{p,q}}(\mathbb{C})$	$\{a,b,c\} = \frac{1}{2}(ab^{\dagger}c + cb^{\dagger}a)$	$\mathrm{SU(p+q)/S(U(p)\times U(q))}$
$\mathfrak{s}_{\mathrm{n}}(\mathbb{C})$	$\{a,b,c\} = \frac{1}{2}(ab^{*}c + cb^{*}a)$	$\mathrm{Sp(n)/U(n)}$
$\mathfrak{a}_{\mathrm{n}}(\mathbb{C})$	$\{a,b,c\} = \frac{1}{2}(ab^{*}c + cb^{*}a)$	$\mathrm{SO(2n)/U(n)}$
$\mathbb{C}^{n}$	$\{a,b,c\}$ $= (\mathrm{a}\cdot b^{*})\mathrm{c} + (\mathrm{c}\cdot b^{*})\mathrm{a} - (\mathrm{a}\cdot \mathrm{c})b^{*}$	$\mathrm{SO(n+2)/(SO(n)\times SO(2))}$
$\mathfrak{h}_{3}(\mathbb{O}_{\mathbb{C}})$	$\{a,b,c\} = \frac{1}{2}(a(b^{*}c) + c(b^{*}a))$	$\mathrm{E_7/((E_6\times U(1))/\mathbb{Z}_3)}$
${\mathbb{O}_{\mathbb{C}}}^{2}$	$\{a,b,c\} = \frac{1}{2}(a(\bar{\mathrm{b}}^{\dagger}c) + c(\bar{\mathrm{b}}^{\dagger}a))$	$\mathrm{E_6/((Spin(10)\times U(1))/\mathbb{Z}_4)}$

Tripotents and Peirce decomposition

Result I: the Standard Model gauge group

The inner automorphisms of ${\mathbb{O}_{\mathbb{C}}}^2$ triple system that:

- have determinant 1 action
- stabilise a minimal tripotent
- stabilise up to phase another colinear tripotent

$(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4 \to \ \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$

form the Standard Model gauge group.

The inner automorphisms form the connected part of all possible automorphisms for finite, simple Jordan triple systems, e.g. complex conjugation is not an inner automorphism.

Manifold of minimal unit circles

- Tripotents form a compact submanifold $M_{\text{Tri}} \subset V$
- Rank n tripotents form connected components for simple Jordan triples
- Consider the connected compact submanifold of minimal tripotents $M \subset V$ which is transitive under $\mathrm{Aut}_0(V)$
- Define an equivalence relation $e \sim f \leftrightarrow \exists \lambda \in \mathrm{U}(1)[e = \lambda f]$
- Equivalence class is a 'circle' $\mathrm{C(e)} \subset \mathrm{V_1(e)}$
- The **circles of minimal tripotents form a connected compact Hermitian symmetric space** S**,** with tangent space isomorphic to $\mathrm{V}_{1/2}(\mathrm{e})$

➢S is transitive under $\mathrm{Aut}_0(V)/\mathrm{U}(1)$

➢S is the same compact Hermitian space as the one associated to$\mathrm{V}_{1/2}(\mathrm{e})$

Fixing two points on S

- Take $\mathrm{V} = {\mathbb{O}_{\mathbb{C}}}^2$ with $\mathrm{Aut}_0(V) = (\mathrm{Spin}(10) \times \mathrm{U}(1))/\mathbb{Z}_4$ where $\mathrm{U}(1)$ is formed by uniform phase transformations
- $V_{1/2}(e) \cong \mathfrak{a}_5(\mathbb{C}) \rightarrow \mathrm{S} = \mathrm{SO}(10)/\mathrm{U}(5)$
- Stabilise a point $C(e) \in \mathrm{S}$: $\mathrm{Spin}(10) \rightarrow \mathrm{U}(5)$
- Consider now the circles $\mathrm{C}(\mathrm{f}) \subset V_{1/2}(e)$ that form $S' \subset \mathrm{S}$
- $(\mathfrak{a}_5(\mathbb{C}))_{1/2}(\mathrm{f}) \cong \mathrm{M}_{\mathrm{p,q}}(\mathbb{C}) \rightarrow \mathrm{S}' = \mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$
- The $\mathrm{U}(5)$ symmetry has a $\mathrm{U}(1)$ subgroup that preserves the points of S':
- Peirce scaling: $\mathrm{V} \rightarrow \lambda^5 V_1(e) + \lambda^1 V_{1/2}(e) + \lambda^{-3} V_0(e)$
- This $\mathrm{U}(1)$ is removed by stabilising e not just $\mathrm{C}(\mathrm{e})$ and by stabilising a point $\mathrm{C}(\mathrm{f}) \in \mathrm{S}'$ we get $\mathrm{SU}(5) \rightarrow \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$

Result II: natural Standard Model representation

Take: $e_1 = (e, 0)$ and $e_2 = (0, e)$ $\longrightarrow$ where $e = \frac{1}{2}(1 + i\ell) \in \mathbb{O}_{\mathbb{C}}$ (with $i \in \mathbb{C}$ and $\ell \in \mathbb{O}$) is an idempotent in $\mathbb{O}_{\mathbb{C}}$ and f is its complementary idempotent $f = \frac{1}{2}(1 - i\ell) \in \mathbb{O}_{\mathbb{C}}$. We have

$$e^2 = e, \quad f^2 = f, \quad ef = fe = 0, \quad e + f = 1.$$

Single Peirce decomposition: $16 \to 10 + \bar{5} + 1$

$$\begin{aligned} P_1(e_1)(\alpha,\beta) &= (e\alpha e, 0), & 1 \\ P_{1/2}(e_1)(\alpha,\beta) &= (e\alpha f + f\alpha e, e\beta), & 10 \\ P_0(e_1)(\alpha,\beta) &= (f\alpha f, f\beta), & \bar{5} \end{aligned}$$

$$\begin{aligned} P_1(e_2)(\alpha,\beta) &= (0, e\beta e), \\ P_{1/2}(e_2)(\alpha,\beta) &= (e\alpha, e\beta f + f\beta e), \\ P_0(e_2)(\alpha,\beta) &= (f\alpha, f\beta f). \end{aligned}$$

Common Peirce decomposition

$$\begin{aligned} P_1(e_2)P_{\frac{1}{2}}(e_1)(\alpha,\beta) &= (0, e\beta e) &&\to (1,1,+1) = \bar{e}_R \\ P_{\frac{1}{2}}(e_2)P_1(e_1)(\alpha,\beta) &= (e\alpha e, 0) &&\to (1,1,\ 0\ ) = \bar{\nu}_R \\ P_{\frac{1}{2}}(e_2)P_{\frac{1}{2}}(e_1)(\alpha,\beta) &= (e\alpha f, e\beta f) &&\to (3,2,+\tfrac{1}{6}) = q_L \\ P_{\frac{1}{2}}(e_2)P_0(e_1)(\alpha,\beta) &= (0, f\beta e) &&\to (\bar{3},1,+\tfrac{1}{3}) = \bar{d}_R \\ P_0(e_2)P_{\frac{1}{2}}(e_1)(\alpha,\beta) &= (f\alpha e, 0) &&\to (\bar{3},1,-\tfrac{2}{3}) = \bar{u}_R \\ P_0(e_2)P_0(e_1)(\alpha,\beta) &= (f\alpha f, f\beta f) &&\to (1,2,-\tfrac{1}{2}) = \ell_L. \end{aligned}$$