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Exceptional Structures in Cosmological Billiards

Exceptional Structures and the Standard Model

Higgs Centre, Edinburgh University,

13 - 17 July 2026

Hermann Nicolai

MPI für Gravitationsphysik, Potsdam

(Albert-Einstein-Institut)

Based on work done in collaboration with Axel Kleinschmidt, Michael Koehn and Jakob Palmkvist.

Main Message(s)

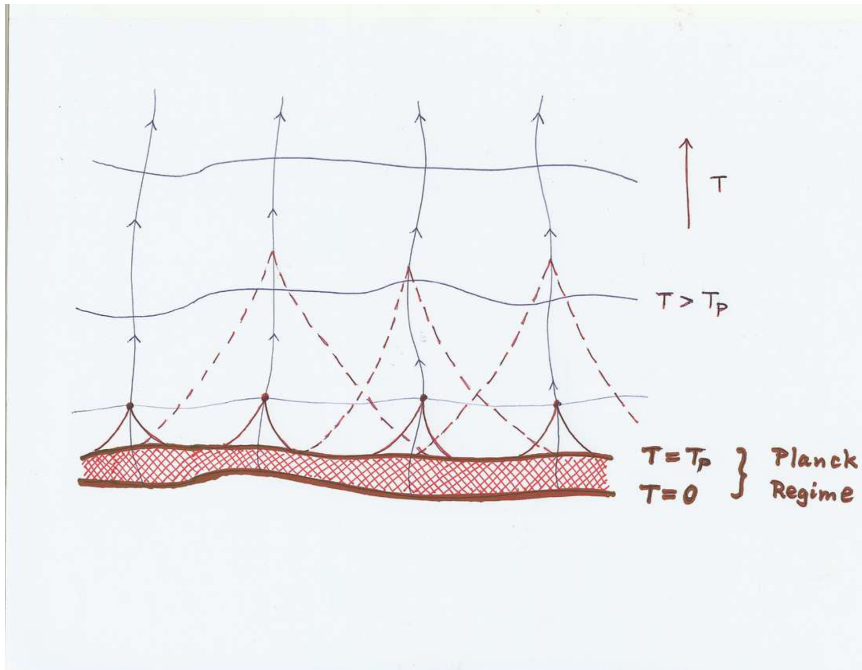
- **This talk: exceptional structures appear already in mini-superspace quantum cosmology**
  - **Octonions and octavians**
  - **Exceptional groups: E_8, E_9 and E_{10}**
  - **New types of modular forms**

- **E_{10} and $K(E_{10})$ are absolutely distinguished $\Rightarrow$**

- **expected to play a key role in understanding**
  - **Quantum gravity and unification**
  - **Structure of fermionic sector of Standard Model with $48 = 3 \times 16$ fundamental spin-$\frac{1}{2}$ fermions (quarks and leptons)**
  - **... and eight supermassive gravitinos as the only new fermionic degrees of freedom $\rightarrow$ a testable prediction (JUNO, DUNE,...)**

... the last two items (based on joint work with A. Kleinschmidt and K.A. Meissner) for some other time!

Basics of BKL

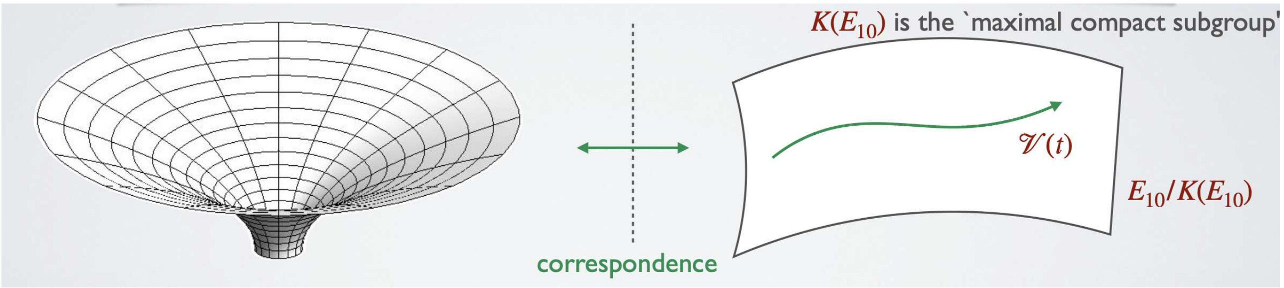


For $T \to 0$ spatial points decouple and the system is effectively described by a continuous *superposition of one-dimensional systems* $\to$ effective dimensional reduction to $D = 1$! [Belinski,Khalatnikov,Lifshitz (1972)]

Space(-time) emergence from $\mathrm{E}_{10}/\mathrm{K}(\mathrm{E}_{10})$

$\Rightarrow$ reverse viewpoint: not E_{10} from dimensional reduction, but space-time based field theory from E_{10}!

$\rightarrow$ map time evolution of bosonic field equations to null geodesic motion on $\mathrm{E}_{10}/\mathrm{K}(\mathrm{E}_{10})$. [Damour,Henneaux,HN(2002)]



$\rightarrow$ analogous to describing evolution of spatial geometry as point particle motion in moduli space $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ of $(D-1)$-geometries.

A concrete scenario for the emergence of space and time from a purely algebraic construct!

For a conceptually different ansatz based on E_{11} see [P.C.West(2001)]

Quantum BKL

[BKL = Belinski,Khalatnikov,Lifshitz(1972);Damour,Henneaux,HN,CQG20(2003)R145]

Metric ansatz with *diagonal* metric components

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \sum_{a=1}^{d} e^{-2\beta^a(t)}\mathrm{d}x_a^2 \qquad \textbf{(Kasner metric)}$$

Substitute ansatz into Einstein-Hilbert action →

$$\mathcal{L} = n^{-1}G_{ab}\dot\beta^a\dot\beta^b \equiv \sum_a (\dot\beta^a)^2 - \left(\sum_a \dot\beta^a\right)^2 \quad \Rightarrow \quad \mathcal{H} = G^{ab}\pi_a\pi_b \approx 0$$

Induced *DeWitt metric* G_{ab} in β-space coincides with the *Cartan-Killing metric* of some indefinite Kac-Moody algebra restricted to CSA (for maximal supergravity $d = 10$ and KMA $= \mathrm{E}_{10}$).

Project motion onto *unit hyperboloid* in β-space

$$\beta^a = \rho\omega^a \quad , \quad \omega^a G_{ab}\omega^b = -1 \quad , \quad \rho^2 = -\beta^a G_{ab}\beta^b$$

with $\omega^a = \omega^a(z)$ parametrizing unit hyperboloid.

⇒ BKL limit: singularity is reached for $\rho \to \infty$ ('Zeno time').

'Integrating out' matter, off-diagonal and spatial curvature degrees of freedom → Effective potential $V(\beta)$ with Toda walls. Drastic simplification in BKL limit: sharp walls partition β-space into *chambers* → metric oscillations towards singularity.

Quantize ⇒ Wheeler-DeWitt (WDW) operator $(d > 2)$

$$\mathcal{H} \equiv G^{ab}\partial_a\partial_b = -\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\triangle_{\mathrm{LB}} \qquad \Rightarrow \ \rho = \text{'time'}$$

with Laplace-Beltrami $\triangle_{LB}$ on $(d-1)$-dimensional unit hyperboloid → wave function obeys WDW equation

$$\mathcal{H}\Psi(\rho, z) = 0$$

To solve separate as $\Psi(\rho, z) = R(\rho)F(z)$ with eigenfunctions $-\triangle_{\mathrm{LB}}F(z) = EF(z)$. Then [Kleinschmidt,Köhn,HN:PRD80(2009)061701]

$$R_{\pm}(\rho) = \rho^{-\frac{d-2}{2}} \exp\left(\pm i\sqrt{E - \frac{(d-2)^2}{4}} \cdot \ln\rho\right) \qquad (d > 2)$$

For Laplace-Beltrami use $(d-1)$-dimensional upper half plane $z = (v, \mathbf{u})$ with $v \in \mathbb{R}_{>}$ and $\mathbf{u} \in \mathbb{R}^{d-2}$ ⇒

$$\triangle_{\mathrm{LB}} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial_{\mathbf{u}}^2$$

With Dirichlet boundary conditions use inequality

[e.g.:Iwaniec:Spectral Methods of Automorphic Forms,AMS(2000) $\to$ KKN(2009)]

$$-(\triangle_{\rm LB}F, F) \geq \int \mathrm{d}v\, \mathrm{d}^{d-2}u\, v^{3-d}(\partial_v F)^2$$

and rewrite

$$(F, F) = \int \mathrm{d}v\, \mathrm{d}^{d-2}u\;v^{1-d}F^2 = \frac{2}{d-2}\int \mathrm{d}v\, \mathrm{d}^{d-2}u\, v^{2-d}F\partial_v F$$

Cauchy-Schwarz inequality (with $-\triangle_{\rm LB}F = EF$) $\Rightarrow$

$$E \;\geq\; \frac{(d-2)^2}{4}$$

$\Rightarrow$ wave function vanishes at singularity ($\rho = \infty$)!

This is DeWitt's proposed mechanism for resolving classical singularities in quantum gravity! [DeWitt(1964)]

Complex* wave function $\Psi(\rho, z) \to$ creates *arrow of time

... could also be relevant for resolution of BH information paradox (if information does not get crushed in singularity it cannot be lost) [Perry(2021):2106.03715]

Octavians and E_8 root lattice

Simple E_8 roots $\varepsilon_1, \cdots, \varepsilon_8$ as unit octavians

[Conway,Smith(2003); see also Kleinschmidt,Palmkvist,HN, math.NT1010.2212]

$$\varepsilon_1 = \frac{1}{2}(1 - e_1 - e_5 - e_6) \qquad \varepsilon_2 = e_1$$
$$\varepsilon_3 = \frac{1}{2}(-e_1 - e_2 + e_6 + e_7) \qquad \varepsilon_4 = e_2$$
$$\varepsilon_5 = (-e_2 - e_3 - e_4 - e_7) \qquad \varepsilon_6 = e_3$$
$$\varepsilon_7 = \frac{1}{2}(-e_3 + e_5 - e_6 + e_7) \qquad \varepsilon_8 = e_4$$

with octonionic units $e_i, e_j, \ldots$ (i, j, k counted mod 7)

$$e_i e_j = e_k \Rightarrow e_{i+1} e_{j+1} = e_{k+1} \quad \textbf{and} \quad e_{2i} e_{2j} = e_{2k}$$

Thus, the E_8 root lattice can be endowed with the structure of a *non-commutative* and *non-associative* ring such that E_8 roots are *units* of this ring!

E_8 Weyl group generated by fundamental reflections $z \to w_j(z) = -\varepsilon_j z \varepsilon_j$ (associative, but $\varepsilon_i(\varepsilon_j z \varepsilon_j)\varepsilon_i \neq (\varepsilon_i\varepsilon_j) z (\varepsilon_j \varepsilon_i)$)

Automorphic properties of wave function

Reflective walls imply boundary conditions:

$$\Psi(\rho, z) = \pm\Psi(\rho, w_I \cdot z)$$

for fundamental Weyl reflections w_I **with** $-(+)$ **for Dirichlet (Neumann) boundary conditions. For** E_{10} **we take**

$$z = \mathbf{u} + \mathrm{i}v \; , \;\; \bar{z} = \bar{\mathbf{u}} - \mathrm{i}v \qquad (\mathbf{u} \in \mathbb{O} \, , \; v > 0 \; ; \;\; \mathrm{i}\mathbf{u} = \bar{\mathbf{u}}\mathrm{i})$$

$\rightarrow$ ***octonionic upper half plane.*** **This leads to 'modular' action of fundamental Weyl reflections in** $\mathbf{W}(\mathbf{E}_{10})$

$$w_{-1}(z) = \frac{1}{\bar{z}} \;\; , \quad w_0(z) = -\theta\bar{z}\theta \quad , \quad w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

where θ **highest root of** E_8 **and** ε_j **simple roots of** E_8**.**

The above action extends the octavian realization of $\mathbf{W}(\mathbf{E}_8)$ **to a full modular realization of the hyperbolic Weyl group** $\mathbf{W}(\mathbf{E}_{10})$**, with an extra 'sedenionic' imaginary unit for the octonionic upper half plane.** $\Rightarrow$

Weyl group of E_{10} = 'PGL$_2$(O)' [Feingold,Kleinschmidt,HN:JAlg322(2009)1295].

Even subgroup of Weyl group = PSL$_2$(O) is the analog of the modular group PSL$_2(\mathbb{Z})$ ($\rightarrow$ Einstein gravity). [see also L.Forte,CQG26(2009)045001]

'Wave function of the Universe' is a PGL$_2$(O) automorphic form! Mathematical theory to be developed...

Restriction of wave function to fundamental domain = division by modular group for string amplitudes.

These results can be supersymmetrized $\rightarrow$ fermionic Fock space has 2^{160} components even for this simplest of approximations. [Kleinschmidt,Köhn,HN(2009)]

More manageable fermionic Fock space via quantization of $N = 1, D = 4$ supergravity in Bianchi IX background [Damour,Spindel:PRD95(2017)126011] **$\rightarrow$ further evidence for AE_3 and $K(AE_3)$ in ordinary (Einstein) gravity.**

Geodesics on $\mathrm{E}_{10}/\mathrm{K}(\mathrm{E}_{10})$ are infinitely unstable

[Damour,HN:CQG22(2005)2849]

Recall geodesic deviation (Jacobi) equation

$$\delta\ddot{\xi}^{\mu} \;=\; \mathcal{R}^{\mu}{}_{\nu\lambda\rho}\,\dot{\xi}^{\nu}\,\delta\xi^{\lambda}\,\dot{\xi}^{\rho}$$

Choose CSA geodesic $\xi(t) \equiv \xi^a(t)H_a$ **(**$\Rightarrow \ddot{\xi}^a(t) = 0$**) with deviation in direction of** E^s_α ***i.e.*** $\delta\xi \;=\; \sum_{\alpha,s}\delta\xi^{\alpha}_s(E^s_\alpha + E^s_{-\alpha})$**.**
$\Rightarrow$ **sectional curvature with generic** $v \in$ **CSA:**

$$\mathcal{R}(v, E^+_\alpha, v, E^+_\alpha) \;=\; -(v^a\alpha_a)^2 \;<\; 0$$

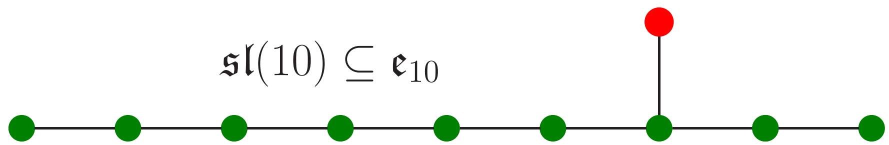
is negative and decreases without bound: for every imaginary root α**, any multiple** $n\alpha$ **is also a root** $\forall n \in \mathbb{N}$
$\Rightarrow\;\; \delta\ddot{\xi}^{n\alpha}_s = n^2(v\cdot\alpha)^2\,\delta\xi^{n\alpha}_s \;\;\Rightarrow\;\; \delta\xi^{n\alpha}_s(t) \propto e^{n|v\cdot\alpha|t}$ **for** $E^+_{n\alpha}$**.**

This strongly suggests that analysis cannot be confined to diagonal metrics and BKL regime, but *must take into account all off-diagonal degrees of freedom.*

Duality symmetries: all in one?

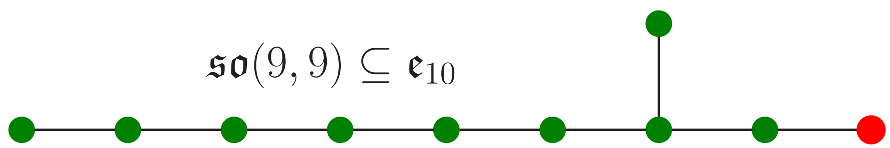
Hyperbolic KMA $\rightarrow$ only finite or affine regular subalgebras!

$\mathfrak{sl}(10) \subseteq \mathfrak{e}_{10}$



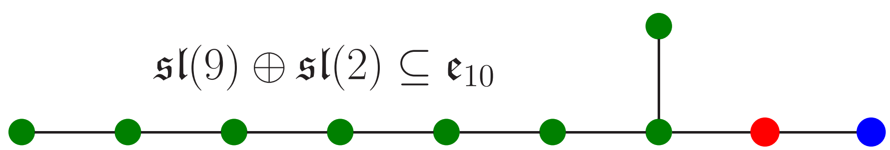
$D = 11$ **SUGRA**

$\mathfrak{so}(9,9) \subseteq \mathfrak{e}_{10}$



mIIA $D = 10$ **SUGRA**

$\mathfrak{sl}(9) \oplus \mathfrak{sl}(2) \subseteq \mathfrak{e}_{10}$



IIB $D = 10$ **SUGRA**

All encompassing: E_{10} contains *all* hyperbolic KMAs with symmetric Cartan matrices as subalgebras. [Viswanath(2008)]

SL(10) decomposition for $\ell \leq 3$

Level decomposition: $\alpha = \ell\alpha_0 + \sum_{j\geq 1} n_j \alpha_j$

- $\ell = 0: \; K^m{}_n h_m{}^n(t) \quad \in$ **GL(10)** $\leftrightarrow$ **graviton**
- $\ell = 1: \; E^{mnp} A_{mnp}(t)$ $\leftrightarrow$ **(electric) 3-form**
- $\ell = 2: \; E^{m_1 \cdots m_6} A_{m_1 \cdots m_6}(t)$ $\leftrightarrow$ **dual (magnetic) 6-form**
- $\ell = 3: \; E^{m_0|m_1 \cdots m_8} A_{m_0|m_1 \cdots m_8}(t)$ $\leftrightarrow$ **dual graviton**

for 'upper triangular' generators $E^{mnp}, \cdots \;\rightarrow$ **parametrize coset**
formally **as** $\mathcal{V}(t) = \exp\left(h_m{}^n(t)K^m{}_n + \frac{1}{3!}A_{mnp}(t)E^{mnp} + \cdots\right) \in \mathbf{E}_{10}/\mathbf{K}(\mathbf{E}_{10})$

Idem for mIIA and IIB (and other) decompositions $\Rightarrow$
$\rightarrow$ different theories by 'slicing' $\mathbf{E}_{10}$ in different ways!

$\Rightarrow$ $\mathbf{E}_{10}$ 'knows all' about supersymmetry!

Conjecture: spatial dependence gets encoded into higher level 'states' $\rightarrow$ space 'de-emerges' near singularity

SL(10) Decomposition for $\ell \leq 18$

Table 1: A_9 representations in E_{10} up to level $\ell = 18$

ℓ	p	m	Λ^2	dim $\mathcal{R}(\Lambda)$	mult(Λ)	μ
1	(001000000)	0 0 0 0 0 0 0 0 0	2	120	1	1
2	(000001000)	1 2 3 2 1 0 0 0 0	2	210	1	1
3	(100000010)	1 3 5 4 3 2 1 0 0	2	440	1	1
	(000000001)	2 4 6 5 4 3 2 1 0	0	10	8	0
4	(001000001)	2 4 6 5 4 3 2 1 0	2	1155	1	1
	(200000000)	1 4 7 6 5 4 3 2 1	2	55	1	1
	(010000000)	2 4 7 6 5 4 3 2 1	0	45	8	0
5	(000001001)	3 6 9 7 5 3 2 1 0	2	1848	1	1
	(100100000)	2 5 8 6 5 4 3 2 1	2	1848	1	1
	(000010000)	3 6 9 7 5 4 3 2 1	0	252	8	0
6	(100000011)	3 7 11 9 7 5 3 1 0	2	3200	1	1
	(000000002)	4 8 12 10 8 6 4 2 0	0	55	8	0
	(010001000)	3 6 10 8 6 4 3 2 1	2	8250	1	1
	(100000100)	3 7 11 9 7 5 3 2 1	0	1155	8	1
	(000000010)	4 8 12 10 8 6 4 2 1	-2	45	44	1

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18	(002000010)	11 22 33 28 23 18 13 8 4	-20	190575	1044218	973
	(200100010)	10 22 34 28 23 18 13 8 4	-20	351000	1044218	1717
	(010100010)	11 22 34 28 23 18 13 8 4	-22	261360	2485020	2674
	(100010010)	11 23 35 29 23 18 13 8 4	-24	83160	5749818	3208
	(000001010)	12 24 36 30 24 18 13 8 4	-26	6930	12970045	1843
	(500000001)	8 21 34 29 24 19 14 9 4	-8	19305	2472	15
	(310000001)	9 21 34 29 24 19 14 9 4	-16	49280	167116	295
	(120000001)	10 21 34 29 24 19 14 9 4	-20	47190	1043926	857
	(201000001)	10 22 34 29 24 19 14 9 4	-22	45045	2485020	1635
	(011000001)	11 22 34 29 24 19 14 9 4	-24	31185	5749818	2255
	(100100001)	11 23 35 29 24 19 14 9 4	-26	17280	12971009	3295
	(000010001)	12 24 36 30 24 19 14 9 4	-28	2310	28592513	2095
	(400000000)	9 22 35 30 25 20 15 10 5	-18	715	425227	90
	(210000000)	10 22 35 30 25 20 15 10 5	-24	1485	5750072	688
	(020000000)	11 22 35 30 25 20 15 10 5	-26	825	12971009	744
	(101000000)	11 23 35 30 25 20 15 10 5	-28	990	28595548	1628
	(000100000)	12 24 36 30 25 20 15 10 5	-30	210	61721165	1168

- **Idem for mIIA and IIB (and other) decompositions**
- ***E.g.*** **for** $\ell = 28$ **we have already 3276 inequivalent irreducible SL(10) representations with maximum outer multiplicity** $\mu = 46450629$ **for** (110100011)**, for** $\ell \leq 28$ **altogether** $\sim 4.5 \times 10^9$ **representations of SL(10)** [T.Fischbacher]**.**
- **No matter where you truncate these level expansions, they keep getting more complicated with increasing level** ℓ **(exponential growth!).**