The Okubo Algebra : stranger than Octonions ?

Alessio Marrani

Centre for Mathematics and Theoretical Physics

University of Hertfordshire UH

Based on joint works with D. Corradetti (Univ. Algarve, PT) and V. Zucconi (Univ. Udine, IT),
J. Phys. A: Math. Theor. **58** 075202 (2025), and *Ann. U. Ferrara* **70** (2024) 681-730

Exceptional Structures & the SM, July 15, 2026

1 The three 8-dimensional, real, division, composition algebras : $\mathbb{O}$, $p\mathbb{O}$, and $\mathcal{O}$

An algebra A is a vector space endowed with a bilinear product. The specific properties of such bilinear product (which we will denote by $\cdot$) lead to various classifications: an algebra A is said to be *commutative* if $x \cdot y = y \cdot x$ for every $x, y \in A$; is *associative* if satisfies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; is *alternative* if $x \cdot (y \cdot y) = (x \cdot y) \cdot y$; and finally, *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. If the algebra is also equipped with a norm n, then it is a normed algebra. Moreover, if the norm respects the multiplicative structure, i.e., $n(x \cdot y) = n(x)\, n(y)$, the algebra is called a *composition algebra*. Thus, a composition algebra is fully identified by the triple $(A, \cdot, n)$.

Composition algebras split into *unital*, i.e. where exists a unit element 1 such that $x \cdot 1 = 1 \cdot x = x$, *para-unital*, i.e. where exists an involution $x \longrightarrow \overline{x}$ and an element called para-unit $\mathbf{1}$ such that $x \cdot \mathbf{1} = \mathbf{1} \cdot x = \overline{x}$, and *non-unital*, namely algebras that do not possess neither a unit element nor a para-unit element. A complete classification is provided by the Generalized Hurwitz Theorem [15], which states that only sixteen composition algebras exist (e.g.on $\mathbb{R}$) : seven are unital and called *Hurwitz algebras*, including the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ along with their split companions $\mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$; another seven are para-unital, closely related to the Hurwitz algebras and termed *para-Hurwitz algebras*; finally, there are two composition algebras, one division and one split, that are both non-unital and 8-dimensional, known as the *Okubo algebras* $\mathcal{O}$ and $\mathcal{O}_s$ [10].

Out of such sixteen composition algebras, in the present treatment we will be interested only in the 8-dimensional division ones, namely in the following three (mutually non-isomorphic) algebras : the octonions $\mathbb{O}$ (which are alternative and unital), the para-octonions $p\mathbb{O}$ (non-alternative and para-unital) and the Okubo algebra $\mathcal{O}$ (non-alternative and non-unital). Their properties are summarized in Table 1.

1 The three 8-dimensional, real, division, composition algebras : $\mathbb{O}$, $p\mathbb{O}$, and $\mathcal{O}$

An algebra A is a vector space endowed with a bilinear product. The specific properties of such bilinear product (which we will denote by $\cdot$) lead to various classifications: an algebra A is said to be *commutative* if $x \cdot y = y \cdot x$ for every $x, y \in A$; is *associative* if satisfies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; is *alternative* if $x \cdot (y \cdot y) = (x \cdot y) \cdot y$; and finally, *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. If the algebra is also equipped with a norm n, then it is a normed algebra. Moreover, if the norm respects the multiplicative structure, i.e., $n(x \cdot y) = n(x)\, n(y)$, the algebra is called a *composition algebra*. Thus, a composition algebra is fully identified by the triple $(A, \cdot, n)$.

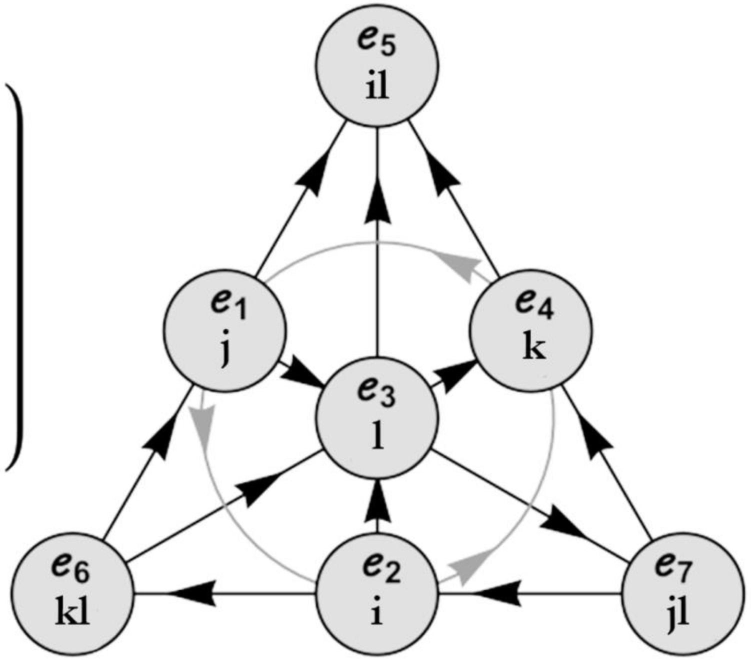
Composition algebras split into *unital*, i.e. where exists a unit element 1 such that $x \cdot 1 = 1 \cdot x = x$, *para-unital*, i.e. where exists an involution $x \longrightarrow \overline{x}$ and an element called para-unit $\mathbf{1}$ such that $x \cdot \mathbf{1} = \mathbf{1} \cdot x = \overline{x}$, and *non-unital*, namely algebras that do not possess neither a unit element nor a para-unit element. A complete classification is provided by the Generalized Hurwitz Theorem [15], which states that only sixteen composition algebras exist (e.g.on $\mathbb{R}$) : seven are unital and called *Hurwitz algebras*, including the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ along with their split companions $\mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$; another seven are para-unital, closely related to the Hurwitz algebras and termed *para-Hurwitz algebras*; finally, there are two composition algebras, one division and one split, that are both non-unital and 8-dimensional, known as the *Okubo algebras* $\mathcal{O}$ and $\mathcal{O}_s$ [10].

Property	$\mathbb{O}$	$p\mathbb{O}$	$\mathcal{O}$
Unital	Yes	No	No
Para-unital	Yes	Yes	No
Alternative	Yes	No	No
Flexible	Yes	Yes	Yes
Composition	Yes	Yes	Yes
Automorphism	G_2	G_2	$\mathrm{SU}(3)$

1.1 The algebra of octonions $\mathbb{O} \equiv (\mathbb{O}, \cdot, n)$

The *octonions* $(\mathbb{O}, \cdot, n)$ are a Hurwitz algebra, and they can be defined as the 8-dimensional real vector space with basis $\{e_0 = 1, e_1, ..., e_7\}$, endowed with a bilinear product $\cdot$ encoded through the *Fano plane* and explained in Fig. 1.1. The resulting algebra is non-associative and non-commutative, but alternative (and thus flexible).

$$\begin{pmatrix}
e_0 \mapsto 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & -1 & e_4 & e_7 & -e_2 & e_6 & -e_5 & -e_3 \\
e_2 & -e_4 & -1 & e_5 & e_1 & -e_3 & e_7 & -e_6 \\
e_3 & -e_7 & -e_5 & -1 & e_6 & e_2 & -e_4 & e_1 \\
e_4 & e_2 & -e_1 & -e_6 & -1 & e_7 & e_3 & -e_5 \\
e_5 & -e_6 & e_3 & -e_2 & -e_7 & -1 & e_1 & e_4 \\
e_6 & e_5 & -e_7 & e_4 & -e_3 & -e_1 & -1 & e_2 \\
e_7 & e_3 & e_6 & -e_1 & e_5 & -e_4 & -e_2 & -1
\end{pmatrix}$$



1.1 The algebra of octonions $\mathbb{O} \equiv (\mathbb{O}, \cdot, n)$

The *octonions* $(\mathbb{O}, \cdot, n)$ are a Hurwitz algebra, and they can be defined as the 8-dimensional real vector space with basis $\{e_0 = 1, e_1, ..., e_7\}$, endowed with a bilinear product $\cdot$ encoded through the *Fano plane* and explained in Fig. 1.1. The resulting algebra is non-associative and non-commutative, but alternative (and thus flexible).

Given an element $x \in \mathbb{O}$ with decomposition $x = x_0 e_0 + x_1 e_1 + ... + x_7 e_7$, one can define its conjugate element as $\overline{x} := x_0 e_0 - x_1 e_1 - ... - x_7 e_7$; as in any unital composition algebra, $\overline{x}$ is the image of x under an order-two homomorphism given by a canonical involution, named *conjugation*. The norm n is the obvious Euclidean one, defined by

$$n(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2, \tag{1.1}$$

and, therefore,

$$n(x) = x \cdot \overline{x} = \overline{x} \cdot x, \tag{1.2}$$

as it holds for every Hurwitz algebra. (1.1) yields that $n(x) = 0 \Leftrightarrow x = 0$ and thus the inverse of a non-zero element of the octonions is easily found as $x^{-1} = \overline{x}/n(x)$. Also, the polarization of (1.2) yields the definition of the octonionic inner product as $\langle x, y \rangle = x \cdot \overline{y} + y \cdot \overline{x}$, so that $\langle x, x \rangle = 2n(x)$. Note that $\mathbb{O}$ is a division algebra, since if $\langle x, y \rangle = 0$, then either x or y are zero.

By exploiting the unitality of $\mathbb{O}$, the octonionic conjugation can equivalently be defined using the orthogonal projection on the unit element 1 as

$$x \mapsto \overline{x} := \langle x, 1 \rangle 1 - x. \tag{1.3}$$

This canonical involution has the distinctive property of being an antihomomorphism with respect to the product, i.e. $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$.

1.2 The algebra of para-octonions $p\mathbb{O} \equiv (\mathbb{O}, \bullet, n)$

Starting from the Hurwitz algebra of octonions $(\mathbb{O}, \cdot, n)$, a para-Hurwitz algebra can be introduced by defining a new product

$$x \bullet y = \overline{x} \cdot \overline{y}, \tag{1.4}$$

for every $x, y \in \mathbb{O}$. The new algebra $(\mathbb{O}, \bullet, n)$ is again a composition algebra, in fact a para-Hurwitz algebra, called *para-octonions* and denoted with $p\mathbb{O}$ (not to be confused with the algebra of split-octonions $\mathbb{O}_s$, which is a Hurwitz algebra with zero divisors) [18, 15]. Para-octonions do not have a unit, but only a para-unit, i.e. $\mathbf{1} \in p\mathbb{O}$ such that $\mathbf{1} \bullet x = x \bullet \mathbf{1} = \overline{x}$. Note that $p\mathbb{O}$ is still a division algebra, since if $x \bullet y = 0$, then either $\overline{x}$ or $\overline{y}$ are zero, thus implying that either x or y are zero.

1.3 The Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$: Petersson's and Okubo's realizations

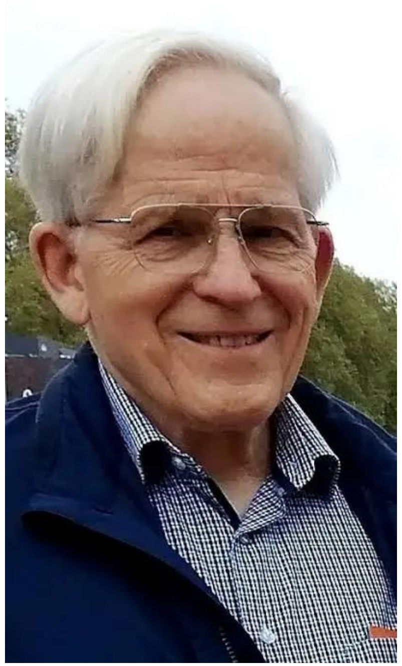
$\mathbb{O}$ also admits an order-three involutive automorphism τ; considering again the basis $\{e_0 = 1, e_1, ..., e_7\}$ of $\mathbb{O}$ as a vector space, τ can be defined as follows [18]:

$$\begin{array}{llll} \tau(e_k) & = e_k, & & k = 0, 1, 3, 7 \\ \tau(e_2) & = -\frac{1}{2}\left(e_2 - \sqrt{3}e_5\right), & \tau(e_4) = -\frac{1}{2}\left(e_4 - \sqrt{3}e_6\right), & \\ \tau(e_5) & = -\frac{1}{2}\left(e_5 + \sqrt{3}e_2\right), & \tau(e_6) = -\frac{1}{2}\left(e_6 + \sqrt{3}e_4\right). & \end{array} \tag{1.5}$$

By starting from the Hurwitz algebra of octonions $(\mathbb{O}, \cdot, n)$, such a map can be exploited in order to obtain a new (non-isomorphic) algebra (belonging to the class of Petersson algebras [19]), whose product (denoted by $*$) is defined as

$$x * y = \tau(\overline{x}) \cdot \tau^2(\overline{y}), \tag{1.6}$$

for every $x, y \in \mathbb{O}$. The new algebra $(\mathbb{O}, *, n)$ is again a composition algebra, called the *Okubo algebra* $\mathcal{O}$. This algebra does not have a unit nor a para-unit, but it has idempotent elements. However, $\mathcal{O}$ is still a division algebra, because if $x * y = \tau(\overline{x}) \cdot \tau^2(\overline{y}) = 0$, then either $\tau(\overline{x})$ or $\tau^2(\overline{y})$ are zero, and recalling that τ is an automorphism, this implies that either x or y are zero.



1.3 The Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$: Petersson's and Okubo's realizations

An independent realisation of $\mathcal{O}$, equivalent to the above realization *à la Petersson* based on τ, was found by Okubo in the late '70s, and it is based on a peculiar deformation of the Jordan product. Following [6] and [10], $\mathcal{O}$ can indeed also be defined as the set of 3×3 Hermitian traceless matrices over the complex numbers $\mathbb{C}$, endowed with the product

$$x * y = \mu xy + \overline{\mu} yx - \frac{1}{3}\mathrm{Tr}\,(xy)\,, \qquad (1.7)$$

non-symmetric, real (over C) deformation of Michel-Radicati version of Jordan product

where $\mu = {}^1/_6\left(3 + \mathrm{i}\sqrt{3}\right)$ and the juxtaposition denotes the ordinary (non-commutative but associative) matrix product. It should be remarked that $*$ is bilinear, but non-symmetric, with its antisymmetric part proportional to $\mathrm{Im}\mu = \frac{\sqrt{3}}{6}$; as anticipated, $*$ can be regarded as a deformation of the Jordan product $x \circ y = \frac{1}{2}xy + \frac{1}{2}yx$, to which it reduces by setting $\mathrm{Im}\mu = 0 \Leftrightarrow \mu = {}^1/_2$ and neglecting the last term in the r.h.s. of (1.7). Nevertheless, the tracelessness property is not preserved under the Jordan product $\circ$, and thus the additional term $-{}^1/_3\mathrm{Tr}\,(xy)$ is actually needed for the closure of the algebra. It is amusing to notice that by simply setting $\mathrm{Im}\mu = 0$ (and retaining the trace term in the r.h.s. of (1.7)), one retrieves the traceless part $\mathfrak{J}_3\,(\mathbb{C})_0$ of the simple, cubic Jordan algebra $\mathfrak{J}_3\,(\mathbb{C})$, whose derivation Lie algebra is $\mathfrak{su}\,(3)$.

Analyzing (1.7), one can realize that the resulting algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$, of real dimension 8, is neither unital, nor associative, nor alternative. However, $\mathcal{O}$ admits idempotent elements, an example of which is

$$e = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (1.8)$$

Nota Bene :
Realization based on the **change of basis** (from vectors in 8 dimensions to matrices in 3 dimensions)

1.3 The Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$: Petersson's and Okubo's realizations

Analyzing (1.7), one can realize that the resulting algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$, of real dimension 8, is neither unital, nor associative, nor alternative. However, $\mathcal{O}$ admits idempotent elements, an example of which is

$$e = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1.8}$$

which indeed satisfies $e * e = e$. By setting $e = e_0$ and defining

$$\begin{gathered} e_1 = \sqrt{3}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \sqrt{3}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \sqrt{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ e_4 = \sqrt{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \sqrt{3}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \sqrt{3}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ e_7 = \sqrt{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \end{gathered} \tag{1.9}$$

Nota Bene :
These are the 8 Gell-Mann matrices, which realize the generators of su(3) in the fundamental representation (**with standard matrix mult.**), and that are also a basis of J3(C)_0 (with the **Michel-Radicati product**)

one obtains an explicit realization of a basis for $\mathcal{O}$. In this realization, the norm n is defined through the matrix trace (denoted as Tr), namely[1]

$$n(x) = \frac{1}{6}\mathrm{Tr}\left(x^2\right), \tag{1.10}$$

where $x^2 \equiv xx$ (i.e., the square of x under the usual matrix product), for every $x \in \mathcal{O}$. It is then easy to see that the (quadratic) norm n defined by (1.10) has Euclidean signature, and furthermore that it is associative and composition over $\mathcal{O}$ itself.

[1] It can be proved that all the treatment is independent on the actual explicit expression of the idempotent e.

1.5 Symmetries

Given an algebra $(A, \cdot, n)$, one can consider :

- The special orthogonal group $\mathrm{SO}(A)$, defined as the identity connected component of the group of endomorphisms of A that preserve the norm n (or equivalently the bilinear and symmetric inner product defined through its polarization), but not necessarily the algebraic structure defined by the product[2] $\cdot$:

$$\mathrm{SO}(A) := \left\{\varphi \in \mathrm{End}^0(A) : n(\varphi(x)) = n(x)\right\}. \tag{1.18}$$

- The automorphism group $\mathrm{Aut}(A)$ is the group of (linear, identity connected) isomorphisms that preserves the algebraic structure :

$$\mathrm{Aut}(A) := \left\{\varphi \in \mathrm{End}^0(A) : \varphi \text{ bijective, linear, } \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\right\}. \tag{1.19}$$

  While an element of $\mathrm{Aut}(A)$ is also an element of $\mathrm{SO}(A)$, the converse is not necessarily true. In fact, in general $\mathrm{Aut}(A)$ is a proper subgroup of $\mathrm{SO}(A)$:

$$\mathrm{Aut}(A) \subsetneq \mathrm{SO}(A). \tag{1.20}$$

- The triality group is the group of triples of orthogonal transformations of A which respect its algebraic structure:

$$\mathrm{Tri}(A) := \left\{(\alpha, \beta, \gamma) \in \mathrm{SO}(A)^{\otimes 3} : \alpha(x \cdot y) = \beta(x) \cdot \gamma(y)\right\}. \tag{1.21}$$

[2]The upperscript 0 denotes the identity connected component of the Lie group.

1.5 Symmetries

In general, $\mathrm{Tri}(A)$ is a (proper) subgroup of the product group $\mathrm{SO}(A)^{\otimes 3}$, and it contains (or coincides with) $\mathrm{SO}(A)$ itself :

$$\mathrm{SO}\,(A) \subseteq \mathrm{Tri}\,(A) \subsetneq \mathrm{SO}\,(A)^{\otimes 3}\,. \tag{1.22}$$

	Octonions $\mathbb{O}$	Para-Octonions $p\mathbb{O}$	Okubo $\mathcal{O}$
$\mathrm{Aut}\,(A)$	$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	$\mathrm{SU}(3)$
$\mathrm{SO}\,(A)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$
$\mathrm{Tri}\,(A)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$

The **Okubonions** are the only algebra [to my knowledge] that lie in the **adjoint representation** of its derivation Lie algebra / automorphism Lie group.

2 $(\mathbb{O}, \cdot, n)$

2.1 Symmetries

Tautologically, any algebra A sits in a linear (not necessarily irreducible) representation of its (linearly realized) automorphism Lie group, whose Lie algebra is the algebra of derivations, i.e. $\mathfrak{der}\,(A) = \mathfrak{Lie}\,(\mathrm{Aut}\,(A))$. Considering $(\mathbb{O}, \cdot, n)$, Table 3 reports that

$$
\begin{aligned}
\mathrm{Aut}\,(\mathbb{O}) &= \mathrm{G}_{2(-14)}, && (2.1)\\
\mathrm{Tri}\,(\mathbb{O}) &= \mathrm{Spin}\,(8)\,, && (2.2)\\
\mathrm{Spin}\,(\mathbb{O}) &= \mathrm{Spin}\,(8)\,, && (2.3)
\end{aligned}
$$

and it holds that

$$
\mathbb{O} \simeq \mathbf{1} \oplus \mathbf{7} \text{ of } \mathrm{G}_{2(-14)}. \tag{2.4}
$$

Both Spin(8) and $\mathrm{G}_{2(-14)}$ have real representations [23]. The relation among these Lie groups is given by the following chain of maximal group embeddings :

$$
\begin{array}{ccccc}
\mathrm{Spin}\,(8) & \supset_{\mathrm{s}} & \mathrm{Spin}\,(7) & \supset_{\mathrm{ns}} & \mathrm{G}_{2(-14)} \\
\mathbf{8}_v & = & \mathbf{7} \oplus \mathbf{1} & = & \mathbf{7} \oplus \mathbf{1} \\
\mathbf{8}_s & = & \mathbf{8} & = & \mathbf{7} \oplus \mathbf{1} \\
\mathbf{8}_c & = & \mathbf{8} & = & \mathbf{7} \oplus \mathbf{1} \\
\mathbf{28} & = & \mathbf{21} \oplus \mathbf{7} & = & \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{7},
\end{array} \tag{2.5}
$$

We remark that the unital nature of $\mathbb{O}$ is related to the fact that Aut($\mathbb{O}$) is a (maximal) subgroup of Spin(7), and that one (say[3], $\mathbf{8}_v$) of the three 8-dimensional representations of Spin(8) contains a singlet (namely, the unit 1 of $\mathbb{O}$) when branched with respect to Spin(7); in fact, (2.5) defines a choice of "polarization" within the triality symmetry of Spin(8) (see e.g. [25]). The existence of a unity implies the following decomposition of $\mathbb{O}$:

$$
\mathbb{O} = \mathbb{R} \oplus \mathbb{O}', \tag{2.6}
$$

2.2 $\mathbf{SU(3)_{\mathbb{O}} \subsetneq Aut(\mathbb{O})}$

The automorphism group of the octonions $\mathrm{Aut}(\mathbb{O})$ is the real compact form of the exceptional Lie group G_2, i.e., $\mathrm{Aut}(\mathbb{O}) = \mathrm{G}_{2(-14)}$. The Lie group $\mathrm{G}_{2(-14)}$ admits three maximal subgroups, two of Borel-de Siebenthal type, i.e., of maximal rank [26], namely $\mathrm{SU}(3)$ and $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$, and one of non-maximal rank, $\mathrm{SU}(2)$. In the present treatment, we are interested in the[4] $\mathrm{SU}(3)$ maximal subgroup of $\mathrm{G}_{2(-14)}$, which we will be denoting as $\mathrm{SU}(3)_{\mathbb{O}}$, and in its representations stemming from the decomposition of the defining and adjoint irreprs. of $\mathrm{G}_{2(-14)}$ itself :

$$\begin{array}{ccc} \mathrm{G}_{2(-14)} & \supset_{\mathrm{ns}} & \mathrm{SU}(3)_{\mathbb{O}} \\ \mathbf{7} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}, \\ \mathbf{14} & = & \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}. \end{array} \tag{2.7}$$

Despite the fact that all representations of $\mathrm{G}_{2(-14)}$ are real, $\mathrm{SU}(3)$ (and thus $\mathrm{SU}(3)_{\mathbb{O}}$ defined in (2.7)) admits complex (i.e., reflexive) representations [23]. In the case under consideration, this implies the existence of a complex structure J in the exceptional (and non-symmetric) presentation S^6_J of the 6-sphere with isotropy group $\mathrm{SU}(3)_{\mathbb{O}}$ (see e.g. [27] and Refs. therein),

$$S^6_J \simeq \frac{\mathrm{G}_{2(-14)}}{\mathrm{SU}(3)_{\mathbb{O}}}. \tag{2.8}$$

Thus, $\mathrm{Aut}(\mathbb{O})$ can also be presented as

$$\mathrm{G}_{2(-14)} \simeq \mathrm{SU}(3)_{\mathbb{O}} \ltimes S^6_J. \tag{2.9}$$

2 $(\mathbb{O}, \cdot, n)$

2.2 $\mathbf{SU}(3)_{\mathbb{O}} \subsetneq \mathbf{Aut}(\mathbb{O})$

The automorphism group of the octonions Aut $(\mathbb{O})$ is the real compact form of the exceptional Lie group G_2, i.e., $\mathrm{Aut}(\mathbb{O}) = \mathrm{G}_{2(-14)}$. The Lie group $\mathrm{G}_{2(-14)}$ admits three maximal subgroups, two of Borel-de Siebenthal type, i.e., of maximal rank [26], namely SU (3) and $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$, and one of non-maximal rank, SU (2). In the present treatment, we are interested in the[4] SU (3) maximal subgroup of $\mathrm{G}_{2(-14)}$, which we will be denoting as $\mathrm{SU}(3)_{\mathbb{O}}$, and in its representations stemming from the decomposition of the defining and adjoint irreprs. of $\mathrm{G}_{2(-14)}$ itself :

$$
\begin{array}{ccc}
\mathrm{G}_{2(-14)} & \supset_{ns} & \mathrm{SU}(3)_{\mathbb{O}} \\
\mathbf{7} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}, \\
\mathbf{14} & = & \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}.
\end{array}
\tag{2.7}
$$

There are seven maximal non-symmetric subgroups $\mathrm{SU}(3)_{\mathbb{O}}$ of $\mathrm{G}_{2(-14)}$, defined as invariance symmetries of one of the seven octonionic imaginary units; they are all equivalent under inner automorphisms of $\mathrm{G}_{2(-14)}$ itself, and they all correspond to the following $\mathrm{SU}(3)_{\mathbb{O}}$-covariant breaking of $\mathbb{O}$:

$$
\begin{array}{ccc}
\mathrm{G}_{2(-14)} & \supset & \mathrm{SU}(3)_{\mathbb{O}} \\
\mathbf{1} \oplus \mathbf{7} & = & \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}, \\
\mathbb{O} = \mathbb{R} \oplus \mathbb{O}' & = & \mathbb{C} \oplus \mathbb{C}^3,
\end{array}
\tag{2.10}
$$

where one can identify

$$
\mathbb{C} \simeq \mathbb{R} \oplus J\mathbb{R},
\tag{2.11}
$$

with J denoting an arbitrary but fixed octonionic imaginary unit, i.e., $J \in \{e_1, ..., e_7\}$, which is $\mathrm{SU}(3)_{\mathbb{O}}$-invariant.

3 $\mathcal{O}$

3.1 Symmetries

As reported in Table 3, the Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$ is characterized by the following symmetries :

$$
\begin{aligned}
\mathrm{Aut}\,(\mathcal{O}) &= \mathrm{SU}(3) \equiv \mathrm{SU}\,(3)_{\mathcal{O}}\,, && (3.1)\\
\mathrm{Tri}\,(\mathcal{O}) &= \mathrm{Spin}\,(8)\,, && (3.2)\\
\mathrm{Spin}\,(\mathcal{O}) &= \mathrm{Spin}\,(8)\,, && (3.3)
\end{aligned}
$$

where we introduced $\mathrm{SU}\,(3)_{\mathcal{O}}$ as the automorphism group $\mathrm{Aut}\,(\mathcal{O})$, in order to distinguish it from the $\mathrm{SU}\,(3)_{\mathbb{O}}$ introduced in (2.7). It also holds that

$$
\mathcal{O} \simeq \mathbf{8} \equiv \mathbf{Adj} \text{ of } \mathrm{SU}\,(3)_{\mathcal{O}}\,. \tag{3.4}
$$

3.2 $\mathbf{SU}\,(3)_{\mathcal{O}} = \mathbf{Aut}(\mathcal{O})$

Spin(8) and[6] $\mathrm{SU}\,(3)_{\mathcal{O}}$ are related by a maximal (and non-symmetric) embedding, under which the defining and the adjoint irrepr. of Spin(8) decompose as follows :

$$\begin{array}{ccc} \mathrm{Spin}\,(8) & \supset_{\mathrm{ns}} & \mathrm{SU}\,(3)_{\mathcal{O}} \\ \mathbf{8}_v & = & \mathbf{8} \\ \mathbf{8}_s & = & \mathbf{8} \\ \mathbf{8}_c & = & \mathbf{8} \\ \mathbf{28} & = & \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}, \end{array} \tag{3.5}$$

where $\mathbf{10} \equiv S^3\mathbf{3}$ is the rank-3 symmetric irrepr. of $\mathrm{SU}\,(3)_{\mathcal{O}}$. The maximal embedding (3.5) exists by virtue of a Theorem of Dynkin (see e.g. Th. 1.5 of [62]), and it is a consequence of the existence of the Cartan-Killing bilinear invariant form in the adjoint irrepr. $\mathbf{8}$ of the Lie group $\mathrm{SU}\,(3)_{\mathcal{O}}$.

3.2 $\mathbf{SU}(3)_{\mathcal{O}} = \mathbf{Aut}(\mathcal{O})$

$$
\begin{array}{ccc}
\mathrm{Spin}(8) & \supset_{\mathrm{ns}} & \mathrm{SU}(3)_{\mathcal{O}} \\
\mathbf{8}_v & = & \mathbf{8} \\
\mathbf{8}_s & = & \mathbf{8} \\
\mathbf{8}_c & = & \mathbf{8} \\
\mathbf{28} & = & \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}},
\end{array}
\tag{3.5}
$$

We stress that the non-unital nature of $\mathcal{O}$ is deeply linked to the fact that $\mathrm{Aut}(\mathcal{O})$ is a *maximal* subgroup of Spin(8), and *not* a subgroup of Spin(7). The lack of a unity implies that $\mathcal{O}$ is *irreducible* under the action of its automorphism group $\mathrm{SU}(3)_{\mathcal{O}}$, as expressed by (3.4). Thus, differently from the (unique) unit element 1 in any unital algebra (which is invariant under the automorphism group of the unital algebra itself), in the basis of the Okubonic 8-dimensional vector space (whose a possible explicit realization is given by(1.9)), the (not unique, but rather threefold[7]) idempotent e is *not* invariant under

$\mathrm{Aut}(\mathcal{O})$, but instead it corresponds to one generator inside the adjoint representation of $\mathrm{Aut}(\mathcal{O})$ itself.

Before discussing the possible relevance of Okubonions in QCD, we would like to comment that, intuitively, a unital algebra (such as $\mathbb{O} \equiv (\mathbb{O}, \cdot, n)$) might turn out to be more amenable at describing theories and phenomena which admit a perturbative description, whereas a non-unital algebra (such as $\mathcal{O} \equiv (\mathbb{O}, *, n)$) might result in a more consistent algebraic modeling of non-perturbative regimes and physical phenomena. For this reason, we will not be dealing with the seamless unital extension of $\mathcal{O}$, defined by $\mathcal{O}^+ := \mathcal{O} \oplus 1$, since this is spoiled of the peculiar property of non-unitality, and, for instance, due to the traceful nature of the unit element, it does *not* admit a realization as $\mathfrak{J}_3(\mathbb{C})_0$ (with suitable deformation of the Jordan product *à la Michel-Radicati*) anymore.

4 $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ as different subgroups of Spin(8)

In order to elucidate the relation between the physical identifications (2.12) and (3.6), in this Section we will show that the two corresponding SU(3), namely $\mathrm{SU}(3)_{\mathbb{O}}$ and $\mathrm{SU}(3)_{\mathcal{O}}$, do *not* coincide, but they are rather totally different. This is a consequence of the different embeddings of $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ in $\mathrm{SO}(\mathbb{O}) \simeq \mathrm{Spin}(8) \simeq \mathrm{SO}(\mathcal{O})$. Indeed, from (2.5), (3.5) and (2.7), one can draw the following chains of maximal group embeddings from Spin(8) to $\mathrm{SU}(3)_{\mathcal{O}}$,

$$\begin{array}{ccc} \mathrm{Spin}(8) & \supset & \mathrm{SU}(3)_{\mathcal{O}} \\ \mathbf{8}_v & = & \mathbf{8} \\ \mathbf{8}_s & = & \mathbf{8} \\ \mathbf{8}_c & = & \mathbf{8} \\ \mathbf{28} & = & \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}, \end{array} \tag{4.1}$$

and to $\mathrm{SU}(3)_{\mathbb{O}}$,

$$\begin{array}{ccccccc} \mathrm{Spin}(8) & \supset & \mathrm{Spin}(7) & \supset & G_{2(-14)} & \supset & \mathrm{SU}(3)_{\mathbb{O}} \\ \mathbf{8}_v & = & \mathbf{7} \oplus \mathbf{1} & = & \mathbf{7} \oplus \mathbf{1} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus 2 \cdot \mathbf{1} \\ \mathbf{8}_s & = & \mathbf{8} & = & \mathbf{7} \oplus \mathbf{1} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus 2 \cdot \mathbf{1} \\ \mathbf{8}_c & = & \mathbf{8} & = & \mathbf{7} \oplus \mathbf{1} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus 2 \cdot \mathbf{1} \\ \mathbf{28} & = & \mathbf{21} \oplus \mathbf{7} & = & \mathbf{14} \oplus 2 \cdot \mathbf{7} & = & \mathbf{8} \oplus 3 \cdot \mathbf{3} \oplus 3 \cdot \overline{\mathbf{3}} \oplus 2 \cdot \mathbf{1}. \end{array} \tag{4.2}$$

Comparing the branchings of the $\mathbf{Adj} \equiv \mathbf{28}$ of Spin(8) along the chains (4.1) and (4.2), one immediately realizes that

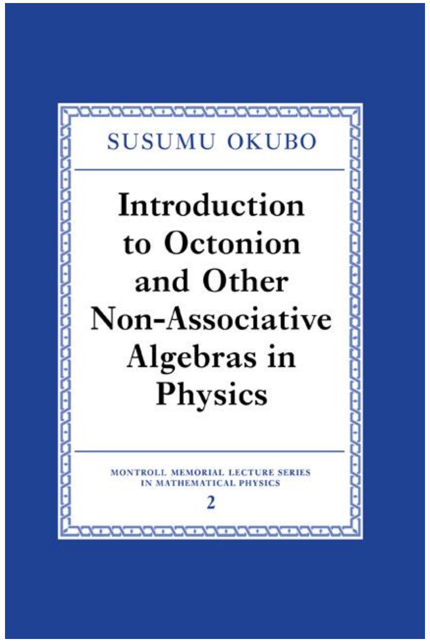
$$\boxed{\mathrm{SU}(3)_{\mathcal{O}} \neq \mathrm{SU}(3)_{\mathbb{O}} .} \tag{4.3}$$

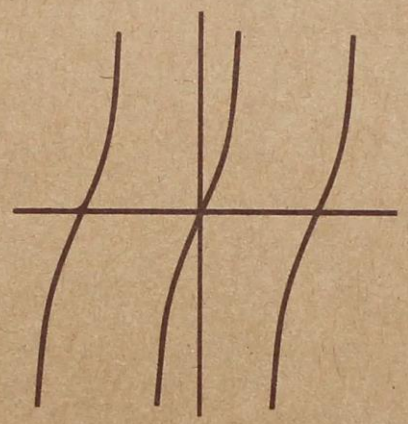
Even more interestingly, $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ do *not* even share a common SU(2) subgroup.

5 Conclusions

Both $\mathbb{O}$ and $\mathcal{O}$ are non-associative, flexible, composition and division, real 8-dimensional algebras. Such two algebras are however very different: for instance, $\mathbb{O}$ is alternative and unital (Hurwitz), whereas $\mathcal{O}$ is non-alternative and non-unital (only admitting idempotent elements, though). Moreover, the Lie group $\mathrm{Aut}(\mathcal{O}) = SU(3)_{\mathcal{O}}$ is much smaller than (and not a subgroup, albeit with the same rank, of) the exceptional Lie group $G_{2(-14)} = \mathrm{Aut}(\mathbb{O})$. More precisely, $\mathrm{Aut}(\mathcal{O}) = SU(3)_{\mathcal{O}}$ and $\mathrm{Aut}(\mathbb{O}) = G_{2(-14)}$ are totally different, rank-2 subgroups of $SO(\mathbb{O}) \simeq \mathrm{Spin}(8) \simeq SO(\mathcal{O})$: $SU(3)_{\mathcal{O}}$ is *maximal* and non-symmetric in Spin(8) (as given by the embedding (4.1)), whereas $SU(3)_{\mathbb{O}}$ is *next-to-maximal* (and non-symmetric) in Spin(8) (as given by (4.2)). We find this fact quite tantalizing, and possibly paving the way to surprising developments. Indeed, as shown in [67] and very recently in [63], by using the Okubonions one can obtain a number of exceptional structures, such as the cubic exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ (*aka* Albert algebra) or various real forms of the exceptional Lie groups of type F_4, E_6 and even E_7 and E_8, without employing the octonions $\mathbb{O}$, nor G_2, at all.

We would like to conclude by citing Susumu Okubo in the introduction of his book [8], dating back to 1995 : "*Octonion algebra may surely be called a beautiful mathematical entity. Nevertheless, it has never been systematically utilized in physics in any fundamental fashion, although some attempts have been made toward this goal. However, it is still possible that non-associative algebras (other than Lie algebras) may play some essential future role in the ultimate theory, yet to be discovered.*" While there have been many advances on the physical applications of octonions since the publication of Okubo's book, non-associative algebras may still be a key player in the formulation of a ultimate theory of Physics. While octonions have been (with very good reason) dubbed *'the strangest numbers in string theory'* [68], in the present paper we have suggested a fierce (possibly even stranger!) opponent : the Okubonions.





tan Q