Split Octonions, Spacetime And E_8

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Who am I?

— Supervised by Prof. Mike Duff at Imperial College

— PhD thesis: "Octonions in Supergravity" — magic squares from YM^2

— Part time teaching physics/part time as a musician, writing papers with Cohl Furey on SM and division algebras

— Postdoc in Environmental Fluid Dynamics at University of Leeds

But octonions keep pulling me back in...

Part I

What is the role of E_8 in particle physics?

(if any...)

Constructing $\mathbf{E}_{8(8)}$

$$
\begin{aligned}
\mathfrak{e}_{8(8)} &= \mathrm{tri}(\mathbb{O}') \oplus \mathrm{tri}(\mathbb{O}') + 3(\mathbb{O}' \otimes \mathbb{O}') \\
&= \mathfrak{spin}(4,4) \oplus \mathfrak{spin}(4,4) + (\underline{8}_v, \underline{8}_v) + (\underline{8}_s, \underline{8}_s) \\
&\qquad + (\underline{8}_c, \underline{8}_c) \\
&= \mathfrak{spin}(8,8) + \underline{128}
\end{aligned}
$$

$$
E = \begin{pmatrix} \hat{\delta}^{\theta}_{\mathrm{s}} + \hat{\delta}^{\lambda}_{\mathrm{s}} & -V^{\dagger} & \Psi_{\mathrm{s}} \\ V & \hat{\delta}^{\theta}_{\mathrm{c}} + \hat{\delta}^{\lambda}_{\mathrm{c}} & \Psi_{\mathrm{c}} \\ -\Psi^{\dagger}_{\mathrm{s}} & -\Psi^{\dagger}_{\mathrm{s}} & \hat{\delta}^{\theta}_{\mathrm{v}} + \hat{\delta}^{\lambda}_{\mathrm{v}} \end{pmatrix}
$$

$\in \mathfrak{e}_{8(8)}$

$V, \Psi_s, \Psi_c \in \mathbb{O}' \otimes \mathbb{O}'$

$$
\begin{aligned}
\hat{\delta}^{\theta}_{\mathrm{v}} V &:= \tfrac{1}{4}\theta^{ab}\left(e_a(e_b^* V) - V(e_a^* e_b)\right) \\
\hat{\delta}^{\theta}_{\mathrm{c}} \Psi_{\mathrm{c}} &:= \tfrac{1}{4}\theta^{ab} e_a(e_b^* \Psi_{\mathrm{c}}) \,, \\
\hat{\delta}^{\theta}_{\mathrm{s}} \Psi_{\mathrm{s}} &:= \tfrac{1}{4}\theta^{ab} e_a^*(e_b \Psi_{\mathrm{s}}) \,,
\end{aligned}
$$

Constructing $\mathrm{E}_{8(-24)}$

$$
\begin{aligned}
\mathfrak{e}_{8(-24)} &= \mathrm{tri}(\mathbb{O}) \oplus \mathrm{tri}(\mathbb{O}') + 3(\mathbb{O}\otimes\mathbb{O}') \\
&= \mathrm{spin}(8) \oplus \mathrm{spin}(4,4) + (\underline{8}_v, \underline{8}_v) + (\underline{8}_s, \underline{8}_s) \\
&\qquad + (\underline{8}_c, \underline{8}_c) \\
&= \mathrm{spin}(12,4) + \underline{128}
\end{aligned}
$$

$$
E = \begin{pmatrix} \hat{\delta}^{\theta}_{\mathrm{s}} + \hat{\delta}^{\lambda}_{\mathrm{s}} & -V^{\dagger} & \Psi_{\mathrm{s}} \\ V & \hat{\delta}^{\theta}_{\mathrm{c}} + \hat{\delta}^{\lambda}_{\mathrm{c}} & \Psi_{\mathrm{c}} \\ -\Psi^{\dagger}_{\mathrm{s}} & -\Psi^{\dagger}_{\mathrm{s}} & \hat{\delta}^{\theta}_{\mathrm{v}} + \hat{\delta}^{\lambda}_{\mathrm{v}} \end{pmatrix}
$$

$$\in \mathfrak{e}_{8(-24)}$$

$$V, \Psi_s, \Psi_c \in \mathbb{O}\otimes\mathbb{O}'$$

$$
\begin{aligned}
\hat{\delta}^{\theta}_{\mathrm{v}} V &:= \tfrac{1}{4}\theta^{ab}\left(e_a(e_b^* V) - V(e_a^* e_b)\right) \\
\hat{\delta}^{\theta}_{\mathrm{c}} \Psi_{\mathrm{c}} &:= \tfrac{1}{4}\theta^{ab} e_a(e_b^* \Psi_{\mathrm{c}}) \,, \\
\hat{\delta}^{\theta}_{\mathrm{s}} \Psi_{\mathrm{s}} &:= \tfrac{1}{4}\theta^{ab} e_a^*(e_b \Psi_{\mathrm{s}}) \,,
\end{aligned}
$$

Embedding the SM Gauge Group

$$E_{8(8)} \supset \mathrm{Spin}(8,8) \supset \mathrm{Spin}(2,4) \times \mathrm{Spin}(6,4)$$

$$\supset \mathrm{Spin}(1,3) \times \mathrm{Spin}(6) \times \mathrm{Spin}(4)$$

And so on...

$$E_{8(-24)} \supset \mathrm{Spin}(12,4) \supset \mathrm{Spin}(2,4) \times \mathrm{Spin}(10)$$

$$\supset \mathrm{Spin}(1,3) \times \mathrm{Spin}(6) \times \mathrm{Spin}(4)$$

And so on...

$$\underline{128} = \Psi_{SM} + \Psi_{SM}^{MIRROR} = \begin{array}{l} (\underline{4}, \underline{2}, \underline{1}; \tfrac{1}{2}, 0) \\ + (\underline{\bar{4}}, \underline{1}, \underline{2}; \tfrac{1}{2}, 0) \\ + (\text{"} \quad \text{"}; 0, \tfrac{1}{2}) \\ + (\text{"} \quad \text{"}; 0, \tfrac{1}{2}) \end{array}$$

1 GEN OF SM FERMIONS + 1 GEN MIRROR FERMIONS

3 Generations from Triality in E_8?

Generation II? Reparameterise the generation of mirror fermions as

$$\Psi^{\text{MIRROR}} = a\Psi$$

where two obvious choices for "a" are:

(1) a = LR symmetric Higgs or mass matrix (swaps $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$)

(2) a valued in the vector rep of Spin(1,3) (swaps (½, 0) and (0, ½))
— position x, momentum p, or covariant derivative?

Generation III? Reparameterise the vector rep $(\mathbf{8}_v, \mathbf{8}_v)$ as an **O'⊗O'** product (or **O⊗O'**) product — or look for bilinears elsewhere in 2 × 2 spin group block

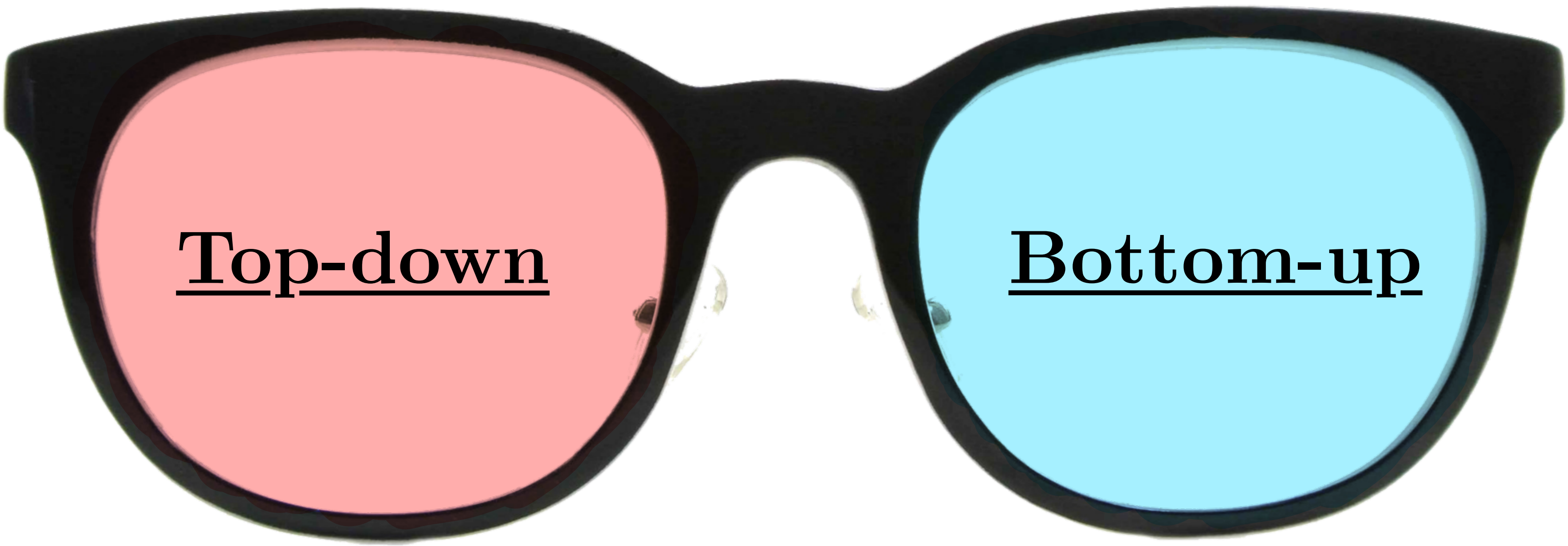
This is all too ad hoc... Needs a guiding principle!

Alternative Viewpoint:
128 = One Generation of Twistors

$$\begin{pmatrix} \pi_\alpha \\ \omega^{\dot\alpha} \end{pmatrix} \overset{\substack{\text{Null} \\ \text{Twistor}}}{=} \begin{pmatrix} \pi_\alpha \\ x^{\dot\alpha\beta}\pi_\beta \end{pmatrix}$$

$$\begin{pmatrix} (\Psi_{SM})_\alpha \\ (\Psi^{MIRROR}_{SM})^{\dot\alpha} \end{pmatrix} \overset{\substack{\text{Null} \\ \text{Twistors}}}{=} \begin{pmatrix} (\Psi_{SM})_\alpha \\ x^{\dot\alpha\beta}(\Psi_{SM})_\beta \end{pmatrix}$$

What does this mean geometrically?

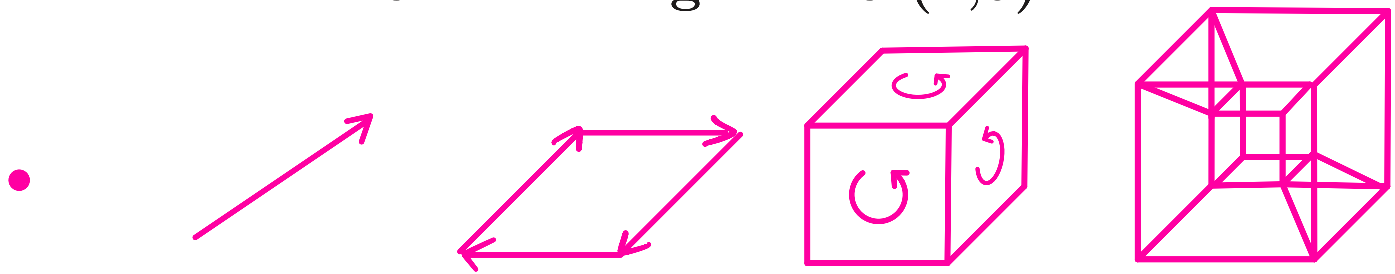


Meet in the middle?

Part II

Can we build $\mathbf{E}_8$ from 3+1 Spacetime Ingredients?

Clifford Algebra Cl(1,3)



$\mathbb{1}$	γ_μ	$\gamma_{\mu\nu}$	$\gamma_{\mu\nu\rho}$	$\gamma_{\mu\nu\rho\sigma}$
1	4	6	4	1
EVEN	ODD	EVEN	ODD	EVEN
$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$	$\begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix}$	$\rightsquigarrow \gamma_* \gamma_\mu$	$\rightsquigarrow \gamma_* = I$
			$\begin{pmatrix} 0 & i\bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}$	$\begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}$

$$\boxed{\begin{aligned} \gamma_*^2 &= -\mathbb{1} \\ I^2 &= -\mathbb{1} \end{aligned}}$$

Dirac Spinors as Even Multivectors

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^{\dot{1}} \\ \bar{\chi}^{\dot{2}} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \psi_1 & \chi_1 \\ \psi_2 & \chi_2 \end{pmatrix}_{\alpha}{}^{\beta} & 0 \\ 0 & \begin{pmatrix} \bar{\chi}^{\dot{1}} & -\bar{\psi}^{\dot{1}} \\ \bar{\chi}^{\dot{2}} & -\bar{\psi}^{\dot{2}} \end{pmatrix}^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Dirac Spinor | Even Multivector | Reference Spinor

$$\Psi_{\text{DIRAC}} = \Psi\, u_{\text{DIRAC}}$$

$$i\Psi_{\text{DIRAC}} = \Psi\, \gamma_{21}\, u_{\text{DIRAC}}$$

$$\boxed{\begin{aligned} \gamma_{21} u_{\text{DIRAC}} &= i\, u_{\text{DIRAC}} \\ \gamma_0 u_{\text{DIRAC}} &= u_{\text{DIRAC}} \end{aligned}}$$

Information Contained in a Dirac Spinor

$$\Psi = \rho^{\frac{1}{2}} e^{\frac{\beta}{2}\gamma_*} R$$

SCALE × DUALITY ROTATION × LORENTZ TRANSFORM

$$\begin{pmatrix} \psi_1 & \chi_1 \\ \psi_2 & \chi_2 \end{pmatrix}_\alpha{}^\beta = \rho^{\frac{1}{2}} e^{i\frac{\beta}{2}} R_\alpha{}^\beta$$

$$\in \mathbb{R}_+ \times U(1)_* \times SL(2,\mathbb{C})$$

Ψ transforms under SL(2,**C**) from the left and an *internal* SL(2,**C**) from the right

Can we geometrise all SM gauge symmetries like this?

Information Contained in a Dirac Spinor

$$\psi \gamma_\mu \tilde{\psi} = \rho R \gamma_\mu \tilde{R}$$

$$= \rho E_\mu = \begin{cases} = \bar{\psi}_D \gamma^\mu \psi_D \\ = -\mathrm{Re}(\bar{\psi}_D \gamma^\mu {\psi_D}^C) \\ = \mathrm{Im}(\bar{\psi}_D \gamma^\mu {\psi_D}^C) \\ = \bar{\psi}_D i \gamma^* \gamma^\mu \psi_D \end{cases}$$

$$\psi \gamma_{\mu\nu} \tilde{\psi} = \rho e^{\beta \gamma_*} R \gamma_{\mu\nu} \tilde{R}$$

$$= \rho \left(\cos\beta \, E_\mu \wedge E_\nu + \sin\beta \, *(E_\mu \wedge E_\nu) \right)$$

Dirac Spinor in O'

$$\mathrm{Spin}(4,4) \supset \mathrm{Spin}(1,3) \times \mathrm{Spin}(3,1)$$

PRESERVES $Cl(1,3)$ INNER PRODUCT: $\langle \Psi \tilde{\Psi} \rangle$

$$\delta\Psi = \frac{1}{4}\lambda^{\mu\nu}\gamma_\mu\gamma_\nu\Psi + \frac{1}{4}\lambda^{\mu'\nu'}\Psi\gamma_{\mu'}\gamma_{\nu'} - \frac{1}{2}\lambda^{\mu\nu'}\gamma_\mu\Psi\gamma_*\gamma_{\nu'}$$

$\Rightarrow$ Define multiplication by internal and external vectors as:

$$\boxed{(v + w\gamma_*)\,\underline{\times}\,\Psi := v\Psi + \Psi w\gamma_*}$$

$$\boxed{(v + w\gamma_*) \,\underline{\times}\, \Psi := v\Psi + \Psi w\gamma_*}$$

$$\underline{8}_v \,\underline{\times}\, \underline{8}_c = \underline{8}_s$$

Equivalent to **O'** multiplication!

$$\chi = V \,\underline{\times}\, \Psi = \text{ODD MULTIVECTOR SPINOR}$$

SCALE × DUALITY ROTATION × IMPROPER LORENTZ TRANSFORM

We can also define a similar product between an odd multivector $v + w\gamma^*$ and an odd multivector spinor:

$$V \,\underline{\times}\, \chi := v\chi + \chi w\gamma_*$$

The Even-Odd Spinor Product

$$t(\Psi, V, \chi) = \left\langle (V \,\underline{\times}\, \chi)\tilde{\Psi} \right\rangle$$

$$= \left\langle \tfrac{1}{2}\left(\chi\tilde{\Psi} + \Psi\tilde{\chi} + \tilde{\Psi}\chi - \tilde{\chi}\Psi\right)\tilde{V} \right\rangle$$

$$\Longrightarrow \boxed{\Psi \,\underline{\times}\, \chi = \tfrac{1}{2}\left(\underbrace{\chi\tilde{\Psi} + \Psi\tilde{\chi}}_{\text{VECTOR}} + \underbrace{\tilde{\Psi}\chi - \tilde{\chi}\Psi}_{\text{TRIVECTOR}}\right)}$$

Summary of O' vs Cl(1,3)

$$\Psi \longleftrightarrow Cl(1,3)_{EVEN} \longleftrightarrow \underline{8}_c$$

$$\chi \longleftrightarrow Cl(1,3)_{ODD} \longleftrightarrow \underline{8}_s$$

$$V \longleftrightarrow Cl(1,3)_{ODD} \longleftrightarrow \underline{8}_v$$

This is everything we need to build $F_{4(4)}$ from Cl(1,3)!

$$\begin{pmatrix} \mathrm{tri}\,\mathbb{O}' & \underline{8}_v \\ \sim & \mathrm{tri}\,\mathbb{O}' \end{pmatrix}, \qquad \begin{pmatrix} \underline{8}_c \\ \underline{8}_s \end{pmatrix} \sim Cl(1,3) \text{ SPINOR}$$

The next step is to complexify and build $E_{6(2)}$ or split complexify to build $E_{6(6)}$ ...

That's all for now...

Thanks for listening!

My "Penrosian" Bias

spacetime geometry needs to be quantised

vs

quantum theory needs to be "spacetime geometrised"

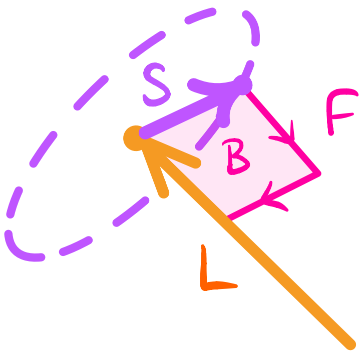
Does the "i" in quantum mechanics have a geometrical meaning?

— Clifford algebra à la Hestenes reveals two purely geometrical (commuting) complex structures "i" and "I" (👀) in Maxwell-Dirac theory:

(1) rotations in the spin plane — like U(1) symmetry in QED;
(2) duality ("chiral") rotations, rotating F into *F

— Penrose/Rindler picture of Weyl spinors as lightlike flags is consistent with the Hestenes picture! LH and RH subspaces from $\mathbf{C} \otimes \mathbf{C} = \mathbf{C} \oplus \mathbf{C}$

Weyl Spinors as Flags



FLAG: $B = L \wedge S =$ NULL BIVECTOR $\sim \psi_\alpha \psi_\beta$

FLAGPOLE: $L =$ NULL VECTOR $\sim \psi_\alpha \bar{\psi}_{\dot{\beta}}$

$$\psi_\alpha \in \mathbb{C}^2 \sim \mathbb{R}_+ \times SU(2)$$

Scale in a frame

Axis and flag angle in frame