A New Residual Distribution Hydro Solver for Astrophysics

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Why we are here ...

- What is a residual distribution solver?
  - Residual calculation
  - Distribution mechanism
- Why invest the time to implement one?
  - Fundamental advantages - Truly Multidimensional
  - Early results
- Future work for the solver
  - Extension to 3D, sources, adaptive time stepping, moving mesh

Universe

Cold Dark Matter

Gravity

N-Body

Highly efficient

Highly accurate

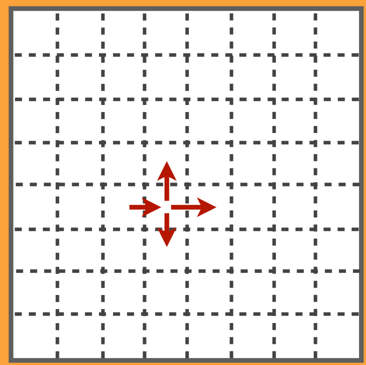
Highly successful

Reproduces large scale structure

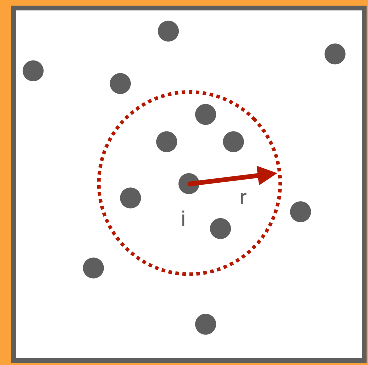
Baryonic Gas

Hydrodynamics (with self gravity)

Eulerian Schemes



Lagrangian Schemes



Moving mesh schemes

Combine shock handling of grid schemes with the resolution adaption of the particle schemes

Fluid on an Unstructured Mesh

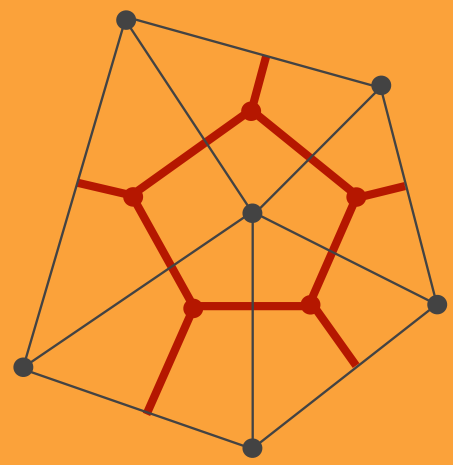
Standard Approaches

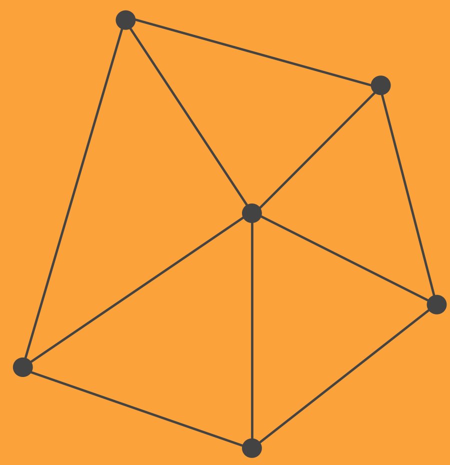
Space broken into cells

Calculate movement of material through faces

Update state of fluid in cells

Residual Distribution (RD) Approach





Fluid state stored at vertices of triangular mesh

Calculate contribution to change from each triangle

The Residual Distribution Hydro Solver

1. The Basic Equations

Fluid Equations

2D Eulerian, inviscid fluid equations in compact form

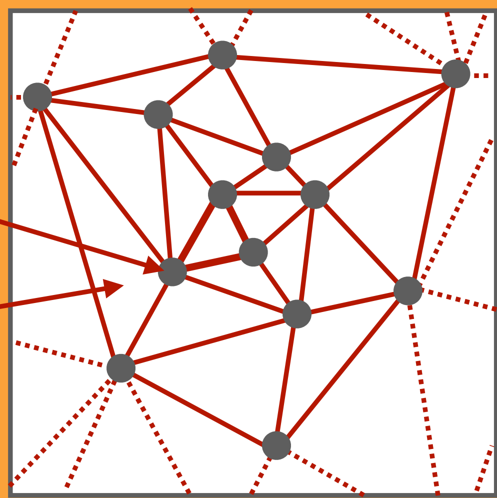
$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0, \quad U = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho E \end{pmatrix}, \quad F_x(U) = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ \rho v H \end{pmatrix}, \quad F_y(U) = \begin{pmatrix} \rho v_y \\ \rho v_x v_y \\ \rho v_y^2 + p \\ \rho v_y H \end{pmatrix},$$

Solve for time evolution

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i \in T} \phi_i$$

Fluid state stored at vertices

Calculate evolution of state based on residual from all triangles



The Residual Distribution Hydro Solver

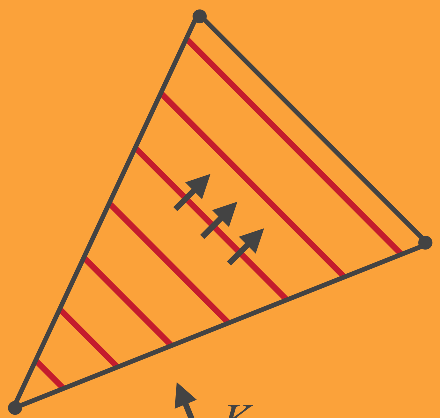
1. The Basic Equations

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i \in T} \phi_i$$

2. The Residual

The Residual

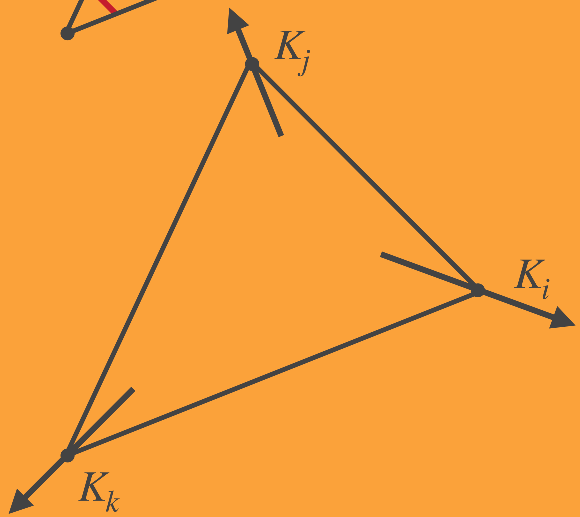
Represents **net flow** of material into and out of triangular element



$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0,$$

$$\phi(U_h) = \int_T \nabla \cdot F_h(U_h) dA$$

ϕ^T = Element residual



$$\phi^T = \sum_{i \in T} K_i \bar{U}_i$$

$\bar{U}_i$ = Fluid state at vertex weighted by **average state** of triangle

The Residual Distribution Hydro Solver

1. The Basic Equations

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i \in T} \phi_i$$

2. The Residual

$\phi^T =$ **Net flow through triangle**

3. The Distribution

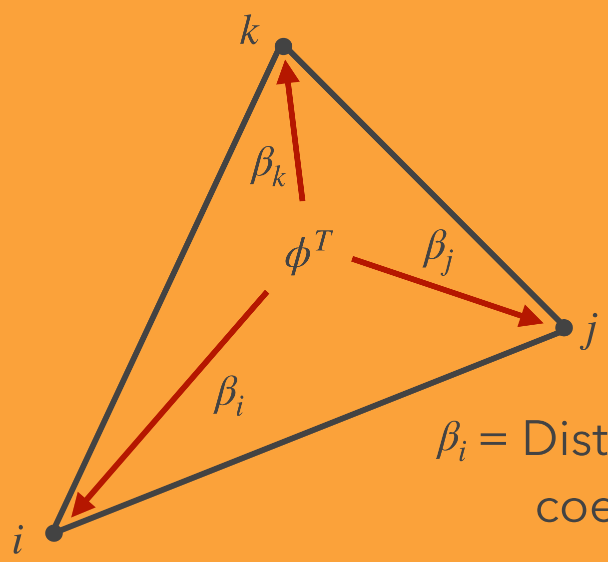
The Distribution

- Split residual amongst vertices of triangle
- Different distribution schemes for different conditions
  - Low Diffusion A (LDA) -> smooth flows
  - N-scheme -> shocks
  - Blended -> combination

Distribution based on **geometry** of triangle and **fluid state** at each vertex

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i \in T} \phi_i$$

$$\phi_i^{LDA} = \beta_i \phi^T$$



β_i = Distribution coefficient

The Residual Distribution Hydro Solver

1. The Basic Equations

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i\in T} \phi_i$$

2. The Residual

$\phi^T =$ **Net flow through triangle**

3. The Distribution

$$\phi_i^{LDA} = \beta_i \phi^T$$

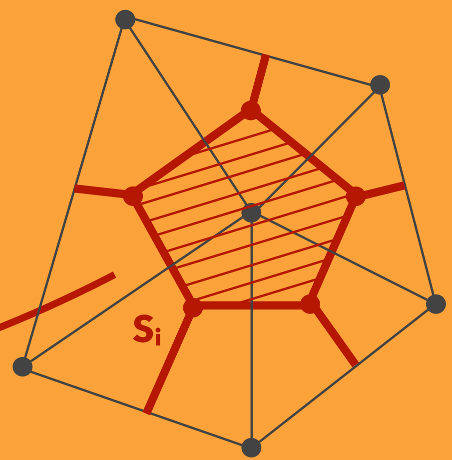
4. The Evolution

Calculating the evolution

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0, \quad U = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho E \end{pmatrix}, \quad F_x(U) = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ \rho v H \end{pmatrix}, \quad F_y(U) = \begin{pmatrix} \rho v_y \\ \rho v_x v_y \\ \rho v_y^2 + p \\ \rho v_y H \end{pmatrix},$$

CFL condition constrained timestep

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|S_i|} \sum_{T|i \in T} \phi_i^T$$



Dual area

Noh Problem

Initial Conditions:

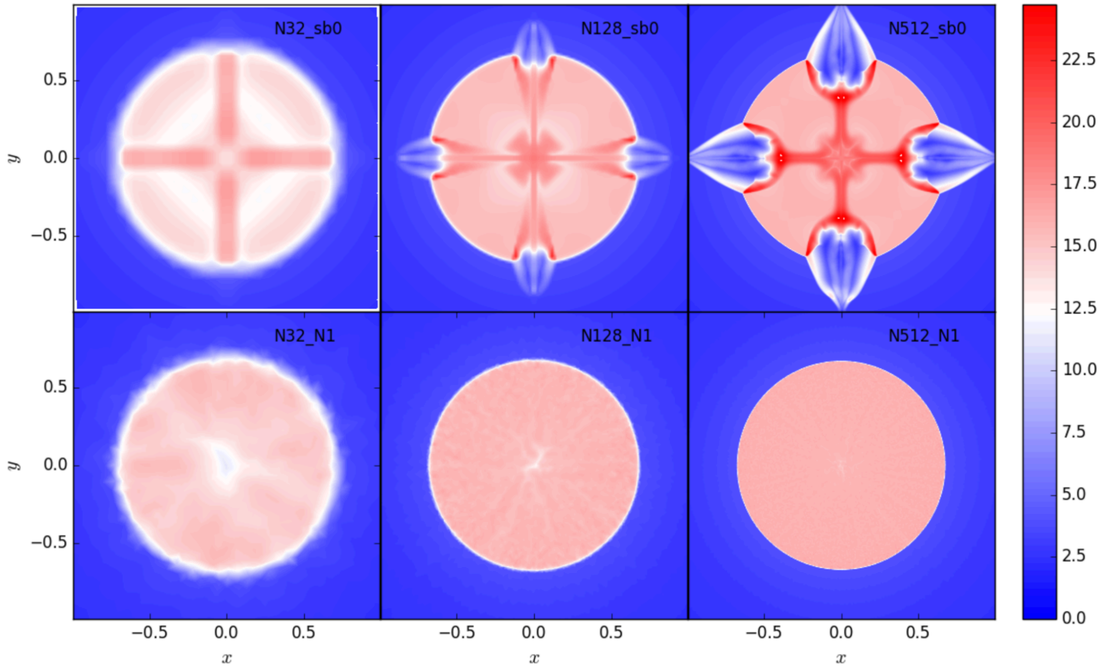
- Uniform density
- Velocity radially inward

Solution:

- Expanding shock

Roe

RD



Density

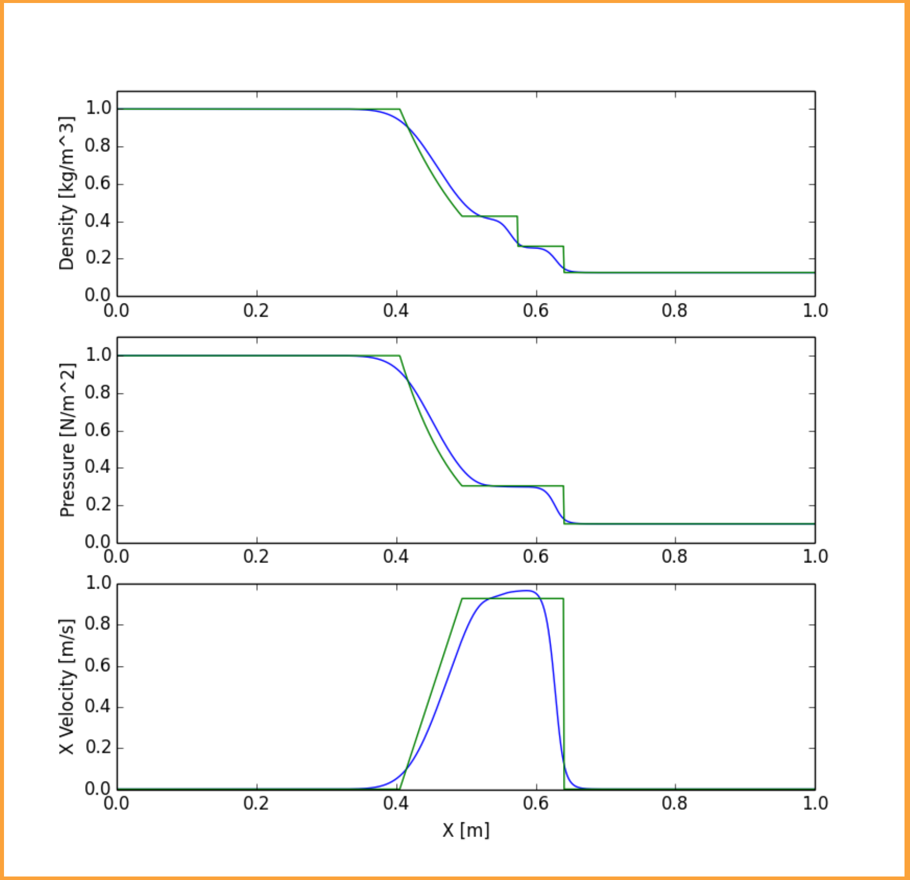
Figure 21. Density at $t = 3$ for the Noh problem. Top panels: results obtained with the first-order Roe solver; bottom panels: results obtained with the first-order N scheme. Resolution increases from left to right: 32×32, 128×128, 512×512.

(S.-J. Paardekooper, 2017) Increasing resolution

My own RD implementation ...

- 2D RD solver on static grid
- Preliminary testing completed
- Reproduces basic hydro tests

  (e.g. Sod shock tube)
- Still in development:
  - Extension to 3D
  - Adaptive time stepping
  - Moving mesh



Summary

- Implementing a new class of hydro solver - the RD solver
  - **Truly multidimensional** 2nd order hydro solver
  - Takes advantage of the innate form of unstructured meshes
- Promising results so far
- Future work for the solver
  - Extension to 3D, sources, adaptive time stepping, moving mesh

References

- M. Ricchiuto and R. Abgrall. *Explicit runge-kutta residual distribution schemes for time dependent problems: Second order case*. Journal of Computational Physics, 229:5653–5691, 2010.

- S.-J. Paardekooper. *Multidimensional upwind hydrodynamics on unstructured meshes using graphics processing units - I. Two-dimensional uniform meshes*. MNRAS, 469:4306–4340, 2017

History and Background

- Family of methods developed to solve sets of PDEs
- Developed over the past 30 years
- Used for a range of fluid dynamics applications
  - Jet engine design to shallow wave modelling
- Not widely used in astrophysics (S.-J. Paardekooper, 2017)
- Key advantages
  - Truely multidimensional
  - 2nd order accurate on narrow stencil
  - Natural source term inclusion