Extreme Value Statistics for Random Matrices

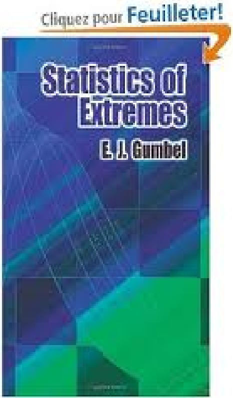
Satya N. Majumdar

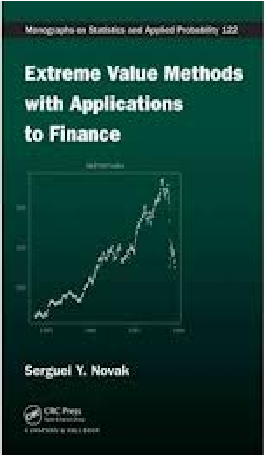
Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Sud, France

Extreme Value Statistics

Extreme Events: rare but devastating

Applications

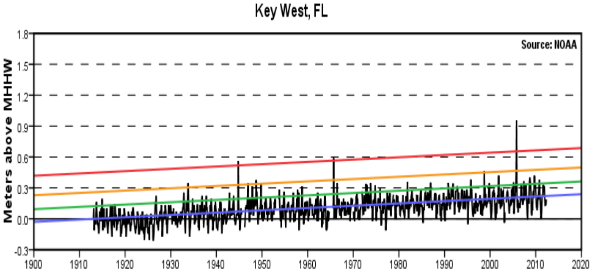


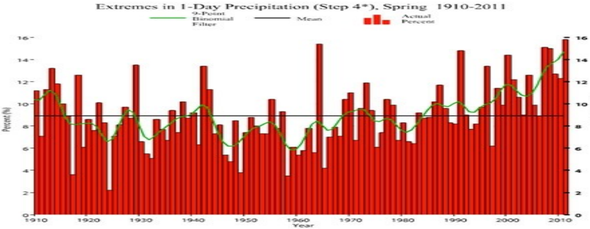


Climate studies, finance and economics, hydrology, sports,....

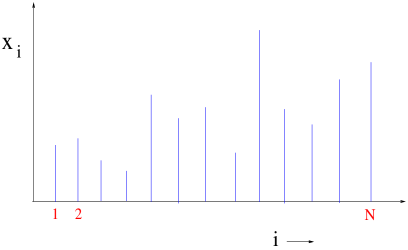
Random walks, disordered systems, random matrices, number theory, ...

Average vs. Extreme





General setting:



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$$P(x_1, x_2, \ldots, x_N)$$

independent or correlated

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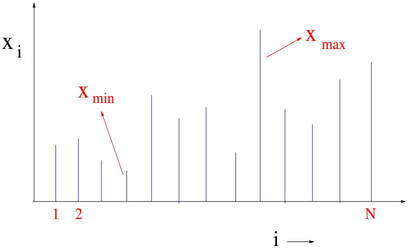

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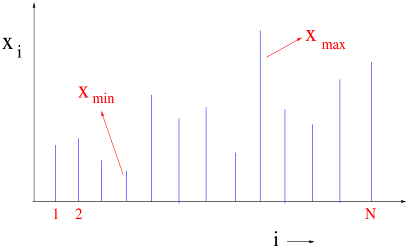

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Q: Given $P(x_1, x_2, \ldots, x_N)$, what can we say about the statistics of $x_{\max}$ and $x_{\min}$?

Extreme statistics of i.i.d random variables

A particularly simple case is when

$\{x_1, x_2, \ldots, x_N\} \Longrightarrow$ set of N i.i.d random variables

each drawn from $p(x) \rightarrow P(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} p(x_i)$

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Scaling limit: N large, x large: $\boxed{Q_N(x) \to F\left[(x - a_N)/b_N\right]}$

Three universal extreme value distributions

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Several applications $\Longrightarrow$ Climate, Finance, Oceanography, Disordered Systems (Random Energy Model of Derrida)

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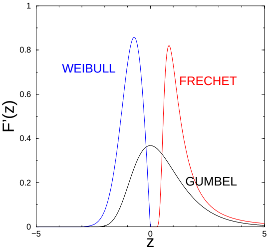

Extreme statistics of correlated variables

In many situations, however, the underlying random variables

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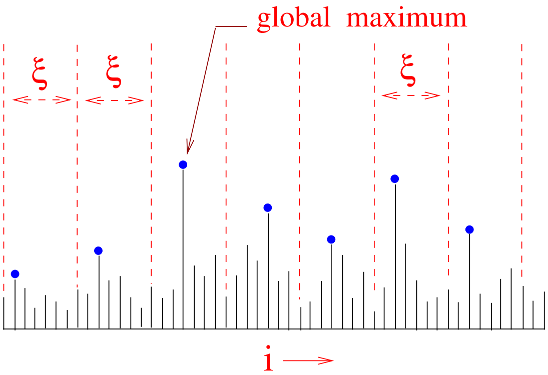
Extreme statistics of correlated variables $\Rightarrow$ nontrivial

Extreme statistics in weakly correlated systems

Weakly correlated variables $\{x_1, x_2, \ldots, x_N\}$
$\rightarrow$ finite correlation length $\xi << N$



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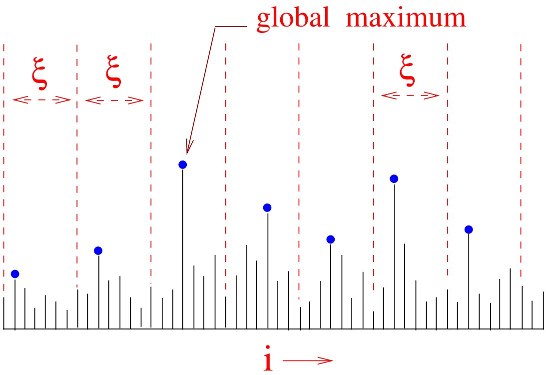

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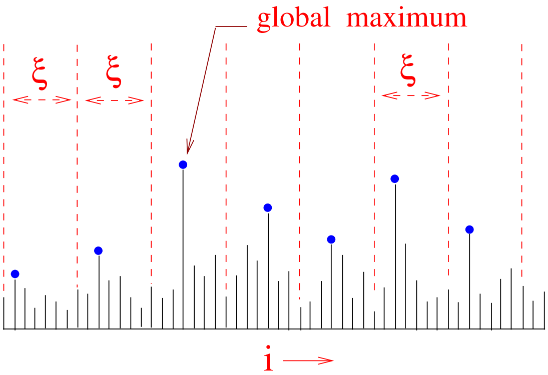

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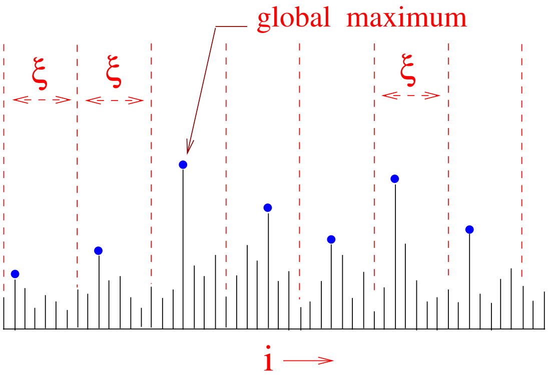

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$\Longrightarrow$ Fréchet, Gumbel or Weibull

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Universality classes for extreme-value statistics

Jean-Philippe Bouchaud†§ and Marc Mézard‡‖

† Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

‡ Laboratoire de Physique Théorique de l'Ecole Normale Supérieure¶, 24 rue Lhomond, 75231 Paris Cedex 05, France

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Abstract. The equilibrium low-temperature physics of disordered systems is governed by the statistics of extremely low-energy states. It is thus relevant to discuss the possible universality classes for extreme-value statistics. We compare the usual probabilistic classification to the results of the replica approach. We show in detail for several problems (including the random energy model and the decaying Burgers turbulence) that one class of independent variables corresponds exactly to the so-called *one step replica symmetry breaking* solution in the replica language. We argue that this universality class holds if the correlations are sufficiently weak, and propose a conjecture on the level of correlations which leads to different universality classes.

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For strongly correlated $\{x_1, x_2, \ldots, x_N\}$: correlation length $\xi \sim O(N)$

Extreme statistics $\rightarrow$ nontrivial $\Longrightarrow$ few exact results

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Examples:

- Random walks and Lévy flights
- Watermelons (vicious random walkers)
- Branching processes
- Mass transport/condensation processes
- Directed polymer & sequence matching
- Fluctuating interfaces (KPZ)
- **Top eigenvalue of a Gaussian random matrix**

Gaussian Random Matrices: A brief reminder

Spectral Statistics in Random Matrix Theory (RMT)

Consider $N \times N$ Gaussian random matrix $J \equiv [J_{ij}]$

(i) real symmetric (ii) complex Hermitian (iii) complex quaternionic

$$J = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1N} \\ J_{12} & J_{22} & \dots & J_{2N} \\ \dots & \dots & \dots & \dots \\ J_{1N} & J_{2N} & \dots & J_{NN} \end{pmatrix}$$

$$\mathrm{Prob.}[J] \propto \exp\left[-\beta \frac{N}{2} \sum_{i,j} |J_{ij}|^2\right]$$

$$\propto \exp\left[-\beta \tfrac{N}{2} \mathrm{Tr}\left(J^\dagger J\right)\right]$$

$\longrightarrow$ invariant under rotation

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N real eigenvalues: $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \longrightarrow$ **strongly correlated**

Spectral statistics in RMT $\Rightarrow$ statistics of $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$

Joint distribution of eigenvalues

Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)

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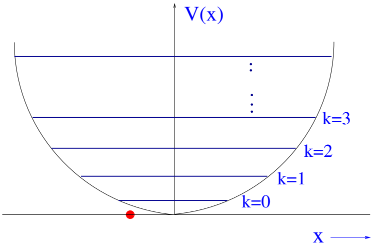
The Vandermonde term $\prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$ makes $\{\lambda_i\}'s$

$\Longrightarrow$ **strongly correlated**

A physical realization of GUE eigenvalues



A single quantum particle in a harmonic potential: $V(x) = \frac{1}{2} m\omega^2 x^2$

Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2} m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x)$$

with $\varphi_k(x \to \pm\infty) = 0$

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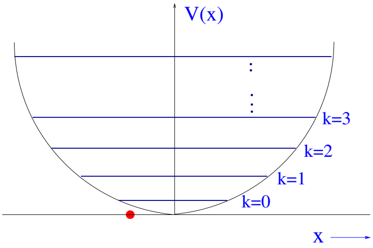

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with energy levels: $\epsilon_k = (k + 1/2)\, \hbar\, \omega \quad k = 0, 1, 2, 3 \ldots$

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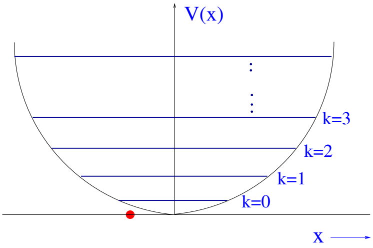

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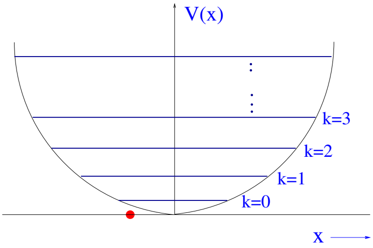

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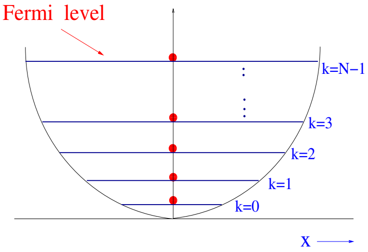
$H_k(x) \to$ Hermite polynomials

For example, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, etc.

N spinless Fermions in a harmonic trap: T=0



ground state many-body wavefunction → Slater determinant

$$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)]$$

with $0 \le i \le (N-1)$, $1 \le j \le N$

ground state energy: $E_0 = \hbar\,\omega\, N^2/2$

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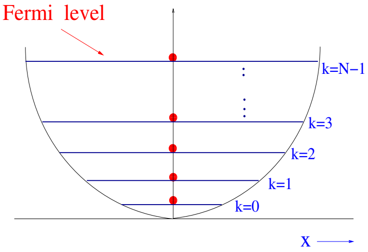

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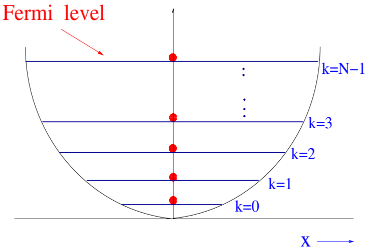

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$$\Longrightarrow \quad \boxed{|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N}\, e^{-\alpha^2 \sum_{i=1}^N x_i^2} \prod_{j<k}(x_j - x_k)^2}$$

Free fermions at T=0 ≡ GUE eigenvalues

- Fermions: squared many-body wave function at $T = 0$ (quantum probability density)

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- GUE eigenvalues: joint probability distribution

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^2$$

$\Rightarrow$ The positions of free fermions in a harmonic trap at $T=0$ behave statistically as the eigenvalues of a GUE random matrix

$$(\alpha\, x_1,\, \alpha\, x_2,\, \ldots, \alpha\, x_N) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)$$

Dean, Le Doussal, S.M. & Schehr, PRL, 114, 110402 (2015); PRA, 94, 063622 (2016); J. Stat. Mech. P063301 (2017); Recent Review: arXiv: 1810.12583

Joint distribution of eigenvalues

Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

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Coulomb gas interpretation: (Dyson, 1962)

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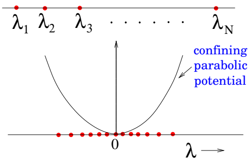
Boltzmann weight of a gas of N pairwise repelling charges (log-repulsion) in an external harmonic potential $V(\lambda) = \lambda^2$



Spectral Density: Wigner's Semicircle Law

- Av. density of eigenvalues (normalized to unity):

$$\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$$

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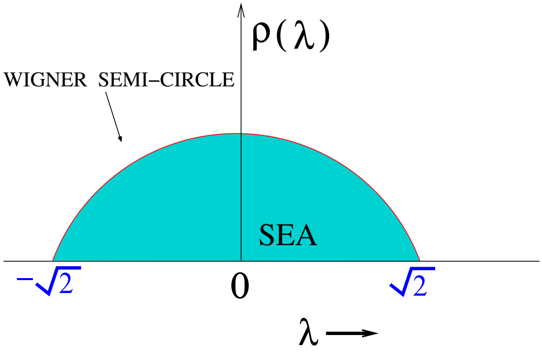
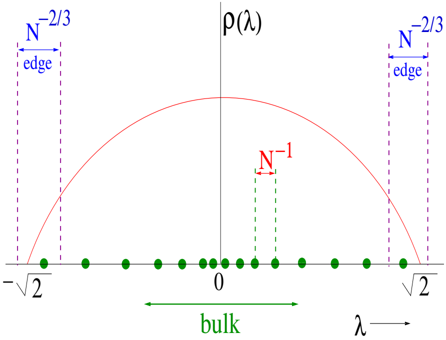


Average density of eigenvalues



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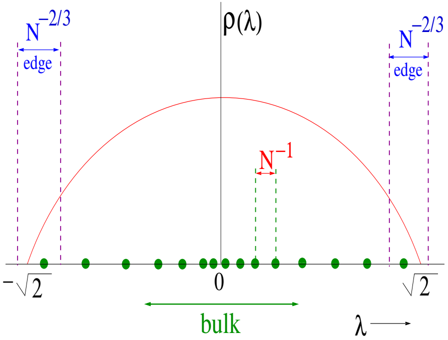

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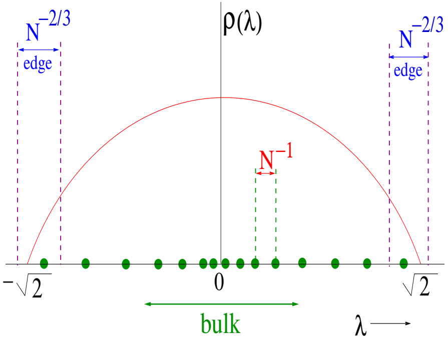

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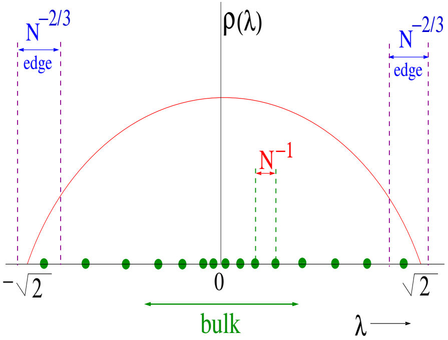

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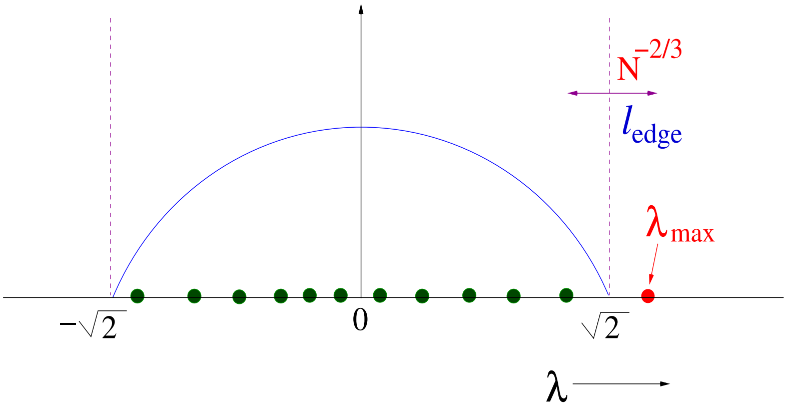
$$l_{\rm edge} >> l_{\rm bulk}$$

[Bowick & Brezin '91, Forrester '93]

Top eigenvalue: $\lambda_{\max}$

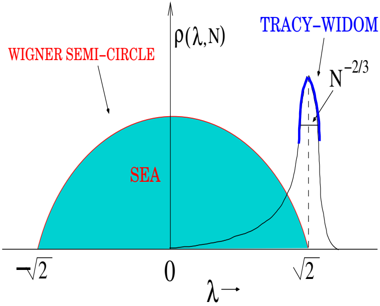
Top Eigenvalue of a random matrix $\lambda_{\max}$

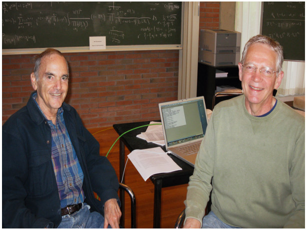


Recent excitements in statistical physics & mathematics on

$\lambda_{\max} \Rightarrow$ the top eigenvalue of a random matrix

Top Eigenvalue of a random matrix $\lambda_{\max}$





Average: $\langle \lambda_{\max} \rangle = \sqrt{2}$; Typical fluctuations: $|\lambda_{\max} - \sqrt{2}| \sim l_{\rm edge} \sim N^{-2/3}$

typical fluctuations, for large N, are distributed via Tracy-Widom ('94)

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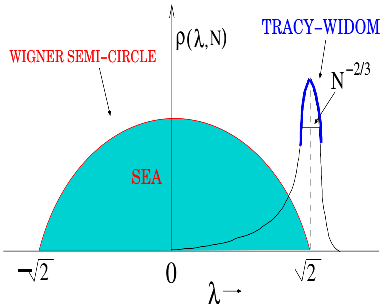

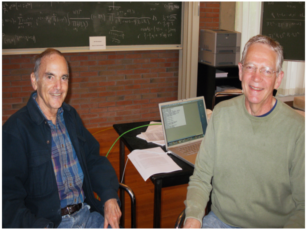

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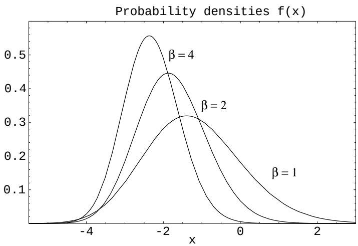
$$f_\beta(x) \to \text{Painlevé-II}$$

Tracy-Widom Distribution for $\lambda_{\max}$



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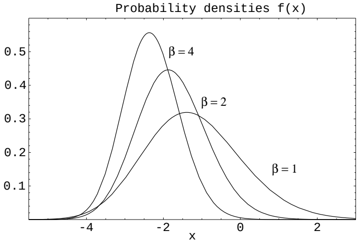

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- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right] \quad \text{as} \quad x \to -\infty$

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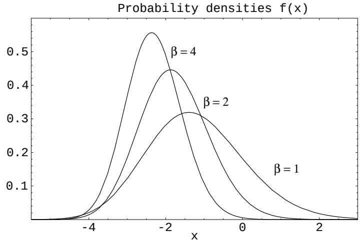

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Typical fluctuations (small) $\Rightarrow$ Tracy-Widom distribution $\rightarrow$ ubiquitous

directed polymer, random permutation, growth models–KPZ equation, sequence alignment, large N gauge theory, liquid crystals, spin glasses,...

Tracy-Widom distribution: Experiments

PRL **104**, 230601 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

Kazumasa A. Takeuchi* and Masaki Sano
Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **85**, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

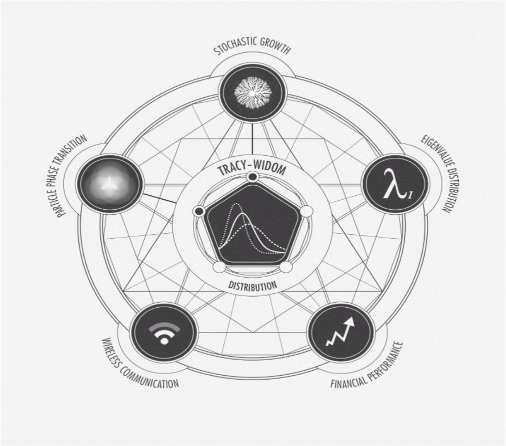
Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Department of Physics of Complex Systems, Rehovot 76100, Israel

PHYSICAL REVIEW B **87**, 184509 (2013)

Universal scaling of the order-parameter distribution in strongly disordered superconductors

G. Lemarié,[1,2] A. Kamlapure,[3] D. Bucheli,[2] L. Benfatto,[2] J. Lorenzana,[2] G. Seibold,[4] S. C. Ganguli,[3] P. Raychaudhuri,[3] and C. Castellani[2]
[1]*Laboratoire de Physique Théorique UMR-5152, CNRS and Université de Toulouse, F-31062 France*
[2]*ISC-CNR and Department of Physics, Sapienza University of Rome, P.le A. Moro 2, 00185 Rome, Italy*
[3]*Tata Institute of Fundamental Research, Homi Bhabha Rd., Colaba, Mumbai 400005, India*
[4]*Institut Für Physik, BTU Cottbus, P.O. Box 101344, 03013 Cottbus, Germany*

Ubiquity of Tracy-Widom distribution



Olena Shmahalo/Quanta Magazine

"Equivalence Principle", M. Buchanan, Nature Phys. 10, 543 (2014)

"At the far ends of a new universal law", N. Wolchover, Quanta Magazine (October, 2014)

Why is Tracy-Widom **ubiquitous**?

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Critical Phenomena : universality $\Longleftrightarrow$ phase transition

microscopic details become irrelevant near a critical point

Questions:

Where to look for a phase transition?

Where is the critical point ?

What are the two phases ?

Large deviations and 3-rd order phase transition

typical fluctuations of size $\sim N^{-2/3} \rightarrow$ Tracy-Widom distributed

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Atypical rare fluctuations of size $\sim O(1)$

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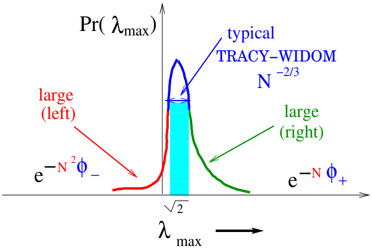

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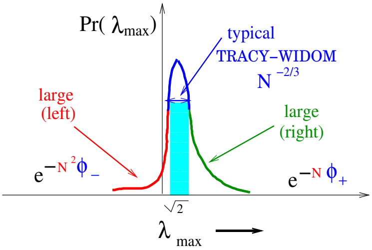

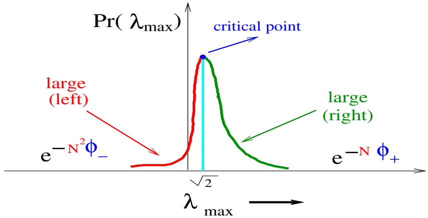

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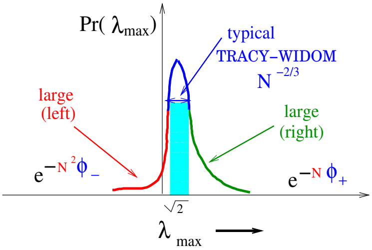

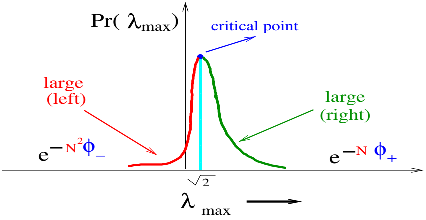

- General large deviation principle: Ben-Arous & Guionnet 1997

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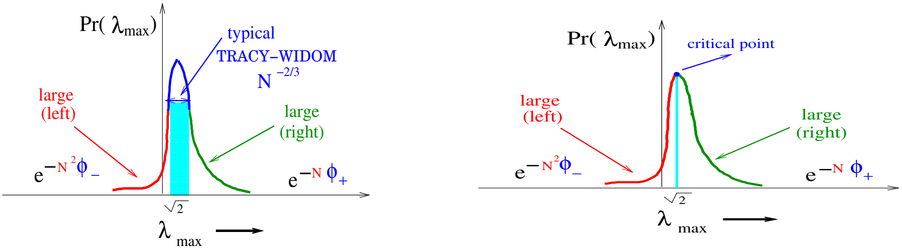

- General large deviation principle: Ben-Arous & Guionnet 1997
- nonanalytic behavior of the large deviation functions at the critical point $\sqrt{2} \Longrightarrow$ 3-**rd order phase transition**

  $\Longrightarrow$ Review: S.M. & G. Schehr, J. Stat. Mech. P01012 (2014)

Large deviations of $\lambda_{\max}$ via Coulomb Gas

Distribution of $\lambda_{\max}$ via Coulomb gas

$$\mathrm{Prob}[\lambda_{\max} \leq w,\ N] = \mathrm{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)}$$

$$\boxed{Z_N(w) = \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log|\lambda_j - \lambda_k|\right\}\right]}$$

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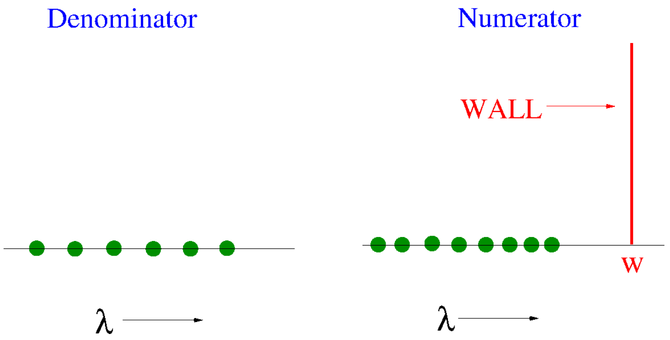
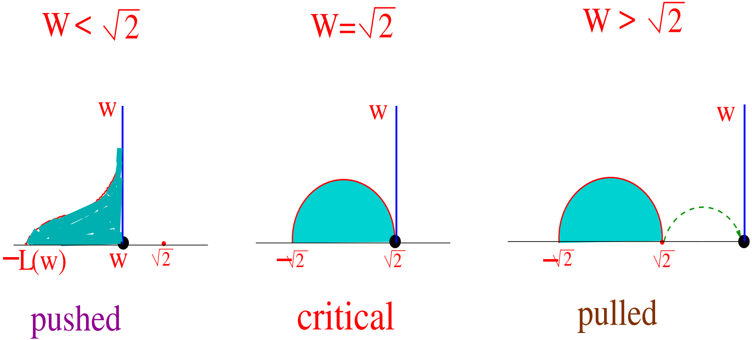


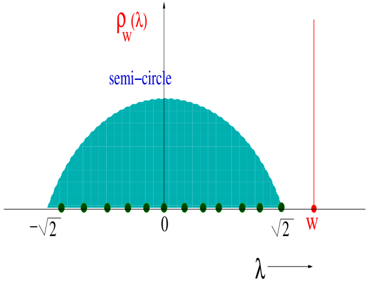
Phase transition

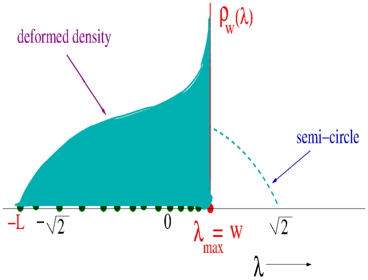
$$\lambda_{max} = W$$



$W = \sqrt{2}$ ⟶ CRITICAL POINT

Saddle point density





- For $w \geq \sqrt{2}$, the density is given by Wigner semicircle law:

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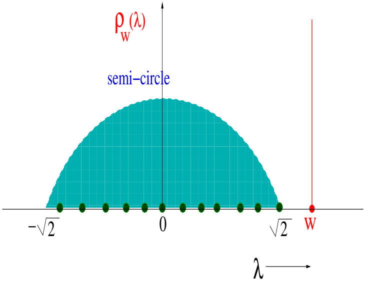
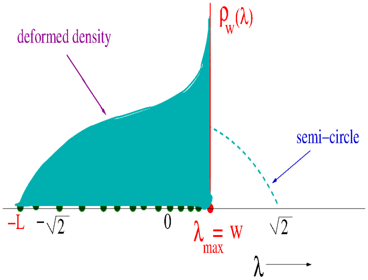
Saddle point density





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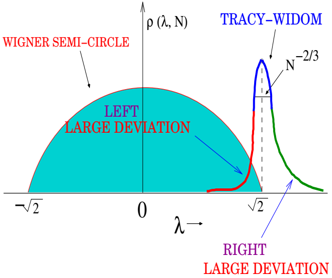
$$\rho_w(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$

- For $w < \sqrt{2}$, the deformed density: $\rho_w(\lambda) = \frac{\sqrt{\lambda+L(w)}}{2\pi\sqrt{w-\lambda}}\,[w + L(w) - 2\lambda]$

where $L(w) = [2\sqrt{w^2+6} - w]/3$

D.S. Dean and S.M., PRL, 97, 160201 (2006); PRE, 77, 041108 (2008)

Large Deviation Tails of $\lambda_{\max}$: Summary



- $f_\beta(x) \longrightarrow$ Tracy-Widom
- $\Phi_\mp(w) \longrightarrow$ left and right 'rate' functions

$\Longrightarrow$ exactly computable

Dean & S.M., PRL, 97, 160201 (2006)

S.M & Vergassola, PRL, 102, 060601 (2009)

Borot, Eynard, S.M., Nadal, JSTAT P11024 (2011)

Prob. density of the top eigenvalue: $\mathrm{Prob.}\,[\lambda_{\max} = w, N]$ behaves as:

$$P(w,\ N) \sim \exp\left[-\beta\, N^2\, \Phi_-(w)\right] \quad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim N^{2/3} f_\beta\left[\sqrt{2}\, N^{2/3}\left(w - \sqrt{2}\right)\right] \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \exp\left[-\beta\, N\, \Phi_+(w)\right] \quad \text{for} \quad w - \sqrt{2} \sim O(1)$$

Exact Left and Right Large Deviation Function

Using Coulomb gas + Saddle point method for large N:

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- Left large deviation function:

$$\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\Big] \quad \text{where} \quad w < \sqrt{2}$$

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

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In particular, as $w \to \sqrt{2}$ (from left), $\Phi_-(w) \to \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3$

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In particular, as $w \to \sqrt{2}$ (from left), $\Phi_-(w) \to \frac{1}{6\sqrt{2}}(\sqrt{2}-w)^3$

- Right large deviation function:

$$\Phi_+(w) = \frac{1}{2}w\sqrt{w^2-2} + \ln\left[\frac{w-\sqrt{w^2-2}}{\sqrt{2}}\right] \quad \text{where} \quad w > \sqrt{2}$$

[Ben Arous, Dembo, Guionnet 2001, S.M. & Vergassola, PRL, 102, 060601 (2009)]

Exact Left and Right Large Deviation Function

Using Coulomb gas + Saddle point method for large N:

- Left large deviation function:

$$\begin{aligned}\Phi_-(w) &= \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} \\ &+ 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\Big] \quad \text{where} \quad w < \sqrt{2}\end{aligned}$$

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

In particular, as $w \to \sqrt{2}$ (from left), $\Phi_-(w) \to \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3$

- Right large deviation function:

$$\Phi_+(w) = \frac{1}{2} w\sqrt{w^2-2} + \ln\left[\frac{w - \sqrt{w^2-2}}{\sqrt{2}}\right] \quad \text{where} \quad w > \sqrt{2}$$

[Ben Arous, Dembo, Guionnet 2001, S.M. & Vergassola, PRL, 102, 060601 (2009)]

As $w \to \sqrt{2}$ (from right), $\Phi_+(w) \to \frac{2^{7/4}}{3}(w - \sqrt{2})^{3/2}$

Large Deviation Functions

These large deviation functions $\Phi_{\pm}(w)$ have been found useful in a large variety of problems:

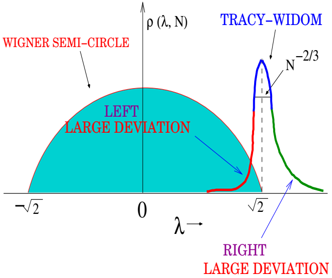
[Fyodorov 2004, Fyodorov & Williams 2007, Bray & Dean 2007, Auffinger, Ben Arous & Cerny 2010, Fydorov & Nadal 2012.... —— stationary points on random Gaussian surfaces and spin glass landscapes]

[Cavagna, Garrahan, Giardina 2000, Parisi & Rizzo 2008,... —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006, Marsh et. al. 2011, ...—— String theory & Cosmology]

[Beltrani 2007, Dedieu & Malajovich, 2007, Houdre 2011, Chiani 2012...—— Random Polynomials, Random Words (Young diagrams)]

3-rd order phase transition

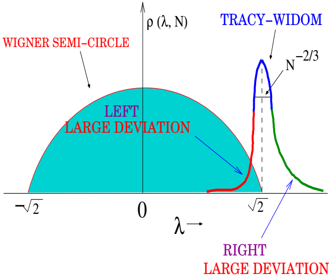


Cumulative distribution:

$$\mathrm{Prob.}[\lambda_{\max} \leq w, N] \sim e^{-\beta N^2 \Phi_-(w)}$$

$\Phi_-(w) \rightarrow$ energy cost in pushing the gas of Coulomb charges to the left of $\sqrt{2}$

3-rd order phase transition



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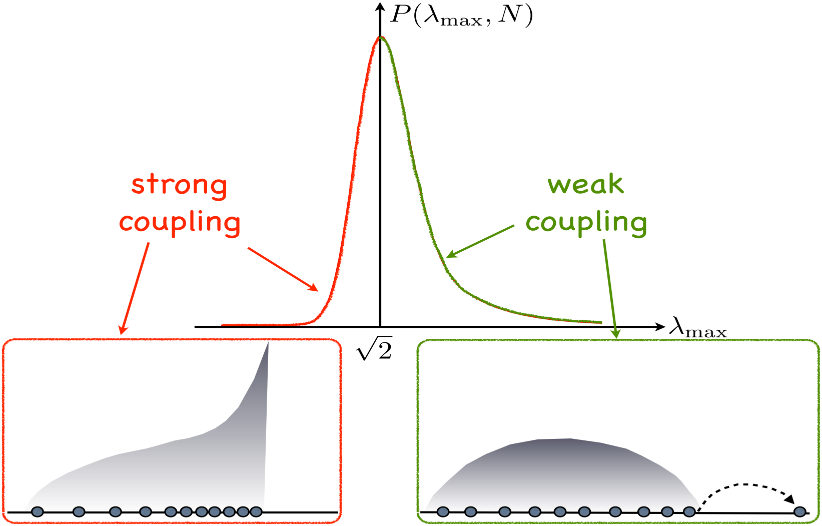
3-rd order phase transition:

$$\lim_{N\to\infty} -\frac{1}{\beta N^2} \ln\left[P(\lambda_{\max} \le w, N)\right] = \begin{cases} \Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\ 0 & \text{as } w \to \sqrt{2}^+ \end{cases}$$

$\longrightarrow$ analogue of the free energy difference

3-rd derivative $\to$ discontinuous

Transition between Strong and Weak phases



PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-N limit of the two-dimensional U(N) (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES ★

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

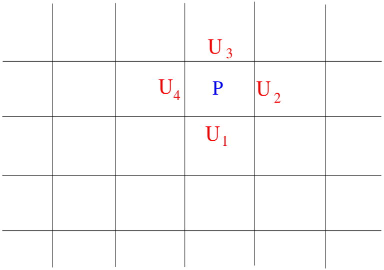
Review of large-N gauge theory: M. Marino, arXiv:1206.6272

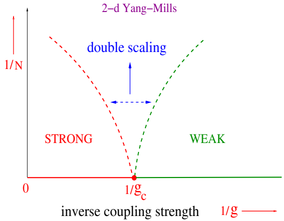
Gross-Witten-Wadia model (1980)

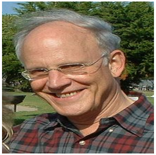
$$Z = \int [DU] \exp\left[S(U)\right]$$

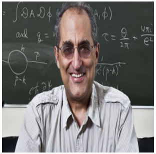
$$S(U) = \frac{N}{g^2} \sum_P \mathrm{Tr}\left(\prod_P U + h.c\right)$$

$g \to$ coupling strength

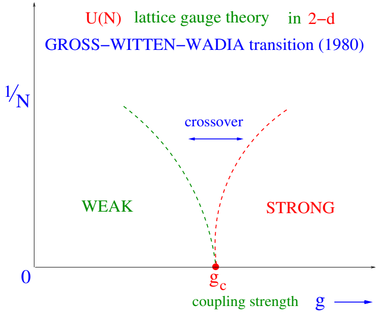




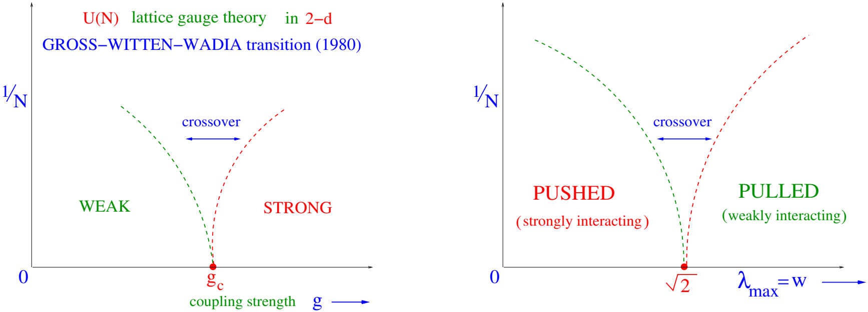




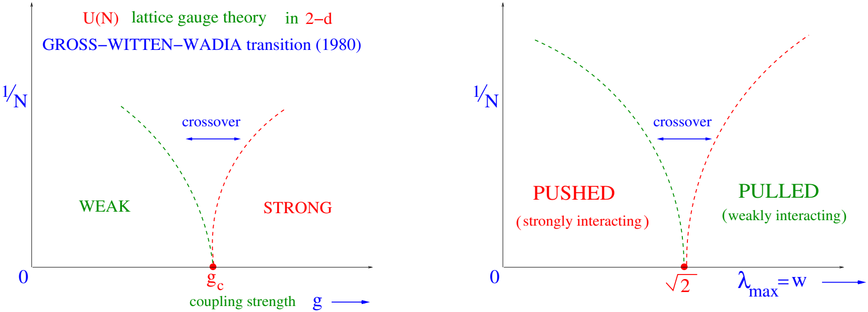
Large N Phase Transition: Phase Diagram



Large N Phase Transition: Phase Diagram



Large N Phase Transition: Phase Diagram



PUSHED phase $\equiv$ Strong coupling phase of Yang-Mills gauge theory
PULLED phase $\equiv$ Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom $\Rightarrow$ crossover function in the double scaling regime (for finite but large N)

Tracy-Widom and 3-rd order phase transition

Tracy-Widom is accompanied by a 3-rd order phase transition between a strong coupling and a weak coupling phase

Tracy-Widom and 3-rd order phase transition

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- large N YM-gauge theory in 2-d Gross-Witten-Wadia '80, Douglas-Kazakov '93, Gross-Matytsin, '94
- Complexity in spin glass models Auffinger, Ben Arous, Cerny '10, Fyodorov, Nadal '13

Tracy-Widom and 3-rd order phase transition

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In addition, we have found similar 3-rd order phase transition in:

- Conductance and Shot Noise in Mesoscopic Cavities
- Entanglement entropy of a random pure state in a bipartite system
- Maximum displacement in Vicious walker problem
- Cold atoms: free fermions in a 1-d harmonic trap at $T = 0$
- Height distribution in $(1+1)$-d KPZ growth models
- 1d Plasma: position of the rightmost charge

Bohigas, Comtet, Dean, Forrester, Le Doussal, Nadal, Schehr, Texier, Vergassola, Vivo+S.M. ('08-'17)

Review: S.M. & G. Schehr, J. Stat. Mech. P01012 (2014)

Main Conjecture:

Wherever Tracy-Widom distribution occurs, there is an accompanying 3-rd order phase transition

TW $\longrightarrow$ finite-size crossover function at a 3-rd order critical point

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Review: S.M. & G. Schehr, J. Stat. Mech. P01012 (2014)

Not all 3-rd phase transitions are accompanied by Tracy-Widom crossover

- 1-d jellium model

  Dhar, Kundu, S.M., Sabhapandit, Schehr, PRL, 119, 060601 (2017)

- Complex Ginibre ensemble

  Cunden, Mezzadri, Vivo, J. Stat. Phys. 164, 1062 (2016); Cunden, Facchi, Ligabo, Vivo, J. Phys. A: Math. Theor. 51, 35LT01 (2018).

Collaborators

Students: A. Grabsch, J. Bun, T. Gautie, B. Lacroix-a-Chez-Toine, R. Marino, C. Nadal, J. Randon-Furling

Postdocs: A. Kundu (ICTS, Bangalore), D. Villamaina (CFM, Paris), J. Grela (LPTMS)

Collaborators:

O. Bohigas, A. Comtet, G. Schehr, C. Texier (LPTMS, Orsay, France), R. Allez (Berlin, Germany), G. Borot (Geneva, Switzerland), J.-P. Bouchaud & M. Potters (CFM, Paris, France), P. Calabrese (Sissa, Trieste, Italy), K. Damle & V. Tripathi (Tata Institute, Bombay, India), A. Dhar (ICTS, Bangalore), F.D. Cunden and N. O'Connell (University College of Dublin, Ireland), D.S. Dean (Bordeaux, France), B. Eynard (Saclay, France), P. J. Forrester (Melbourne, Australia), A. Lakshminarayan (IIT Madras, India), P. Le Doussal (ENS, Paris, France), S. Sabhapandit (RRI, Bangalore), A. Scardichio (ICTP, Trieste, Italy), S. Tomsovic (Washington State Univ., USA), M. Vergassola (UCSD, USA), P. Vivo (King's College, UK)

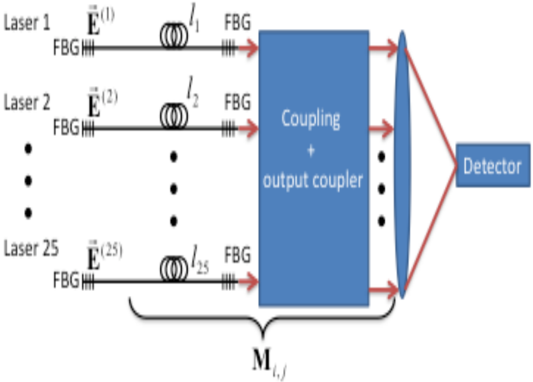
Experimental Verification with Coupled Lasers

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500,000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.

Experimental Verification with Coupled Lasers

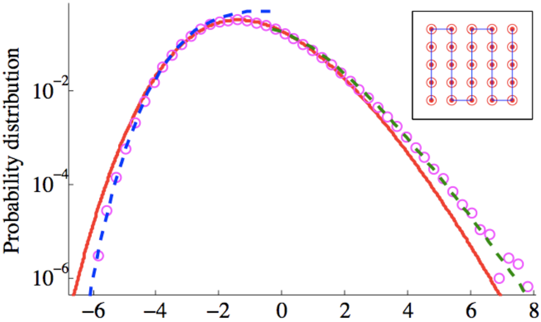


combined output power from fiber lasers $\propto \lambda_{\max}$

$\lambda_{\max} \rightarrow$ top eigenvalue of the Wishart matrix $W = X^t X$

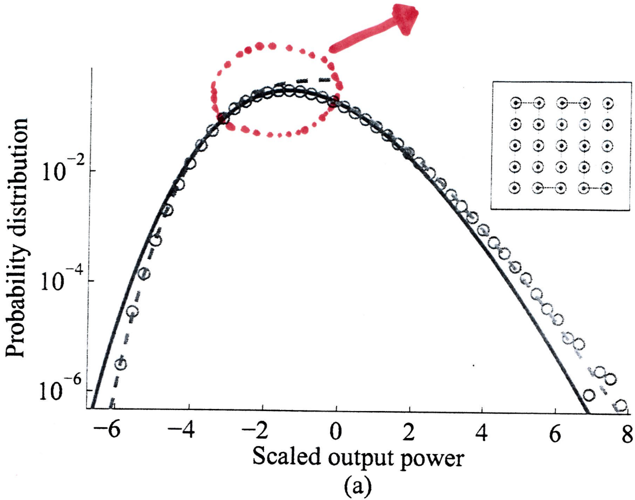
where $X \rightarrow$ real symmetric Gaussian matrix $(\beta = 1)$

Experimental Verification with Coupled Lasers



(a)

Experimental Verification with coupled lasers



Tracy-Widom density with $\beta = 1$

Fridman et. al. arXiv: 1012.1282

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