Entanglement and tensor networks for quantum many body systems

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The quantum many body problem

- Quantum physics of the 1920's:

$$-i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

  - Hilbert space is endowed with a tensor product structure

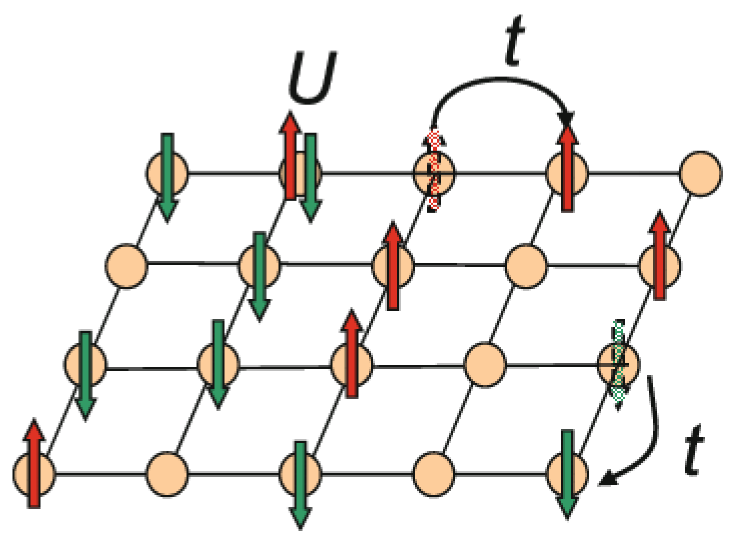
$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_n} \psi_{i_1 i_2 i_3 \cdots} |i_1\rangle|i_2\rangle$$

  - "The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble. It therefore becomes desirable that approximate practical methods of applying quantum mechanics should be developed, which can lead to an explanation of the main features of complex atomic systems without too much computation." [Dirac '29]

Quantum many body problem

- Large part of theoretical physics /chemistry in last 90 years has focused on realizing Diracs dream:
  - Hartree-Fock
  - Perturbation theory, Feynman diagrams
  - Coupled cluster theory
  - Density Functional Theory
    - All of those methods rely on the existence of a good fiducial noninteracting state
- Many strongly interacting quantum many body systems of interest do not have such a fiducial state
  - E.g. Hubbard model:

$$\hat{H} = -t \sum_{\langle i,j \rangle,\sigma} (\hat{c}^\dagger_{i,\sigma} \hat{c}_{j,\sigma} + \hat{c}^\dagger_{j,\sigma} \hat{c}_{i,\sigma}) + U \sum_{i=1}^{N} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$



  - Quantum Monte Carlo can sometimes help, but in many cases of interest sign problem

- Finding ground states of (quasi-)local quantum Hamiltonians is very different from diagonalizing a random matrix:

$$H = \sum_{ij} O_{ij}$$

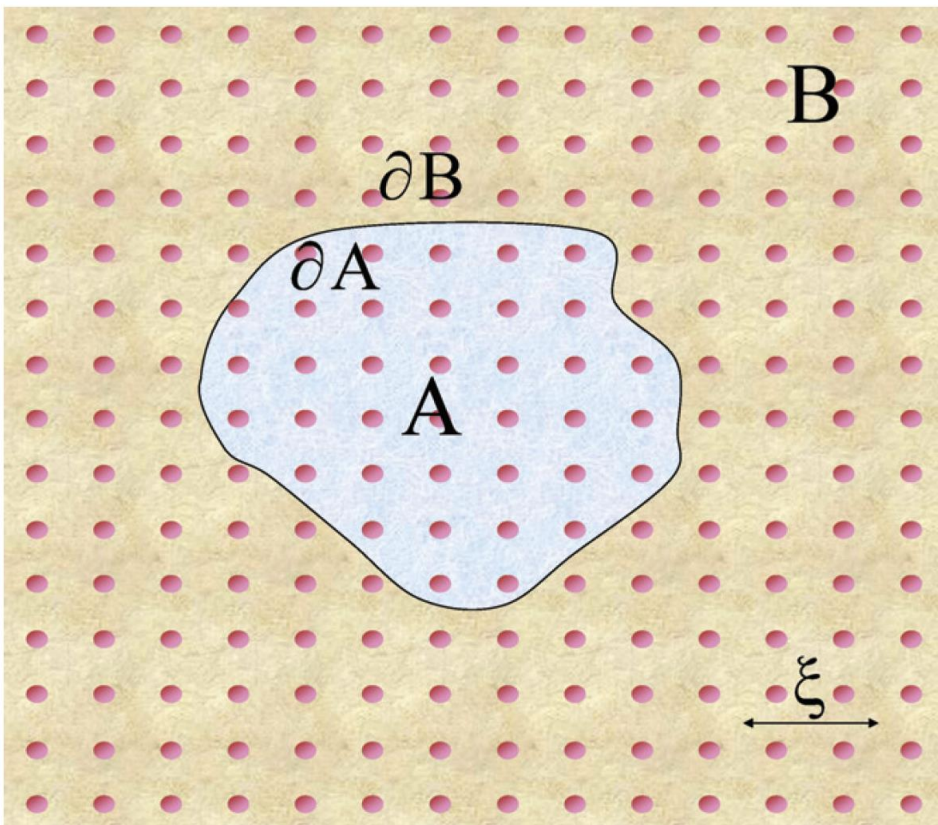
  - LOCALITY of interactions impose that the ground state of my system has extremal local density matrices compatible with the symmetries of the system (translation invariance, SU(N), …)
  - Unlike the random matrix case, once I know the eigenvector corresponding to extremal eigenvalues (= ground state), I basically know the full spectrum of my system (quasiparticles / correlation functions)

- Why is many body physics possible at all? Because there is a lot of structure in the ground states
  - They exhibit an area law for the entanglement entropy

    => Possible to compress the description!

Area Laws for the entanglement entropy

- Ground and Gibbs states of interacting quantum many body Hamiltonians with local interactions have very peculiar properties
  - Area law for the entanglement entropy (ground states) or for mutual information (Gibbs states)



1. Ground states:

$$S(\rho_A) = c.\partial A$$

Srednicki '93; Hastings '07; …

$$S(\rho_A) = \frac{c}{6}.\log(A/\epsilon)$$

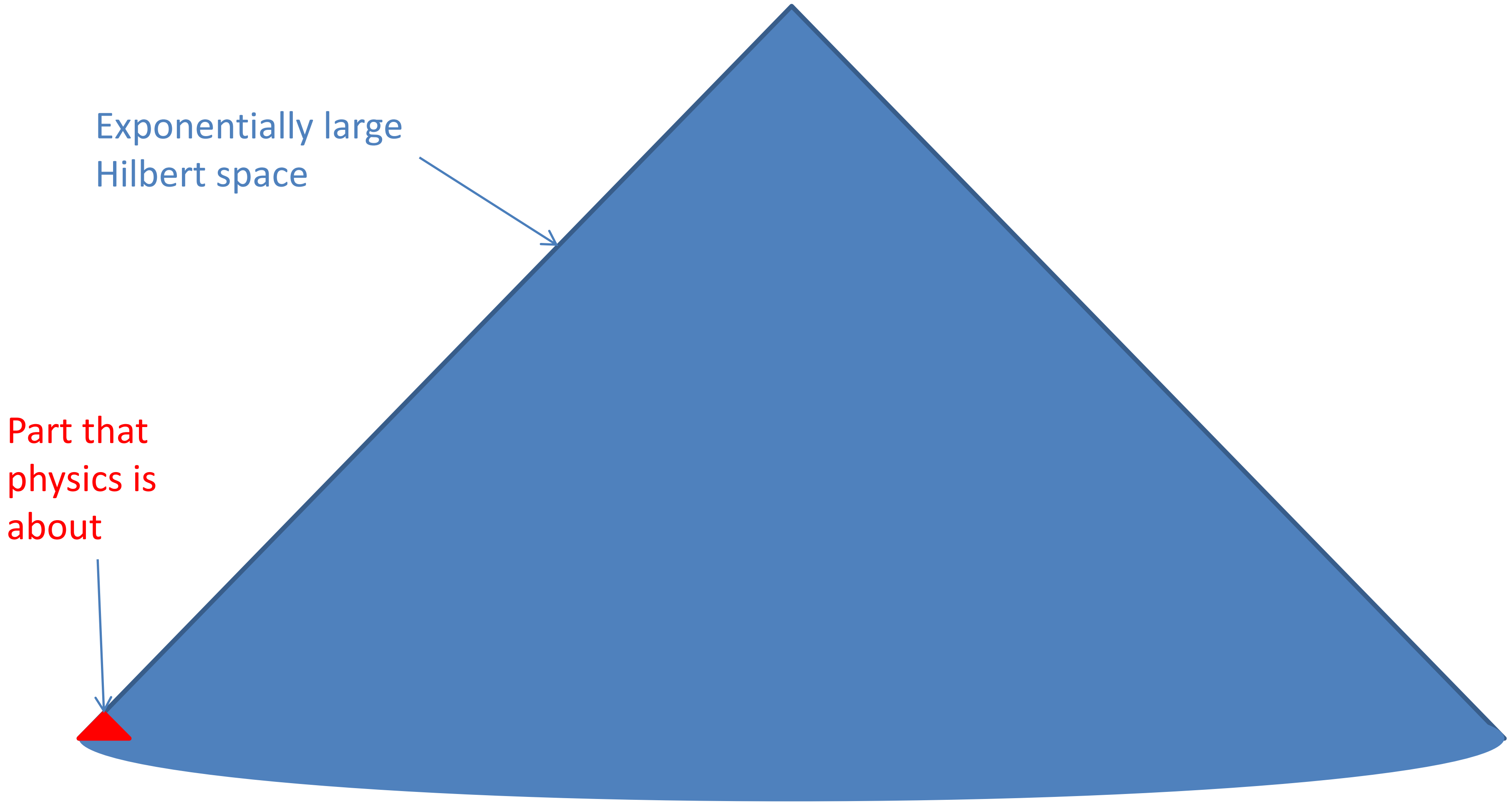
Holzhey, Larsen, Wilczek '94; …

2. Gibbs states:

$$\begin{aligned} I(A,B) &= S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \\ &= c.\partial A \end{aligned}$$

Wolf, Hastings, Cirac, FV '08

The illusion of Hilbert space



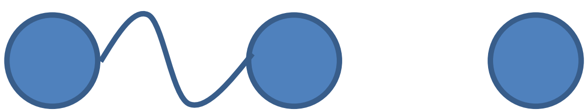
Poulin, Qarry, Somma, FV: PRL ‘11

Monogamy of Entanglement

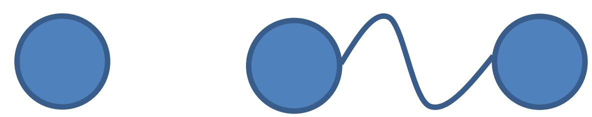
- Consider the Heisenberg antiferromagnet on 3 spin ½'s:

$$\mathcal{H} = \sum_i \vec{S}_i \vec{S}_{i+1}$$

- What is the ground state?



$$(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)\,|\uparrow\rangle$$



$$|\uparrow\rangle\,(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

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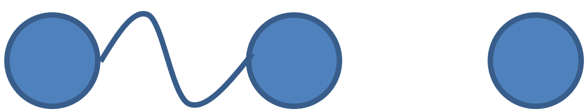

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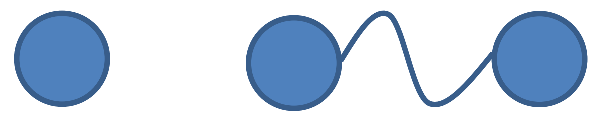

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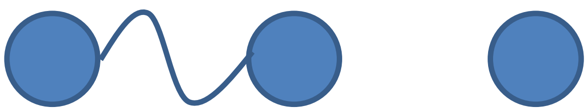
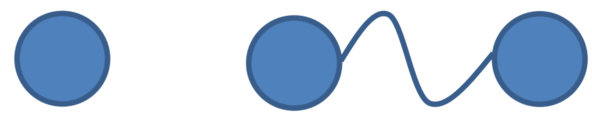


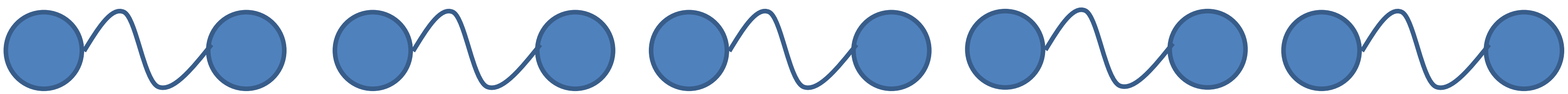


$$(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)\,|\uparrow\rangle \quad - \quad |\uparrow\rangle\,(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

- Monogamy: impossibility of sharing a singlet with two spin ½'s
  - Mathematically: there does not exist a density matrix $\rho_{123} \geq 0$ which is positive such that its marginals $\rho_{12} = \mathrm{Tr}_3\,(\rho_{123})$ and ρ_{23} are singlets
  - All interesting long range physics / entanglement in quantum spin systems is a consequence of this optimal trade-off in local marginals

Tensor Calculus for Quantum Spin Chains

- Systematic way of creating states which have extremal local marginals but keep translational invariance: *matrix product states*



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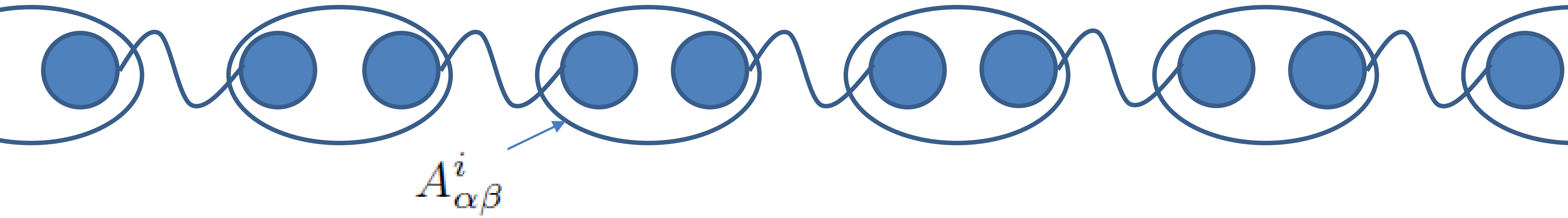
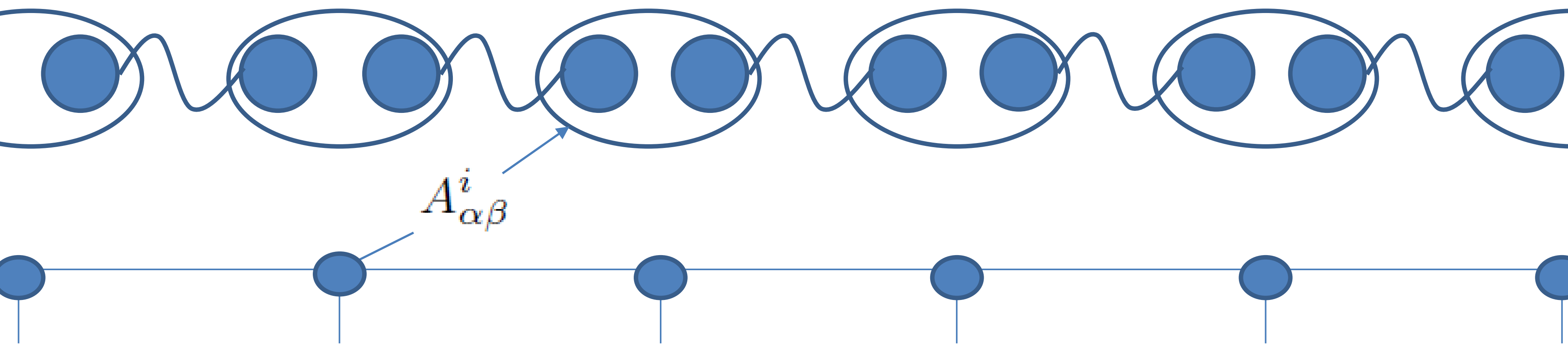


Tensors for Quantum Spin Chains

- Systematic way of creating states which have extremal local marginals but keep translational invariance: *matrix product states*



$$|\psi\rangle = \sum_{i_1 i_2 i_3 \cdots} \mathrm{Tr}\left(A^{i_1} A^{i_2} A^{i_3} \cdots\right) |i_1\rangle |i_2\rangle |i_3\rangle \cdots$$

I. Affleck, T. Kennedy,
E. Lieb, H. Tasaki '87

M. Fannes, B. Nachtergaele,
R. Werner '91

K. Wilson '70s

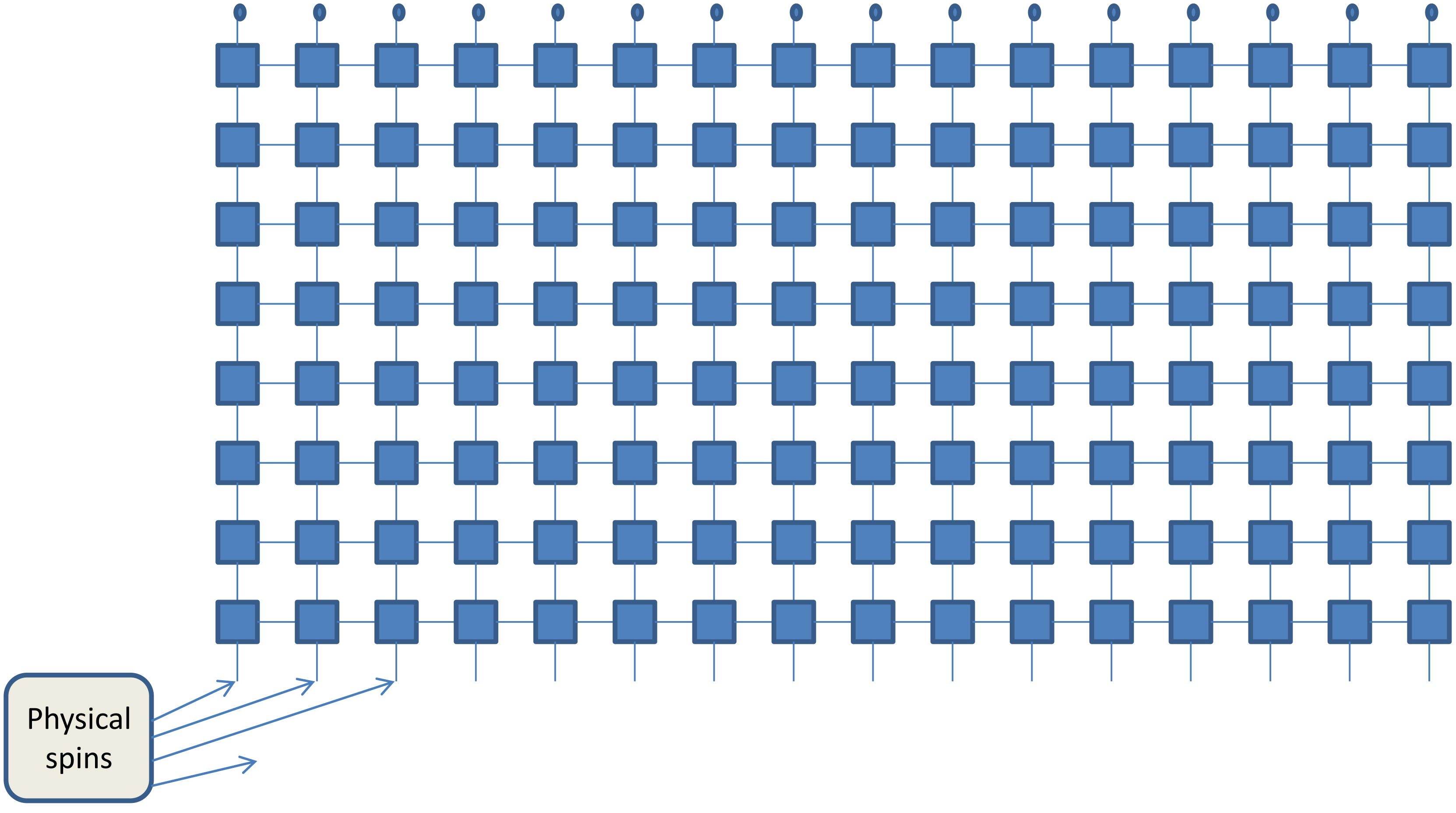
S. White '92

FV, I. Cirac '04

G. Vidal '04

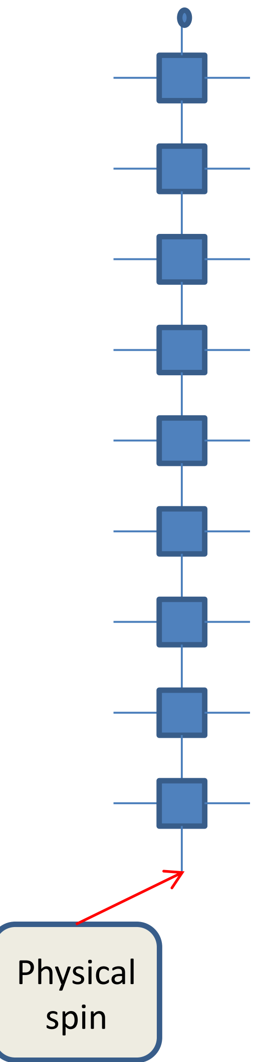
Path Integral representation of ground states

- Let us consider an arbitrarily Hamiltonian of a quantum spin system, and a path integral $\exp\left(-\beta\mathcal{H}\right)|\psi_0\rangle$ representing the ground state for $\beta \to \infty$



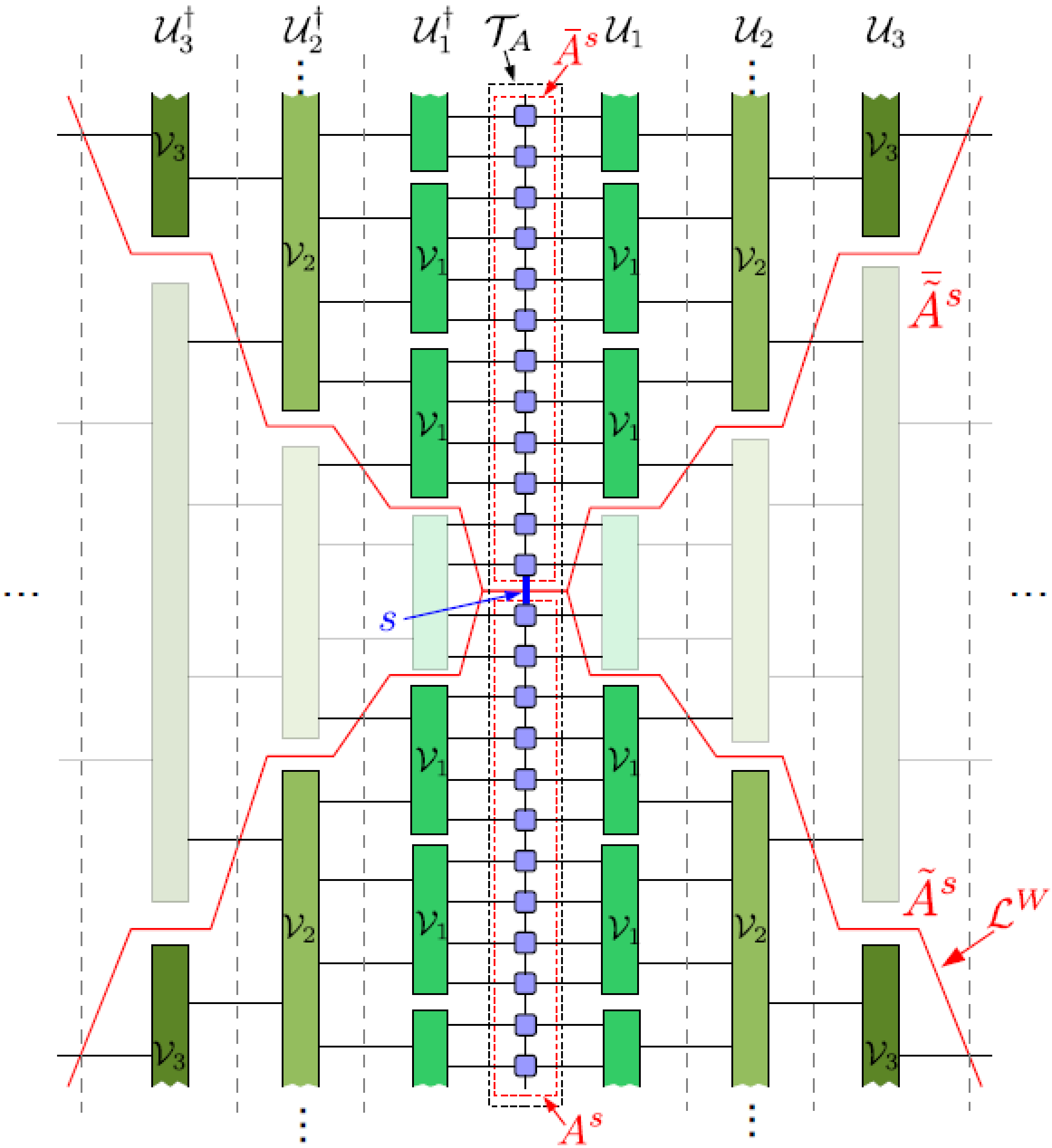
Wilson's RG for quantum impurities

- Let us consider 1 column in this transfer matrix picture:



- The physical spin can be understood as a Kondo like impurity attached to a translational invariant system ("conduction electrons")

- The crucial question that we tackle here: can we compress the information on the "virtual" links such as to obtain a more economical representation of the ground state?
  - Just like Wilson, we can indeed envision devising an RG transformation "compressing" the information of the conduction electrons
  - The dimension of the compressed Hilbert space will be related to the amount of quantum entanglement in the system

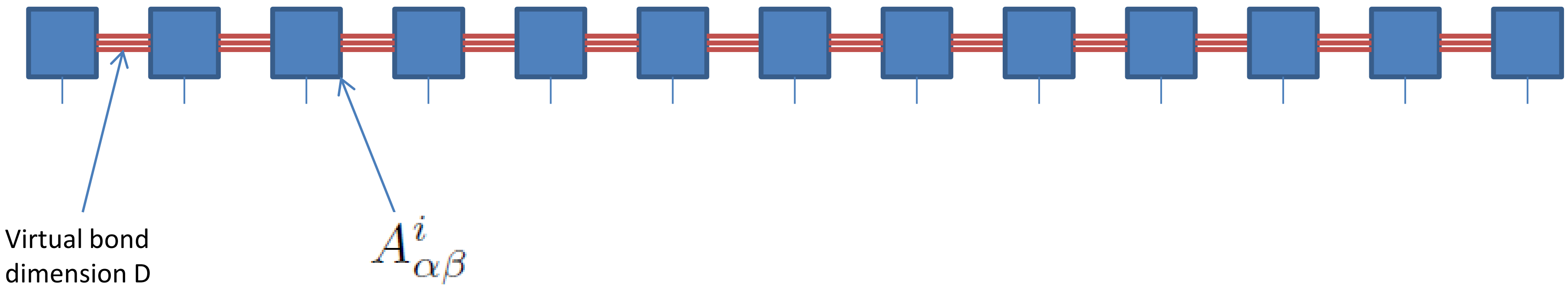


Wilson RG for impurity: Finite bond dimension corresponds to introducing a cut-off, but for a gapped system this cut-off is given by the gap and hence does not introduce an extra approximation

M. Bal et al., arXiv:1408.5140

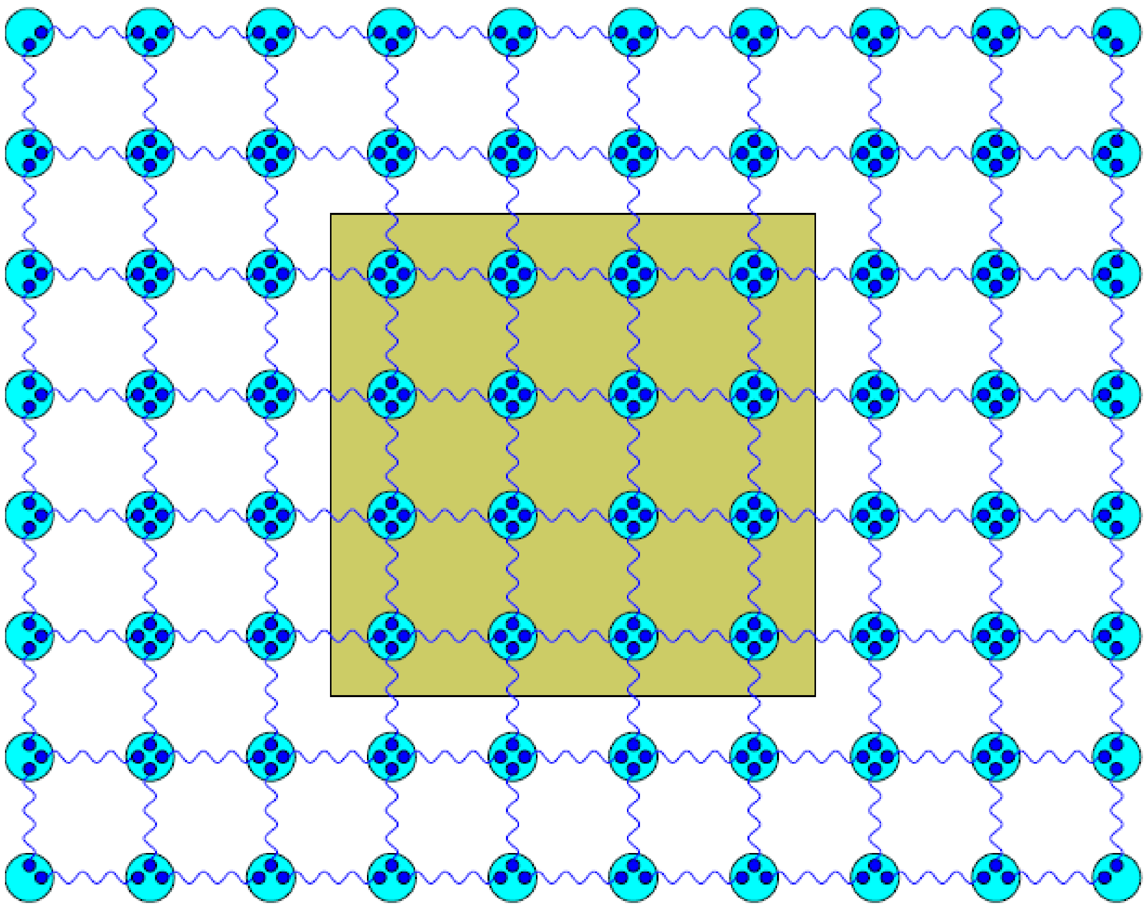
Matrix Product States

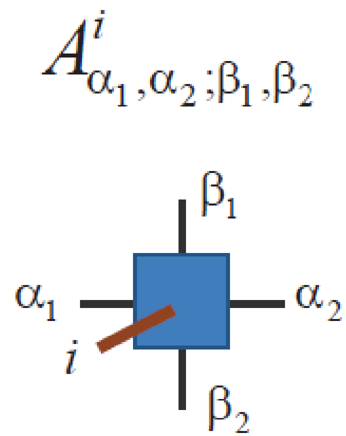
- The picture that is emerging is that any ground state of a quantum spin chain can be presented by a MPS:



$$|\psi\rangle = \sum_{i_1 i_2 \cdots} \mathrm{Tr}\left(A^{i_1}_{\alpha_1\alpha_2} A^{i_2}_{\alpha_2\alpha_3} \cdots\right) |i_1\rangle |i_2\rangle \cdots$$

Higher dimensions: Projected Entangled Pair States





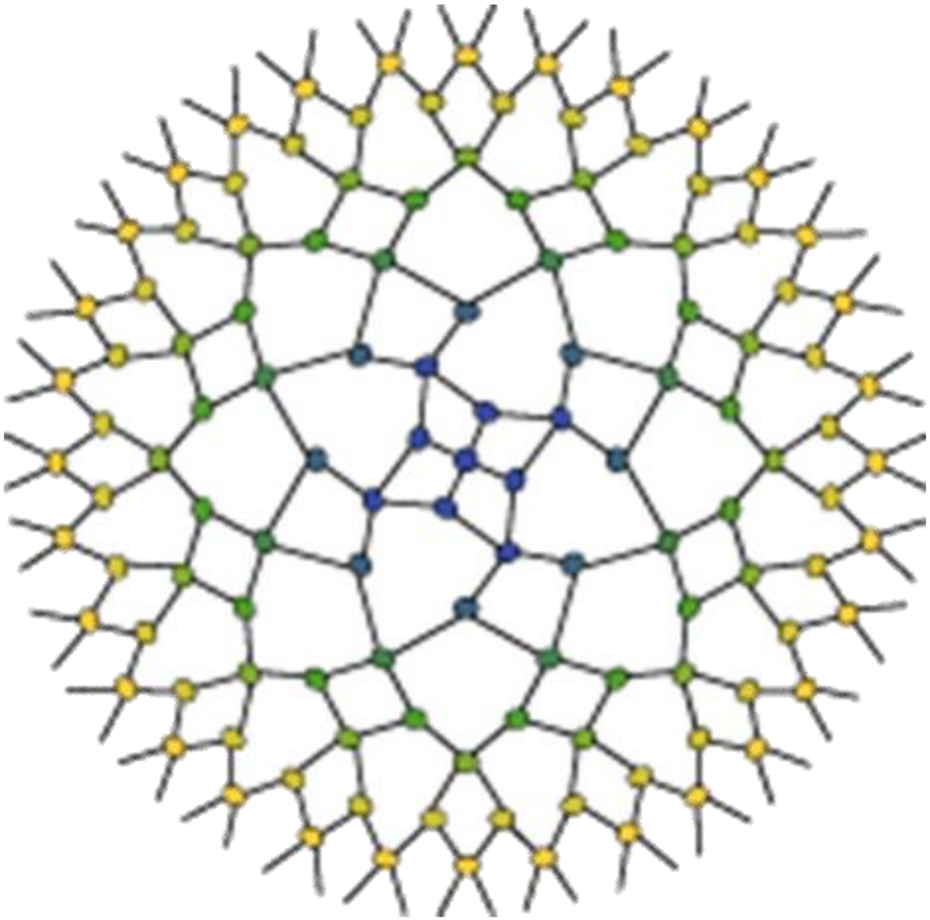
- Key: all many body information is stored in single tensor acting on correlation space

FV, Cirac ‘04

Multiscale Entanglement Renormalization Ansatz

G. Vidal '07

Scale invariant (critical) systems



- Crucial ideas in tensor networks:

  - Tensors model the entanglement structure: modelling correlations makes much more sense than modelling wavefunction directly
    - Tensors dictate the ***entanglement patterns***

  - Tensor networks can be efficiently contracted due to holographic property: map quantum 3D -> 2D -> 1D -> 0D problems, and this can be done efficiently due to area laws

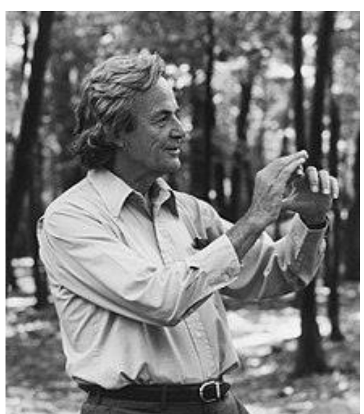
  - States are defined in thermodynamic limit; finite size scaling is replaced by finite entanglement scaling

  - Local tensor contains all global information about quantum many body state
    - different phases of matter can be distinguished by symmetries of those local tensor, including topological phases
    - Tensor networks provide a natural way of dealing with gauge theories: enforcing symmetries

Feynman on the quantum many body problem:

"Now, in field theory, what's going on over here and what's going on over there and all over space is more or less the same. Why do we have to keep track in our functional of all things going on over there while we are looking at the things that are going on over here? . . . It's really quite insane, actually: we are trying to find the energy by taking the expectation value of an operator which is located here and we present ourselves with a functional which is dependent on everything all over the map. That's something wrong. Maybe there is some way to surround the object, or the region where we want to calculate things, by a surface and describe what things are coming in across the surface. It tells us everything that's going on outside."

"I think it should be possible some day to describe field theory in some other way than with the wave functions and amplitudes. It might be something like the density matrices where you concentrate on quantities in a given locality and in order to start to talk about it you don't immediately have to talk about what's going on everywhere else . . . "

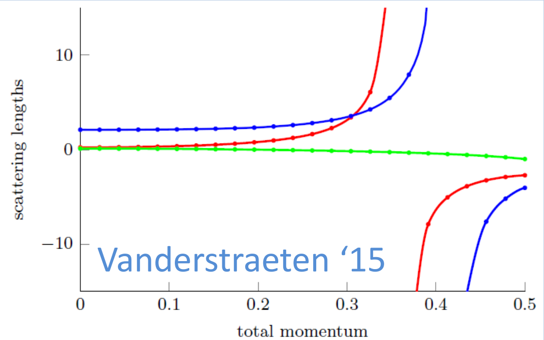


One of Feynman's last talks: "Difficulties in Applying the Variational Principle to Quantum Field Theories" (Wangerooge '87)

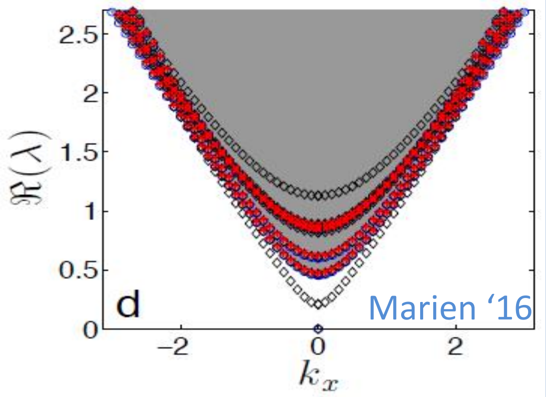
Tensor Networks

Computational aspects:

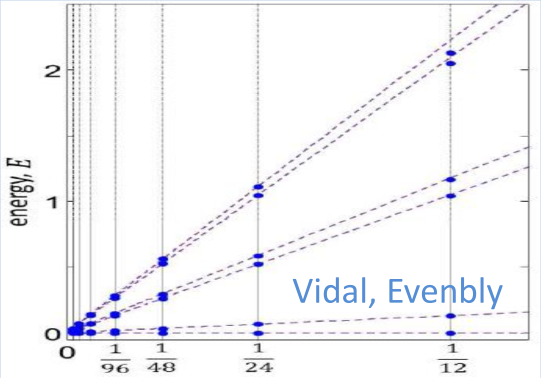
- MPS



- PEPS

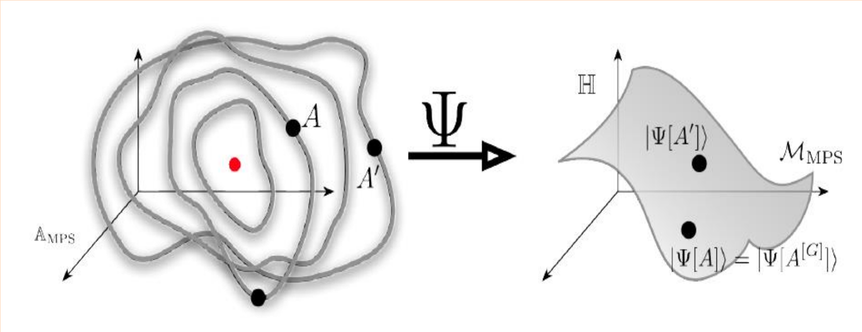


- MERA



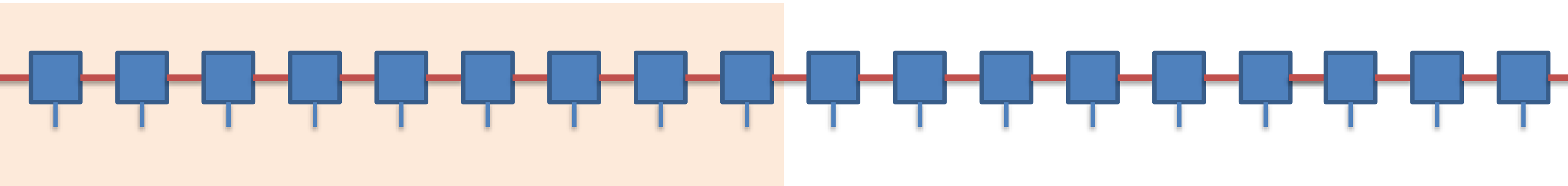
Conceptual aspects: the shadow world

- Area laws and the corner of Hilbert space: the manifold of ground states of local Hamiltonians



- Modelling the entanglement degrees of freedom: symmetries
  - Symmetry fractionalization, classification of SPT phases
  - Topologically ordered matter: holographic Landau type order parameters

1D systems: matrix product states



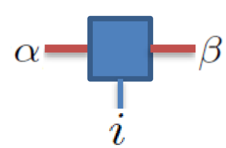
- MPS form a representation of the manifold of all states in 1D satisfying an ***AREA law***

- Ground states of quantum spin chains satisfy such an area law:
  - Entropy of the reduced density matrix of a halve infinite chain is finite (gapped) or exponentially smaller than number of spins (critical)
  - Hence MPS parameterize the corner in Hilbert space representing all ground states of gapped spin chains

FV, Cirac '05, Hastings '07

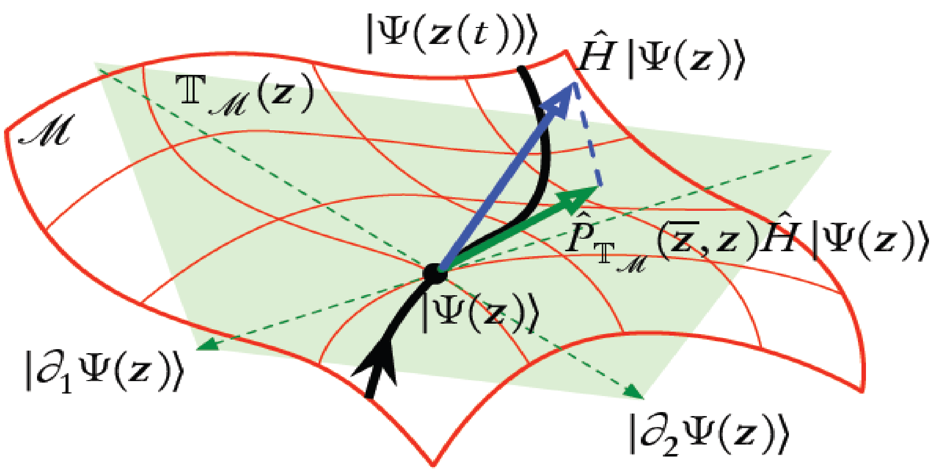
- This means that the MPS formalism allows for an exponential compression of ground states in terms of 1 simple tensor (uniform MPS or uMPS)

$A^{i}_{\alpha\beta}$ =



- Instead of working in the exponentially large Hilbert space, we do the physics on the low-dimensional manifold of MPS

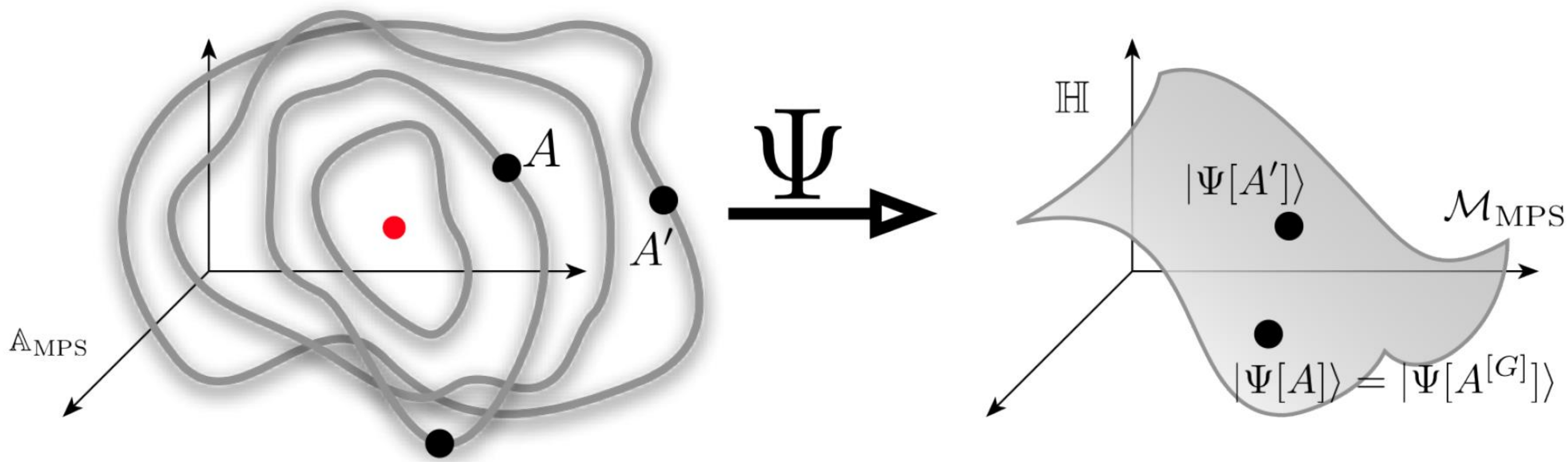
$$-i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad \rightarrow \qquad -i\frac{d}{dt}|\psi(A)\rangle = P_T(A)H|\psi(A)\rangle \Leftrightarrow \frac{d}{dt}A^i = F\left(A^i\right)$$



  - We get effective nonlinear differential equation: symplectic, Poisson brackets, …
  - DMRG, TEBD, TDVP, iPEPS…: variations on how to split this differential equation in terms of different Trotter steps
  - Tangent planes: effective Hamiltonians and elementary excitations

Haegeman et al. '10

Manifold of MPS



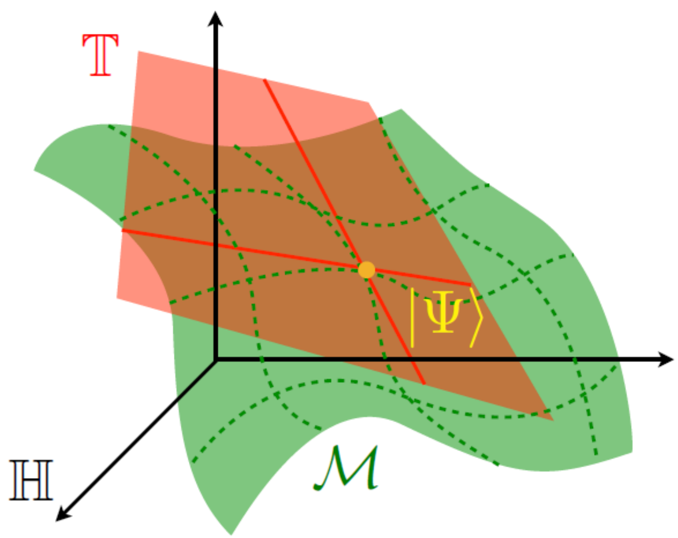
- There is a clear redundancy in the MPS parameterization

$$A^i \leftrightarrow X A^i X^{-1}$$

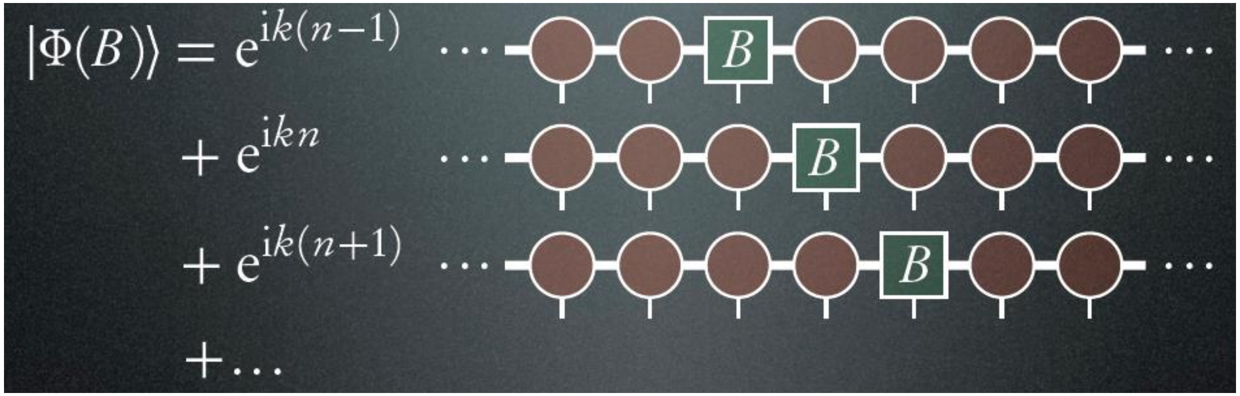
- Due to the parameter this “gauge” redundancy in the matrix product state representation, matrix product states have the mathematical structure of a (principal) fiber bundle, giving rise to a so-called Kahler manifold

Elementary excitations: post-MPS methods

- The tangent plane on the manifold around the ground state on the system defines a linear subspace of interest; we can project the many body Hamiltonian on that subspace, and get an effective Hamiltonian of dimension $(d-1)D^2$. This allows to extract dispersion relations to an unreasonable accuracy

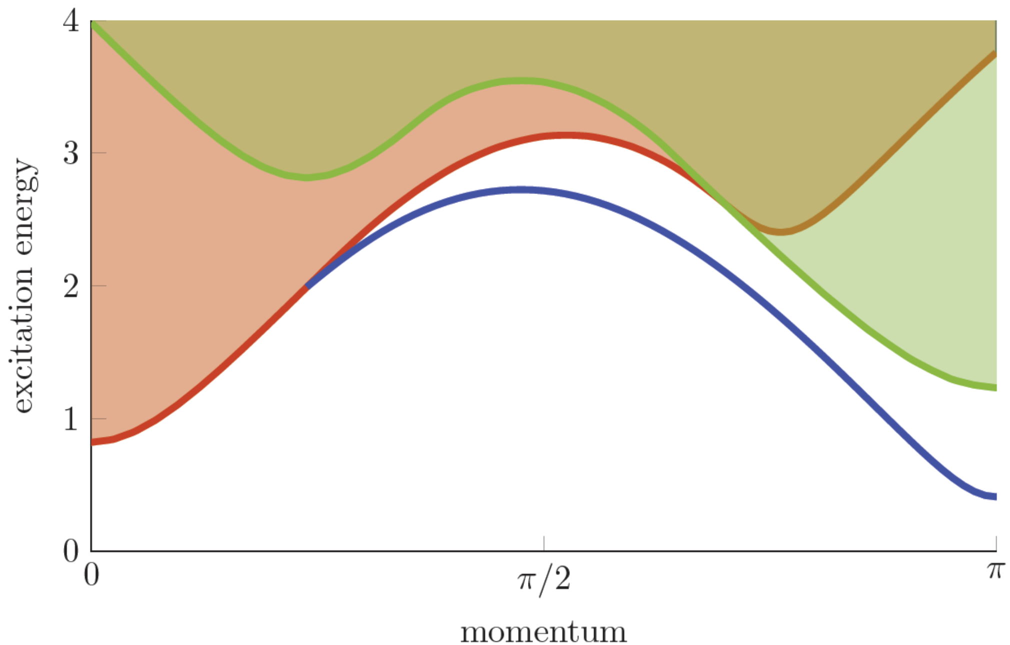


- Lieb-Robinson bounds yield a proof that elementary excitations in gapped systems must indeed be of that form: follows from locality of interactions

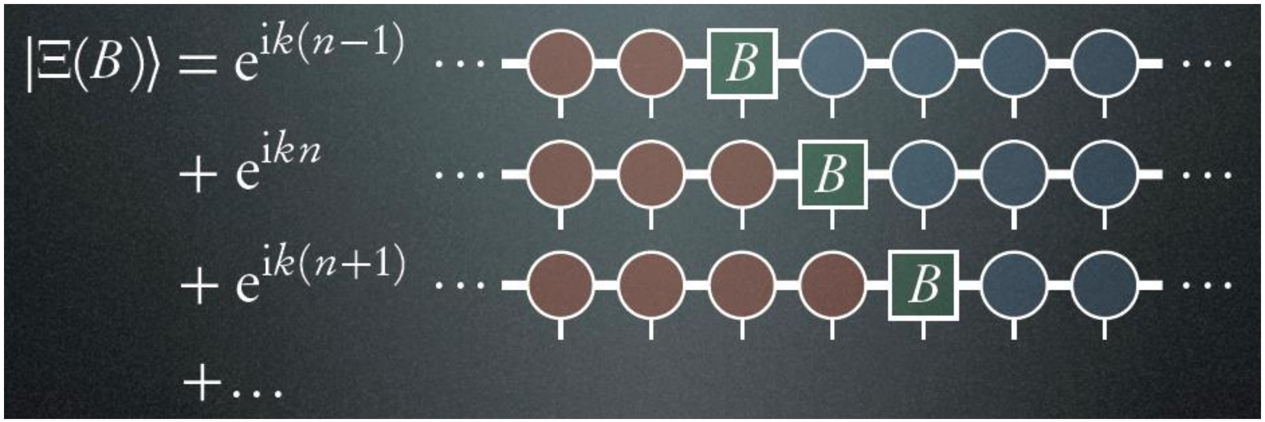


Spin 1 Heisenberg model:

$$\Delta^{(\infty)}_{\mathrm{Haldane}} = 0.410479248463^{+6\times10^{-12}}_{-3\times10^{-12}}.$$



Topological nontrivial excitations

- 

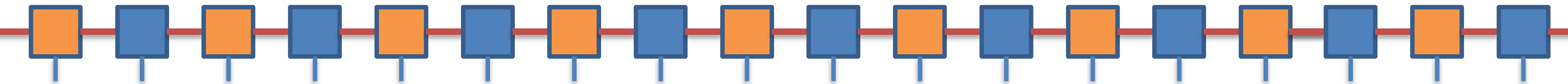
- If there is symmetry breaking, then the elementary excitations will be “domain walls”: topologically nontrivial excitations (cfr. Solitons in Mandelstam ansatz or eg. Jordan-Wigner) where the operators u transform one vacuum in the other one

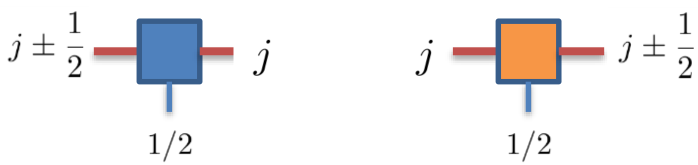
$$\hat{O}(n) = \hat{o}_n \prod_{m>n} \hat{u}_n$$

  – Although topologically nontrivial, this is still “local” : the only relevant information is at the end point; what happens is of course that those excitations always come in pairs (cfr. spinons in Heisenberg antiferromagnet)

Haegeman et al. ‘13

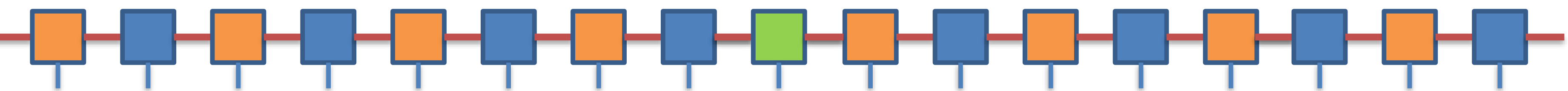
What is the spin of a spin wave?

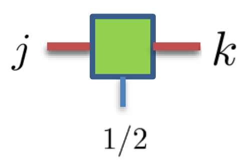




Singlets (Wigner 3j-symbols)

- Spinon:





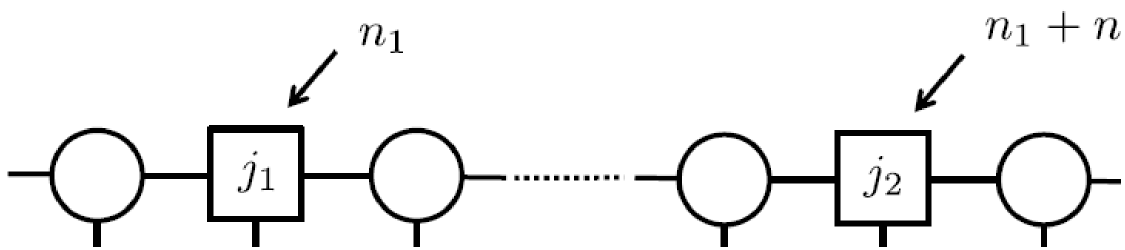
Transforms according to half-integer!

Zauner-Stauber et al., “On the nature of holons and spinons”, PRB ‘18

Scattering of elementary excitations

- How do those elementary excitations interact with each other?
  - Natural MPS ansatz for scattering:

$$|\chi_{(j_1,j_2),K}(n)\rangle = \sum_{n_1} \mathrm{e}^{iKn_1}$$



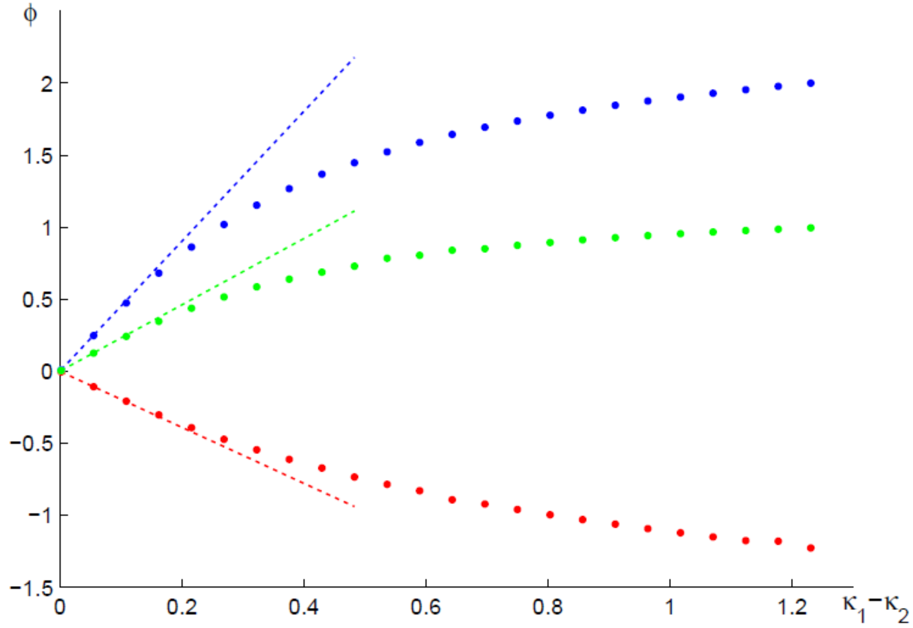
$$|\Upsilon(K)\rangle = \sum_{n=0}^{+\infty} \sum_{j=1}^{L_n} c^j(n) \, |\chi_{K,j}(n)\rangle$$

$$|\chi_{K,j}(n=0)\rangle = \sum_{n_1=-\infty}^{+\infty} \mathrm{e}^{iKn_1} \sum_{\{s\}=1}^{d} \boldsymbol{v}_{\boldsymbol{L}}^\dagger \left[\prod_{m<n_1} A^{s_m}\right] B_{(j)}^{s_{n_1}} \left[\prod_{m>n_1} A^{s_m}\right] \boldsymbol{v}_{\boldsymbol{R}} \, |\{s\}\rangle$$

$$|\chi_{K,(j_1,j_2)}(n>0)\rangle = \sum_{n_1=-\infty}^{+\infty} \mathrm{e}^{iKn_1} \sum_{\{s\}=1}^{d} \boldsymbol{v}_{\boldsymbol{L}}^\dagger \left[\prod_{m<n_1} A^{s_m}\right] B_{(j_1)}^{s_{n_1}} \left[\prod_{n_1<m<n_1+n} A^{s_m}\right] B_{(j_2)}^{s_{n_1+n}} \left[\prod_{m>n_1+n} A^{s_m}\right] \boldsymbol{v}_{\boldsymbol{R}} \, |\{s\}\rangle$$

Vanderstraeten et al.,'15

- Example: scattering of magnons in the spin 1 Heisenberg model



	a_0	a_1	a_2
$D = 120$	1.94475	-4.51330	-2.35951
$D = 142$	1.94777	-4.51535	-2.30559
$D = 162$	1.94493	-4.51561	-2.30491
$D = 192$	1.94454	-4.51527	-2.30586
$D = 208$	1.94470	-4.50912	-2.30587
$D = 220$	1.94492	-4.51537	-2.30598
	1.945	-4.515	-2.306

Table I. Convergence of the scattering length for different values of the MPS bond dimension D.

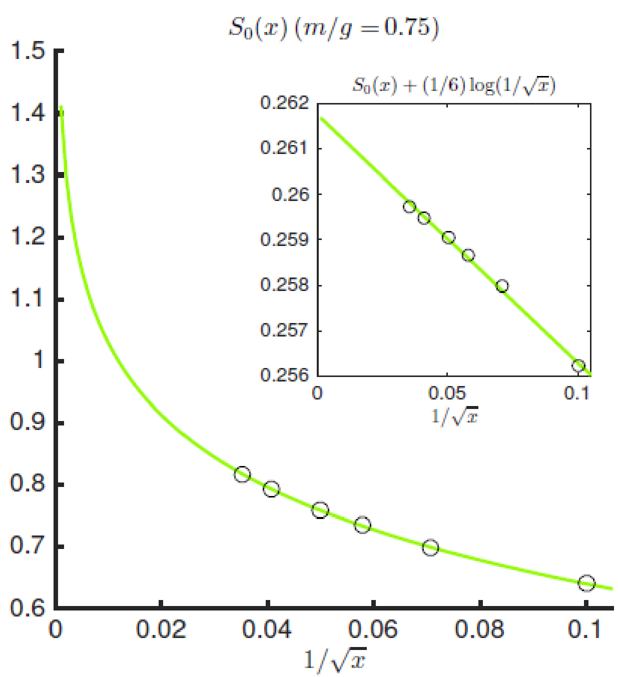
Tensor networks at work: Schwinger model

Buyens, Van Acoleyen, FV '13-'17

- Kogut-Susskind staggered formulation with $x = 1/g^2a^2$

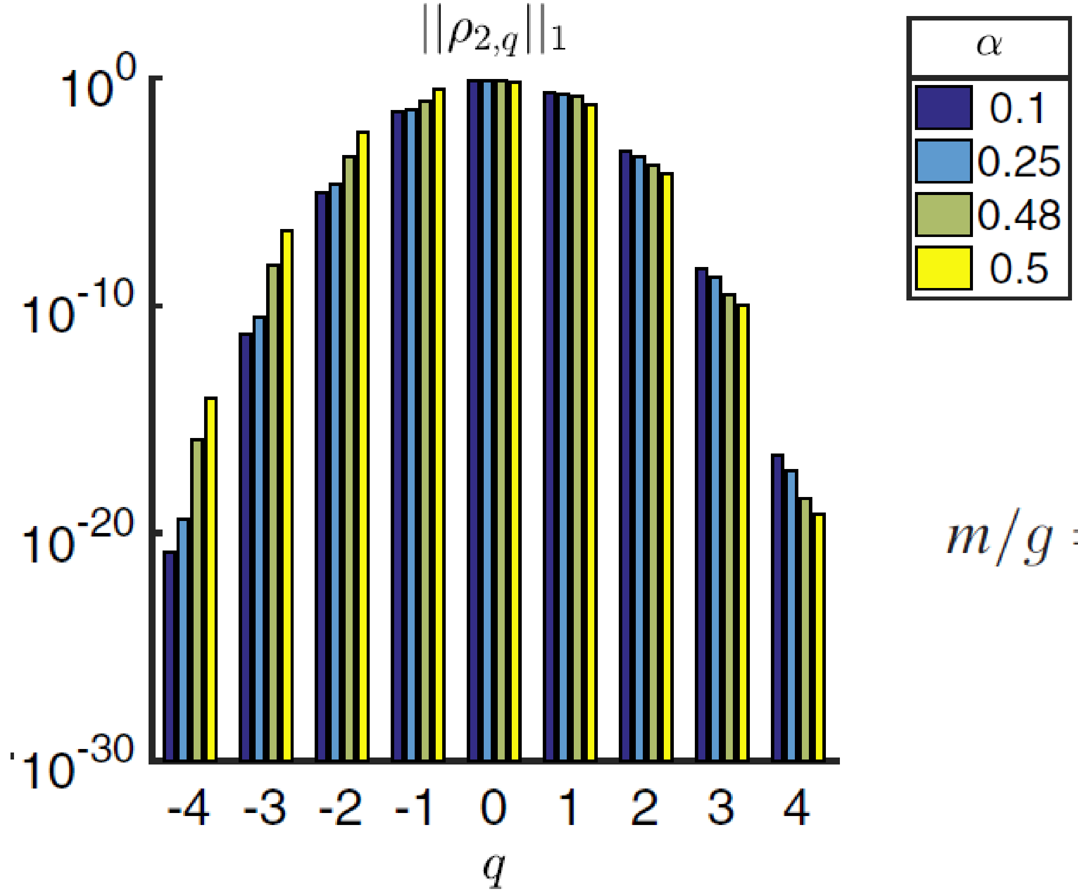
$$\mathcal{H} = \frac{g}{2\sqrt{x}}\left(\sum_{n=1}^{2N}[L(n)+\alpha(n)]^2 + \frac{\sqrt{x}}{g}m\sum_{n=1}^{2N}(-1)^n\left(\sigma_z(n)+(-1)^n\right) \quad + x\sum_{n=1}^{2N-1}\left(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1)+h.c.\right)\right)$$

- Entanglement spectrum in continuum limit



$$f_1(x) = A_0 + B_0 \log\left(\frac{1}{\sqrt{x}}\right) + C_0\frac{1}{\sqrt{x}}$$

- Cutting off the electrical field:

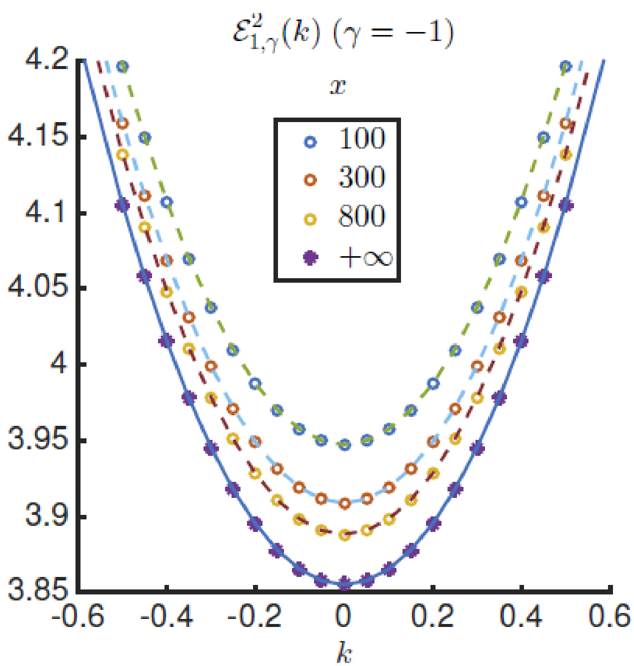


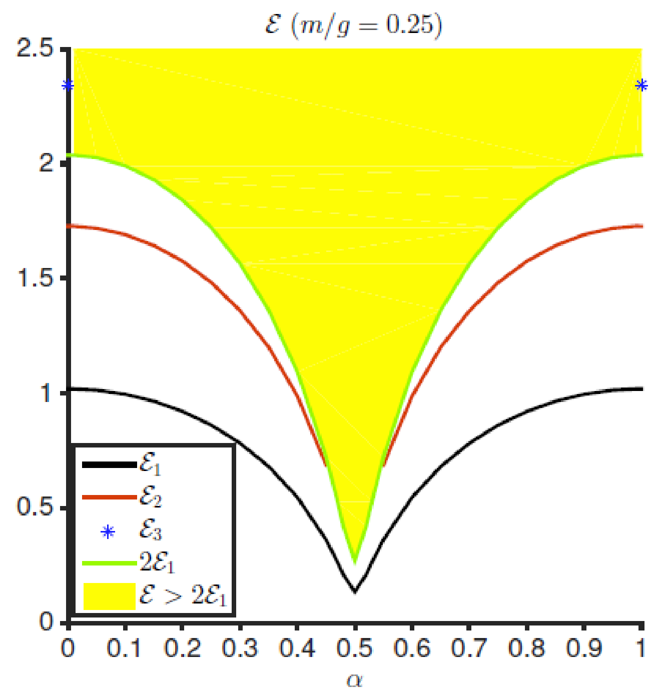
$$m/g = 0.3,\ x = 100$$

  – This justifies “qubit” approach to Schwinger model

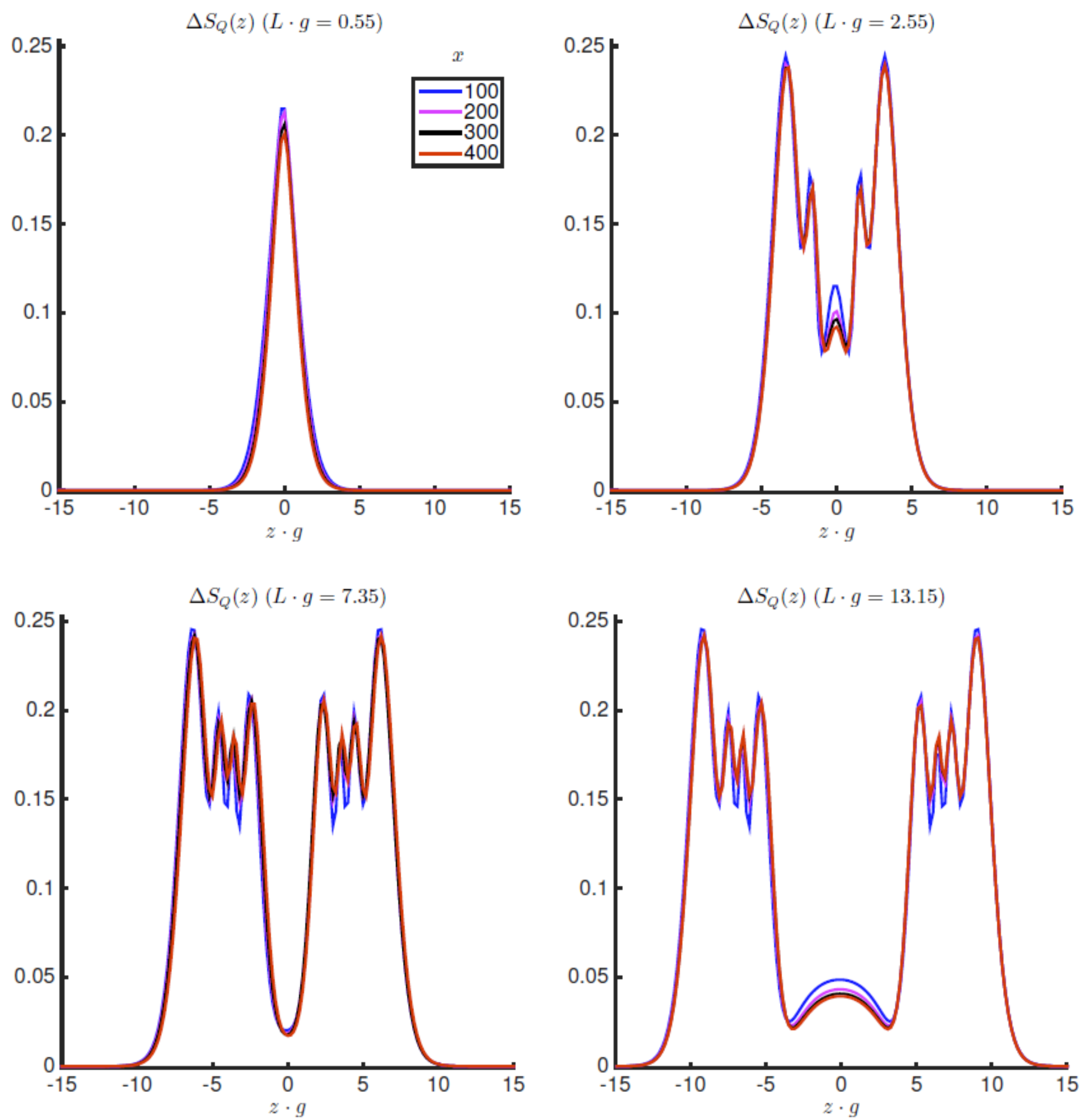
- Excitation spectrum: continuum limit

m/g	ω_0	$\mathcal{E}_{1,v}$	$\mathcal{E}_{1,s}$	$\mathcal{E}_{2,v}$
0	-0.318320(4)	0.56418(2)		
0.125	-0.318319(4)	0.789491 (8)	1.472 (4)	2.10 (2)
0.25	-0.318316(3)	1.01917 (2)	1.7282 (4)	2.339(3)
0.3	-0.318316(3)	1.11210 (8)	1.82547 (3)	2.4285 (3)
0.5	-0.318305(2)	1.487473 (7)	2.2004 (1)	2.778 (2)
0.75	-0.318285 (9)	1.96347 (3)	2.658943(6)	3.2043(2)
1	-0.31826 (2)	2.44441 (1)	3.1182 (1)	3.640(4)

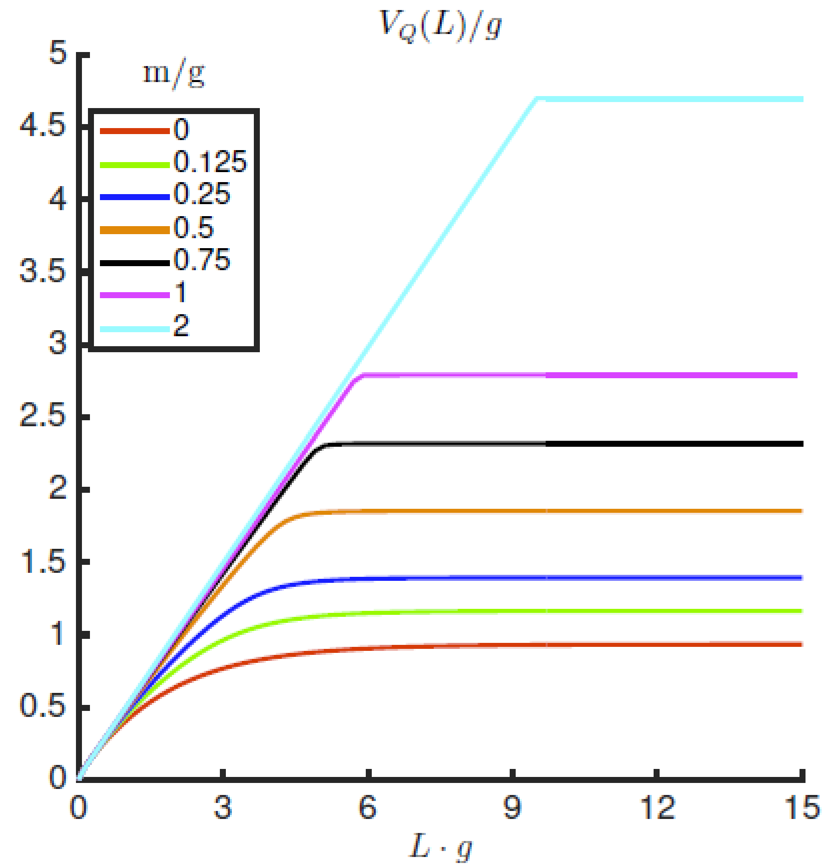




- Entanglement entropy in presence of test charges Q=4.5 as function of L



- Quark-antiquark potential for Q=1



Continuous MPS

$$|\chi_\epsilon\rangle = \sum_{i_1\cdots i_N} \mathrm{Tr}\left[A^{i_1}\cdots A^{i_N}\right]\left(\hat{\psi}_1^\dagger\right)^{i_1}\cdots\left(\hat{\psi}_N^\dagger\right)^{i_N}|\Omega\rangle$$

$$\begin{array}{rcl} A^0 & = & I + \epsilon Q \\ A^\alpha & = & \sqrt{\epsilon} R_\alpha \end{array} \qquad \{R_\alpha, R_\beta\}_\pm = 0 \qquad \hat{\psi}_i = \frac{\hat{a}_i}{\sqrt{\epsilon}}$$

- The scaling is chosen such as to guarantee a finite density
- We can explicitely take the limit epsilon -> 0:

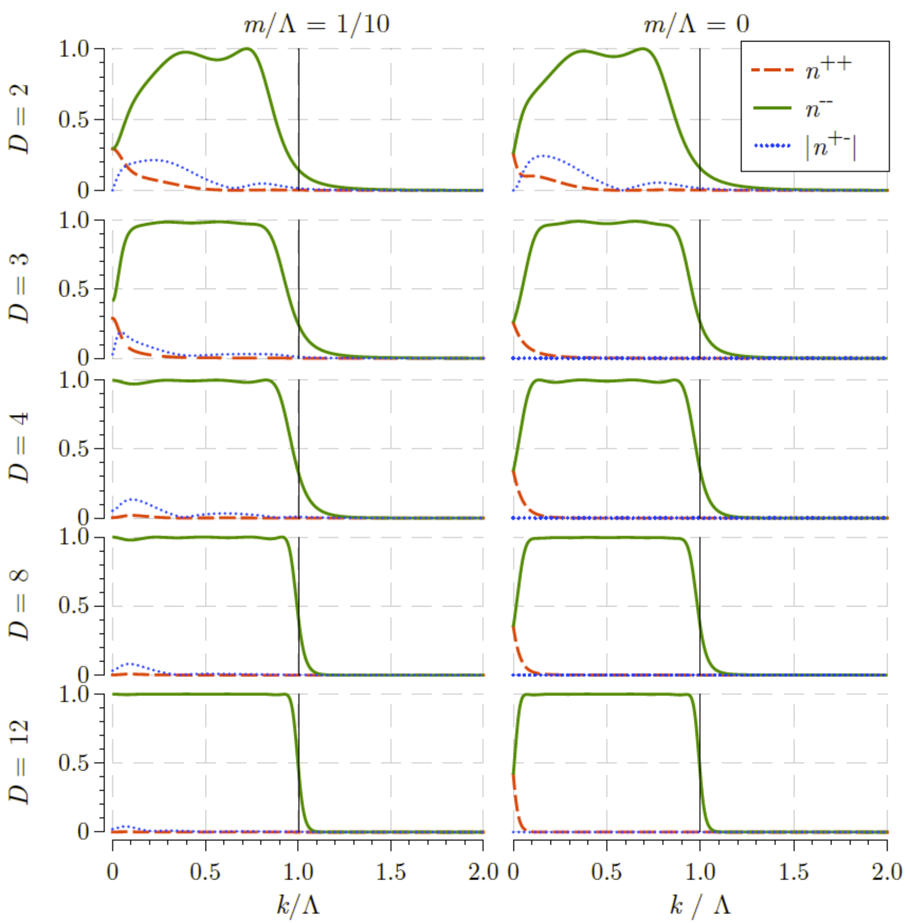
$$|\Psi[Q, R_1, \ldots, R_q]\rangle \triangleq \mathrm{tr}\left\{B \mathcal{P}\mathrm{exp}\left[\int_{-L/2}^{+L/2} dx\; Q(x) \otimes \hat{\mathbb{1}} + \sum_{\alpha=1}^{q} R_\alpha(x) \otimes \hat{\psi}_\alpha^\dagger(x)\right]\right\}|\Omega\rangle$$

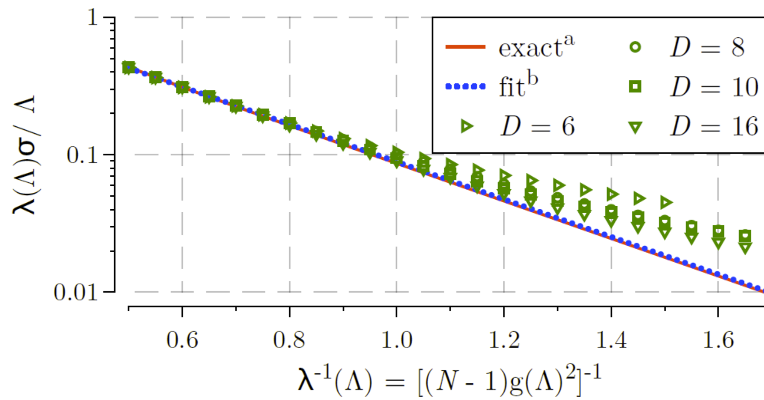
– The variational parameters are the matrix fields Q(x),R_α(x) and B

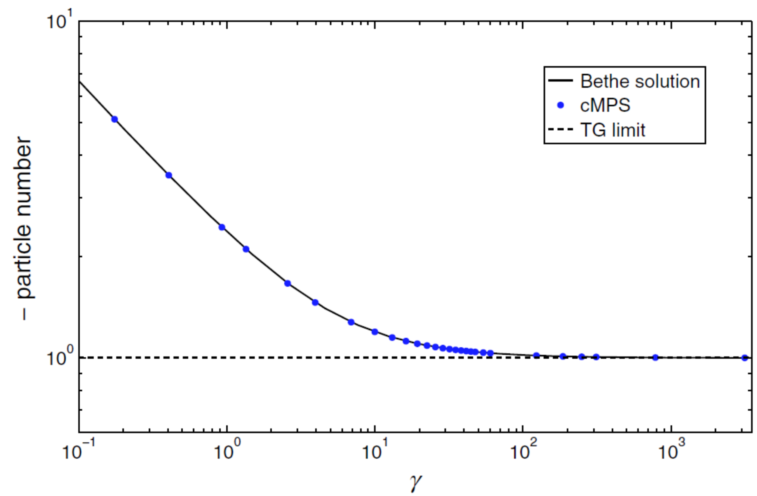
FV, Cirac '10; Haegeman, FV '11; …

Continuous MPS: variational methods in the continuum

$$|\Psi\rangle = \mathrm{Tr}_{\mathrm{aux}}\left[\mathscr{P} e^{\int_{-\infty}^{+\infty} \mathrm{d}x\, Q\otimes \mathbb{1} + \sum_\alpha R_\alpha \otimes \hat{\psi}_\alpha^\dagger(x)}\right]|\Omega\rangle$$







PEPS as variational ansatz

- PEPS is higher D version of MPS

  - Many problems in condensed matter remain unsolved: Hubbard, …
  - Big advantage over Monte Carlo: no sign problem
  - State of the art variational energies
  - Biggest asset of PEPS: once the optimal tensors are found, we can start doing physics with them (symmetries, entanglement spectrum, …)

- Order parameter for 2D Heisenberg antiferromagnet

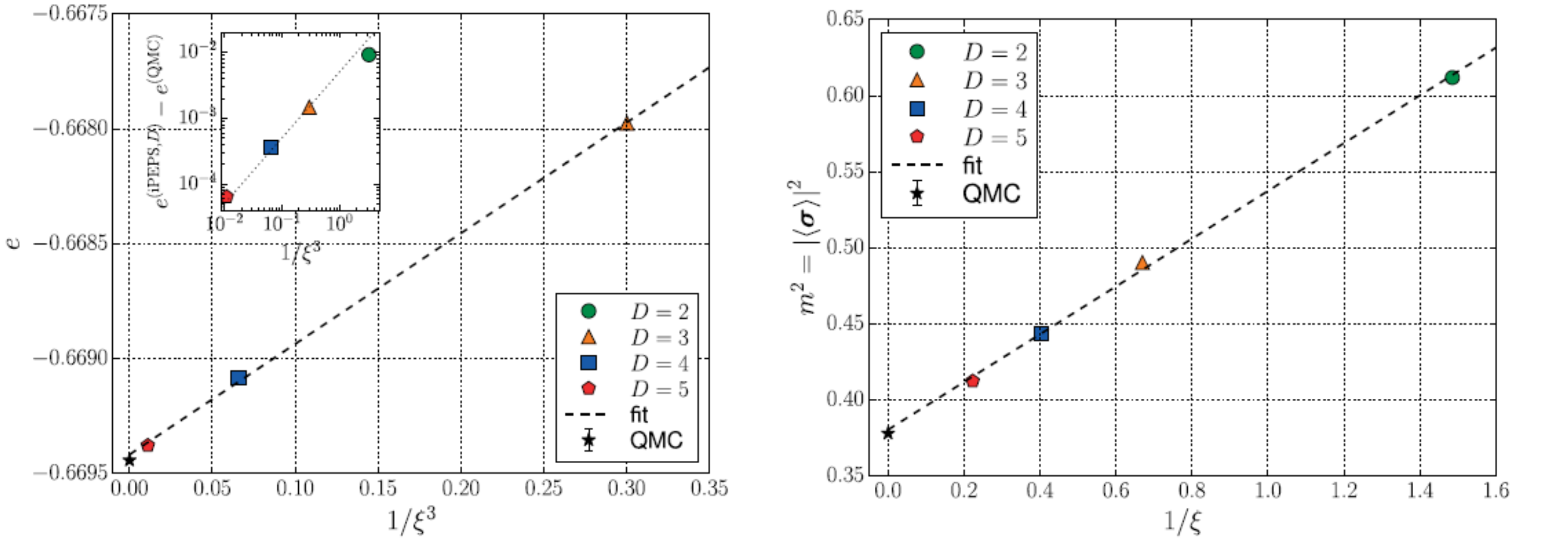


FIG. 6. $S = 1/2$ Heisenberg antiferromagnet: iPEPS data for the ground-state energy per site e (left-hand panel) and the order parameter squared m^2 (right-hand panel). We plot the data as a function of the expected $1/\xi^3$ (for e) and $1/\xi$ (for m^2) dependence. The linear fits to the $D = 3, 4, 5$ results in the left-hand and all $D \geq 2$ in the right-hand panel extrapolate closely to the high-precision quantum Monte Carlo reference results [56]. The inset in the left-hand panel highlights the overall $1/\xi^3$ convergence of the energy per site.

Magnetization of $SrCu_2(BO_3)_2$ in Ultrahigh Magnetic Fields up to 118 T

Y. H. Matsuda,[1,*] N. Abe,[1] S. Takeyama,[1] H. Kageyama,[2] P. Corboz,[3] A. Honecker,[4,5] S. R. Manmana,[4] G. R. Foltin,[6] K. P. Schmidt,[6] and F. Mila[7]

Shastry model: $$H = J' \sum_{\langle i,j \rangle} S_i S_j + J \sum_{\langle\langle i,j \rangle\rangle} S_i S_j - h \sum_i S_i^z$$

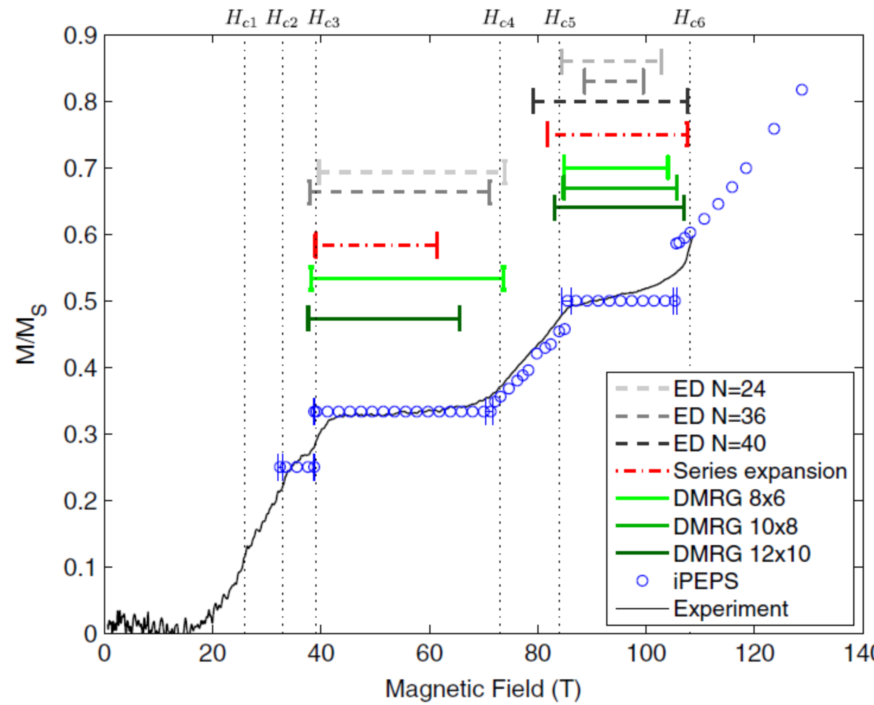


FIG. 4 (color online). Comparison between the experimental magnetization curve and the iPEPS simulation results for $J'/J = 0.63$. The extent of the 1/3 and 1/2 plateaus predicted by the other methods is shown on top of the plateaus.

PHYSICS

Stripe order in the underdoped region of the two-dimensional Hubbard model

Bo-Xiao Zheng,[1,2*†] Chia-Min Chung,[3*] Philippe Corboz,[4,5*] Georg Ehlers,[6*] Ming-Pu Qin,[7*] Reinhard M. Noack,[6] Hao Shi,[7*] Steven R. White,[3] Shiwei Zhang,[7] Garnet Kin-Lic Chan[1]†

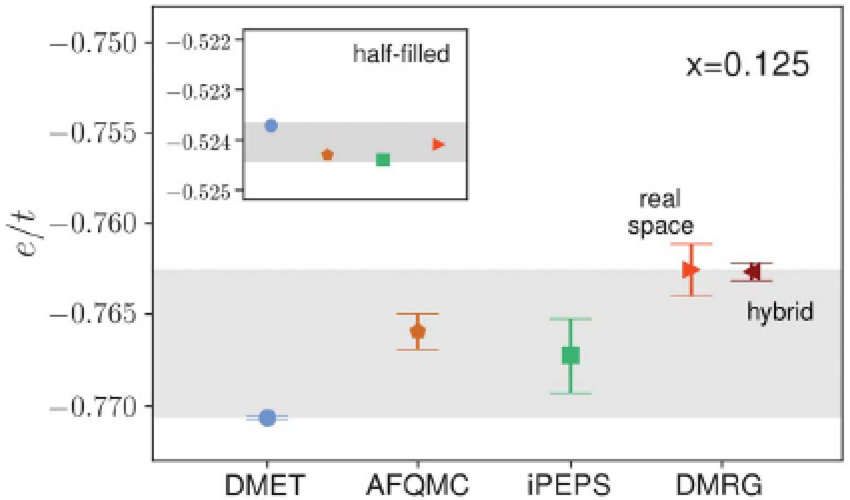


Fig. 1. Ground-state energies. Best estimates of ground-state energy for the 1/8-doped 2D Hubbard model at $U/t = 8$ from DMET, AFQMC, iPEPS, and DMRG in units of t. Inset shows best estimates of ground-state energy for the half-filled 2D Hubbard model at $U/t = 8$. Here and elsewhere, error bars indicate only the estimable numerical errors of each method; uncontrolled systematic errors are not included. For details, see (*30*).

Simulating excitation spectra with projected entangled-pair states

Laurens Vanderstraeten,[1,*] Jutho Haegeman,[1] and Frank Verstraete[1,2]

2D Heisenberg antiferromagnet : spin wave velocity

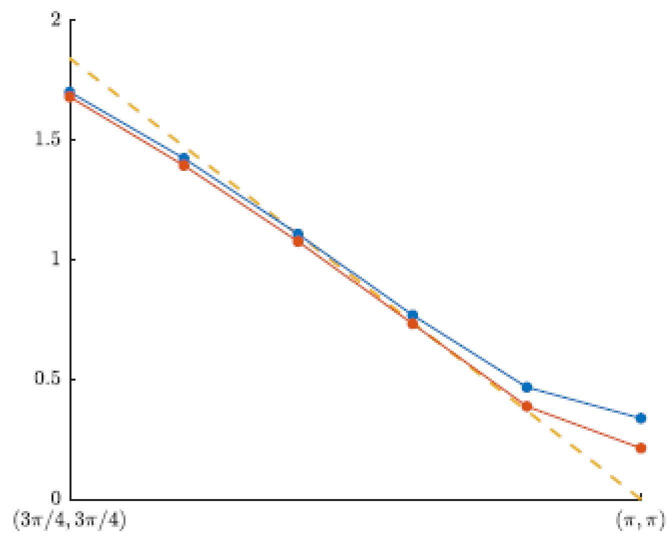
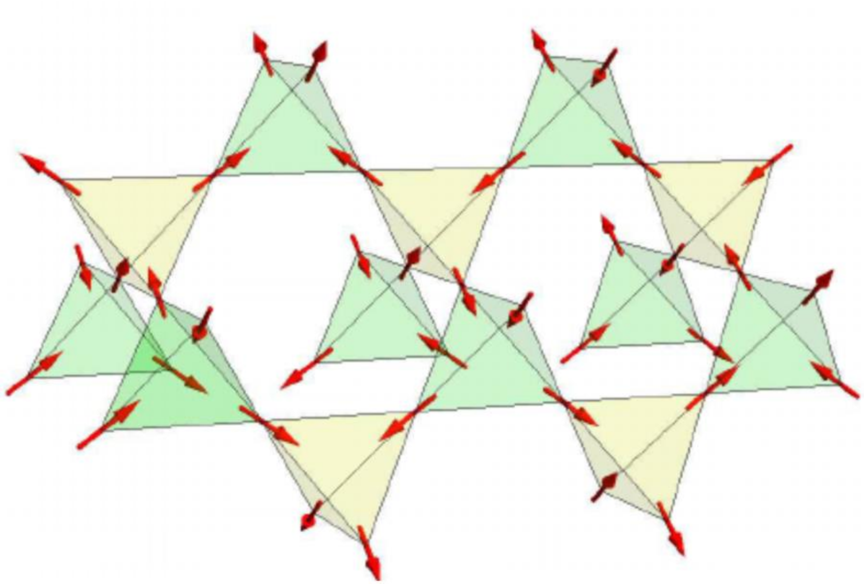


FIG. 3. The dispersion relation of the Heisenberg model approaching the gapless point $(\pi, 0)$ with bond dimension $D = 3$ (blue) and $D = 4$ (red) and environment bond dimension up to $\chi = 100$, as compared to the linear dispersion relation with $v_s \approx 1.65847$[32] (yellow). Clearly, the finite bond dimension of the PEPS induces an artificial gap, which grows smaller as D increases. If we estimate the spin-wave velocity as the slope at the inflection point in the $D = 4$ curve, we find $v_s \approx 1.638$.

Residual entropies for three-dimensional frustrated spin systems with tensor networks

Laurens Vanderstraeten,[1,*] Bram Vanhecke,[1] and Frank Verstraete[1,2]

- We can equally well use PEPS to calculate free energies for 3D classical statistical mechanical models
- Example: residual entropy of ice



Pauling[13]	mean field	1.5
Nagle[15]	series expansion	1.50685(15)
Berg et.al.[19]	multicanonical	1.507117(35)
Herrero et.al.[16]	num. integration	1.50786(12)
Kolafa[17]	num. integration	1.5074660(36)
PEPS	$D = 2$	1.50735
	$D = 3$	1.507451
	$D = 4$	1.507456

Table II. The residual entropies for ice I_h as computed from a mean-field approach, series expansion, multicanonical Monte Carlo and numerical integration using Monte Carlo, compared to our variational PEPS results.

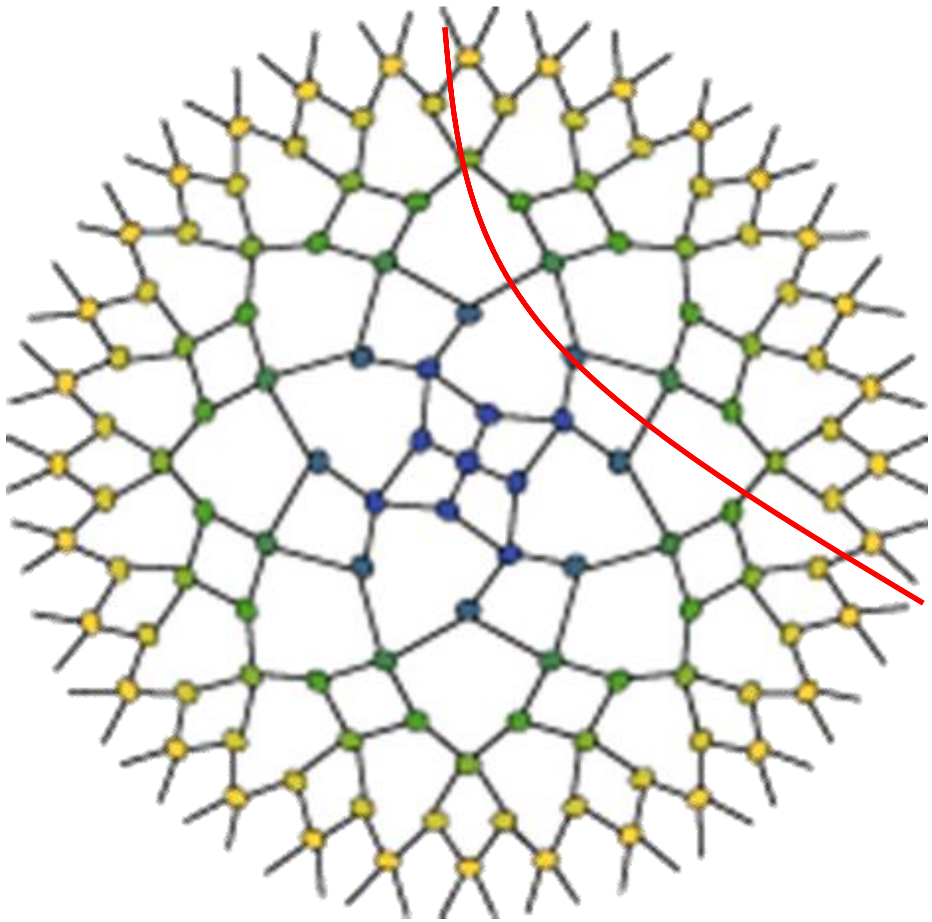
Vanderstraeten et al, ‘18

Tensor networks: theoretical aspects

- Holographic property:
  - All the interesting physics of quantum many body ground states can be described by looking at “effective” theories arising on the virtual degrees of freedom of the MPS/PEPS/…
  - Physics of virtual degrees of freedom describe entanglement Hamiltonians in a lower dimensional space: dimensional reduction

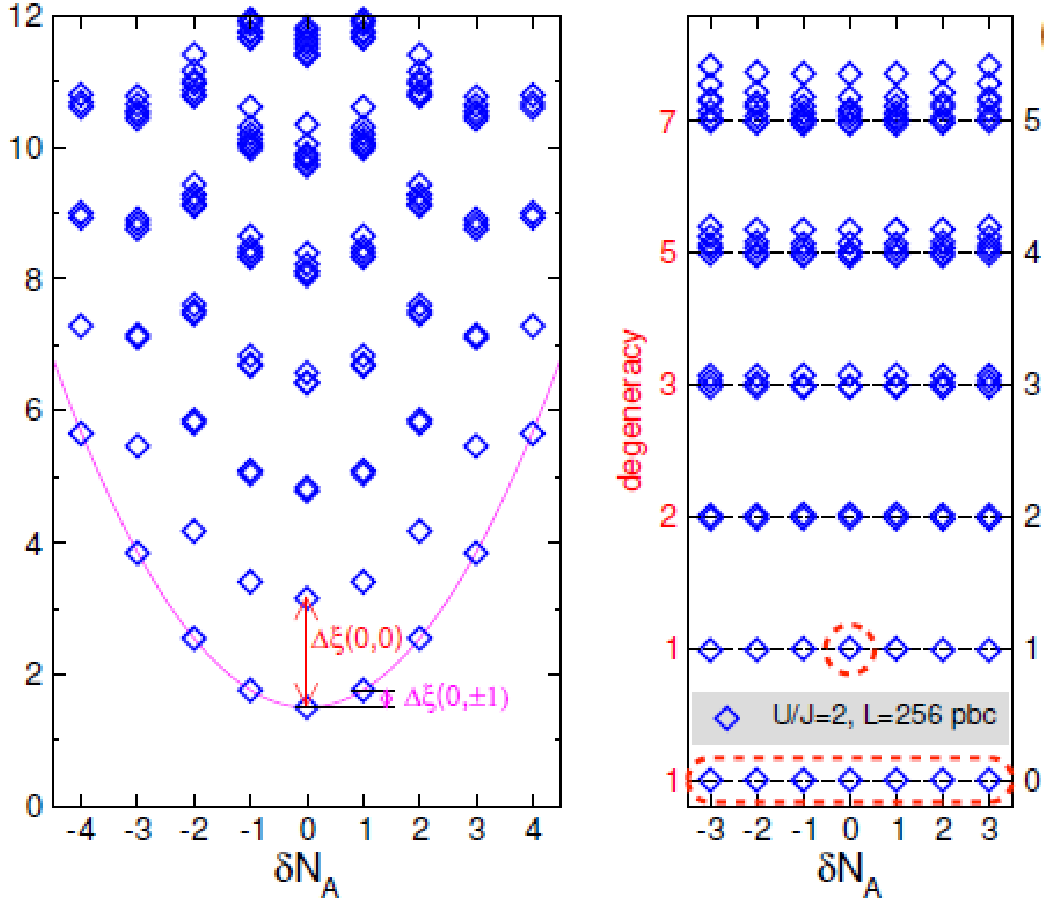
AdS/CFT correspondence and Ryu-Takayanagi conjecture

- Connection between geometry and entanglement entropy: minimal cut



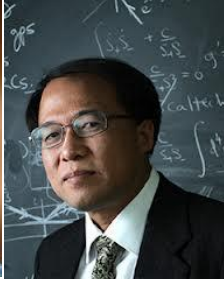
Entanglement Spectroscopy

- CFT content from the Schmidt eigenvalues of a Bose-Hubbard spin chain



Lauchli '13

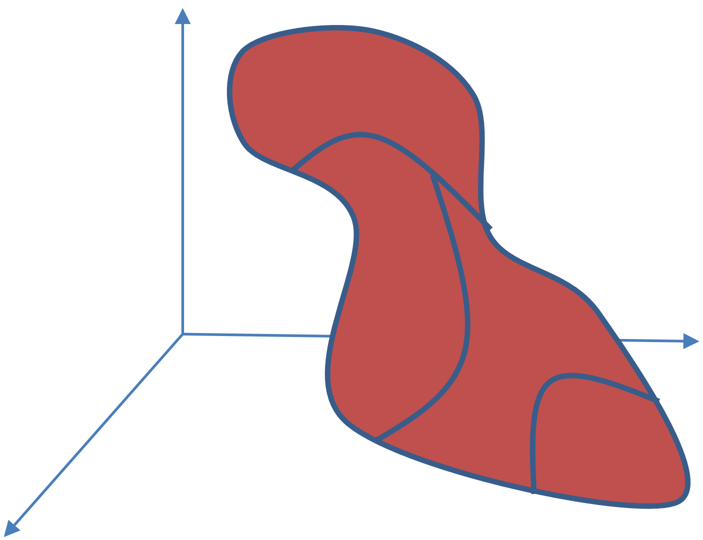
Topological phases of Matter

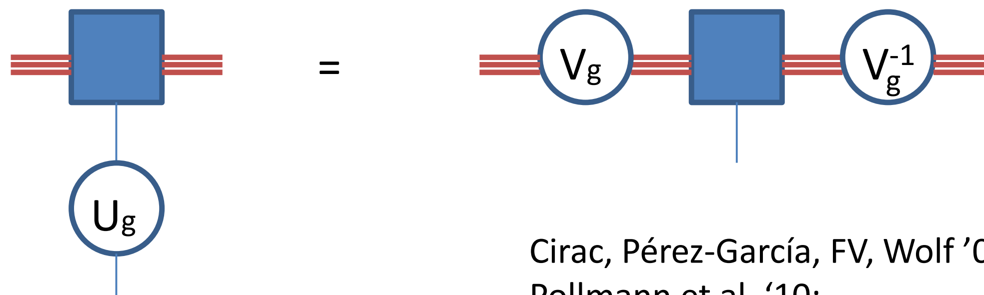



- Last decades has seen a revolution understanding topological phases of matter
  - Realization in Quantum Hall systems, observation of Majorana fermions, …
  - Topological phases of matter: there is no LOCAL order parameter distinguishing topological phases from trivial ones
    - Phase is characterized by <u>*long range entanglement*</u>
    - This entanglement can be used to built a fault-tolerant quantum computer
- Tensor network approach: topological order is all about symmetries of the entanglement degrees of freedom
  - Landau paradigm of order parameters is recovered in symmetries of LOCAL tensors

1D interacting SPT phases of matter: MPS

- Classification of phases of matter of 1-D spin chains under adiabatic paths preserving a symmetry: manifold breaks into pieces (symmetry protected topological order)



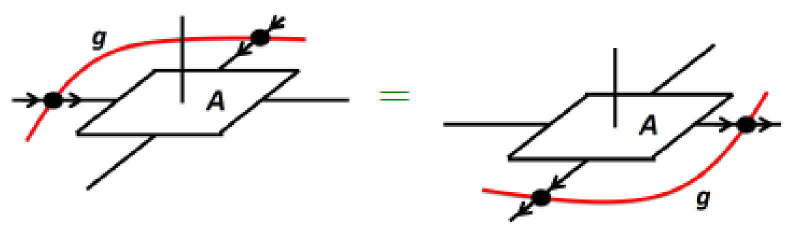


Cirac, Pérez-García, FV, Wolf '08
Pollmann et al. '10;
Chen, Wen '11
Cirac, Pérez-García, Schuch '11

  - Different phases are characterized by projective representations of physical symmetry group ($H^2(G,U(1))$)
  - In case of fermions: graded tensor algebras, and already topological phases without imposing symmetries (Majorana / Kitaev spin chain)

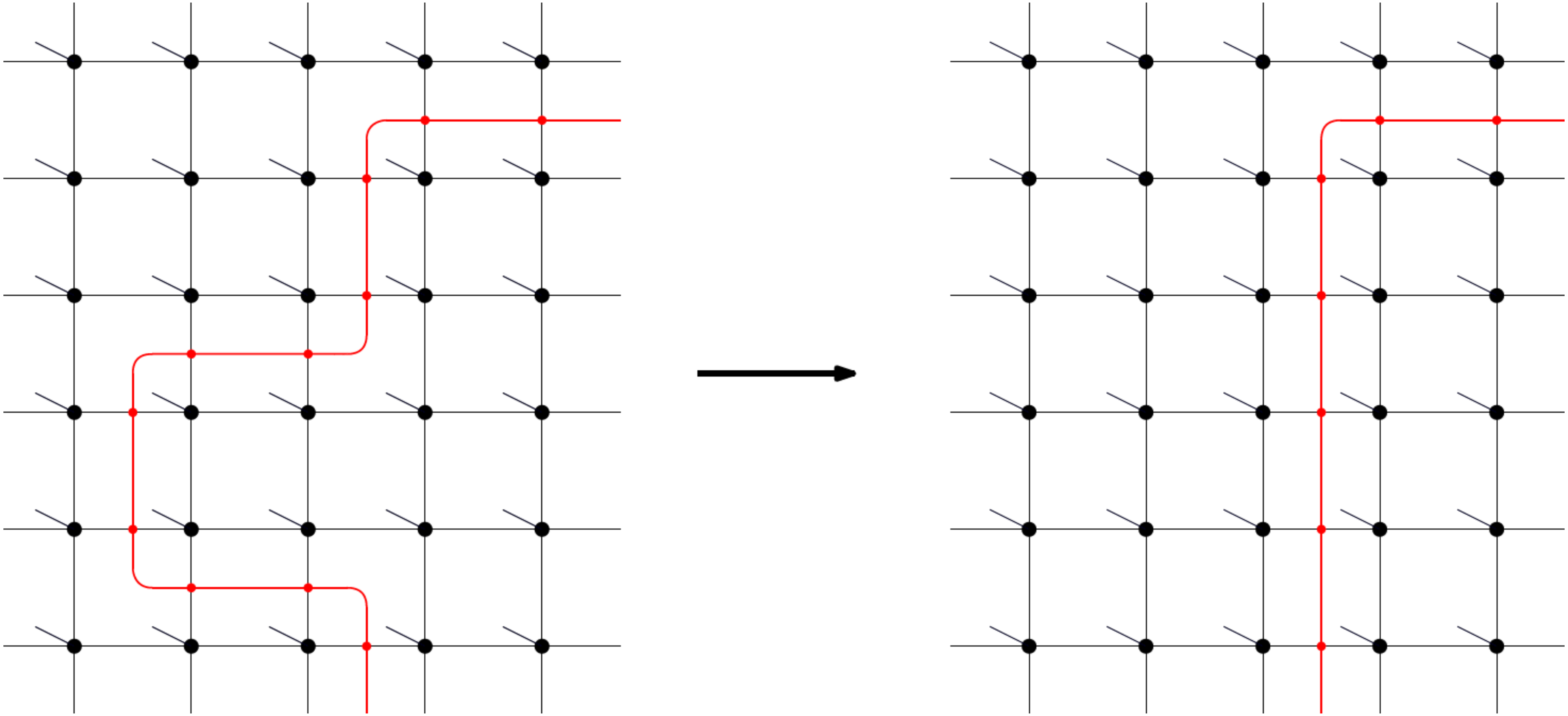
Symmetries in PEPS

- Symmetries and topological order is much richer in 2 dimensions: existence of anyons, Wilson loop operators, …
  - 2 dimensions is where the most surprising things can happen: 2 is low enough to have a lot of entanglement (3 dimensions is already much closer to mean field theory), but 2 is large enough to have nontrivial statistics (e.g. fractional quantum Hall effect)
  - All those exotic materials exhibit a special entanglement structure which is locally reflected in symmetries of the microscopic tensors

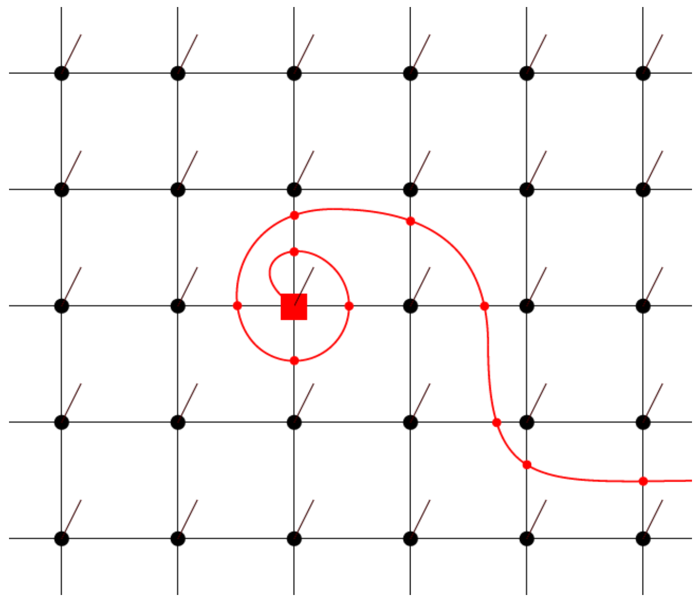


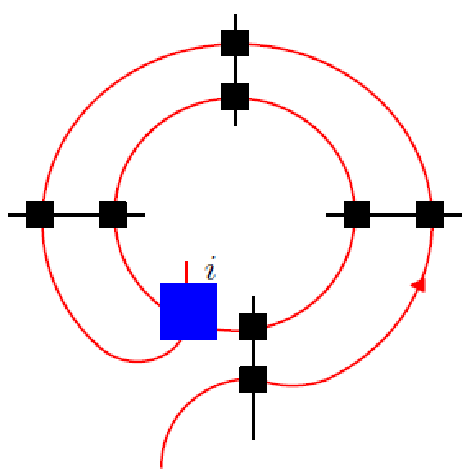
  - Probing entanglement reveals nonlocal order parameters: Landau symmetry breaking, but now on the entanglement degrees of freedom

- Those symmetries give rise to Wilson loops that can be pulled through the tensor network:

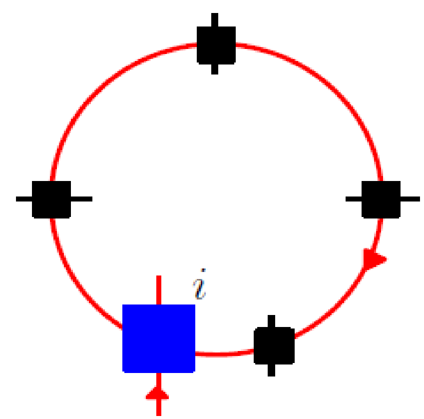


- Elementary excitations (anyons) in the system consist of end points of those strings: those necessarily come in pairs (cfr. Fermions: simplest type of anyon)

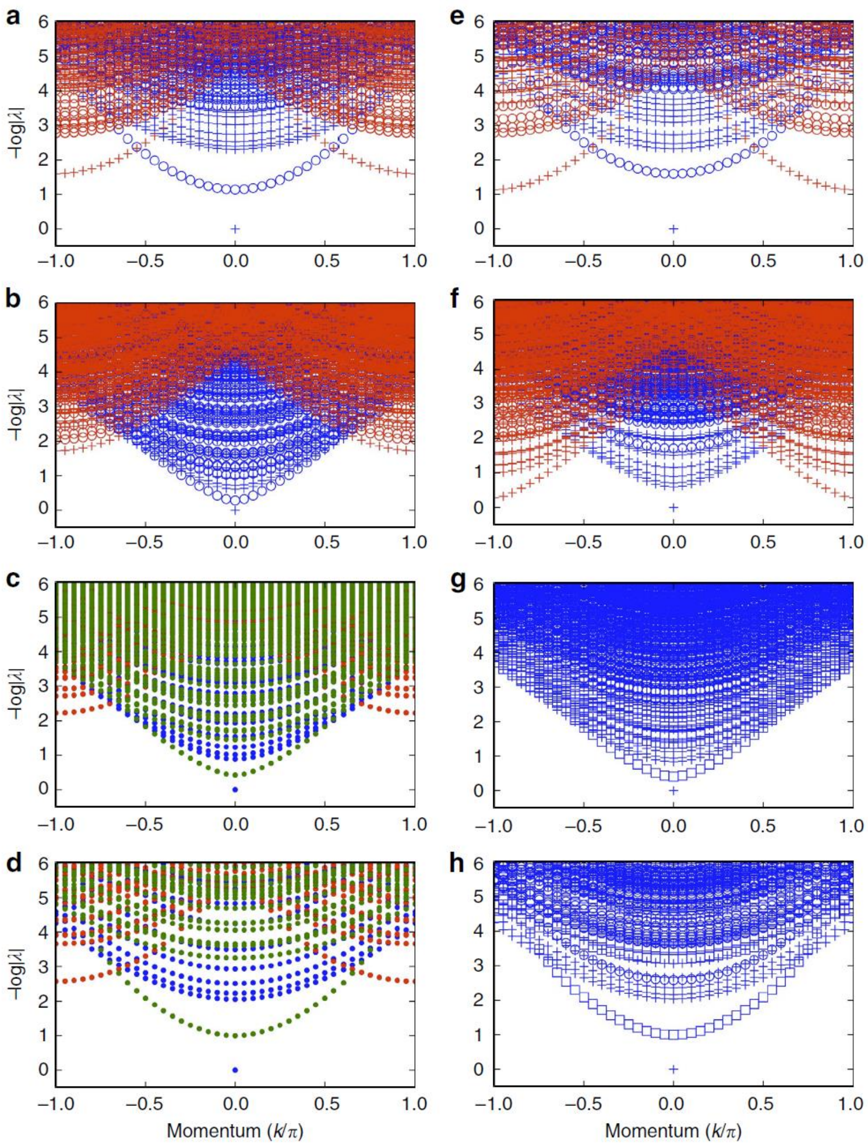




$$= e^{i\phi}$$



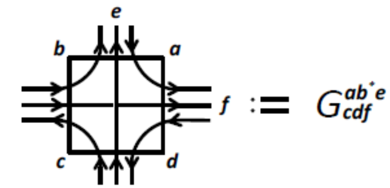
- Entanglement spectrum and confinement/deconfinement phase transition by anyon condensation in a Z2 gauge theory (Shenker/Fradkin == toric code with string tension)

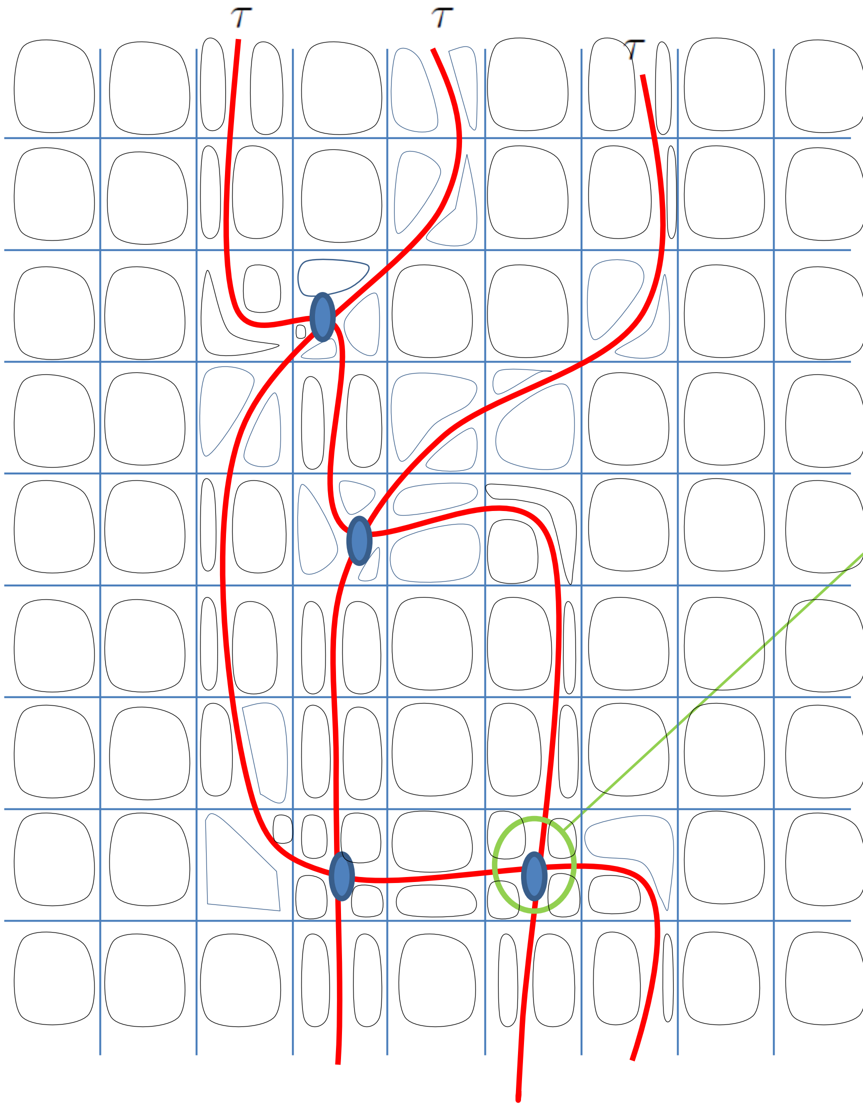


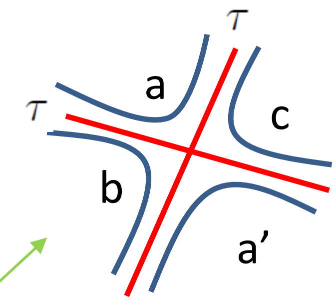
Haegeman, Zauner, Schuch, FV '15

Topological Quantum Computation in the shadow world

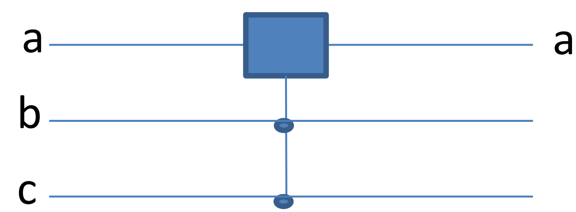
We can identify a tensor product structure of logical qubits with the entanglement (virtual) degrees of freedom; e.g. Fibonacci string net







Braiding tensor is F-symbol



Controlled-Controlled-U gate

$$\begin{pmatrix} \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

Freedman, Kitaev, Wang, …

From TQFT to CFT

- There is a very rich mathematical structure underlying the algebra of those symmetry operators: they form a representation of so-called tensor fusion categories.

- Those tensor fusion categories form the mathematical foundation of both topological quantum field theory and of conformal field theory

  – From the point of view of tensor networks, TQFT = CFT : same tensor network, just a different interpretation!

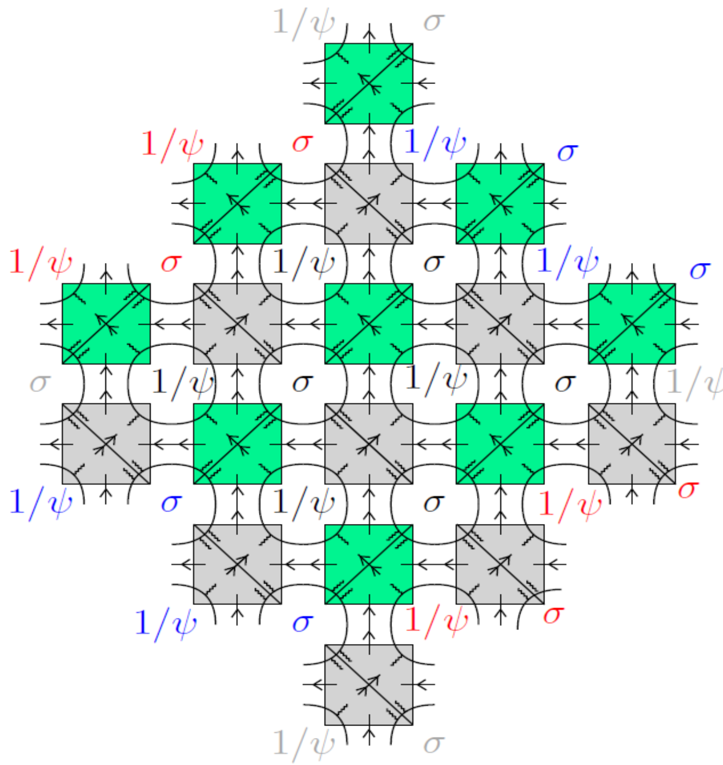
  – Having nontrivial MPO order implies critical and/or topological

  – Provides a new perspective on modular transformations, boundary conformal field theory, …: it's all about symmetries in the entanglement degrees of freedom

Vanhove et. al., PRL '18

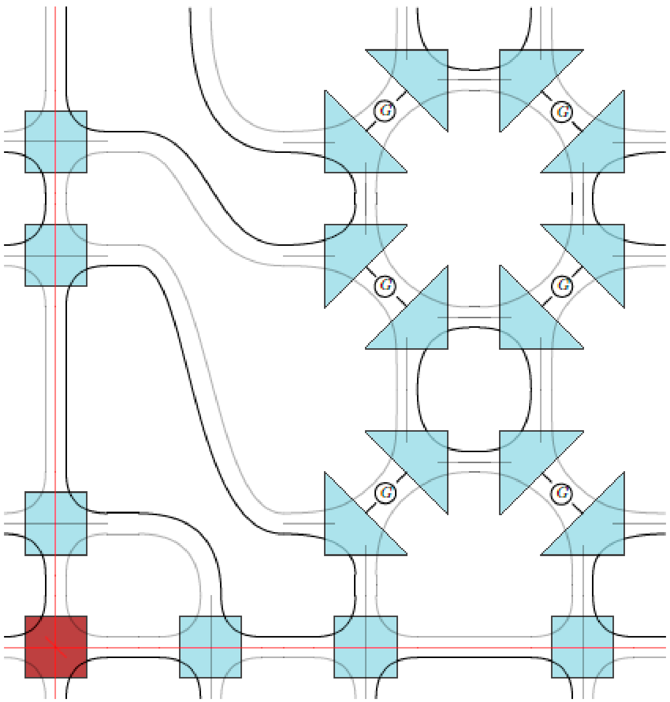
Classical Ising model from SET

- Partition function of classical Ising can exactly be written as a strange correlator of the Ising SET, including all defects (duality defects)
  - The product state by the requirement of the Z_2 symmetry



  - By including MPOs, we can realize all 9 topological/conformal sectors
  - All information about scaling exponents, primary fields, etc. is encoded algebraically in the MPO tube-algebras

$$\sigma \times \sigma = 1 + \psi$$



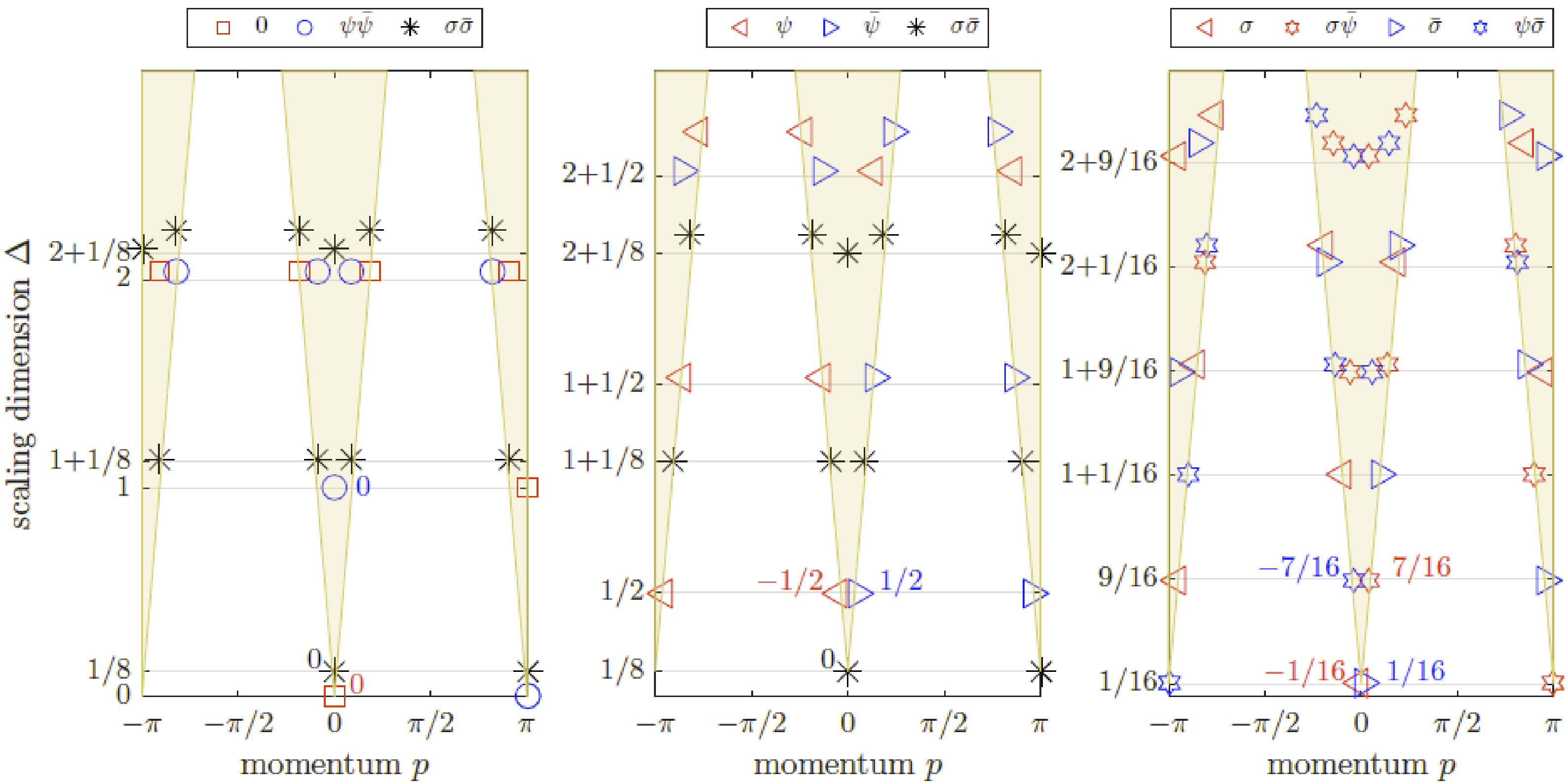
	$\mathcal{T}^{1}_{11}$	$\mathcal{T}^{\psi}_{1\psi}$	$\mathcal{T}^{\psi}_{\psi 1}$	$\mathcal{T}^{1}_{\psi\psi}$	$\mathcal{T}^{\sigma}_{\sigma 1}$	$\mathcal{T}^{\sigma}_{\sigma\psi}$	$\mathcal{T}^{\sigma}_{1\sigma}$	$\mathcal{T}^{\sigma}_{\psi\sigma}$	$\mathcal{T}^{1}_{\sigma\sigma}$	$\mathcal{T}^{\psi}_{\sigma\sigma}$
$\chi_1\chi_1^*$	1	1					$\sqrt{2}$			
$\chi_\psi\chi_\psi^*$	1	1					$-\sqrt{2}$			
$\chi_\psi\chi_1^*$			1	-1				$-i\sqrt{2}$		
$\chi_1\chi_\psi^*$			1	-1				$i\sqrt{2}$		
$(\chi_\sigma\chi_\sigma^*)_{00}$	2	-2								
$(\chi_\sigma\chi_\sigma^*)_{11}$			2	2						
$\chi_\sigma\chi_1^*$					1	$-i$			$e^{-i\pi/8}$	$e^{3i\pi/8}$
$\chi_\sigma\chi_\psi^*$					1	$-i$			$e^{7i\pi/8}$	$e^{-5i\pi/8}$
$\chi_1\chi_\sigma^*$					1	i			$e^{i\pi/8}$	$e^{-3i\pi/8}$
$\chi_\psi\chi_\sigma^*$					1	i			$e^{-7i\pi/8}$	$e^{5i\pi/8}$

$$Z^{1}_{11} = \chi_1\chi_1^* + \chi_\psi\chi_\psi^* + \chi_\sigma\chi_\sigma^*$$

$$Z^{\psi}_{\psi 1} = \chi_\sigma\chi_\sigma^* + \chi_\psi\chi_1^* + \chi_1\chi_\psi^*$$

$$Z^{\sigma}_{\sigma 1} = \chi_\sigma\chi_1^* + \chi_1\chi_\sigma^* + \chi_\sigma\chi_\psi^* + \chi_\psi\chi_\sigma^*$$

$$[F^{\sigma\sigma\sigma}_{\sigma}]_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [F^{\sigma\psi\sigma}_{\psi}]^{\sigma}_{\sigma} = [F^{\psi\sigma\psi}_{\sigma}]^{\sigma}_{\sigma} = -1$$



- All characters of the CFT can now be expressed as a linear combination of the ones of the idempotents; Dehn twists etc give rise to transformation properties of those charactrers (Verlinde S matrix, T-matrix, …)

Entanglement Matters and Tensor Networks

Quantum Computation

Projected Entangled Pair States

Multiscale Entanglement Renormalization Ansatz

Matrix Product States

Lieb-Robinson bounds

Entanglement

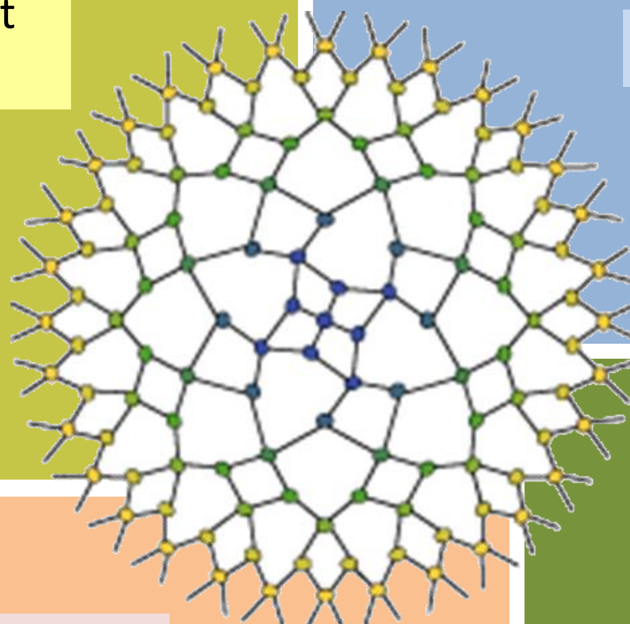
Lattice Gauge Theories

Anyon Condensation

Holographic Principle

Quantum Topological Order

Renormalization Group



Quantum Quenches

Cold Atomic Gases

Quasi-Particles

Quantum Phase Transitions

Non-Commutative Gross-Pitaevskii

Fractional Quantum Hall

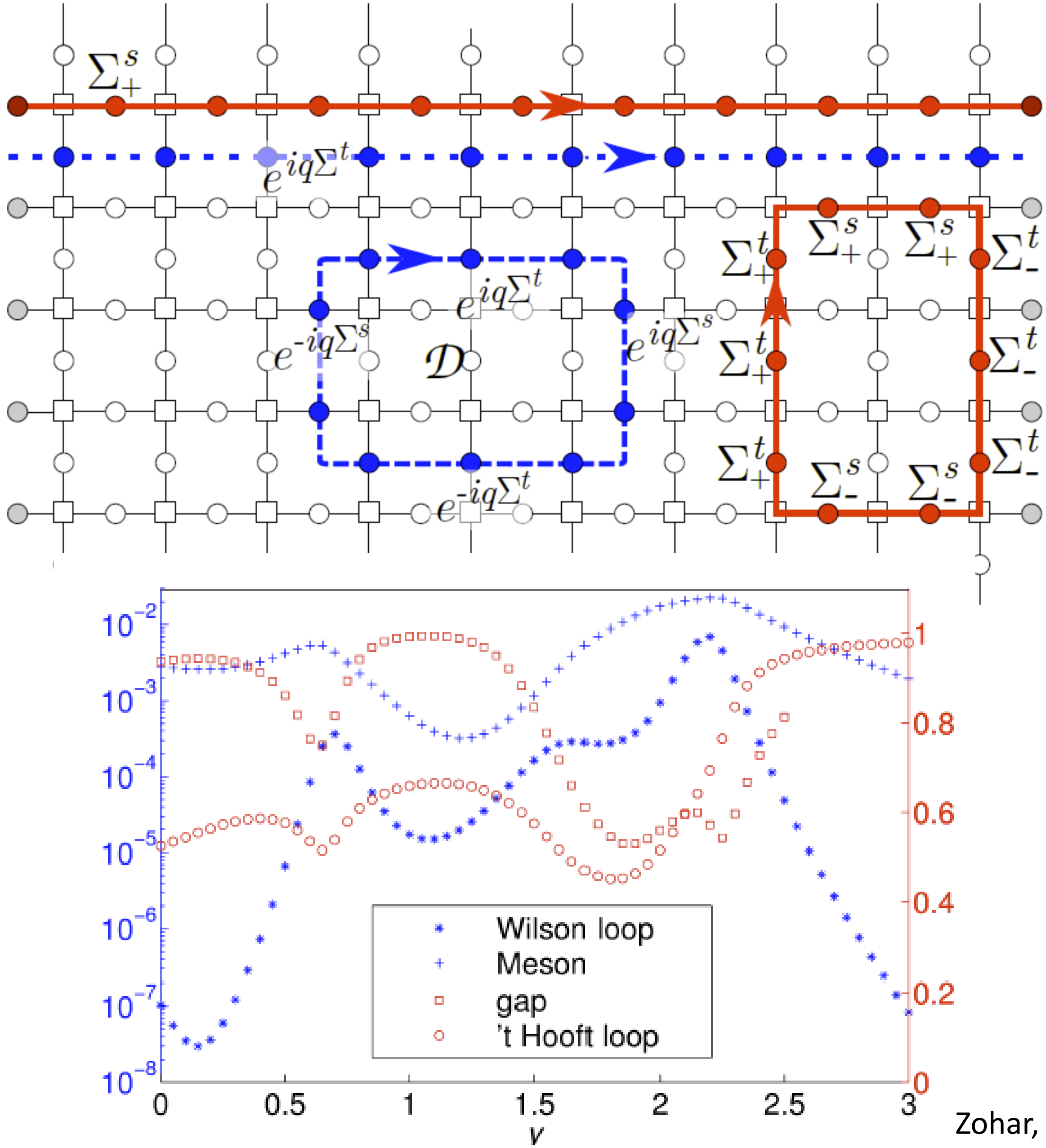
Bosonic SPT phases

Hubbard Model

(Virtual) Order Parameter

Quantum Spin Liquids

QED in 2+1 D



Zohar, Burrello, Wahl, Cirac '16