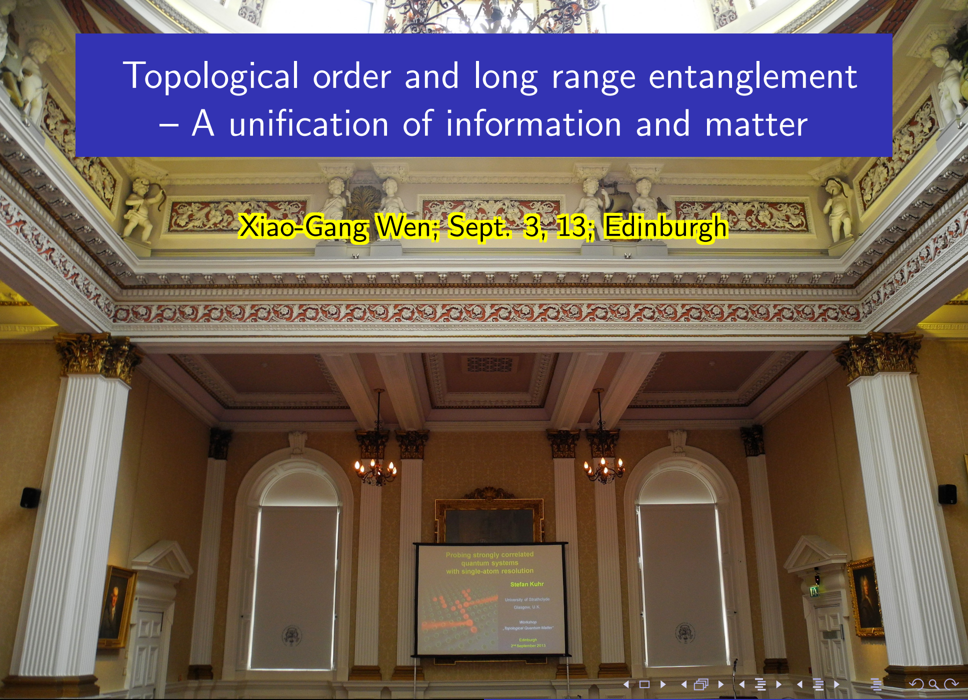
Topological order and long range entanglement – A unification of information and matter

Xiao-Gang Wen; Sept. 3, 13; Edinburgh



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After the discovery of FQH effect and high Tc superconductivity 30 years ago, the study of topological quantum matter slowly became a very active field.

Such a study in condensed matter may also has an impact on elementary particle physics.

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Such a study in condensed matter may also has an impact on elementary particle physics.

Matters (elementary particles) are described by standard model which contain vector gauge fields and Grassman spinor fields

Can those fields actually come from qubits?

Can we use qubit model to simulate the standard model?

Information $=?=$ **Matter**

The essence of quantum theory

- Quantum theory $\rightarrow$ a unification between **matter** and **information**
  Information: Changing information (qubits) $\rightarrow$ frequency
  According to quantum physics: frequency $\rightarrow$ energy
  According relativity: energy $\rightarrow$ mass $\rightarrow$ **Matter**

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- Quantum theory → a unification between **matter** and **information**
  Information: Changing information (qubits) → frequency
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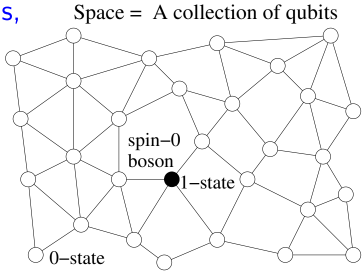
  According relativity: energy → mass → **Matter**
- But can simple qubits (quantum information) really produce all kinds of matter (and all the elementary particles)?
- If matter was formed by spin-0 bosons, then qubits could produce all kinds of matter and all spin-0 bosons: The space is formed by a collection of qubits and the 0-state of a qubit represents the vacuum. Then the 1-state of a qubit represents a spin-0 boson in space.



- *Ground state of the space-forming qubits = vacuum*
  Collective excitations above the ground state = elementary particles

Can space-forming qubits unifies all elementary particles?

- But our elementary particles have very strange properties:
  1. Identical particles
  2. Spin-1 bosons with only two-components $\to$ gauge bosons
  3. Fractional angular momentum (spin-1/2)
  4. Fermi statistics
  5. Only right-hand fermions couple the $SU(2)$-gauge-bosons
  6. Lorentz symmetry
  7. Spin-2 bosons with only two-components $\to$ gravitons?

  Can simple qubits produce the above strange properties?

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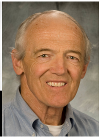
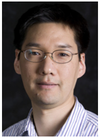
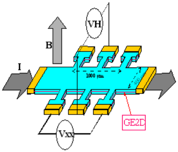
  Can simple qubits produce the above strange properties?
- **Unification**: The above unrelated phenomena (1-6) have *a unified origin*: **long-range entanglement** of the space-forming qubits
- **New world view**: Elementary particles and matter come from the fluctuations of **long range entangled** qubits that form space.
- **New math:** Many-qubit entanglement can be extremely rich and complex. It can have **long range entanglement**. It requires new math like tensor category, group cohomology, ... to describe.

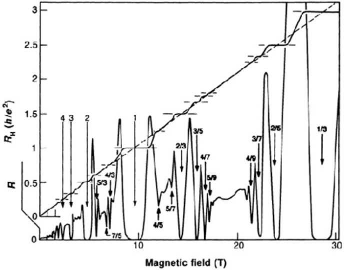
The experimental discovery of long-range entanglement

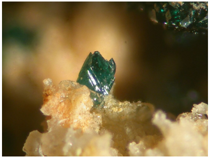
- We used to believe that **all phases and phase transitions are described by symmetry breaking**
- Counter examples:
  - Quantum Hall states $\sigma_{xy} = \frac{m}{n}\frac{e^2}{h}$
  - Spin liquid states









- FQH states and spin-liquid states have different phases with no symmetry breaking, no crystal order, no spin order, ... so they must have a new order – **topological order**

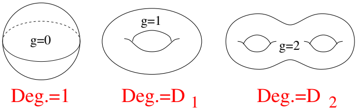
What is topological order: a def. based on physical probe

- In 1989 when the concept of topological order was first introduced, I only understood topological order through its macroscopic topological properties. Wen 89

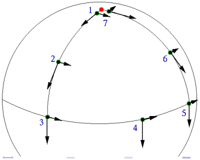
  We conjectured that topological order can be completely defined by using only two topological properties (at least in 2D):

  (1) **Topology-dependent ground state degeneracy** D_g Wen 89

  that is robust against any impurities!



  (2) **Non-Abelian Berry's phases** of the degenerate ground state from deforming the torus: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$ Wen 90



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Real experimental probe of topological orders: No finite-temperature phase transition (why it is a probe?)

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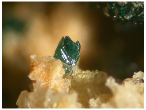
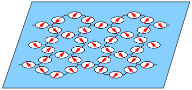
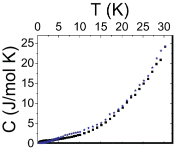
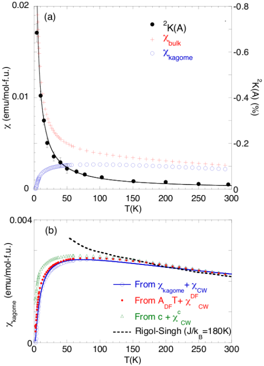
Herbertsmithsite: spin-1/2 on Kagome lattice $H = J \sum \mathbf{S}_i \cdot \mathbf{S}_j$.



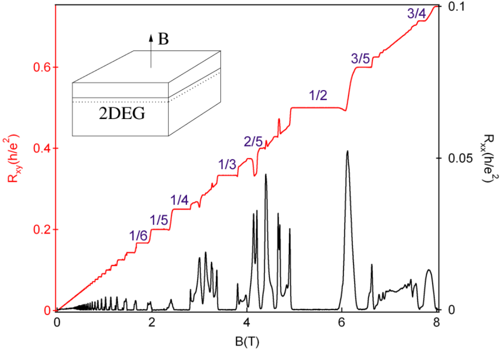


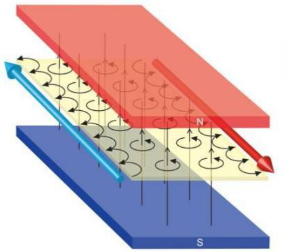




- $J \sim 200K$, no phase trans. down to $50mK$ $\to$ spin liquid Helton etal 06
- Numerical calculations Misguich-Bernu-Lhuillier-Waldtmann 98; Jiang-Weng-Sheng 08; Yan-Huse-White 10 $\to$ Z_2 topological order Read-Sachdev 91, Wen 91

Real experimental probe of topological orders: Robust gapless boundary excitations against any perturbations

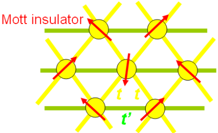


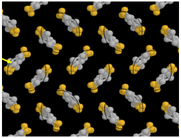


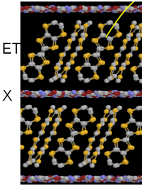
- Chiral edge transport Halperin 85 for IQH, Wen 89 for FQH $\rightarrow$ perfect conductor
- Different bulk topological orders $\rightarrow$ different edge states Wen 89
- Abelian FQH states with $\nu = 1/3, 2/3, 4/3, 5/3, \ldots$ have an integral number of edge branches. Non-Abelian FQH states with $\nu = 5/2, \ldots$ have a **factional** number of edge branches

Real experimental probe of topological orders: fractional charge/quantum-number and fractional statistics

Hubbard model on triangular lattice:

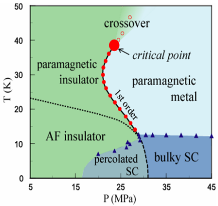


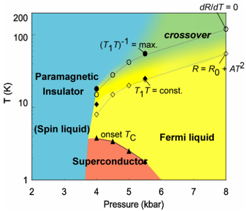




$t'/t = 0.5 \sim 1.1$

X=Cu[N(CN)$_2$]Cl, Cu$_2$(CN)$_3$,...





Spin interaction
$J = 250K$
But no AF order down to $35mK$

Cu[N(CN)$_2$]Cl $t'/t = .75$ Cu$_2$(CN)$_3$ $t'/t = 1.06$

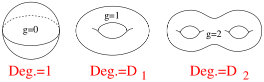
- Spin-charge separation +emergent fermion → spinon Fermi surface

Symmetry-breaking/topological orders through experiments

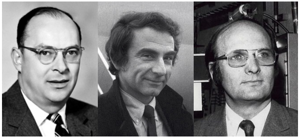
Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (examples: FQH, spin liquid) (none example: topo. insulator)	Topological degeneracy, Non-Abelian geometric phases T, S No finite temperature phase trans, Topological gapless boundary states, Fractional charge/statistics

- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** T, S define a completely new class of order – **topological order**. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

What is the microscopic origin of topological order?



- Why the macroscopic properties, D_g and $S\&T$, are independent of any local perturbations? What is the microscopic understanding?
- Zero-resistance and Meissner effect $\to$ macroscopic definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

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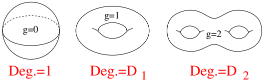

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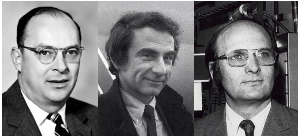
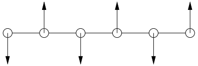


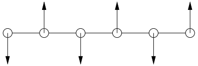
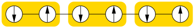
Bardeen-Cooper-Schrieffer 57

- It took 20 years to gain a microscopic understanding of topological order: **long-range entanglements** Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06,Levin-Wen 06

What is NOT long-range entanglement?

-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\rightarrow$ unentangled

What is NOT long-range entanglement?

-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ $\to$ short-range entangled (SRE) entangled

- Symmetry breaking states are short-range entangled
- **Topologically ordered states are beyond symmetry breaking and beyond short-range entanglement**
  $\to$ **long-range entanglement**

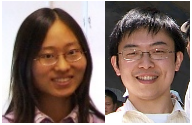
- But **what IS long-range entanglement**?

Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\to$ all systems belong to one trivial phase

Pattern of long-range entanglements = topological order



For gapped systems with no symmetry:

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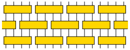
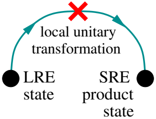
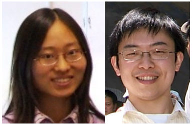
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$

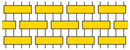


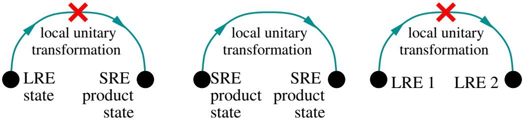
Pattern of long-range entanglements = topological order

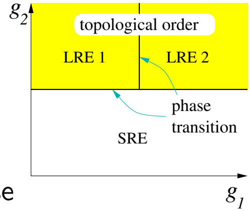


For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\to$ all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states** $\to$ many phases
  - There are **short range entangled (SRE) states** $\to$ one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders** Wen 1989
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

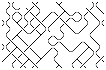
How to make long range entanglements (topo. orders)

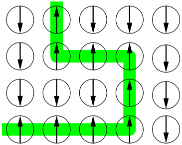
To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin configurations}} | \uparrow\downarrow\downarrow\uparrow \ldots \rangle = | \rightarrow\rightarrow\rightarrow\rightarrow \ldots \rangle$

- Sum over a subset of spin configurations: sum over all the "string states", where the up-spins form strings:

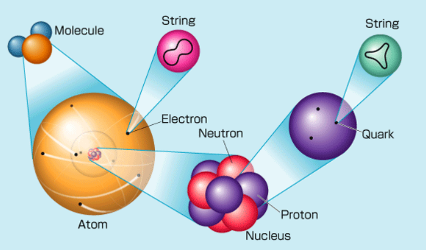
$|\Phi_{\text{closed string}}\rangle = \sum_{\text{all loops}} \Big|$  $\Big\rangle$

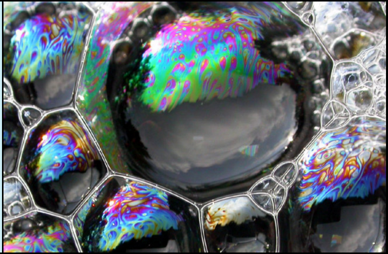
$|\Phi_{\text{string-net}}\rangle = \sum_{\text{all string-nets}} \Big|$  $\Big\rangle$



$\rightarrow$ string-net condensation Levin-Wen 05 (string-net liquid).

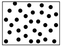
$\rightarrow$ $|\Phi_{\text{string}}\rangle$ has long-range entanglement and a non-trivial **topological order**

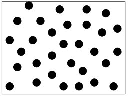




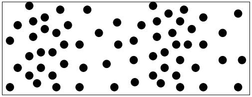
Long range entanglements are source of many wonders

Long range entanglements (closed oriented strings) $\to$ emergence of electromagnetic waves (photons)

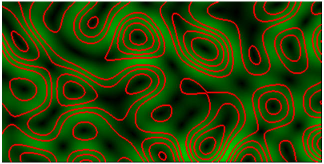
- Wave in superfluid state $|\Phi_{\text{SF}}\rangle = \sum_{\text{all position conf.}}$ $\Big|$$\Big\rangle$:

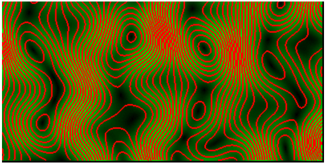


density fluctuations:
Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
$\to$ Longitudinal wave



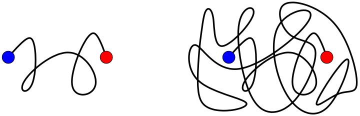
- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}}$ $\Big|$$\Big\rangle$:

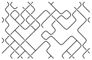


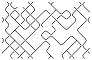


String density $\mathbf{E}(\mathbf{x})$ fluctuations $\to$ waves in string condensed state." Strings have no ends $\to$ $\partial \cdot \mathbf{E} = 0$ $\to$ **only two transverse modes**." Equation of motion for string density $\to$ Maxwell equation: $\dot{\mathbf{E}} - \partial \times \mathbf{B} = \dot{\mathbf{B}} + \partial \times \mathbf{E} = \partial \cdot \mathbf{B} = \partial \cdot \mathbf{E} = 0$. ($\mathbf{E}$ electric field)

Long range entanglements (non-positive amplitudes) → Emergence of Fermi statistics, fractional statistics, etc



- In string condensed states, the ends of string be have like point particles (gauge charges), with certain statistics:
- For string condensed state $|\Phi\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

  The end of strings are bosons.
- For string condensed state $|\Phi\rangle = \sum_{\text{all conf.}} \pm \Big|$  $\Big\rangle$

  The end of strings are fermion. Levin-Wen 2003
- **String-net/entanglement provides a way to unify gauge interactions and Fermi statistics in 3D**

Long-range entangled qubits (topologically ordered qubits) can produce

1. Identical particles
2. Spin-1 bosons with only two-components $\to$ gauge bosons
3. Fractional angular momentum (spin-1/2)
4. Fermi statistics
5. Chiral gauge coupling (Parity violation in weak interaction)
6. Lorentz symmetry
7. Spin-2 bosons with only two-components $\to$ gravitons?

- **Can entangled qubits produce chiral gauge coupling** (*ie* chiral-fermion/chiral-gauge theory)?

$\to$ **a non-perturbative lattice definition of standard model** *(For a long time, we believe that lattice gauge theory can only produce equal gauge coupling to right- and left-hand fermions.)*

- This long standing problem can be solved by studying **entanglement with symmetry**, since such a study gives rise to a **classification of gauge anomaly**.

Gapped states with symmetry and the SPT states

For gapped systems with a symmetry G (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ one phase

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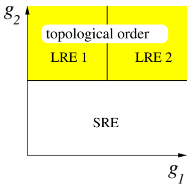
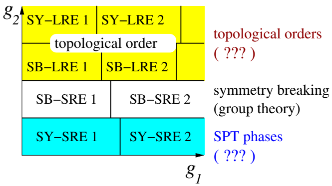
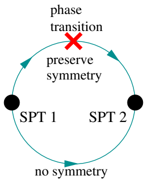
We may call them **symmetry protected** **trivial** **(SPT)** phase







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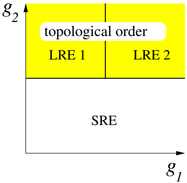
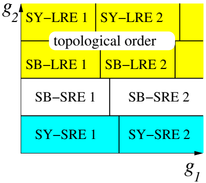
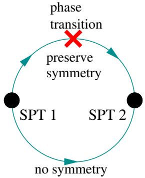
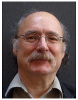






- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

1D 

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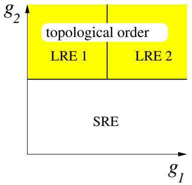
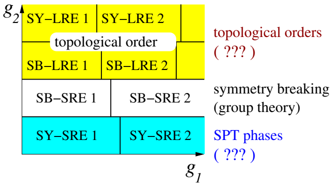
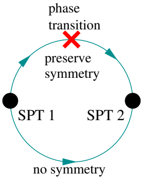
or **symmetry protected topological (SPT)** phase







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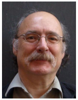
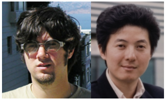
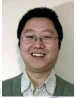
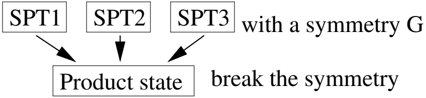
- Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06
  and 3D Moore-Balents 07; Fu-Kane-Mele 07

1D  2D   3D  

SRE states with symmetry → SPT orders

- *Symmetry protected topological (SPT) phases* are gapped quantum phases with certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry.



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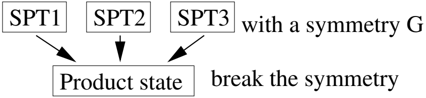

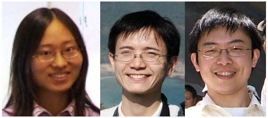

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- **A classification of (all?) SPT phase**: Chen-Gu-Liu-Wen 11
  For a system in d spatial dimension with an on-site symmetry G, its SPT phases that do not break the symmetry G are classified by the elements in $\mathcal{H}^{d+1}[G, U(1)]$ – the $d+1$ cohomology class of the symmetry group G with G-module $U(1)$ as coefficient.

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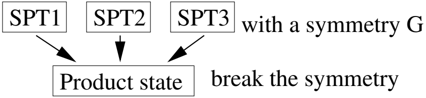

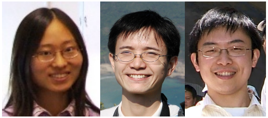

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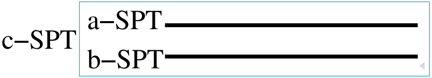
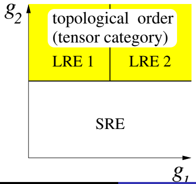
- $\mathcal{H}^{d+1}[G, U(1)]$ form an Abelian group: $a + b = c$,
  - Stacking a-SPT state and b-SPT state give us a c-SPT state.

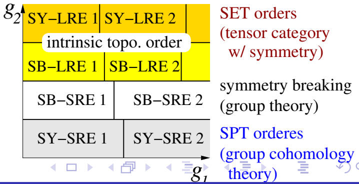


Bosonic SPT phases in any dim. and for any symmetry

Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U(1) \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U(1) \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times$ trans	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SO(3)$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$

Table of $\mathcal{H}^{d+1}[G, U_T(1)]$
"Z_2^T": time reversal,
"trans": translation,
others: on-site symm.
$0 \rightarrow$ only trivial phase.
$(\mathbb{Z}_2) \rightarrow$ free fermion result





Compare topological order and SPT order

- **Topological order** = *patterns of long-range entanglement*
  - The essence: long-range entanglement = "topology"
- **SPT order** = *short-range entanglement + symmetry* G.
  - The essence: trivial "topology" entangled with symmetry

	Topo. order (FQH states)	SPT order (Topo. ins.)	trivial (Band ins.)
Entanglements	long range	short range	short range
Fractional charges	Yes	No	No
Fractional statistics	Yes	No	No
Non-Abelian/Fermi stat. of **extrinsic defects**	Yes $>$ Majorana/Fermi	Yes Majorana/Fermi	Yes Majorana/Fermi
Gapless boundary	topo. protected	symm. protected anomalous symm.	not protected

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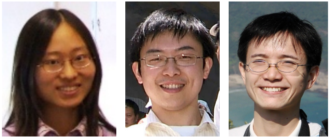
- If we think SPT order is "topological" → it is hard to classify it
  If we think SPT order is "trivial" → we can classify it with interaction

Interacting bosonic SPT phase: A group-cohomolgy theory

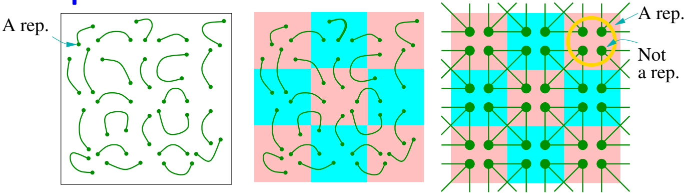
Chen-Liu-Wen 11, Chen-Gu-Liu-Wen 11

For any symmetry group G
and in any dimensions d

Two key observations:



- **Short-range-entangled states have a simple canonical form**:



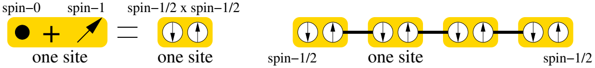
after we treat each block as an effective site.

- *Each effective site has several independent degrees of freedoms entangled with its neighbors.*
- *The combined degrees of freedoms on a site form a rep. of G*

- **Each degree of freedoms on the effective site may not form a representation of G, but the whole state is still inv. under G**

Non-trivial short-range entangled states w/ symmetry

- **Haldane phase w/ $SO(3)$ symm.**: spin-1/2 is not a rep. of $SO(3)$



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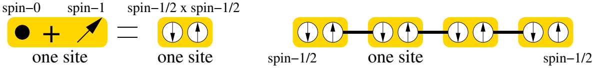
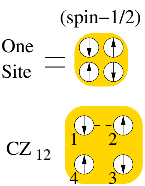
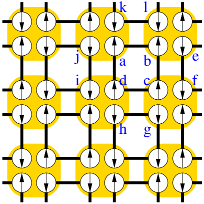


- **2D SPT phase w/ Z_2 symm.**:

  Chen-Liu-Wen 2011

  - Physical states on each site: $(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$
  - The ground state wave function: $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$





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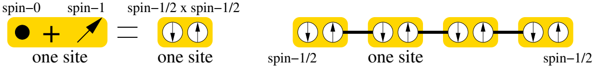

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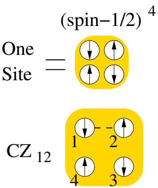

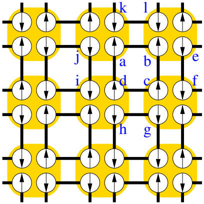

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- The on-site Z_2 symmetry: (acting on each site $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$):

$$U_{CZX} = U_{CZ} U_X, \quad U_X = X_1 X_2 X_3 X_4, \quad U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}$$

$$CZ : |\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle, \; |\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle, \; |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle, \; |\downarrow\downarrow\rangle \to -|\downarrow\downarrow\rangle$$

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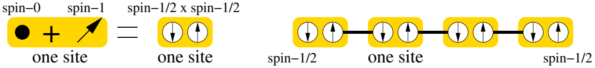

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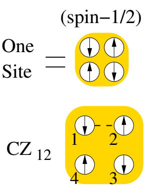

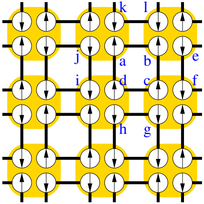

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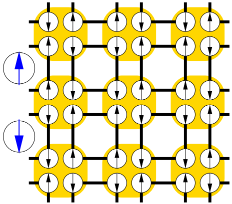
- Z_2 symm. Hamiltonian $H = \sum_{\Box} H_p, \ H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,
  $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|, \ P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

Edge excitations for the 2D Z_2 SPT state

- Bulk Hamiltonian $H = \sum_{\square} H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij}, P_{kl}$,
  $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$, $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

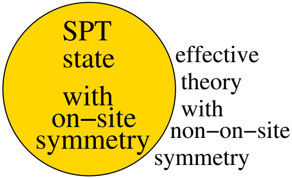


- Edge excitations: *gapless or break the $\mathbb{Z}_2$ symmetry, robust against any perturbations that do not break the $\mathbb{Z}_2$ symmetry.*
- Edge effective spin $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$.
- Edge effective Z_2 symmetry : $U_{Z_2} = \mathrm{e}^{\mathrm{i}(\sum_i \frac{1}{4}(\tilde{Z}_i \tilde{Z}_{i+1} - 1))} \prod_i \tilde{X}_i$ *which cannot be written as $U_{Z_2} = \prod_i O_i$, such as $U_{Z_2} = \prod \tilde{X}_i$. Not an on-site symmetry!*
- Edge effective Hamiltonian ($c = 1$ gapless if the Z_2 is not broken)

$$H_{\text{edge}} = -J \sum \tilde{Z}_i \tilde{Z}_{i+1} + B_x \sum [\tilde{X}_i + \tilde{Z}_{i-1} \tilde{X}_i \tilde{Z}_{i+1}]$$
$$+ B_y \sum [\tilde{Y}_i - \tilde{Z}_{i-1} \tilde{Y}_i \tilde{Z}_{i+1}]$$

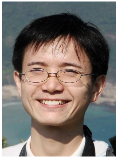
Non-on-site symmetry at the boundary of SPT phases

- What is an **on-site symmetry**?
It is the usual global symmetry:
$U_{\text{tot}}(g) = \otimes_{\text{sites}} U_i(g), g \in G$
- What is a **non-on-site symmetry**?
It is an unusual global symmetry
that cannot be realized as an on-site symmetry in a lattice.



- The G symmetry of the **boundary effective theory** of a SPT state is a **non-on-site symmetry**
→ *cannot find a lattice realization of the boundary effective theory in the same dimension where G is an on-site symmetry*
(But can find a lattice realization of the boundary effective theory in one-higher dimension where G is an on-site symmetry)
The physical consequence of on non-on-site symmetry
the boundary ground states must be gapless or degenerate:
(1) Gapless without breaking the symmetry
(2) Degenerate with the symmetry due to topological order
(3) Degenerate due to spontaneous symmetry breaking

A field theory example: 2+1D $SU(2)$ SPT states Liu-Wen 12

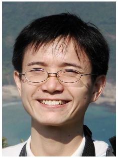


For $SU(2)$, $\mathcal{H}^3[SU(2), U(1)] = \mathbb{Z}$
$\to$ We can use the topological terms in field theory to realize all the $SU(2)$-SPT states classified by $\mathbb{Z}$:

$$S_{\text{bulk}} = \int_M \frac{1}{\lambda}\text{Tr}(\partial g^{-1}\partial g) - i\frac{k}{12\pi}\int_M \text{Tr}(g^{-1}dg)^3, \quad k \in \mathbb{Z}, \; g \in SU(2)$$

The (left) $SU(2)$ symmetry: $g(x) \to hg(x), \quad h, g(x) \in SU(2)$

A field theory example: 2+1D $SU(2)$ SPT states Liu-Wen 12



For $SU(2)$, $\mathcal{H}^3[SU(2), U(1)] = \mathbb{Z}$
$\to$ We can use the topological terms in field theory to realize all the $SU(2)$-SPT states classified by $\mathbb{Z}$:

$$S_{\text{bulk}} = \int_M \frac{1}{\lambda}\text{Tr}(\partial g^{-1}\partial g) - i\frac{k}{12\pi}\int_M \text{Tr}(g^{-1}dg)^3, \quad k \in \mathbb{Z},\ g \in SU(2)$$

The (left) $SU(2)$ symmetry: $g(x) \to hg(x), \quad h, g(x) \in SU(2)$

- The edge excitations are gapless (described by fixed-point WZW):

$$S_{\text{Bnd}} = \int_{\partial M} \frac{k}{8\pi}\text{Tr}(\partial g^{-1}\partial g) - i\int_M \frac{k}{12\pi}\text{Tr}(g^{-1}dg)^3,$$

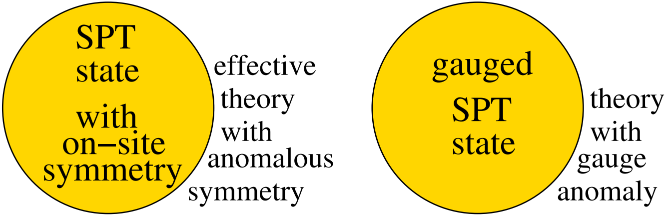
- At the fixed point, we have a equation of motion Witten 84

$$\partial_{\bar{z}}[(\partial_z g)g^{-1}] = 0, \quad \partial_z[(\partial_{\bar{z}} g^{-1})g] = 0, \quad z = x + \mathrm{i}\,t.$$

Right movers $[(\partial_z g)g^{-1}](z) \to SU(2)$-charges
Left movers $[(\partial_{\bar{z}} g^{-1})g](\bar{z}) \to$ No $SU(2)$-charge for $g(x) \to hg(x)$
Chiral level-k $SU(2)$ Kac-Moody algebra $\to SU(2)$ **ABJ anomaly**

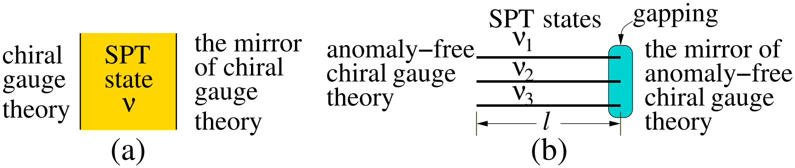
A classification of gauge anomalies by SPT phases

- A theory with non-site-symmetry cannot be gauged in the same dimensions → **non-site-symmetry = anomalous symmetry**.
- But it can be gauge if we view the theory as a boundary theory of a lattice model in one-higher dimension → **anomaly in-flow**



- Bosonic gauge anomalies with gauge group G in d space-time dimensions are classified by G-symmetric SPT states in one-higher dimensions (*ie* by $\mathcal{H}^{d+1}[G, U(1)]$)
- Such a classification of gauge anomalies allows us to solve the long standing chiral-ferimon/chiral-gauge problem.

A lattice definition of any anomaly-free gauge theories



- Anomaly-free gauge theories $\sum \nu_i = 0 \rightarrow$
- trivial SPT state in one-higher dimensions $\rightarrow$
- the edge states can be gapped without symmetry breaking $\rightarrow$
- **We can use lattice gauge theories to define any anomaly-free gauge theories as long as we allow direct interactions in the matter sector.**

Long-range entangled qubits on discrete space lattice (*ie* topologically ordered qubits) can produce:

1. Identical particles
2. Spin-1 bosons with only two-components → gauge bosons
3. Fractional angular momentum (spin-1/2)
4. Fermi statistics
5. Chiral gauge coupling (Parity violation in weak interaction)
6. Lorentz symmetry (assume discrete time → space-time lattice)
7. Spin-2 bosons with only two-components → gravitons?

Information unifies matter and interactions

We can realize/simulate the standard model using condensed matter and cold atom systems.