Axions

Lectures at
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of Theoretical Physics

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Plan:

- Strong CP problem
- Dynamical solution of Strong CP problem
- Laboratory and stellar constraints
- Axion dark matter
- Axion models

Lecture I

Strong

CP

Problem

Theta parameter in Yang Mills theory

YM theory

non-Abelian generalisation of Maxwell theory

Consider general compact simple Lie group G

Much of our discussion will hold for general compact, simple Lie group G. Recall that there is a finite classification of these objects. The possible options for the group G, together with the dimension of G and the dimension of the fundamental (or minimal) representation F, are given by

G	$\dim G$	$\dim F$
$SU(N)$	N^2-1	N
$SO(N)$	$\frac{1}{2}N(N-1)$	N
$Sp(N)$	$N(2N+1)$	$2N$
E_6	78	27
E_7	133	56
E_8	248	248
F_4	52	6
G_2	14	7

where we're using the convention $Sp(1) = SU(2)$. (Other authors sometimes write $Sp(2n)$, or even $USp(2n)$ to refer to what we've called $Sp(N)$, preferring the argument to refer to the dimension of F rather than the rank of the Lie algebra $\mathfrak{g}$.)

G has an underlying Lie algebra $\mathfrak{g}$ whose generators T^a satisfy

$$[T^a, T^b] = i f^{abc} T^c$$

Here $a, b, c = 1, \ldots, \dim G$ and f^{abc} are the fully anti-symmetric structure constants. The factor of i on the right-hand side is taken to ensure that the generators are Hermitian: $(T^a)^\dagger = T^a$.

We will need to normalise our Lie algebra generators. We require that the generators in the fundamental (i.e. minimal) representation F satisfy

$$\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab} \tag{2.2}$$

In what follows, we use T^a to refer to the fundamental representation, and will refer to generators in other representations R as $T^a(R)$. Note that, having fixed the normalisation (2.2) in the fundamental representation, other $T^a(R)$ will have different normalisations. We will discuss this in more detail in Section 2.5 where we'll extract

For each element of the Lie algebra $\mathfrak{g}$ we introduce a gauge field A_μ^a. These are then packaged into Lie-algebra-valued gauge field

$$A_\mu \equiv A_\mu^a T^a$$

From the gauge field we construct a Lie algebra valued field strength

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]$$

The YM theory defined via large symmetry group.
These come from space-time dependent functions of the Lie group G

$$\Omega(x) \in G$$

gauge group = set of all such transformations
$\Updownarrow$
maps of space-time into G

action on gauge field

$$A_\mu \to \Omega(x) A_\mu \Omega^{-1}(x) + i\,\Omega(x)\,\partial_\mu \Omega^{-1}(x)$$

This induces ~~the~~ the action

$$F_{\mu\nu} \to \Omega(x) F_{\mu\nu} \Omega^{-1}(x)$$

For $G = U(1)$, $\Omega = e^{i\omega}$ and then

$$A_\mu \to A_\mu + \partial_\mu \omega$$

- In YM theory, the „electric" and „magnetic" fields

$$E_i = F_{0i} \quad ; \quad B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk}$$

are not gauge invariant

- Building blocks of YM action:
  - Lorentz invariant
  - gauge invariant
  - quadratic in field strength

$$\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$$

$$\mathrm{tr}\, F_{\mu\nu} {}^{*}F^{\mu\nu}$$

where

$$ {}^{*}F^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$S_{YM} = \int d^4x \, -\frac{1}{2g^2} \, tr(F_{\mu\nu} F^{\mu\nu})$$

- g^2 YM coupling

using $F_{\mu\nu} = F^a_{\mu\nu} T^a$

and $tr(T^a T^b) = \frac{1}{2} \delta^{ab}$

$$S_{YM} = \int d^4x \, -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$$

- up to $\frac{1}{g^2}$ similar to Maxwell action

However, in the Yang-Mills action, all terms appear with fixed coefficients determined by the definition of the field strength (2.4). Instead, we've chosen to write the (inverse) coupling as multiplying the entire action. This difference can be accounted for by a trivial rescaling. We define

$$\tilde{A}_\mu = \frac{1}{g} A_\mu \quad \text{and} \quad \tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - ig[\tilde{A}_\mu, \tilde{A}_\nu]$$

Then, in terms of this rescaled field, the Yang-Mills action is

$$S_{\rm YM} = -\frac{1}{2g^2} \int d^4x \; {\rm tr}\, F^{\mu\nu} F_{\mu\nu} = -\frac{1}{2} \int d^4x \; {\rm tr}\, \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$$

In the second version of the action, the coupling constant is buried inside the definition of the field strength, where it multiplies the non-linear terms in the equation of motion as expected.

– „θ term"

$$S_\theta = \frac{\theta}{16\pi^2} \int d^4x \; \mathrm{tr}\, F_{\mu\nu} {}^*F^{\mu\nu}$$

- Integrand is a total derivative:

$$\frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}\, {}^{*}F^{\mu\nu} = \partial_\mu K^\mu$$

with

$$K^\mu \equiv \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(A_\nu \partial_\rho A_\sigma - \frac{2i}{3} A_\nu A_\rho A_\sigma\Big)$$

i.e.

→ Theta-term does not change classical equations of motion $\left[\begin{array}{l} D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0 \\ D_\mu {}^{*}F^{\mu\nu} = 0 \text{ (Bianchi)} \end{array}\right.$

→ theta term depends only on boundary information

→ „topological"

– θ is an angular variable.
For simple gauge groups:

$$\theta \in (-\pi, \pi]$$

- Consider YM quantum field theory
- Quantization via Euclidean path integral

- Wick rotation:

$$S_{YM}^{(M)} = \int d^4x_{(M)} \left(-\frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \right)$$

$$x_0^{(M)} \hat{=} -i x_4^{(E)}$$

$$A_0^{(M)} \to i A_4^{(E)}$$

$\Rightarrow$

$$i S_{YM}^{(M)} = i \int d^4x_{(M)} \, -\frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu})$$

$$\Rightarrow \; - \int d^4x_{(E)} \frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu}^{(E)} F_{\mu\nu}^{(E)})$$

$$i S_\theta^{(M)} \to i \frac{\theta}{16\pi^2} \int d^4x_{(E)} \operatorname{tr}(F_{\mu\nu} {}^*F_{\mu\nu})$$

- Vacuum to vacuum amplitude $\langle 0^+ | 0^- \rangle \equiv Z$ in terms of Euclidean path integral

$$Z = \int [dA]$$

$$\exp\left[-\frac{1}{2g^2} \int d^4x \, \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) + \frac{i\theta}{16\pi^2} \int d^4x \, \mathrm{tr}\left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right)\right]$$

- Integration over fields which approach pure gauge (zero energy) at infinity:

$$A_\mu \to i\,\Omega(x)\,\partial_\mu \Omega^{-1}(x)$$

at $|x| \to \infty$

for $\Omega \in G$

ν ... winding number

(Pontryagin index
topological charge)

of mapping

$$\Omega : S^3_\infty \to G$$

$\uparrow$

$\simeq$ boundary of
Euclidean spacetime

$$\partial \mathbb{R}^4$$

$$\left[G = SU(2): \quad S^3_\infty \to SU(2) \simeq S^3 \right]$$

In fact, mappings

$$\Omega(x) : S^3_\infty \longrightarrow G$$

fall into distinct classes

gauge transformations can „wind" around S^3_∞ such that one gauge transformation cannot be continuously transformed into another

Such maps characterised by homotopy theory

The homotopically distinct maps are classified by the group $\Pi_n(X)$. For us, the relevant formula is

$$\Pi_3(G) = \mathbf{Z}$$

for all simple, compact Lie groups G. In words, this means that the winding of gauge transformations is classified by an integer n. This statement is intuitive for $G = SU(2)$ since $SU(2) \cong \mathbf{S}^3$, so the homotopy group counts the winding of maps from $\mathbf{S}^3 \mapsto \mathbf{S}^3$. For higher dimensional G, it turns out that it's sufficient to pick an $SU(2)$ subgroup of G and consider maps which wind within that. It turns out that these maps cannot be unwound within the larger G. Moreover, all topologically non-trivial maps within G can be deformed to lie within an $SU(2)$ subgroup. It can be shown that this winding

ν computed by

$$\nu(\Omega) = \frac{1}{24\pi^2} \int_{S^3_\infty} dS_\mu \, \varepsilon^{\mu\nu\varphi\sigma} \, \mathrm{tr}\left((\Omega \partial_\nu \Omega^{-1})(\Omega \partial_\varphi \Omega^{-1})(\Omega \partial_\sigma \Omega^{-1}) \right)$$

Example of gauge transformation $\Omega(x) \in SU(2)$ with

$U = 1$:

$$\Omega(x) = \frac{x_\mu \sigma^\mu}{\sqrt{x^2}}$$

where $\sigma^\mu = (1, -i\underline{\sigma})$

Therefore:

$$Z = \sum_{\nu=-\infty}^{+\infty} Z_\nu \exp(i\theta\nu)$$

with

$$Z_\nu = \int_\nu [dA] \exp\left(-\frac{1}{2g^2}\int d^4x \, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\right)$$

- θ angular parameter ✓

Theta parameter in QCD

- Add one massless Dirac fermion:

$$\mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu - i A_\mu) \psi$$

global symmetries:

$$\psi \to e^{i\alpha} \psi$$

$$j^\mu = \bar{\psi} \gamma^\mu \psi \qquad \text{vector current}$$

$$\psi \to e^{i\alpha\gamma^5} \psi$$

$$j^\mu_A = \bar{\psi} \gamma^\mu \gamma^5 \psi \qquad \text{axial current}$$

- classically both conserved
- quantum mechanically axial vector current not conserved:

$$\partial_\mu j^\mu_A = \frac{1}{8\pi^2} \mathrm{tr}(F_{\mu\nu} {}^*F^{\mu\nu})$$

chiral anomaly

ABJ anomaly

Can be understood see

with from non-invariance

of path integral measure

under chiral transformations

[Fujikawa]

$$\delta\psi = i\,\varepsilon(x)\,\gamma^5\psi$$

$$\delta\bar{\psi} = i\,\varepsilon(x)\,\bar{\psi}\gamma^5$$

$$\int [d\psi][d\bar{\psi}] \longrightarrow$$

$$\int [d\psi][d\bar{\psi}]\, \exp\left[-\frac{i}{8\pi^2}\int d^4x\, \varepsilon(x)\, \mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\right]$$

Thus, a chiral rotation

$$\psi \to e^{i\alpha\gamma^5}\psi$$

effectively shifts theta angle by

$$\theta \to \theta - 2\alpha$$

Theta-angle is not physical!

Can be absorbed by changing phase of fermion

- Add mass to fermion

For Dirac fermion, two choices:

$\bar{\psi}\psi$ invariant under P

$i\bar{\psi}\gamma^5\psi$

In terms of Weyl fermions, two mass parameters split into modulus and complex phase

$$\mathcal{L}_{mass} = m\left(e^{i\phi}\psi_+^\dagger\psi_- + e^{-i\phi}\psi_-^\dagger\psi_+\right)$$

only

$$\bar{\theta} \equiv \theta + \phi$$

has physical meaning

- more generally, with n_f fermions, we can have complex mass matrix M and

$$\bar{\theta} = \theta + \arg(\det M)$$

remains invariant under chiral rotation.

Parity, Time-Reversal and All That

- Non-zero value of $\bar{\theta}$ leads to CP violation

- Theta-term does not preserve same symmetries as YM gauge kinetic term

In terms of field strengths:

$$S = S_{YM} + S_{\bar{\theta}}$$

$$= -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$$+ \frac{\bar{\theta}}{16\pi^2}\int d^4x \, \mathrm{tr}\left(F_{\mu\nu} {}^{*}F^{\mu\nu}\right)$$

$$= \frac{1}{g^2}\int d^4x \, \mathrm{tr}\left(\underline{E}^2 - \underline{B}^2\right)$$

$$+ \frac{\bar{\theta}}{4\pi^2}\int d^4x \, \mathrm{tr}\left(\underline{E} \cdot \underline{B}\right)$$

Both gauge invariant and Lorentz invariant

However, S_θ not invariant under parity P and time reversal invariance T

P flips all directions of space

$$P: \underline{x} \to -\underline{x}$$

T flips direction of time:

$$T: t \to -t$$

P and T act on $\underline{E}$ and $\underline{B}$ as

$$P: \underline{E}(\underline{x},t) \to -\underline{E}(-\underline{x},t)$$

$$P: \underline{B}(\underline{x},t) \to \underline{B}(-\underline{x},t)$$

and

$$T: \underline{E}(\underline{x},t) \to \underline{E}(\underline{x},-t)$$

$$T: \underline{B}(\underline{x},t) \to -\underline{B}(\underline{x},-t)$$

i.e.

$\underline{E}$ odd under P and even under T

$\underline{B}$ even under P and odd under T

Correspondingly:

$$P(S_{\bar\theta}) = -S_{\bar\theta} = S_{-\bar\theta}$$

$$T(S_{\bar\theta}) = -S_{\bar\theta} = S_{-\bar\theta}$$

Theta term breaks both parity and time-reversal invariance

"$\bar\theta \to -\bar\theta$" under P and T

P and T conserved for $\bar\theta = 0$ and $\bar\theta = \pi$ $\left(\begin{array}{c}\text{since}\\ S_\pi = S_{-\pi}\end{array}\right)$

High energy theorists usually refer to CP rather than $\mathcal{T}$. Here C is charge conjugation:

$$C: \underline{\underline{E}} \to -\underline{\underline{E}}$$

$$C: \underline{\underline{B}} \to -\underline{\underline{B}}$$

with the consequence

$$CP: \underline{\underline{E}} \to \underline{\underline{E}}$$

$$CP: \underline{\underline{B}} \to -\underline{\underline{B}}$$

rather like $\mathcal{T}$

CP unitary

$\mathcal{T}$ anti-unitary

What is the value of $\bar{\theta}$ in QCD?

- Particular sensitive probe of $\not{P}$ and $\not{T}$ violation: intrinsic electric dipole moment of neutron
- Placed in magnetic and electric field, a neutral non-relativistic particle of spin S can be described by

$$H = -\mu \, \underline{B} \cdot \frac{\underline{S}}{s} - d \, \frac{\underline{E} \cdot \underline{S}}{s}$$

under $\mathcal{P}$:

$$\mathcal{P}(\underline{B} \cdot \underline{S}) = \underline{B} \cdot \underline{S}$$

while

$$\mathcal{P}(\underline{E} \cdot \underline{S}) = -\underline{E} \cdot \underline{S}$$

i.e.

$$d \neq 0 \longleftrightarrow \mathcal{P} \text{ violation}$$

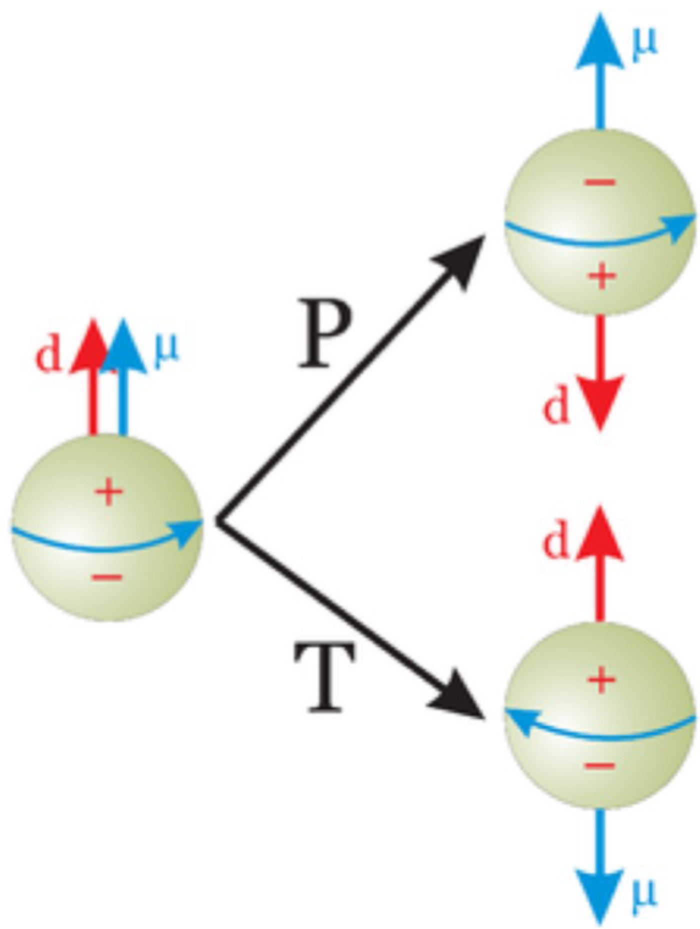
$$\mathcal{T}(\underline{B} \cdot \underline{S}) = \underline{B} \cdot \underline{S}$$

$$\mathcal{T}(\underline{E} \cdot \underline{S}) = -\underline{B} \cdot \underline{S}$$

Follows from

$$\mathcal{P}(\underline{S}) = \underline{S}$$

$$\mathcal{T}(\underline{S}) = -\underline{S}$$



- Theoretical prediction of d_n induced from theta-term:

$$|d_n| = 2.4(1.0) \times 10^{-16} |\bar{\theta}| \, e\,\mathrm{cm}$$

[Pospelov, Ritz 00]

- Can understand size of theoretical prediction from dimensional grounds and fact that theta term should have no effect if one of the quarks is massless:

chiral anomaly!

$$|d_n| \sim \underbrace{\left(\frac{m_u m_d}{m_u + m_d}\right)}_{\text{reduced quark mass}}^{1/2} \frac{1}{m_n^2} e|\bar{\theta}|$$

Experimental result

$$|d_n| < 2.9 \times 10^{-26}\, e\,\mathrm{cm}$$

$$\Rightarrow$$

$$|\bar{\theta}| < 10^{-10}$$

Strong CP problem:
(puzzle)

$$|\bar{\theta}| = |\theta + \arg(\mathrm{Det}\, M)|$$

$$< 10^{-10} \quad !!$$

Exercises Lecture I

Reminder:

$A_\mu \equiv A_\mu^a T^a$ with $[T^a, T^b] = i f^{abc} T^c$

$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$^*F^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$E_i \equiv F_{0i}$$

$$B_i \equiv -\frac{1}{2}\varepsilon_{ijk} F_{jk}$$

1.) Show that

a) $$-\frac{1}{2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \mathrm{tr}\left(\underline{E}^2 - \underline{B}^2\right)$$

and

b) $$\frac{1}{4}\,\mathrm{tr}\left(F_{\mu\nu}\,{}^{*}F^{\mu\nu}\right) = \mathrm{tr}\left(\underline{E}\cdot\underline{B}\right)$$

2.) Show that

$$\frac{1}{2} \operatorname{tr} F_{\mu\nu} {}^*F^{\mu\nu} = \partial_\mu K^\mu$$

with

$$K^\mu \equiv \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2i}{3} A_\nu A_\rho A_\sigma\right)$$

3.) Show that

$$\frac{1}{16\pi^2}\int d^4x \, \mathrm{tr}\left(F_{\mu\nu} \, {}^*F^{\mu\nu}\right)$$

can be written as

$$\frac{1}{24\pi^2}\int_{S^3_\infty} dS_\mu \, \varepsilon^{\mu\nu\varphi\sigma} \, \mathrm{tr}\left((\Omega\partial_\nu\Omega^{-1})(\Omega\partial_\varphi\Omega^{-1})(\Omega\partial_\sigma\Omega^{-1})\right)$$

if

$$A_\mu(x) \to i\,\Omega(x)\,\partial_\mu\Omega^{-1}(x)$$

for $\sqrt{x^2} \to \infty$