Exercises Lecture I

Reminder:

$A_\mu \equiv A_\mu^a T^a$ with $[T^a, T^b] = i f^{abc} T^c$

$\mathrm{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$${}^*F^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$E_i \equiv F_{0i}$$

$$B_i \equiv -\frac{1}{2} \varepsilon_{ijk} F_{jk}$$

1) Show that

a) $$-\frac{1}{2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \mathrm{tr}\left(\underline{E}^2 - \underline{B}^2\right)$$

and

b) $$\frac{1}{4}\,\mathrm{tr}\left(F_{\mu\nu}\,{}^{*}F^{\mu\nu}\right) = \mathrm{tr}\left(\underline{E}\cdot\underline{B}\right)$$

2) Show that

$$\frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}\, {}^{*}F^{\mu\nu}$$

$$= \partial_\mu K^\mu$$

with

$$K^\mu \equiv \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(A_\nu \partial_\rho A_\sigma$$

$$- \frac{2i}{3} A_\nu A_\rho A_\sigma\Big)$$

3) Show that

$$\frac{1}{16\pi^2}\int d^4x \, \mathrm{tr}\left(F_{\mu\nu} \, {}^*F^{\mu\nu}\right)$$

can be written as

$$\frac{1}{24\pi^2}\int_{S^3_\infty} dS_\mu \, \varepsilon^{\mu\nu\varphi\sigma} \, \mathrm{tr}\left((\Omega\partial_\nu\Omega^{-1})(\Omega\partial_\varphi\Omega^{-1})(\Omega\partial_\sigma\Omega^{-1})\right)$$

if

$$A_\mu(x) \to i\,\Omega(x)\partial_\mu\Omega^{-1}(x)$$

for $\sqrt{x^2} \to \infty$