On Quantum Tunneling

Neil Turok
Perimeter Institute

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Outline

- elementary approach to quantum tunneling using complex classical paths

- vast range of applications, from foundational questions to quantum chemistry, the Hawking evaporation of black holes and even the validity of the ‘inflationary multiverse’

- Classically, tunneling through a barrier is not just hard, it's impossible
- Postselection + the semiclassical expansion
- Predictions for real-time weak measurements
- Extension to quantum field theory and gravity
- Implications for inflationary 'multiverse'
- Applications from quantum chemistry to black holes

Feynman path integral

$$\Psi(x_f,t_f) = \mathrm{N}\int Dx \int dx_i \; e^{\frac{i}{\hbar}S(x_f,t_f,x_i,t_i)}\Psi(x_i,t_i)$$

This incorporates ‘pre- and post-selection’
Limit $\hbar \to 0$, perform via saddle point method
$\to$ classical solution(s) dominate

Can introduce weak measuring device to ‘see’ where the particle was between the initial and final times

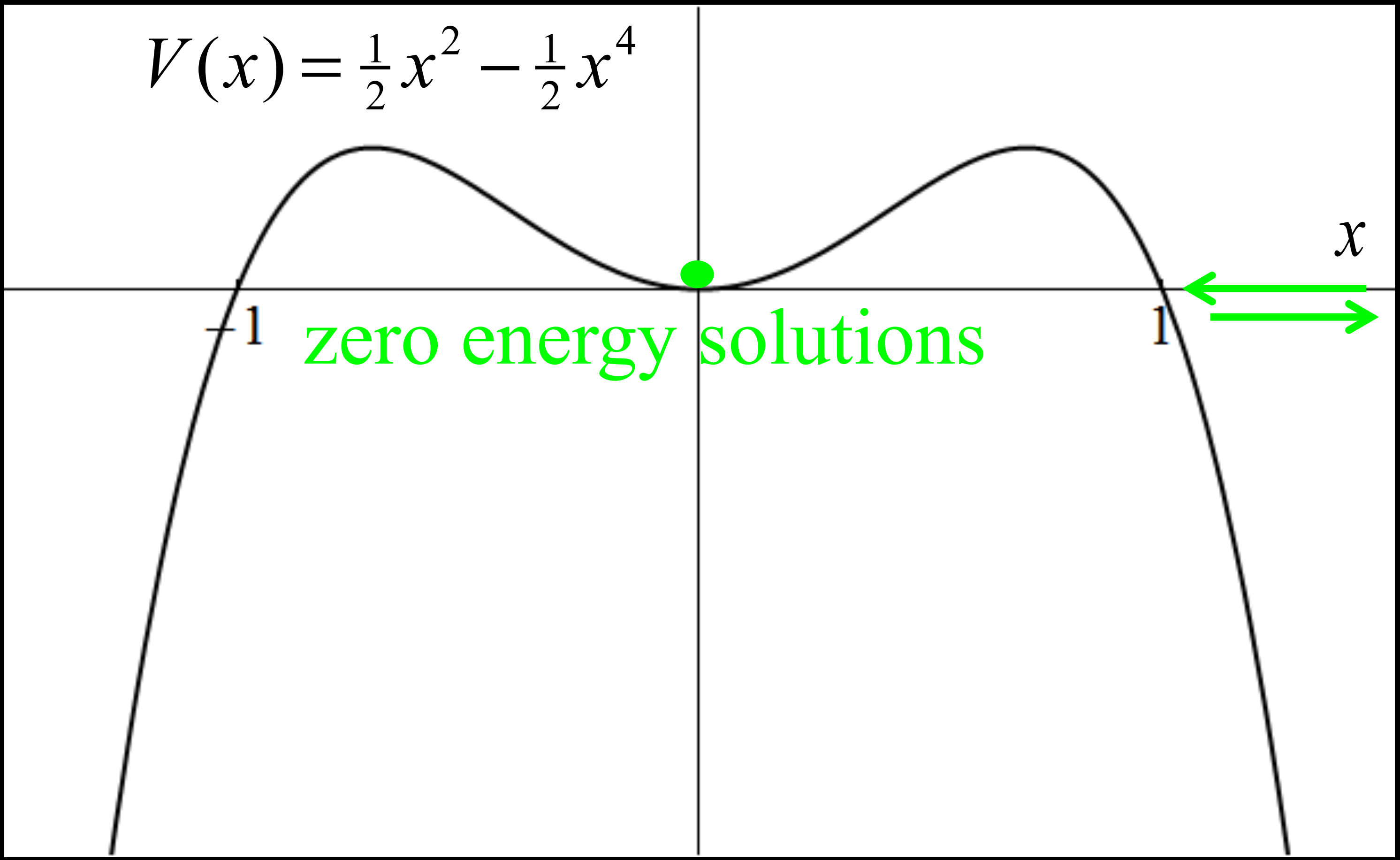
Example: particle in a potential

$$\tfrac{i}{\hbar} S = \tfrac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\tfrac{1}{2} m\dot{x}^2 - V(x) \right), \text{ take } V(x) = \tfrac{1}{2}\kappa x^2 - \tfrac{1}{2}\lambda x^4$$

$$t \rightarrow \sqrt{\tfrac{m}{\kappa}}\, t;\; x \rightarrow \sqrt{\tfrac{\kappa}{\lambda}}\, x$$

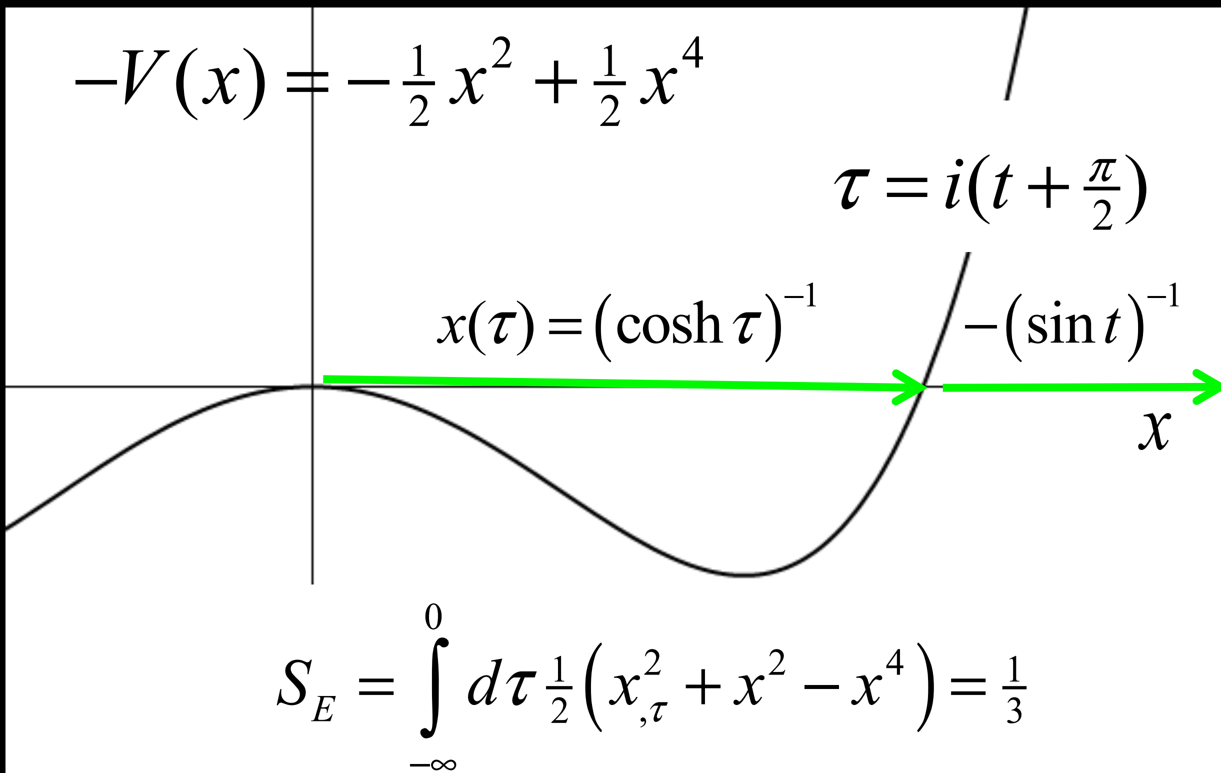
$$\tfrac{i}{\hbar} S = i \frac{\kappa^{\frac{3}{2}} m^{\frac{1}{2}}}{\hbar \lambda} \int_{t_i}^{t_f} dt \;\; \tfrac{1}{2}(\dot{x}^2 - x^2 + x^4)$$

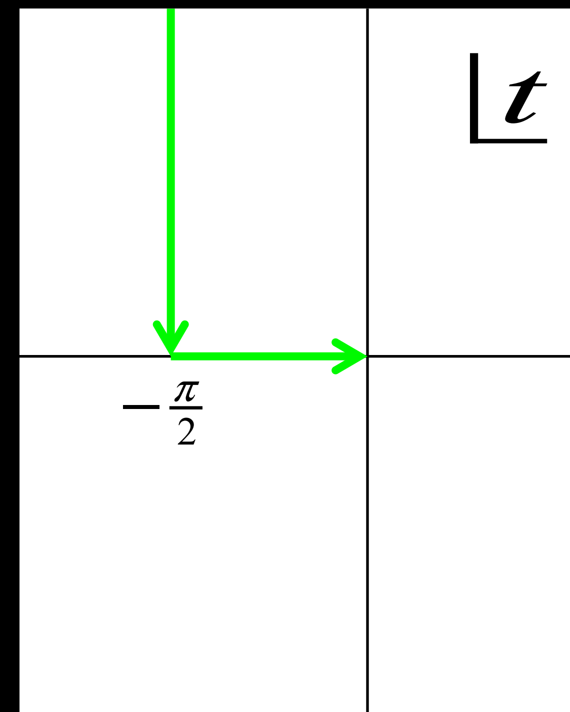
dimensionless


$V(x)=\frac{1}{2}x^2-\frac{1}{2}x^4$
x
−1
zero energy solutions
1

Euclidean "bounce"

Callan/Coleman 70's





Deficiencies of the Euclidean approach:

Dependence on initial state is very implicit

Cannot ask real-time questions e.g. where was the particle at each moment of time? How did it get through the barrier?

Hard to extend to time-dependent situations

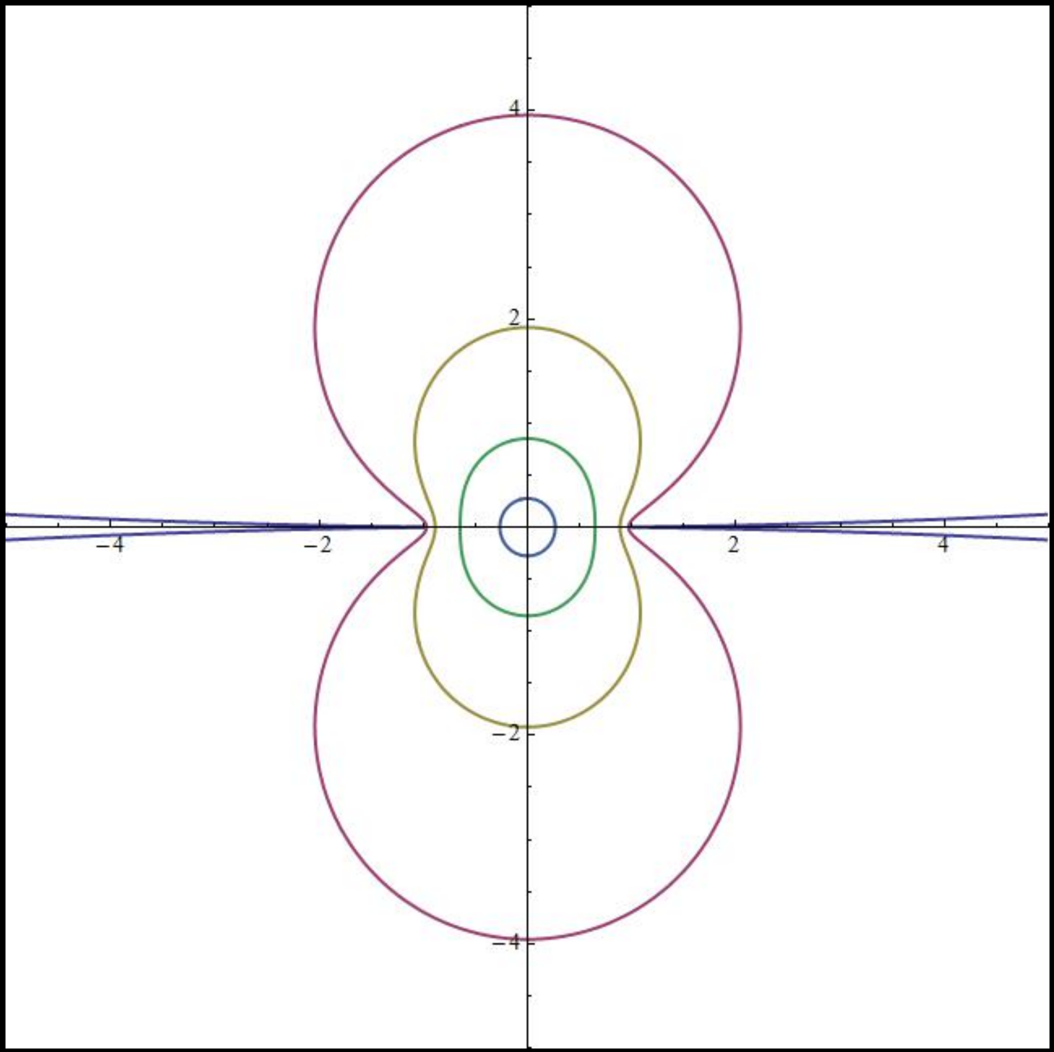
Can we do better?

General classical solution described by two complex numbers:

Energy E and time delay t_0

For real E, solutions are periodic

e.g. $E = 0 \Rightarrow x(t) = \frac{1}{\sin(t_0 - t)}; \quad t_0 = imaginary:$



Small imaginary part of energy will “carry us across” these solutions

General classical solution expressible in terms of a Jacobi elliptic function

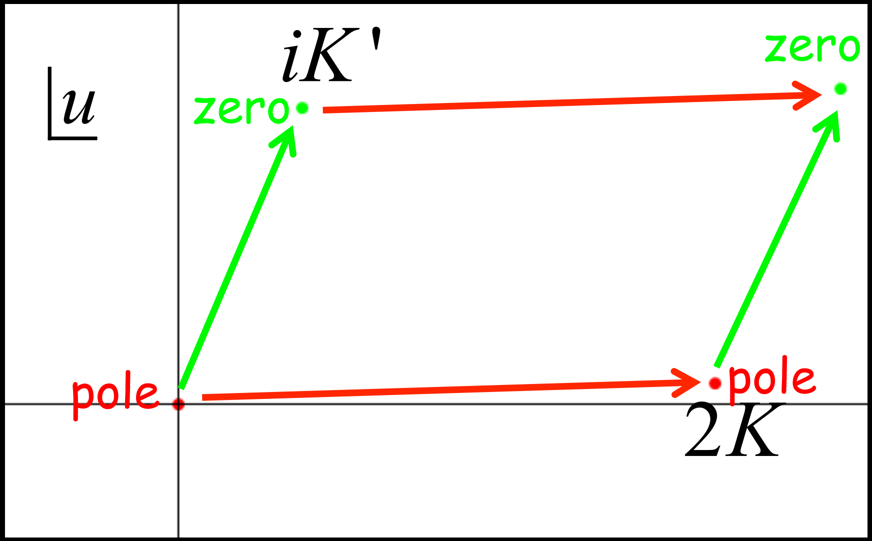
$$x(t) = \frac{1}{\sqrt{1+m}\ sn\left(\frac{t_0 - t}{\sqrt{1+m}}\middle| m\right)}; \ E = \frac{m}{2(1+m)^2}$$

(we shall be interested in small complex values of the energy)

Double periodicity in complex t-plane

$$x(t) = \frac{1}{\sqrt{1+m}\ sn\left(\frac{t_0 - t}{\sqrt{1+m}}\middle| m\right)};\ u = \frac{t_0 - t}{\sqrt{1+m}}$$

$$K(m) = \text{"quarter period"};\ K'(m) = K(1-m);$$



For small complex energy, i.e. small m

$$K = \frac{\pi}{2}\left(1 + \frac{m}{4} + \frac{9m^2}{64}\ldots\right);\ K' = -\frac{1}{2}\ln\frac{m}{16} + o(m\ln m);$$

Expansion in nome $q = e^{-\pi K'/K}$; $q = \frac{m}{16} + \frac{m^2}{32} + \ldots$

Define $u = \frac{\pi}{2K\sqrt{1+m}}(t_0 - t)$

$$x(t) = \frac{\pi}{2K\sqrt{1+m}}\left(\frac{1}{\sin u} + 4\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}\sin(2n+1)u\right)$$

Initial state: gaussian wavepacket

$$\Psi(x_i,t_i) \propto e^{-\frac{x_i^2}{4L^2}} \Rightarrow \frac{x_i}{L} + i\frac{2Lp_i}{\hbar} = 0$$

For false vacuum "ground state," $L = 1/\sqrt{2}$

Boundary conditions for classical solution

$$x + i\dot{x} = 0, \qquad t = t_i$$

$$x = x_f, \qquad t = t_f$$

Assume $T \equiv t_f - t_i \gg 1, \;\; x_f \gg 1$

$$x_f \gg 1 \Rightarrow t_0 - t_f = x_f^{-1} + \tfrac{1}{6} x_f^{-3} + \frac{(3+2m+3m^2)}{40(1+m)^2} x_f^{-5} + \ldots \ll 1$$

Solution has small, nearly imaginary E, *i.e.*, $m = i\varepsilon$

Let $z = e^{iu}$, then $x \approx \frac{2i}{z-z^{-1}} + \frac{m}{4}\frac{z-z^{-1}}{2i}$

$u \approx (1 - \frac{3i\varepsilon}{4})(t_0 - t) \sim -t + \frac{3i\varepsilon}{4} t,$ for large negative t,

z becomes large, $x \sim 2iz^{-1} + 2iz^{-3} + \frac{m}{8i} z \Rightarrow$

$$x + i\dot{x} \sim -4iz^{-3} + \frac{m}{4i} z \Rightarrow 3\varepsilon T e^{3\varepsilon T} \approx 48 i T e^{-4iT}$$

Solutions $\varepsilon_n = \frac{1}{3} T^{-1} W_n(48iTe^{-4iT}),\ n \in \mathbb{Z},$ where

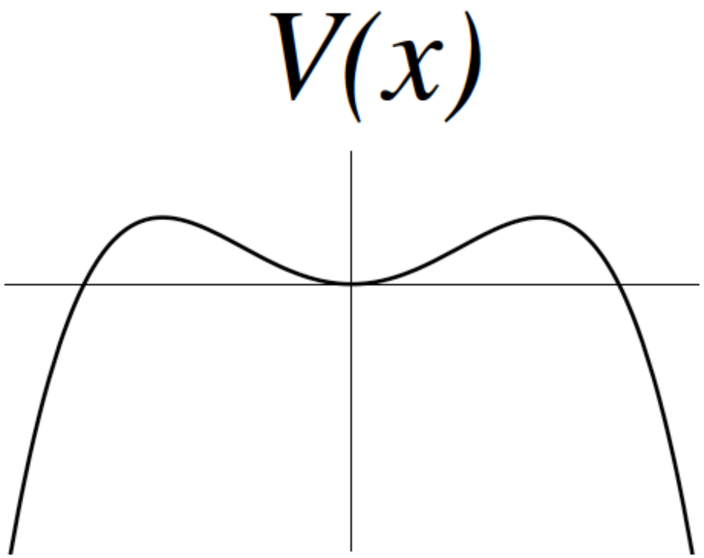
Lambert function $W_n(y)$ solves $xe^x = y$

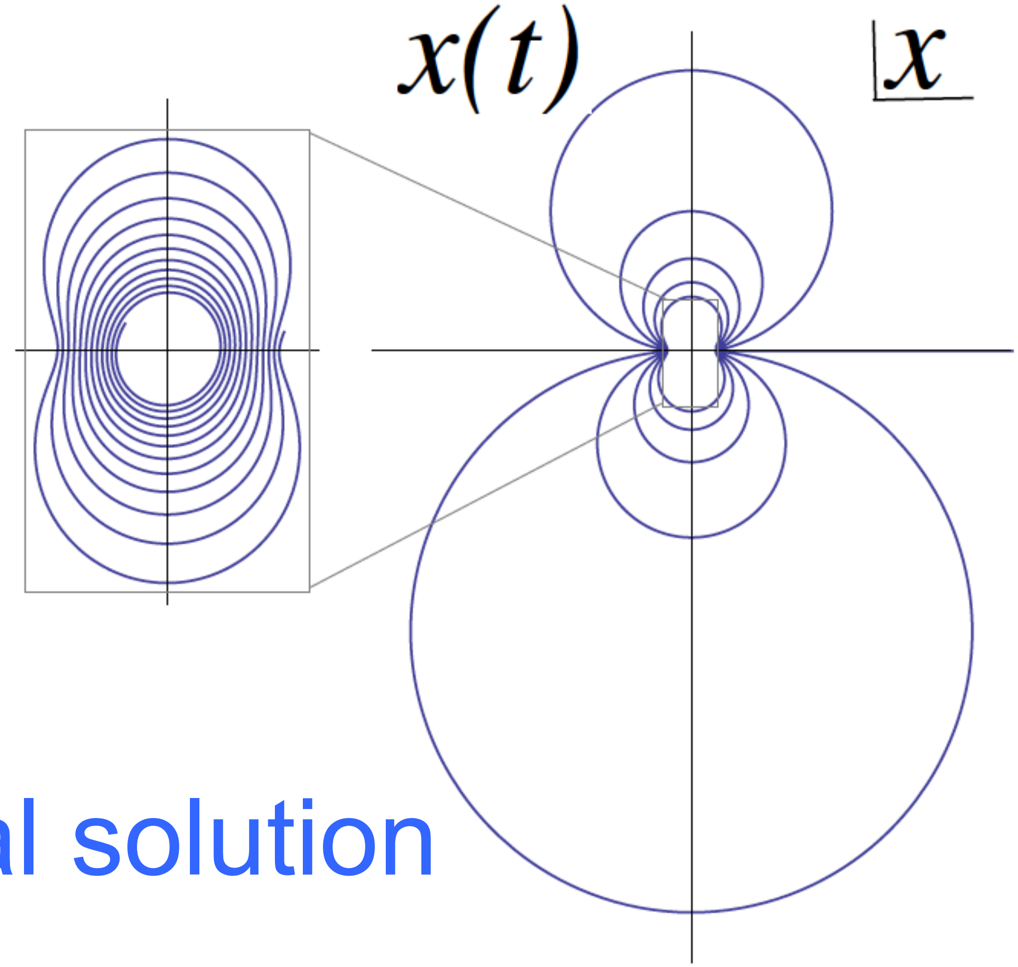
In the same approximation,

$$i(S - S^*) = -\frac{2}{3} + \frac{3}{16}\mathrm{Re}(\varepsilon_n^2) + ..$$

Principal branch ($n = 0$) has greatest semiclassical exponent

Very roughly, $\mathrm{Re}(\varepsilon_0) \sim \frac{\ln T}{3T}$, $\mathrm{Im}(\varepsilon_0) \sim \frac{1}{T}$

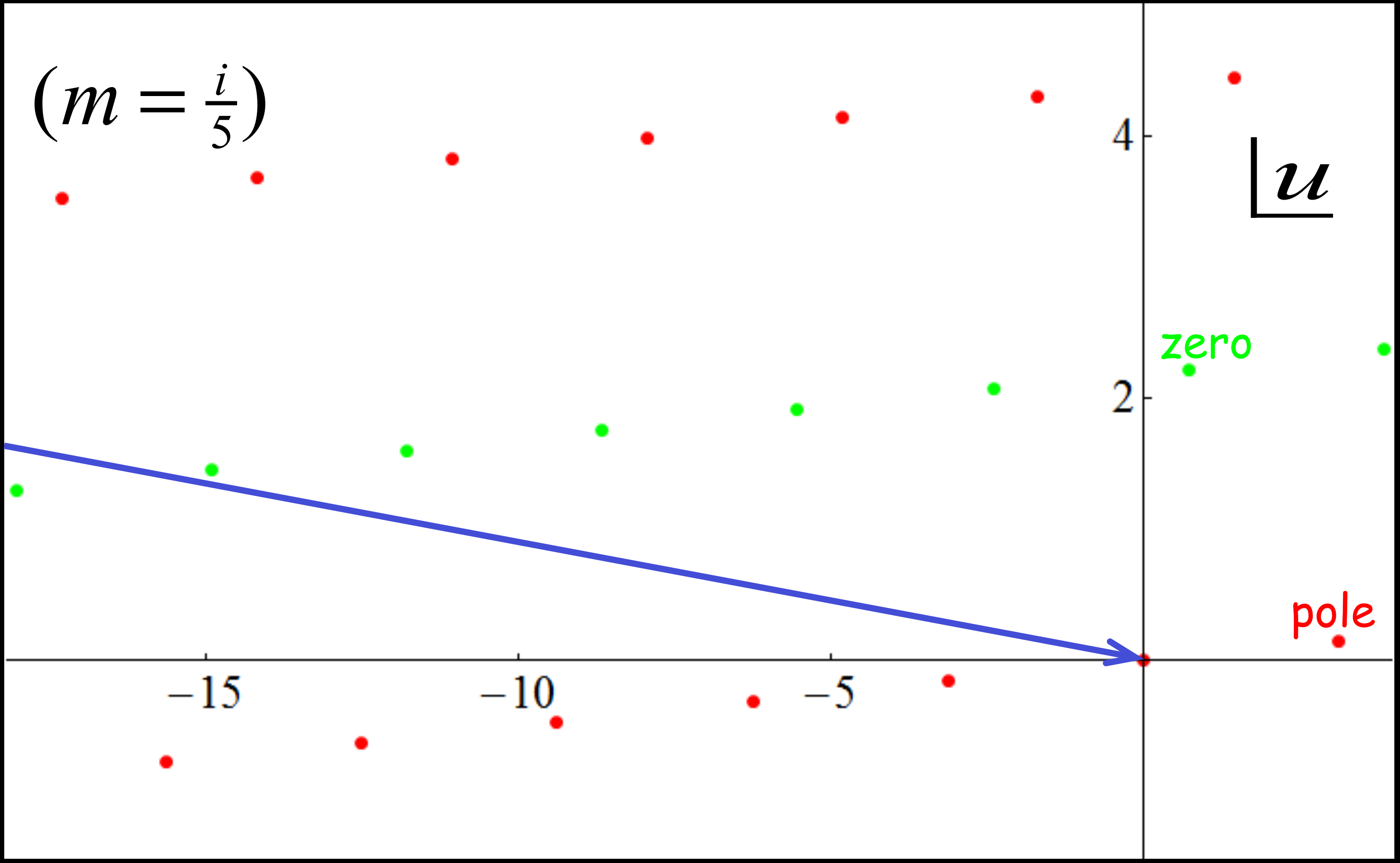




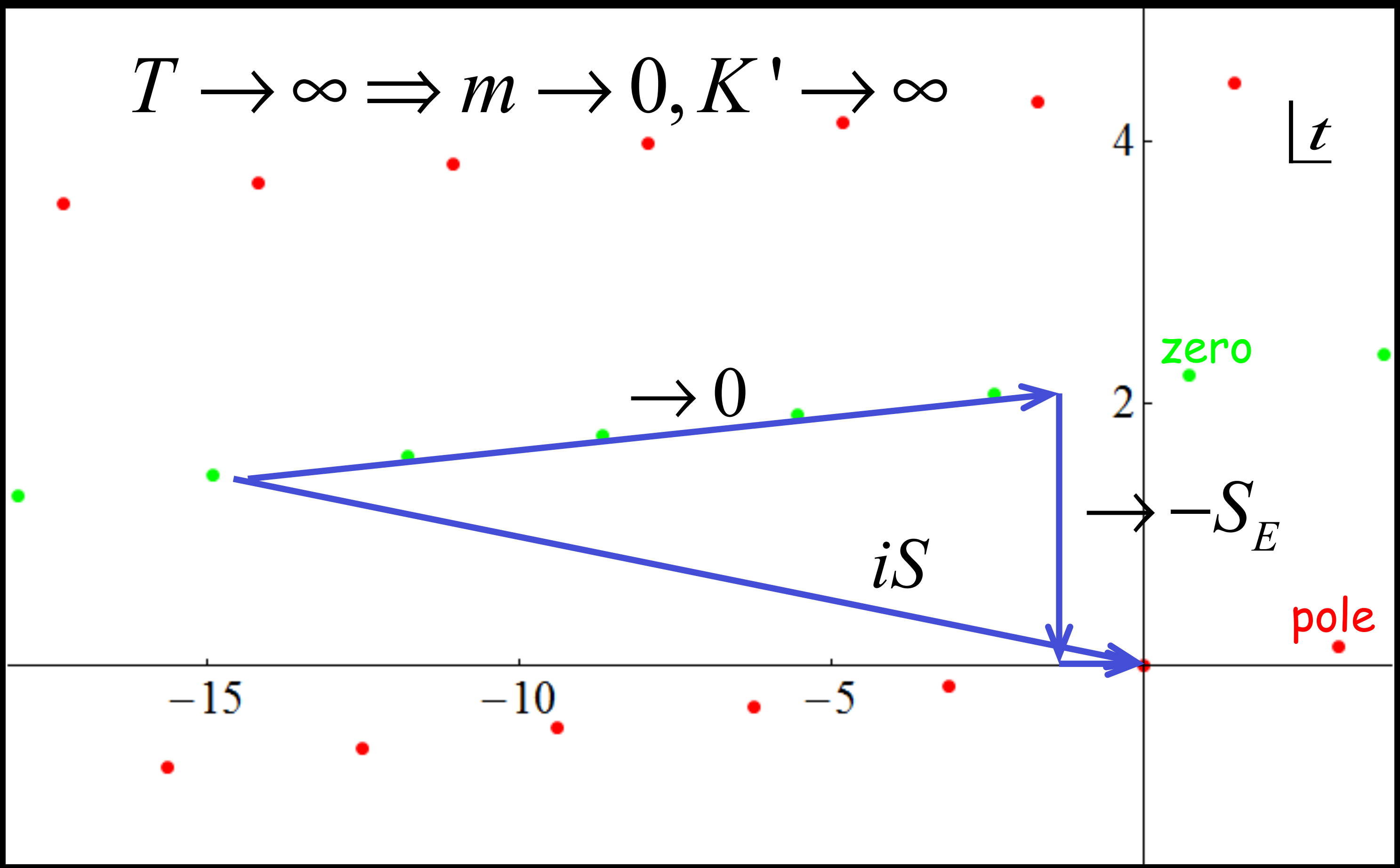
Dominant classical solution

cf Bender, Brody, Hook hep-th 0804.4169

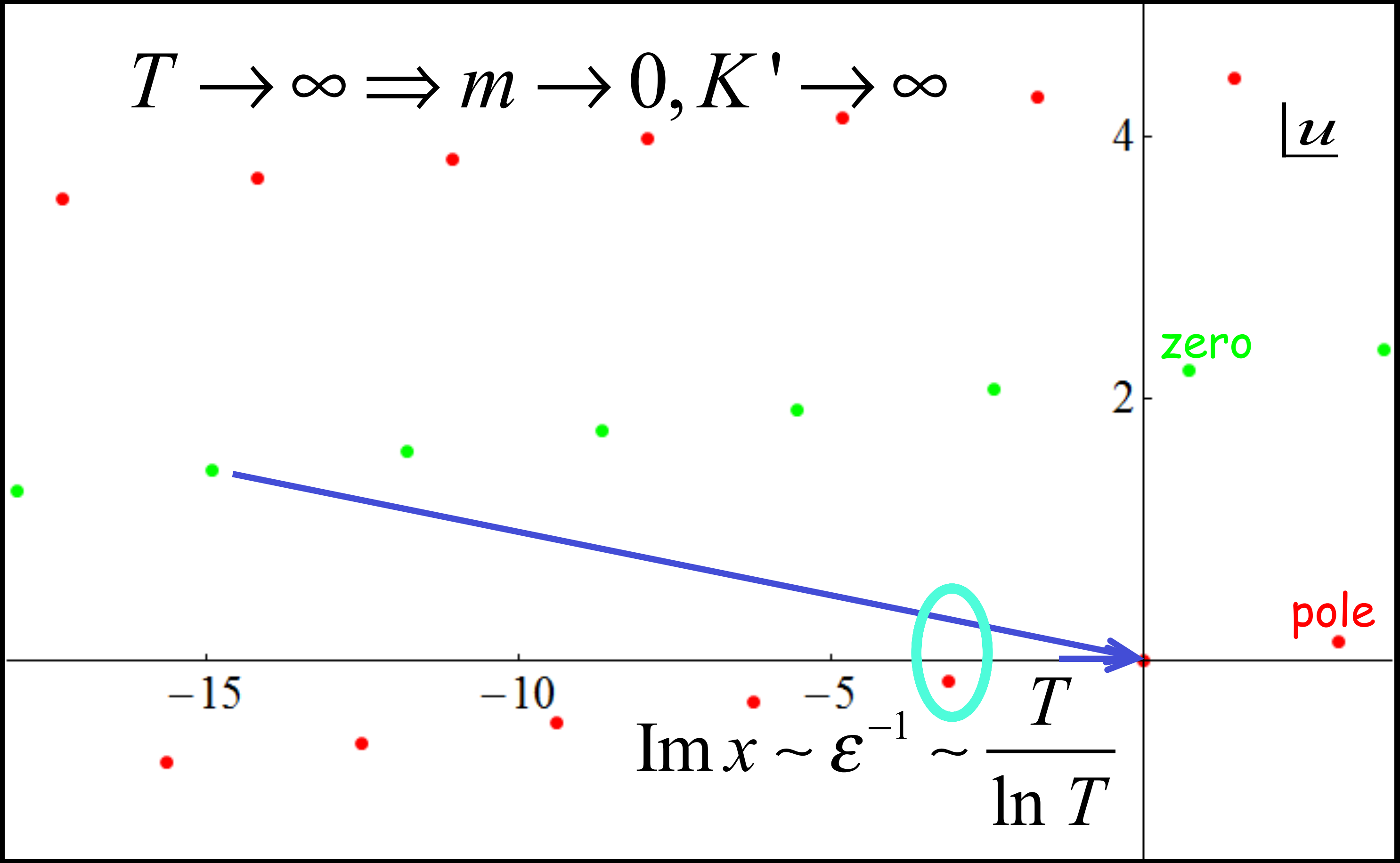
Double periodicity in complex t-plane



Action is a contour integral in t: $\Longrightarrow$ connection with Euclidean bounce



Imaginary part of solution becomes very large, just prior to tunneling



Cubic potential

$$V(x) = \tfrac{1}{2}\kappa x^2 - \tfrac{1}{3}\lambda x^3 \qquad t \to \sqrt{\tfrac{m}{\kappa}}\,t;\; x \to \tfrac{\kappa}{\lambda}\,x$$

$$\tfrac{i}{\hbar} S = i\frac{\kappa^{\frac{5}{2}} m^{\frac{1}{2}}}{\hbar\lambda^2}\int_{t_i}^{t_f} dt\ \left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right)$$

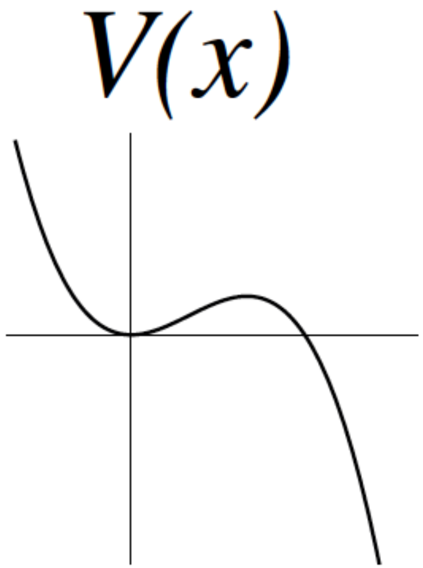
dimensionless

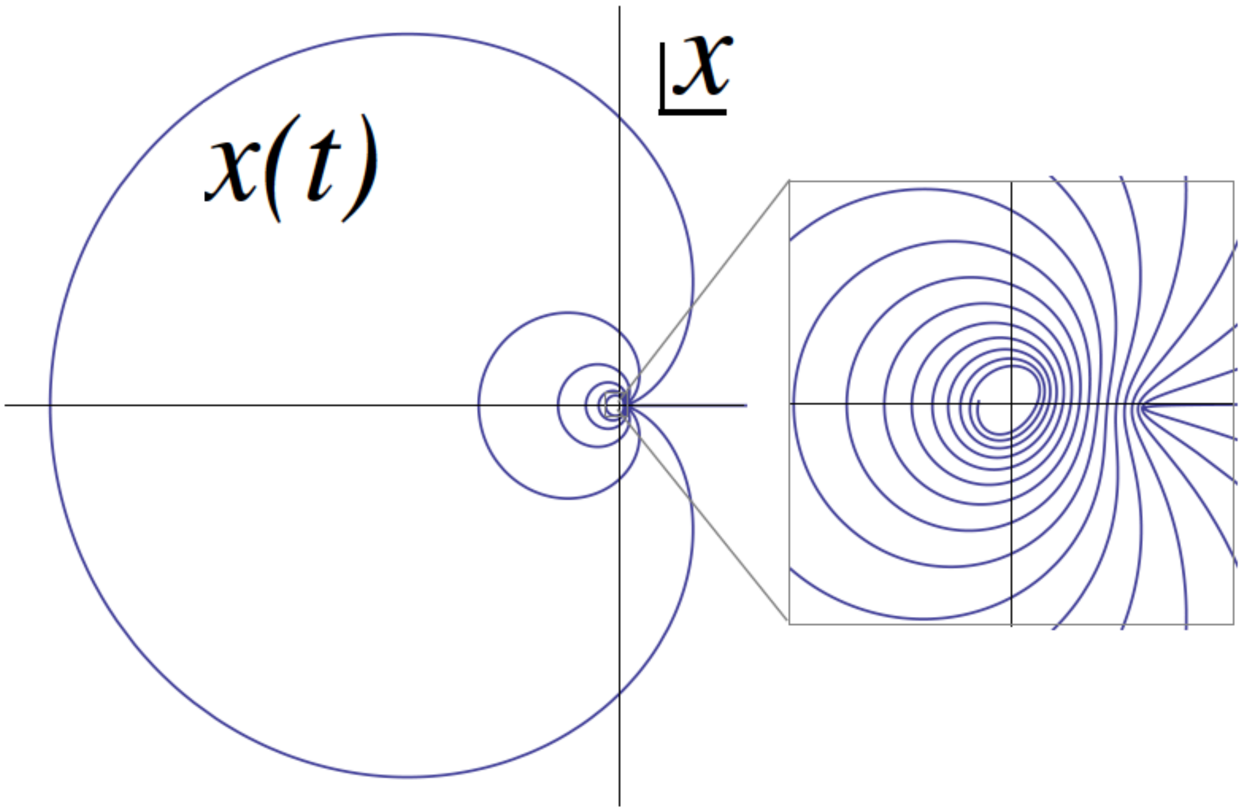
General classical solution (Weierstrass)

$$x(t) = A + \frac{B}{sn\left(C(t_0 - t)\middle| m\right)^2};$$

$$E = \frac{1}{12} - \frac{(2m-1)(m-2)(m+1)}{24\left[1+m(m-1)\right]^{3/2}} \approx \frac{9}{32} m^2, \ |m| \ll 1;$$

$$A = \frac{1}{2}\left(1 - \frac{1+m}{\sqrt{1+m(m-1)}}\right); \ B = \frac{3}{2\sqrt{1+m(m-1)}}; \ C = \frac{1}{2\sqrt[4]{1+m(m-1)}}$$


V(x)


x(t)
x

Even larger $\mathrm{Im}(x)$ just before tunneling due to double poles in complex t-plane

Couple to measuring device (pointer):

$$H_x \rightarrow H_x + P^2/2M + gPx\delta(t - t_m)$$

where $g << 1$.

Pointer momentum P commutes with Hamiltonian $\Rightarrow$ work in momentum basis

Interaction has this effect:

$$\Psi(x, P, t_m^+) = e^{-igPx/\hbar}\Psi(x, P, t_m^-)$$

If Ψ for pointer is Gaussian of width L_{pt}, then for small g, effect on pointer is

$$\langle X \rangle \rightarrow \langle X \rangle + g\,\mathrm{Re}\big(x(t_m)\big)$$

$$\langle P \rangle \rightarrow \langle P \rangle + \frac{g\hbar}{2L_{pt}^2}\,\mathrm{Im}\big(x(t_m)\big)$$

quantum post-selection bias

For $g\,\mathrm{Im}(x) \ll L_{pt}$ this shift in $\langle P \rangle$ is a small fraction of the quantum uncertainty in P, *i.e.*, L_{pt}.

Nevertheless, it can be measured with arbitrary accuracy if the experiment and the weak measurement are repeated a sufficiently large number of times

(Aharonov et al.)

For a measurement performed a quarter-period before the particle tunnels,

$$\mathrm{Im}(x) \sim \hbar e^{S_E/\hbar} \rightarrow \infty \text{ as } \hbar \rightarrow 0 \text{ !}$$

Experimental tests may be possible in quantum dots

(w/ J. Taylor, UMD/NIST)

Extensions and generalisations:

* vary initial Gaussian: L, x_c, p_c
* vary shape of initial wavepacket
* include time-dependent forcing

* higher dimensions
* quantum field theory
* electroweak vacuum stability
* black hole evaporation

Quantum Field Theory

* harder: infinite number of degrees of freedom

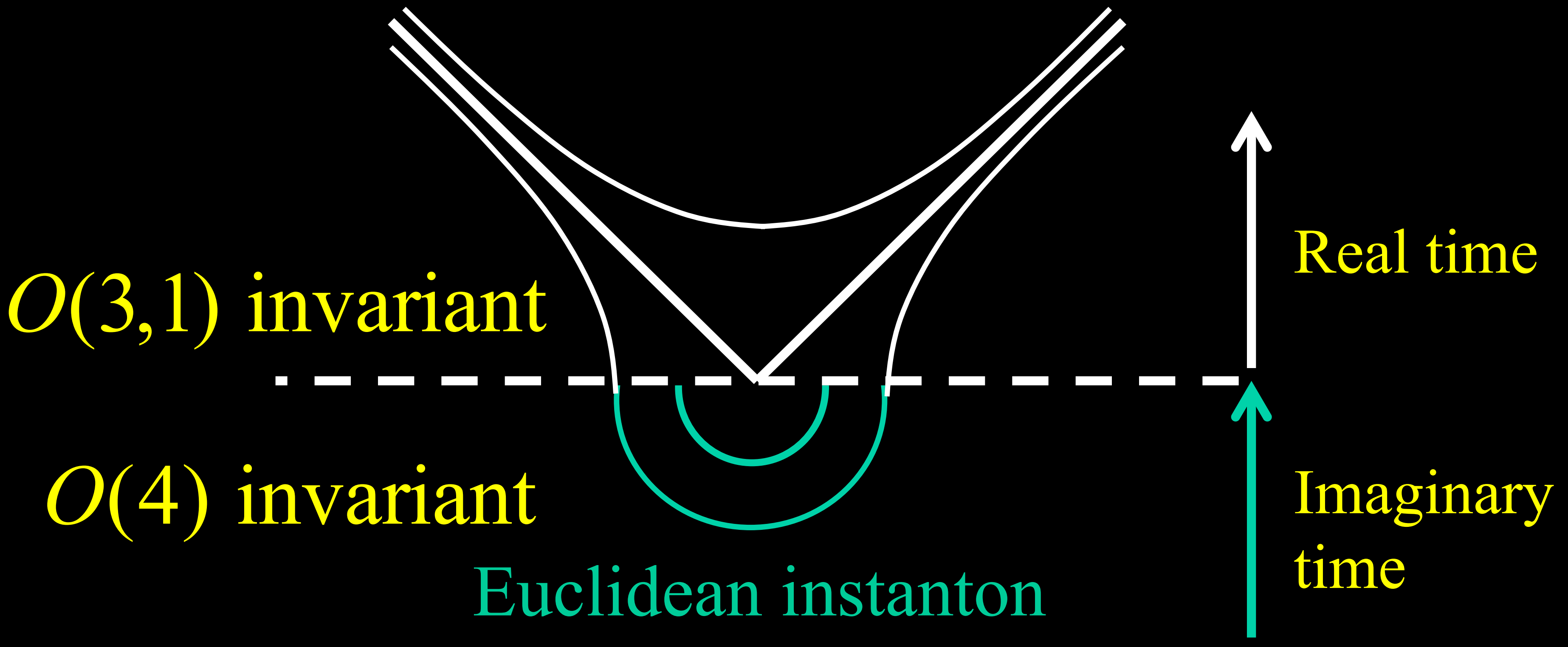
* initial ‘false vacuum’ wavefunctional

* IMPORTANT: this state defines a preferred frame, because it is not the true, Lorentz-invariant ground state

* A Lorentz-invariant solution (of the Callan-Coleman type) is necessarily time-reversal invariant and hence not the semiclassical solution we seek

* Nonetheless, its spatial profile provides a good ansatz for the emerging bubble in the large tunneling time limit.

Bubble Nucleation: Euclidean Approach

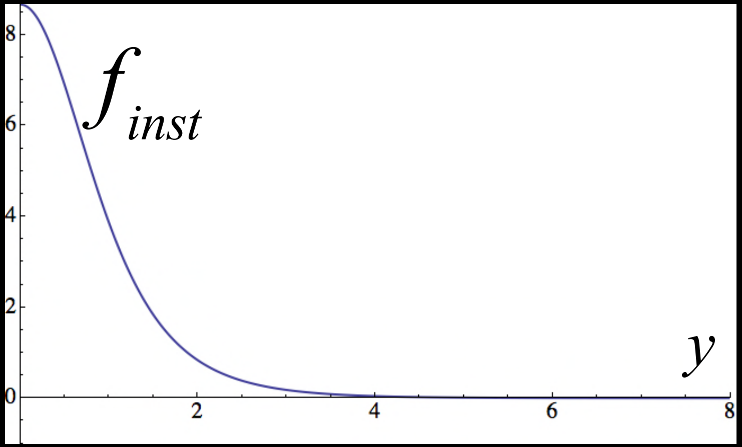


Bubble nucleation in flat spacetime

$$S = \int dt d^3x \left(\tfrac{1}{2}\left(\partial_t \phi\right)^2 - \tfrac{1}{2}\left(\nabla \phi\right)^2 - \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{3} \lambda \phi^3 \right)$$

$$y^\mu = m x^\mu,\ \phi = \tfrac{m^2}{\lambda} f \Rightarrow S_E = \tfrac{m^2}{\lambda^2} \int d^4 y \left(\tfrac{1}{2}\left(\nabla f\right)^2 + \tfrac{1}{2} f^2 - \tfrac{1}{3} f^3 \right)$$

$S =$



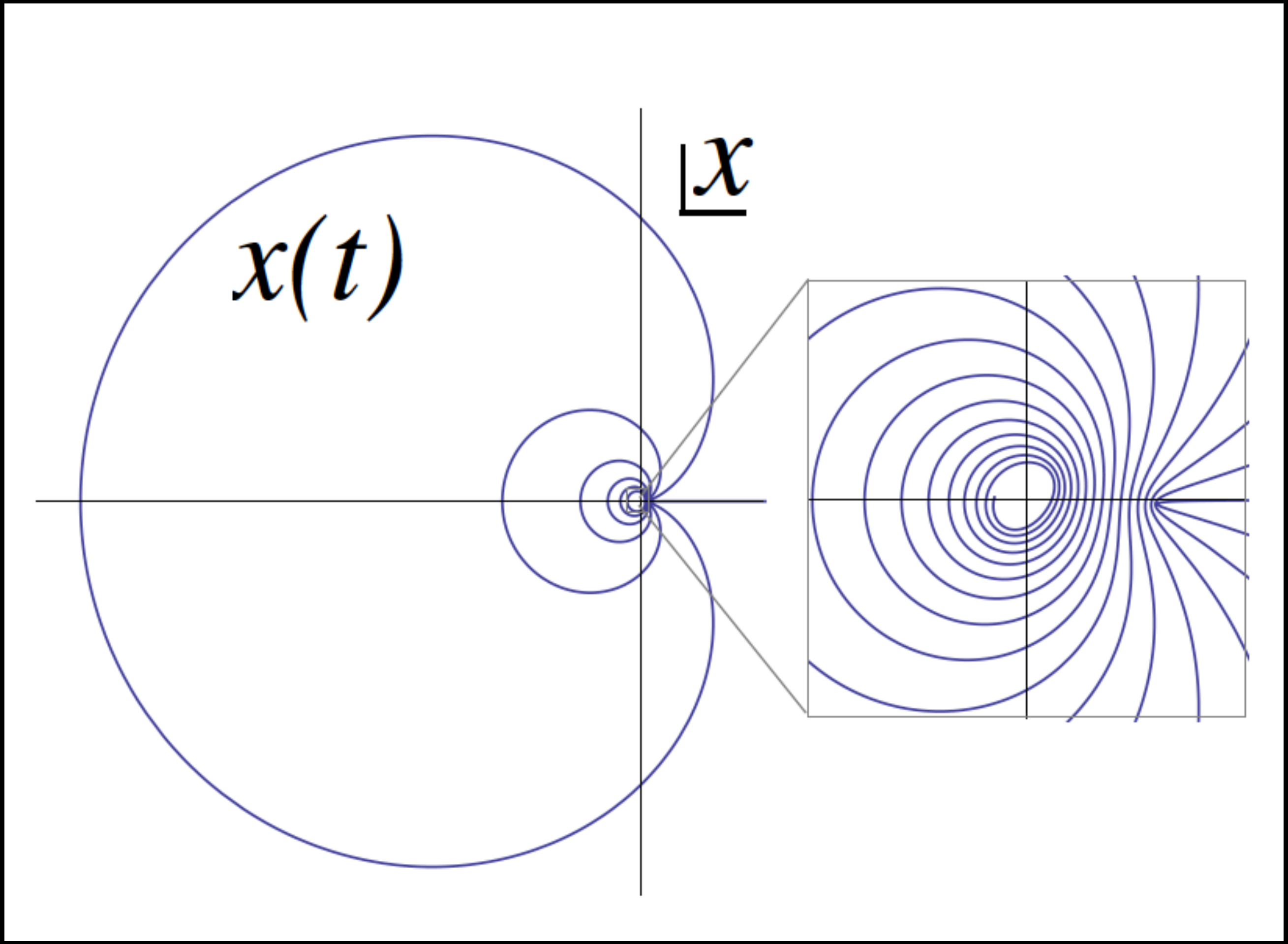
Real-time ansatz

$$\phi = \tfrac{m^2}{\lambda} f_{inst}(mr) x(mt);\ \bar{t} \equiv mt$$

$$S = \tfrac{m^2}{\lambda^2} \int d\bar{t} \left(\tfrac{1}{2} a \dot{x}^2 - \tfrac{1}{2} b x^2 + \tfrac{1}{3} c x^3 \right)$$

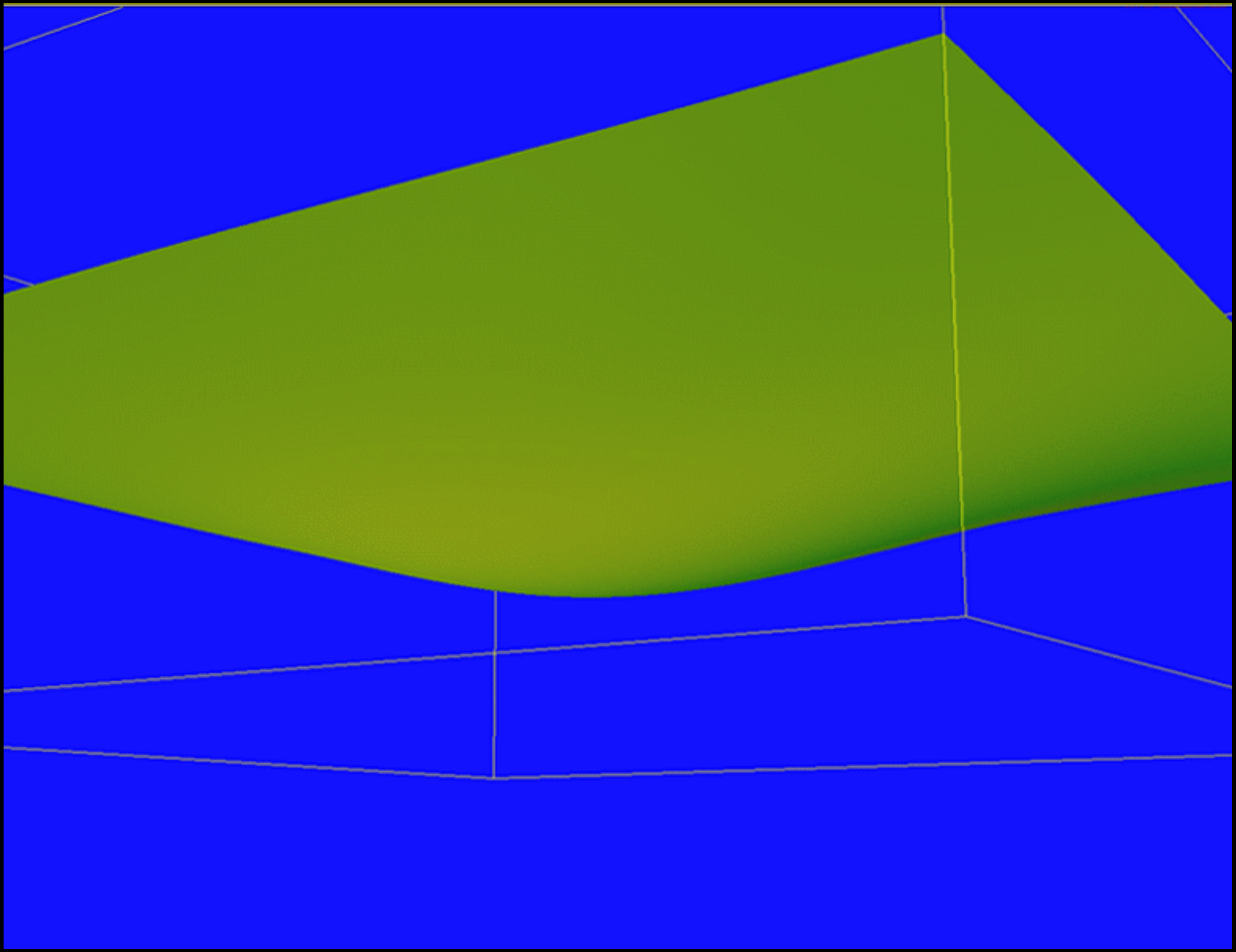
where a, b, c are various moments of $f \Rightarrow S_E \approx 1.04 S_{E,inst}$

Suggests this should be an excellent approximation



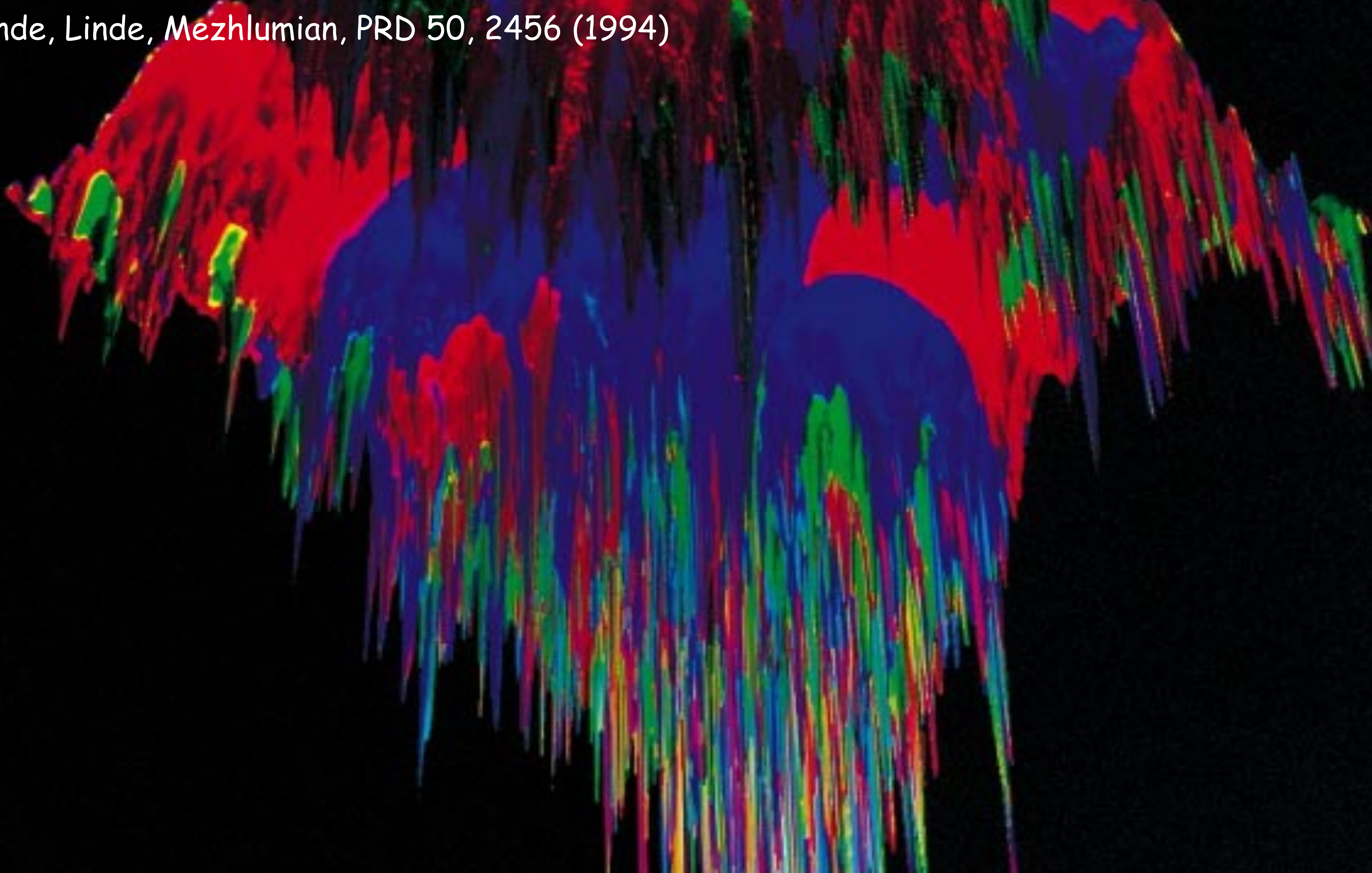
Ansatz may be systematically improved using linear theory response

The Inflationary ‘Multiverse’



Linde, Linde, Mezhlumian, PRD 50, 2456 (1994)

Linde, Linde, Mezhlumian, PRD 50, 2456 (1994)

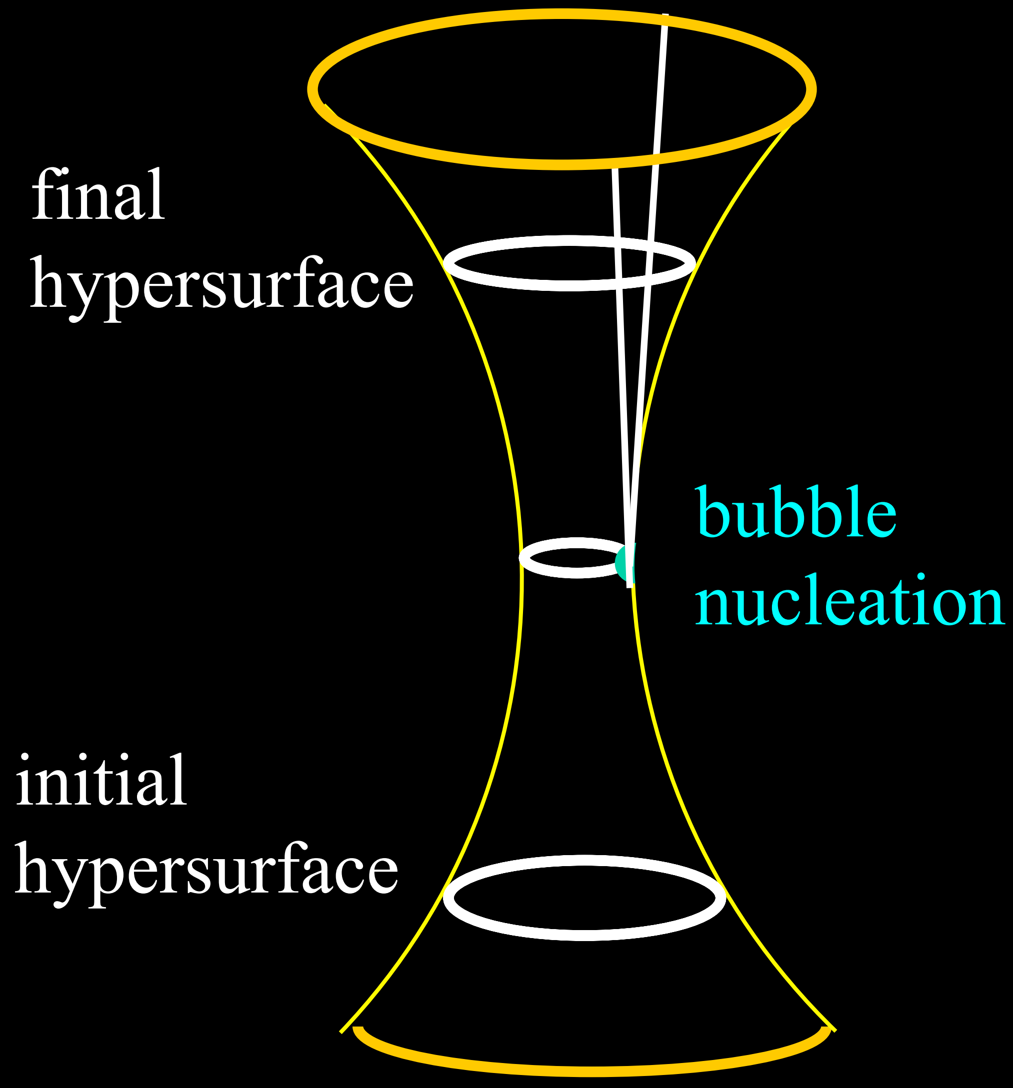


"Anything that can happen will happen
- and it will happen an infinite number of times"

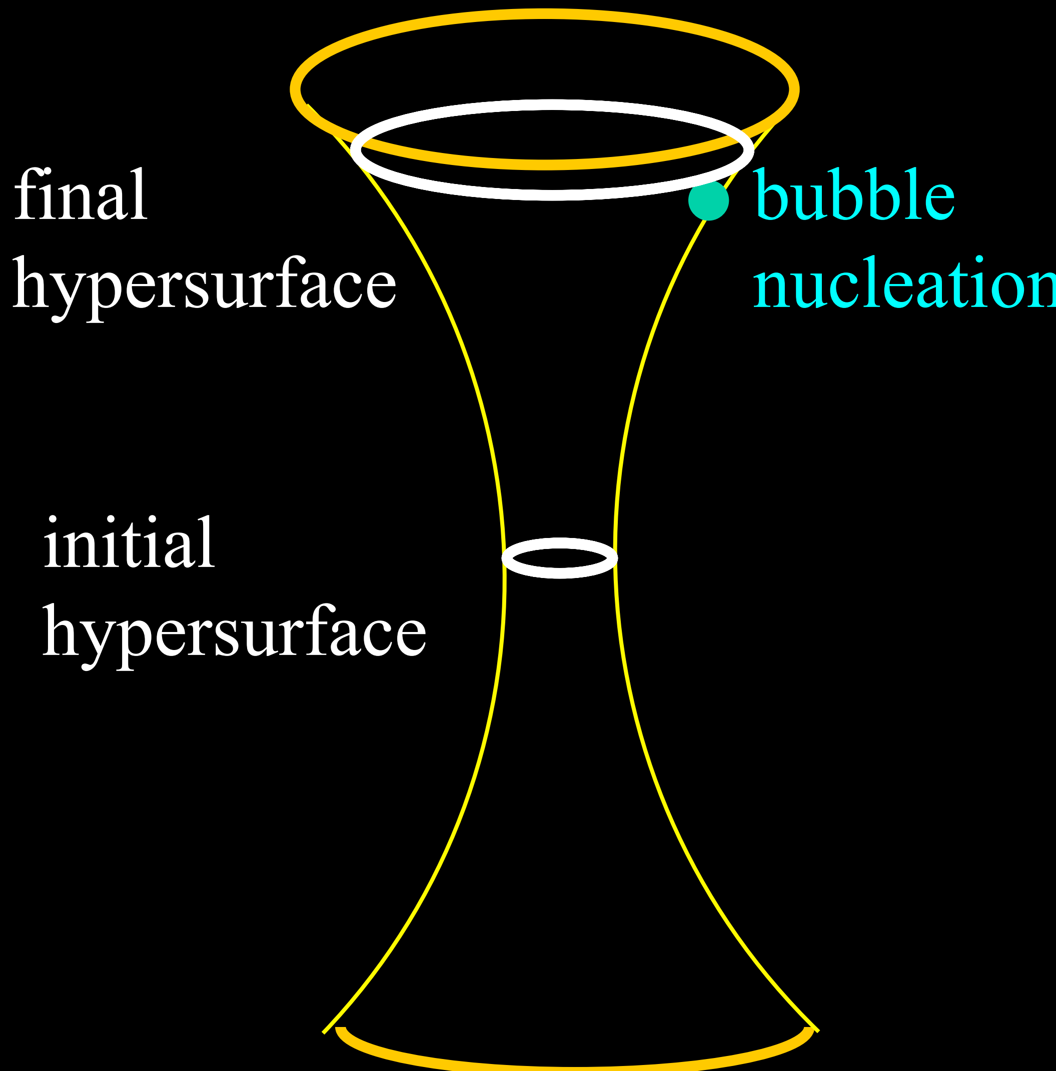
Guth

Note: the treatment of quantum effects is very heuristic in this, and other discussions of the ‘multiverse’ (e.g. Susskind *et al.*)

Bubble nucleation in de Sitter spacetime provides an ideal setting to explore these questions


final
hypersurface
bubble
nucleation
initial
hypersurface

If, instead, the initial hypersurface is chosen ‘at the throat’ and we try to describe a bubble which nucleates much later, then damping of field oscillations due to the exponential expansion of the universe has a big effect. It seems to me likely that no solution of the desired form exists.


final
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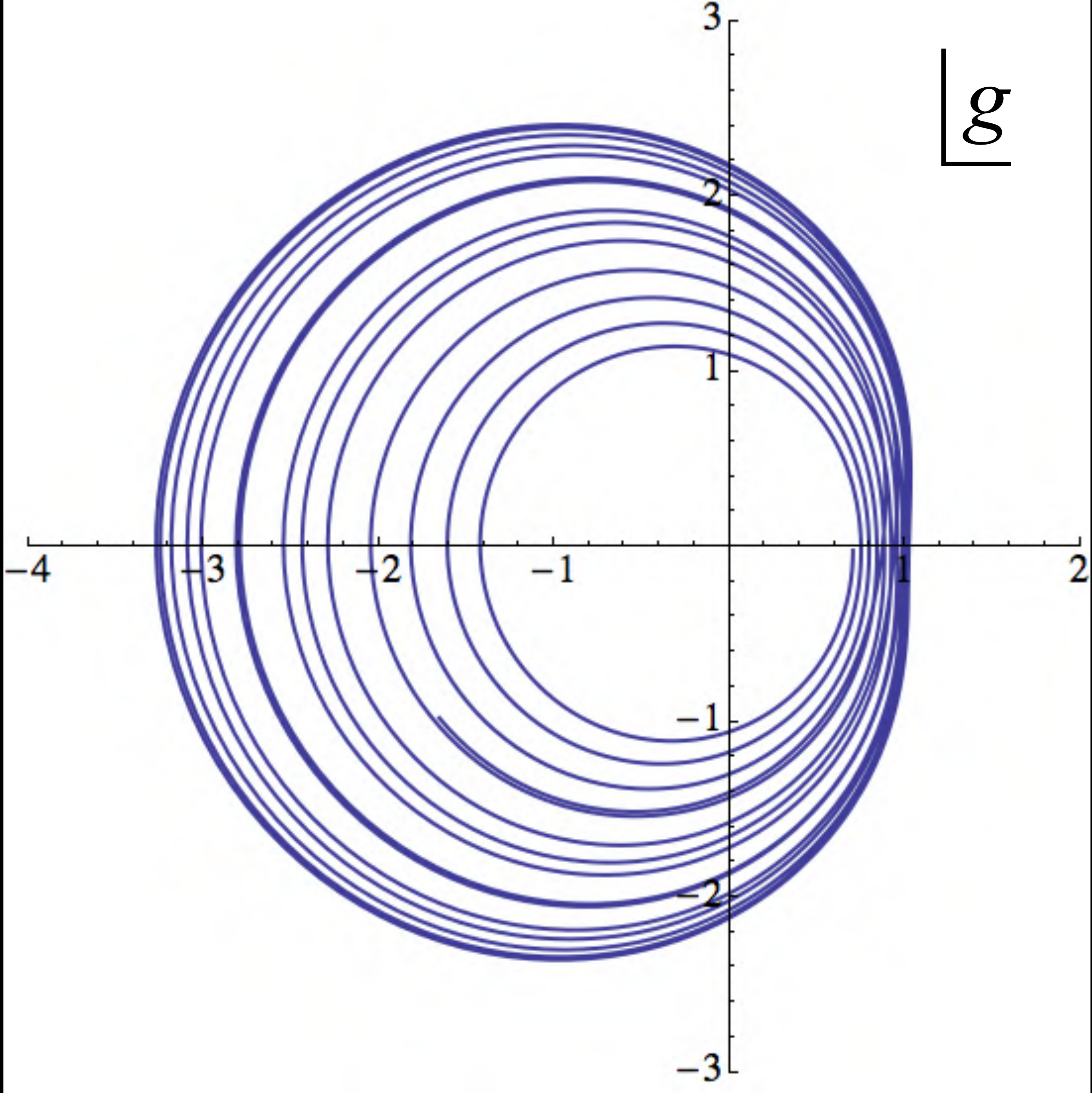
initial
hypersurface


g
3
2
1
−1
−2
−3
−4
−3
−2
−1
1
2

The above discussion suggests that many-bubble 'inflationary multiverse' is inconsistent with the semiclassical approximation because there is no classical solution describing the nucleation of a bubble long after the initial hypersurface.

This is consistent with the Gibbons-Hawking calculation of the entropy of de Sitter spacetime – there are a finite number of states, so you just cannot have infinitely many independent bubbles

Interesting implications for today's metastable Higgs vacuum… and for black holes

Thank you!