PT-symmetry and the double-scaling (correlated) limit in quantum field theory

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Uncorrelated limits:

Suppose you have TWO parameters in a physics problem:

(1) perturbation parameter ε $(\varepsilon \ll 1)$

(2) coupling constant α

For *fixed* α, perturbative solution *S*(ε,α) is conventional *uncorrelated* perturbation series:

$$S(\varepsilon, \alpha) \sim \sum_{n=0}^{\infty} a_n(\alpha)\varepsilon^n$$

Correlated limits:

A nontrivial *correlated limit* arises if α is *not* fixed, but tends to a critical value as ε tends to 0:

$$\alpha \longrightarrow \alpha_{\text{crit}} \text{ as } \varepsilon \longrightarrow 0$$

In a correlated limit:

(1) All terms in perturbation series become comparable as $\varepsilon \longrightarrow 0$

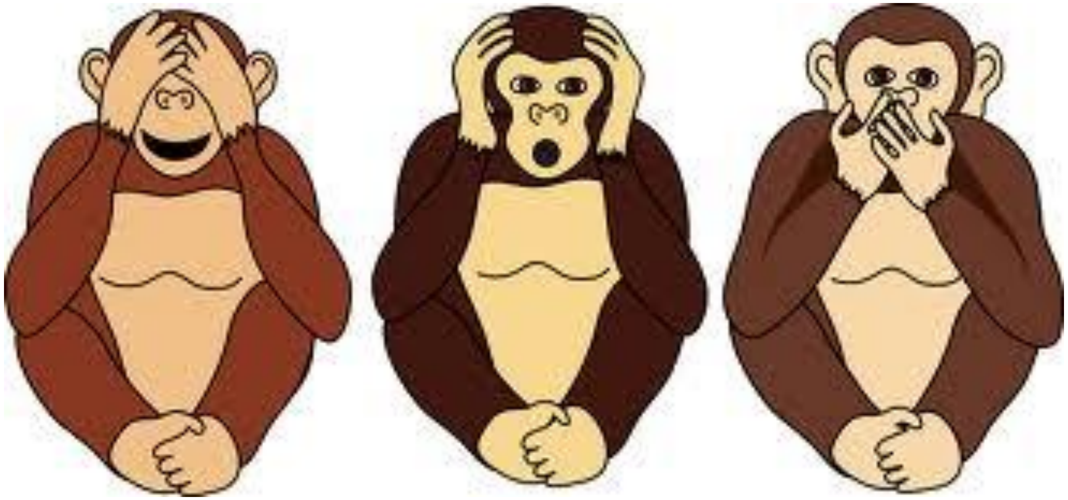
(2) Series *S* undergoes a transmutation -- it depends on *one* parameter γ, which is a combination of ε and α: *S* = *S*(γ)

(3) *S*(γ) still diverges but can be Borel summed

(4) When summed, *S*(γ) is a universal function (describes essentials of problem but is insensitive to specific details)

(5) Often, *S*(γ) is *entire* (analytic for all γ)

Three examples of correlated limits...

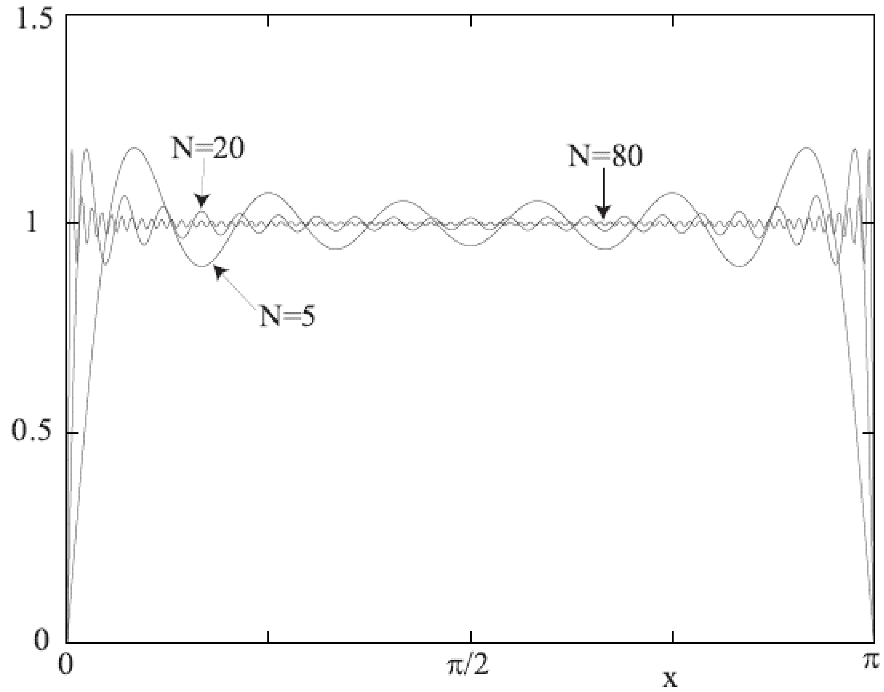


Example 1: Nonuniformly convergent Fourier sine series at edge of interval of convergence as a correlated limit

$$f(x) = \sum_{1}^{\infty} a_n \sin(nx)$$

$(0 < x < \pi)$

Partial sum of first *N* terms for the function *f*(*x*) = 1:



$$\sum_{1}^{N} a_n \sin(nx)$$

$$N \to \infty, \qquad x \to 0, \qquad \gamma \equiv Nx$$

This *correlated limit* is described by the *Gibbs function G*(γ) = Si(2γ)

***G*(γ) is *entire* and *universal*:**
Valid for all functions *f*(*x*) that do not vanish at *x* = 0 and/or *x* = π

Example 2: ***One-turning-point problem***:

Transition in QM wave function from classically allowed to classically forbidden region as a correlated limit

$$\hbar^2 \phi''(x) = Q(x)\phi(x) \qquad Q(x) \sim ax$$

WKB series for wave function away from turning point:

$$\phi_{\mathrm{WKB}}(x) = \exp\left[\frac{1}{\hbar}\int_0^x ds \sum_{n=0}^{\infty} \hbar^n S_n(s)\right] \quad (\hbar \to 0)$$

Correlated limit:

$$\hbar \to 0 \qquad x \to 0 \qquad \gamma = a^{1/2} x^{3/2} / \hbar$$

$$\boxed{\phi(\gamma) = c\mathrm{Ai}(\gamma)}$$

Solution to one-turning-point problem is *Airy* function: Solution is *entire* and *universal* [valid for all potentials *Q*(*x*) that vanish linearly at the turning point]

Example 3: Laplace's method for asymptotic expansion of integrals

Laplace integral $Z(N) = \int_0^\infty dr\, e^{-NS(r)}$ **for large *N*:**

Assume that $S'(r) > 0$

Repeated integration by parts gives complete asymptotic expansion:

$$Z(N) \sim e^{-NS(0)} \sum_{k=1}^{\infty} N^{-k} \left[\frac{1}{S'(r)} \frac{d}{dr} \right]^{k-1} \frac{1}{S'(r)} \Big|_{r=0}$$

(This is an uncorrelated expansion for large *N*)

Suppose $S'(0)$ is small,
but higher derivatives of *S*(*r*) are not small at *r* = 0

As $S'(0)$ → 0, *k*th term in series approximated by

$$N^{-k}[-2S''(0)]^{k-1}[S'(0)]^{1-2k}\Gamma(k-1/2)/\Gamma(1/2)$$

because this has greatest number of
powers of $S'(0)$ in denominator

Correlated limit:

$N \to \infty$, $S'(0) \to 0$, $\gamma^2 \equiv N[S'(0)]^2/S''(0)$ is fixed

Assume that $S''(0) > 0$ so that $\gamma^2 > 0$

$$Z(\gamma) \sim \frac{e^{-NS(0)}}{\sqrt{NS''(0)}} \sum_{k=0}^{\infty} (-2)^k \gamma^{-2k-1} \frac{\Gamma(k+1/2)}{\Gamma(1/2)}$$

Series diverges, but Borel sum is a *parabolic cylinder function*:

$$Z(\gamma) \sim e^{-NS(0)} \exp\left(\gamma^2/4\right) \mathrm{D}_{-1}(\gamma)/\sqrt{NS''(0)}$$

***Z*(γ) is *entire*. It is *universal* -- depends only on two numbers, *S*(0) and $S''(0)$. *Z*(γ) applies *universally* to all functions *S*(*r*) with these two particular values.**

[Uncorrelated series depends on all derivatives of *S*(*r*) at *r* = 0.]

For the special value $\gamma = 0$, $D_{-1}(0) = \sqrt{\pi/2}$ gives the famous result known as Laplace's method

$$Z(N) \sim e^{-NS(0)}\sqrt{\pi/[2NS''(0)]} \quad (N \to \infty)$$

Laplace's method is a limiting case of the correlated limit for which $S'(0)$ = 0 and $S''(0)$ > 0. Correlated limit describes approach of *Z*(γ) to Laplace's formula.

SUMMARY OF THE THREE EXAMPLES:

Laplace's method is a correlated limit that describes in a *universal* fashion what happens as derivative of $S(r)$ approaches 0 at the Laplace point, just as Gibbs function describes in a *universal* fashion how a nonuniformly convergent Fourier series for $f(x)$ behaves as x approaches the boundary of the interval, and just as the Airy function describes the *universal* transition at a turning point.

BIG problem with correlated limit in QFT…

Uncorrelated large-*N* expansion for an O(*N*) QFT in 0 dimensions

Partition function:

$$Z = \int d^{N+1}x \exp\left[-\frac{1}{2}\sum_{n=1}^{N+1} x_n^2 - \frac{\lambda}{4}\left(\sum_{n=1}^{N+1} x_n^2\right)^2\right]$$

Rotational symmetry:

$$\sum_{n=1}^{N+1} x_n^2 = Nr^2$$

$$\lambda = g/N$$

$$Z = \mathcal{A}_{N+1}\int_0^\infty dr\, e^{-NL(r)} \qquad L(r) = r^2/2 + gr^4/4 - \log r$$

Note: *g* must be *positive* so that this integral representation for *Z* converges!!

Laplace's method: Locate the Laplace points – zeros of

$$L'(s) = r + gr^3 - 1/r$$

One Laplace point lies in the range of integration $0 \leq r < \infty$:

$$r_0 = \sqrt{(G-1)/(2g)}$$

$$G \equiv \sqrt{1+4g}$$

$$Z \sim \frac{\mathcal{A}_{N+1} e^{-NL(r_0)}}{\sqrt{NG/\pi}} \sum_{k=0}^{\infty} a_k N^{-k} \quad (N \to \infty)$$

$$a_0 = 1$$

$$a_1 = \frac{5 - 6G^2 - G^3}{24G^3}$$

$$a_2 = \frac{385 - 924G^2 - 10G^3 + 684G^4 + 12G^5 - 143G^6}{1152G^6}$$

This is the uncorelated large-*N* asymptotic expansion of the partition function *Z*

Correlated limit of the large-N expansion

For all terms in the expansion to have the same order of magnitude, the correlated limit must be

$$N \to \infty$$

and

$$g \to g_{\text{crit}} = -1/4$$

(that is, $G \to 0$)

with

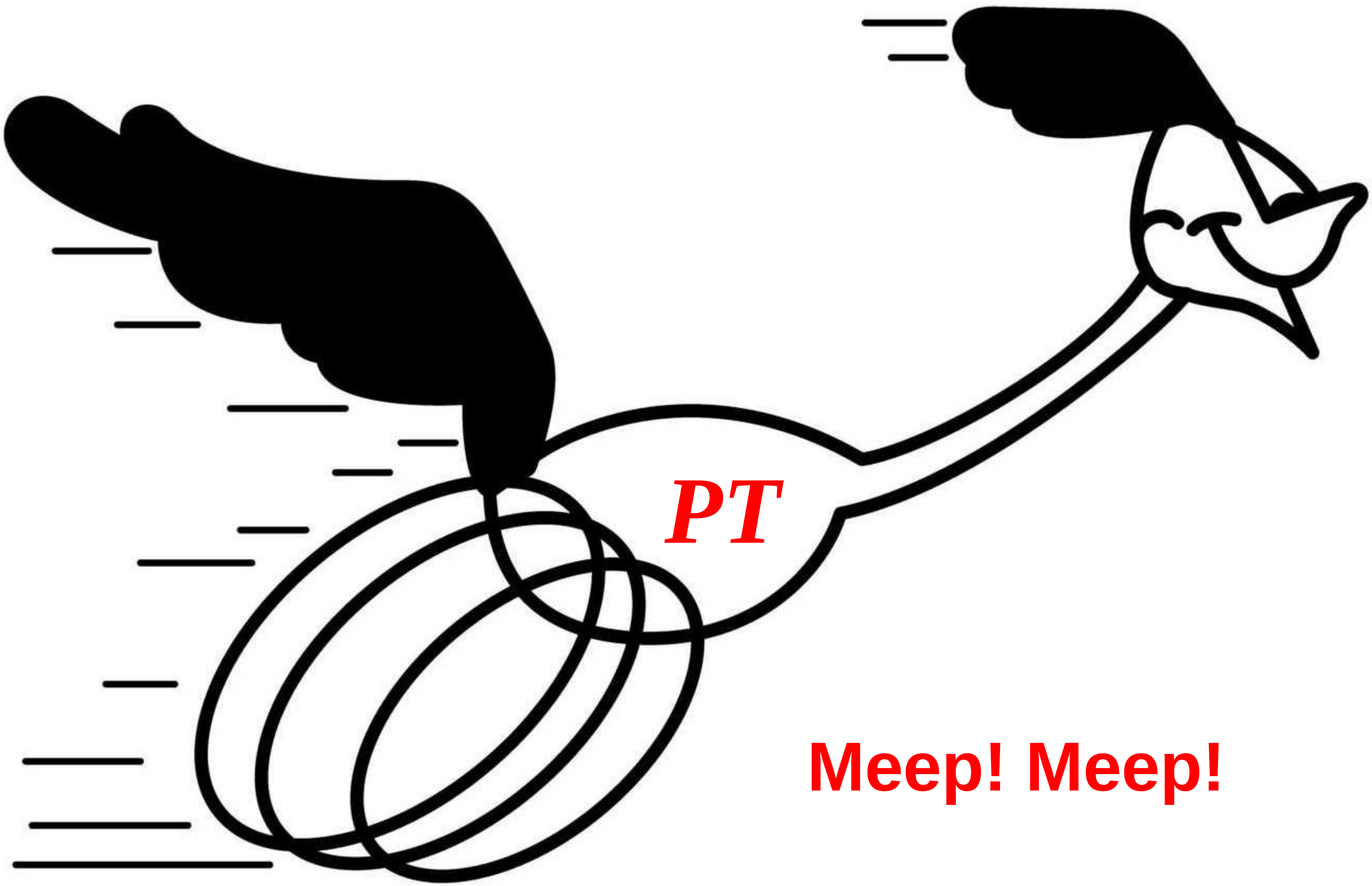
$$\gamma \equiv NG^3/2$$

G **measures departure from critical coupling**

$$Z \sim \frac{\mathcal{A}_{N+1} e^{-NL(r_0)}}{\sqrt{NG/\pi}} \left(1 + \frac{5}{48\gamma} + \frac{385}{4608\gamma^2} + \ldots \right)$$

Disaster! Correlated limit is invalid.
Requires that *g* < 0. Series is a *nonalternating* divergent series and thus not Borel summable.

PT-symmetric quantum mechanics to the rescue…



Meep! Meep!

PT-symmetric quantum mechanics:

Hamiltonian is non-Hermitian, but if it is *PT symmetric* – that is, *invariant under combined space and time reflection* – the eigenvalues can still be entirely real and positive!

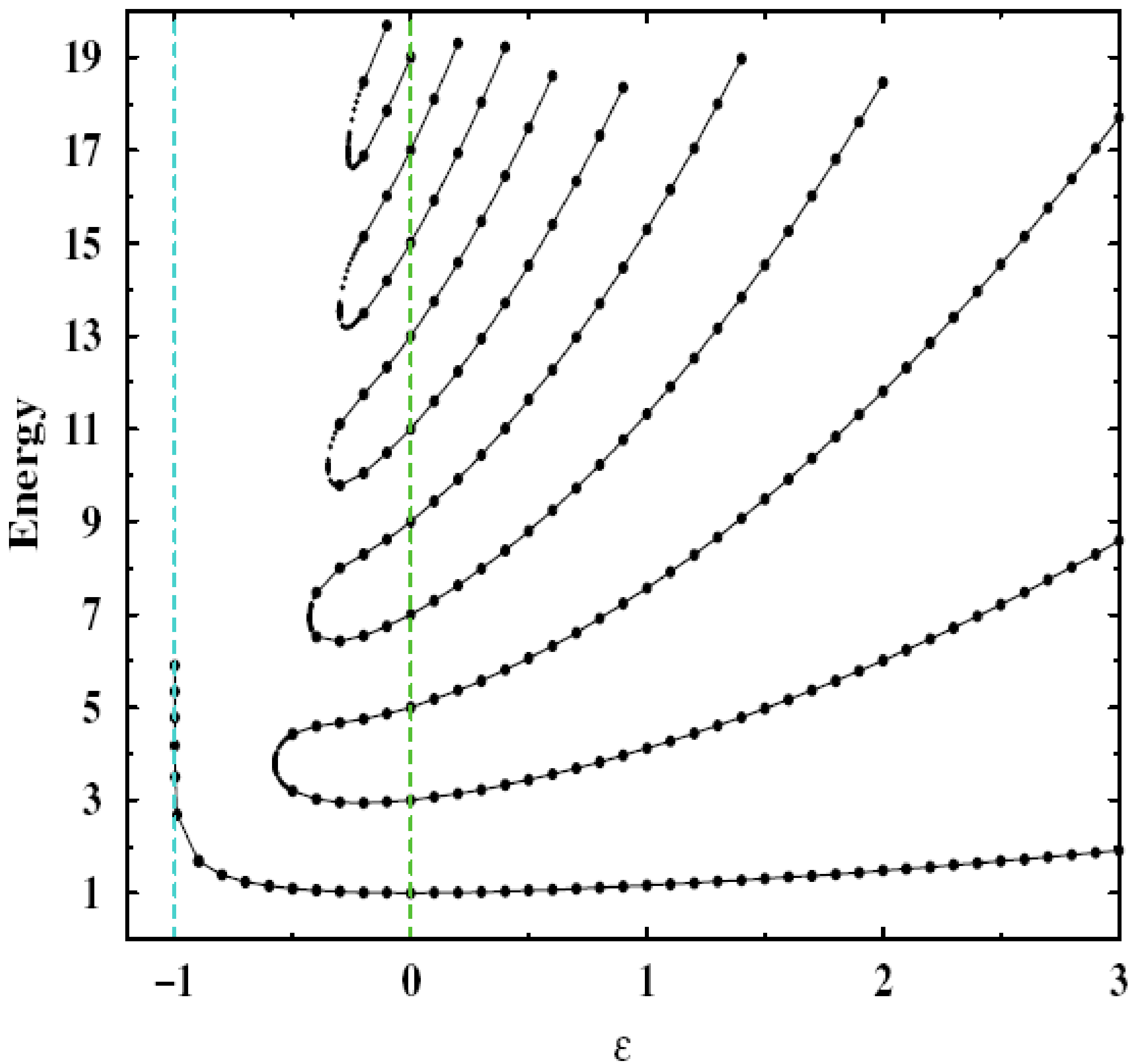
Example: $H = p^2 + ix^3$

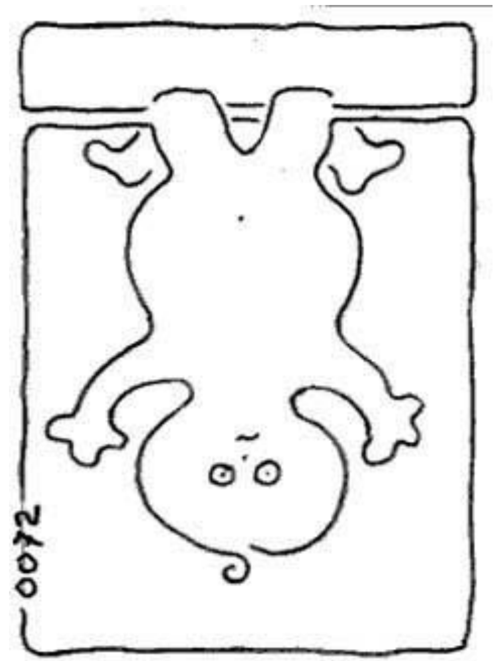
This Hamiltonian has *PT* symmetry!

Moreover, the Hamiltonian is self-adjoint with respect to a new adjoint; namely *CPT*. The Hilbert space metric is positive definite and time evolution is unitary.

A class of *PT*-symmetric Hamiltonians:

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$





Note: ε = 2 gives an upside-down potential with positive discrete eigenvalues!

CMB and S. Boettcher
***Physical Review Letters* 80**, 5243 (1998)

2 x 2 non-Hermitian *PT*-symmetric Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (r,\ s,\ \theta \text{ real})$$

$\mathcal{T}$ is complex conjugation and $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

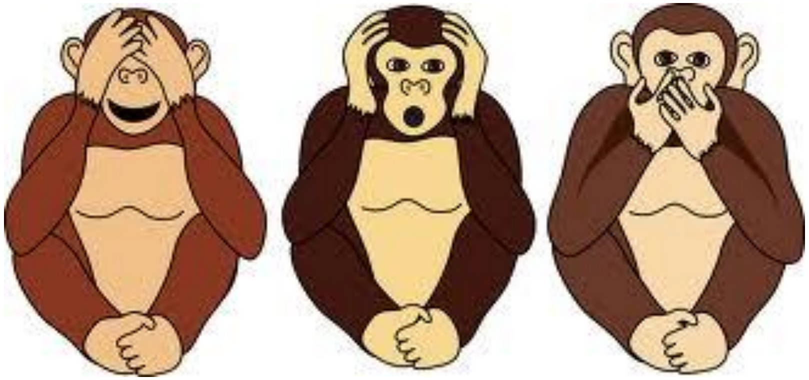
$$E_{\pm} = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \qquad \text{real if } s^2 > r^2\sin^2\theta$$

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}$$

where $\sin\alpha = (r/s)\sin\theta.$

***PT*-symmetric quantum mechanics is fun. You can re-visit what you already know about conventional Hermitian quantum mechanics. And, you can fix problems arising in Hermitian QM!**

Three examples:



1. “Ghost Busting: *PT*-Symmetric Interpretation of the Lee Model,”
CMB, S. Brandt, J.-H. Chen, and Q. Wang
Phys. Rev. D **71,** 025014 (2005) **[arXiv: hep-th/0411064]**

2. “No-ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model,” CMB and P. Mannheim
Phys. Rev. Lett. **100,** 110402 (2008) **[arXiv: hep-th/0706.0207]**

3. “*PT*-Symmetric Interpretation of Double-Scaling”
CMB, M. Moshe, and S. Sarkar
J. Phys. A: Math. Theor. **46,** 102002 (2013) **[arXiv: hep-th/1206.4943]**
and
“Double-Scaling Limit of the O(N)-Symmetric Anharmonic Oscillator”
CMB and S. Sarkar
J. Phys. A: Math. Theor. **46,** 442001 (2013) **[arXiv: hep-th/1307.4348]**

Example 1: Lee Model

$$V \to N + \theta, \qquad N + \theta \to V.$$

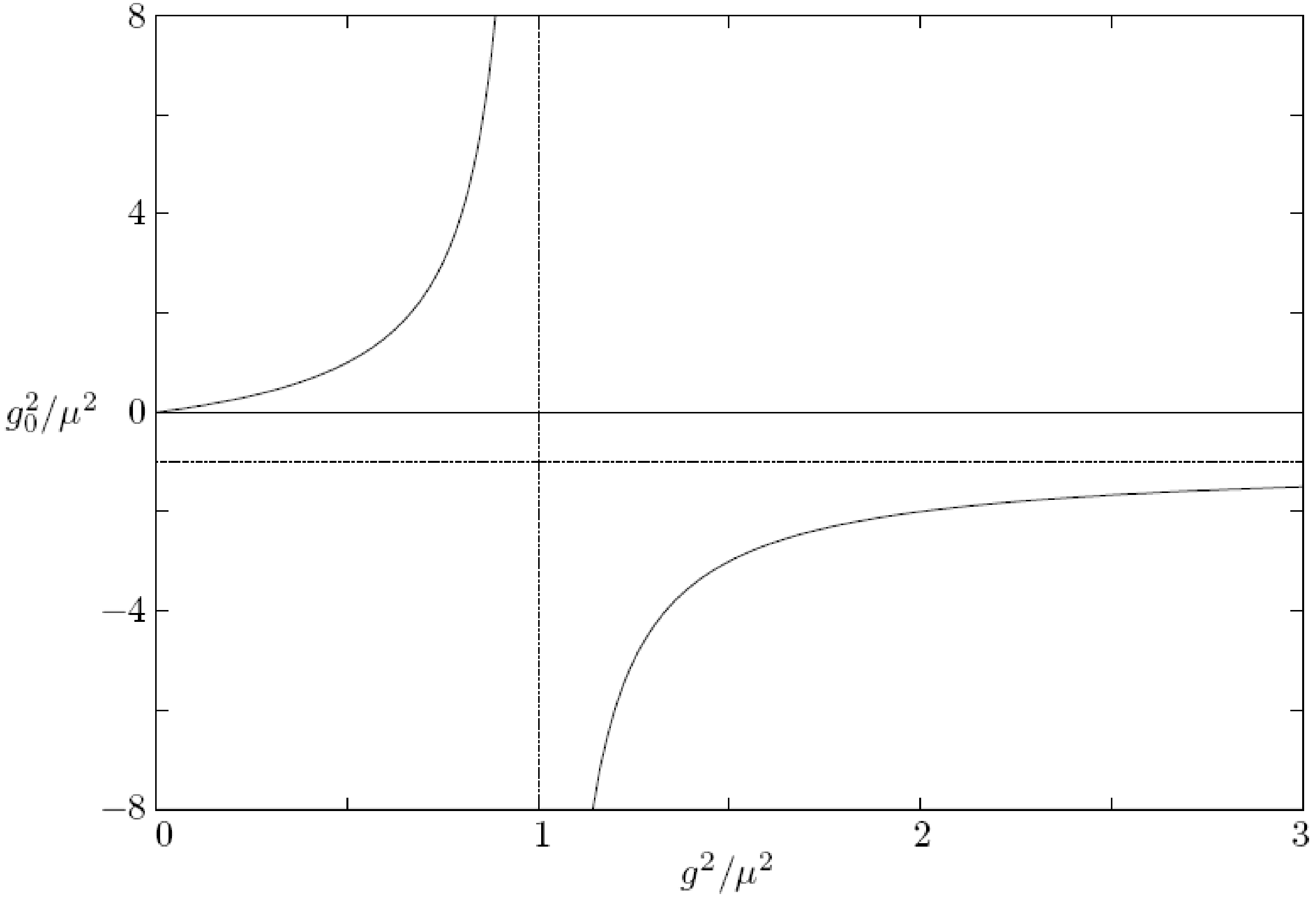
$$H = H_0 + g_0 H_1,$$

$$H_0 = m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a,$$
$$H_1 = V^\dagger N a + a^\dagger N^\dagger V.$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

Problem with the Lee Model:



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

MR0076639 (17,927d) 81.0X
Källén, G.; Pauli, W.
On the mathematical structure of T. D. Lee's model of a renormalizable field theory.
Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), *no. 7, 23 pp.*

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR0064658 (16,317b)] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V \rightleftarrows N+\theta$ but forbidding $N \rightleftarrows V+\theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/{g_0}^2 = 1 - Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio $g^2/{g_0}^2$ should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR0046306 (13,713d); Källén, Helv. Phys. Acta **25** (1952), 417–434; MR0051156 (14,435l); Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR0072756 (17,332e); Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR0064652 (16,315e)]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value g_{crit}. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

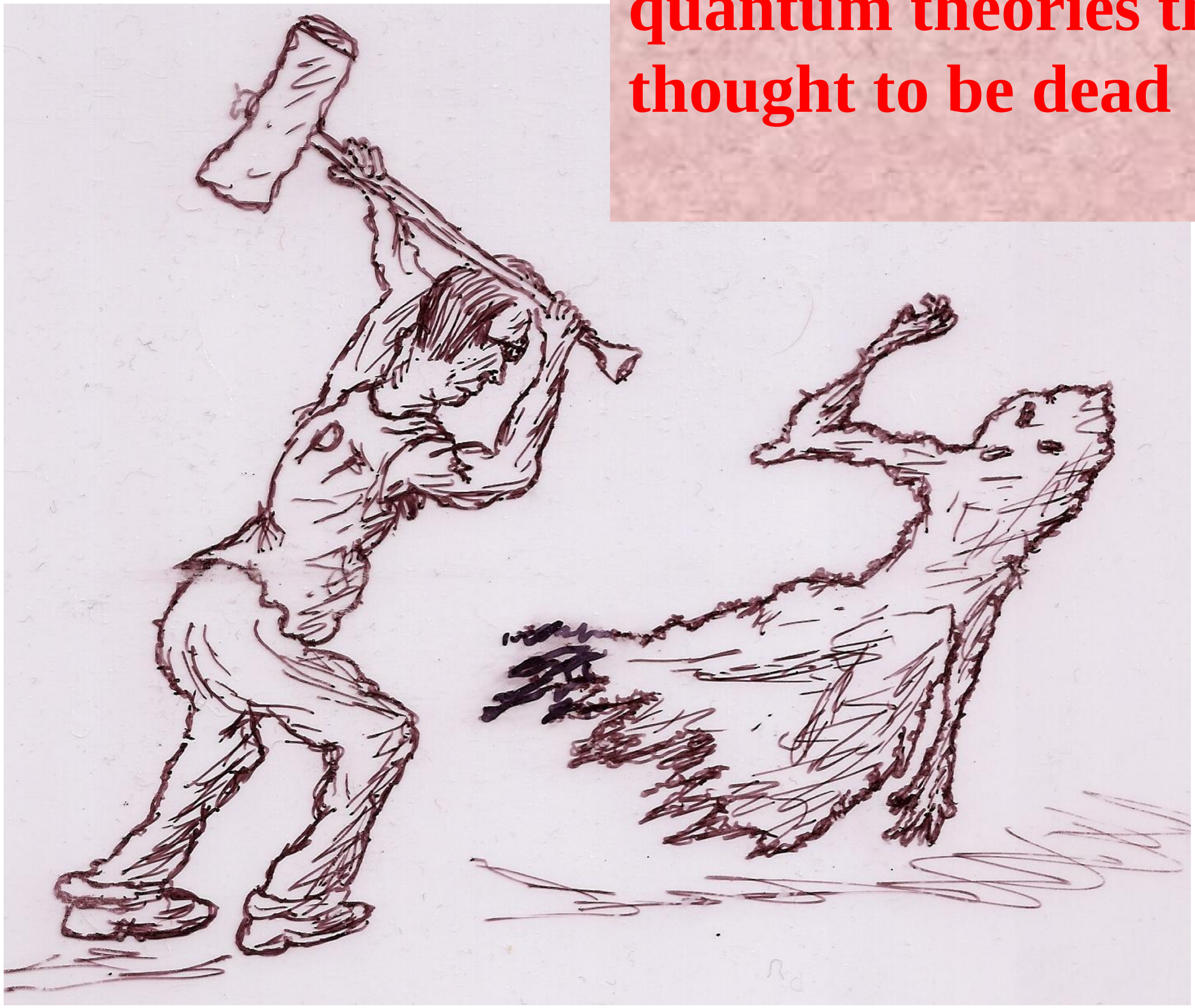
The authors discover the remarkable result that if $g > g_{\text{crit}}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g > g_{\text{crit}}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

Reviewed by *A. Salam*

“A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)

GHOSTBUSTING: Reviving quantum theories that were thought to be dead



“Ghost Busting: ***PT***-Symmetric Interpretation of the Lee Model”
CMB, S. Brandt, J.-H. Chen, and Q. Wang, *Phys. Rev. D* **71**, 025014 (2005) [arXiv: hep-th/0411064]

Example 2: Pais-Uhlenbeck model

$$I = \frac{\gamma}{2} \int dt \left[\ddot{z}^2 - \left(\omega_1^2 + \omega_2^2 \right) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2 \right]$$

Gives a fourth-order field equation:

$$z''''(t) + (\omega_1^2 + \omega_2^2) z''(t) + \omega_1^2 \omega_2^2 z(t) = 0$$

Problem: A fourth-order field equation gives a propagator like

$$G(E) = \frac{1}{(E^2 + m_1^2)(E^2 + m_2^2)}$$

$$G(E) = \frac{1}{m_2^2 - m_1^2}\left(\frac{1}{E^2 + m_1^2} - \frac{1}{E^2 + m_2^2}\right)$$

↑

GHOST!

Two possible realizations…

(I) If a_1 and a_2 annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2|\Omega\rangle = 0,$$

then the energy spectrum is real and bounded below. The state $|\Omega\rangle$ is the ground state of the theory and it has zero-point energy $\frac{1}{2}(\omega_1 + \omega_2)$. The problem with this realization is that the excited state $a_2^\dagger|\Omega\rangle$, whose energy is ω_2 above ground state, has a *negative Dirac norm* given by $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$.

(II) If a_1 and $a_2^\dagger$ annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2^\dagger|\Omega\rangle = 0,$$

then the theory is free of negative-norm states. However, this realization has a different and equally serious problem; namely, that the energy spectrum is unbounded below.

There can be other realizations as well!

Calculate the equivalent Dirac Hermitian Hamiltonian:

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2$$

CMB and P. Mannheim
"No-ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model"
Physical Review Letters **100**, 110402 (2008)

Example 3: Double-scaling limit in QFT

PT-symmetric reformulation of the theory

New O($N+1$)-symmetric partition function

$$Z = \mathrm{Re} \int d^{N+1}x \, e^{-L}$$

Take N to be an *even* integer.

The Lagrangian L is

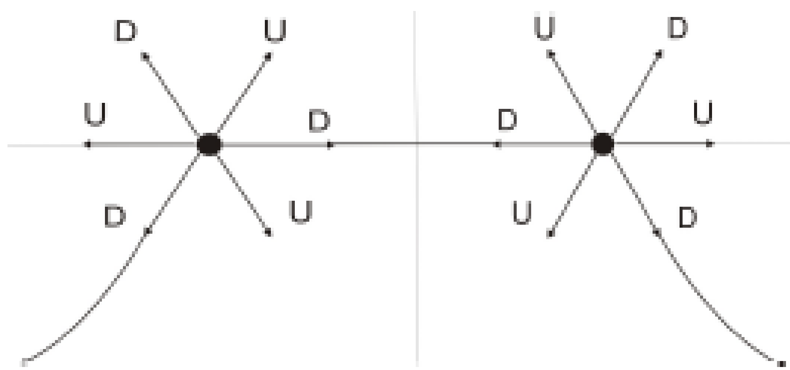
$$L = \frac{1}{2}\sum_{j=1}^{N+1} x_j^2 + \frac{\lambda i^{\varepsilon}}{2+\varepsilon}\Big(\sum_{j=1}^{N+1} x_j^2\Big)^{1+\varepsilon/2}$$

The integral is taken on the real axis and it converges if $\varepsilon < 1$.

We let $\lambda = gN^{-\varepsilon/2}$ and again introduce the radial variable r by $\sum_{n=1}^{N+1} x_n^2 = Nr^2$. The crucial assumption that N is *even* allows us to extend the radial integral to the entire real-r axis:

$$Z = \frac{1}{2}\mathcal{A}_{N+1}\int_{-\infty}^{\infty} dr \, e^{-N\mathcal{L}(r)} \qquad \mathcal{L} = r^2/2 + gr^2(ir)^{\varepsilon}/(2+\varepsilon) - \log r$$

Boundary conditions on integral: Path of integration lies in a pair of $\mathcal{PT}$-symmetric Stokes wedges centered about $-\pi\varepsilon/(4+2\varepsilon)$ and $-(4\pi+\pi\varepsilon)/(4+2\varepsilon)$. The wedges have angular opening $\pi/(2+\varepsilon)$ and contain the real-r axis if $\varepsilon<1$. As ε increases above 1, the wedges rotate downward into the complex plane and become narrower. At $\varepsilon=2$ the wedges are centered about $-\pi/4$ and $-3\pi/4$ and have angular opening $\pi/4$.



$$Z\sim \mathcal{A}_{N+1}e^{NL(\sqrt{2})}2^{-1/6}\pi N^{-1/3}\mathrm{Bi}(\gamma^{2/3})e^{-2\gamma/3}$$

Lots of possible future applications:

1. ***PT*** **Higgs model:** $-g\phi^4$ **theory is asymptotically free, stable, conformally invariant, and** $\langle\phi\rangle \neq 0$

2. ***PT*** **QED** $eA_\mu J^\mu$ **like a theory of magnetic charge, asymptotically free, opposite Coulomb force**

3. ***PT*** **gravity:** $G\phi_{\mu\nu}T^{\mu\nu}$ **has a repulsive force**

4. ***PT*** **Dirac equation allows for massless neutrinos to undergo oscillations**

That’s all – Thanks for listening!

Go away and think about *PT*-symmetric quantum theory.

XXXXXXXXXXXXXXXXX

ONE-dimensional Anharmonic Oscillator

$$H = -\sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2} + \frac{\mu^2}{2} \sum_{j=1}^{N+1} x_j^2 + \frac{\lambda}{4} \left(\sum_{j=1}^{N+1} x_j^2 \right)^2$$

Go to polar coordinates:

$$V(r) = \frac{1}{4r^2} + \frac{\mu^2}{2} r^2 + \frac{g}{4} r^4$$

Double scaling condition:

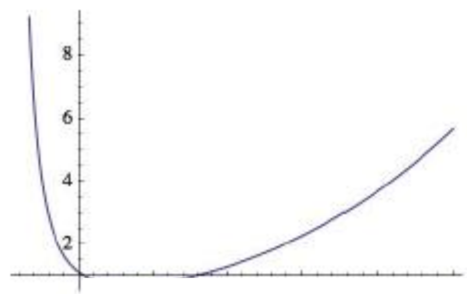
$$V'(r) = -\frac{2}{r^3} + \mu^2 r + g r^3 = 0, \quad \text{and} \quad V''(r) = -\frac{6}{r^4} + \mu^2 + 3 g r^2 = 0$$

The problem:

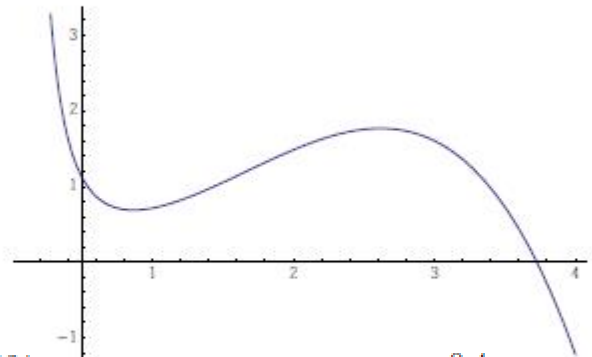
$$g_{\text{crit}} = -(2/3)^{3/2} \mu^3 \approx -0.544331 \mu^3$$

Negative!!

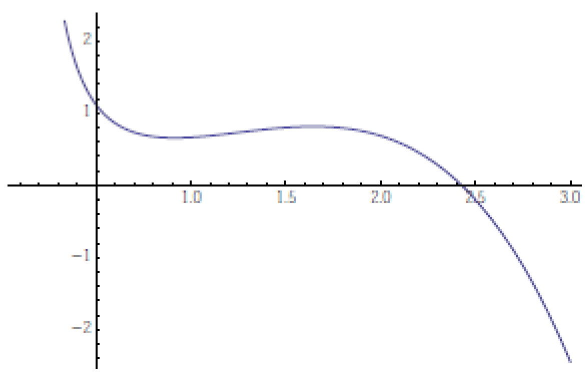
As you approach the critical coupling...



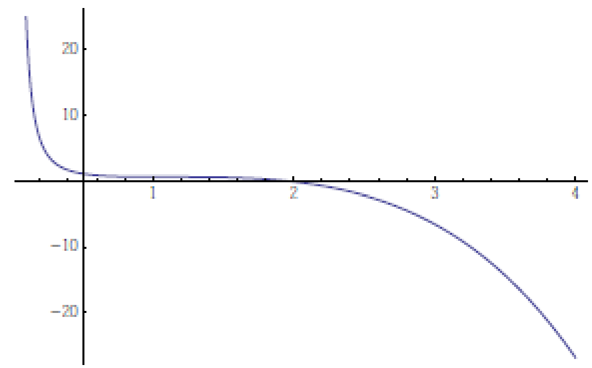
$g_{\rm crit} + 0.6$ where $g_{\rm crit} = -0.544331$



$g_{\rm crit} + 0.4$



$g = g_{\rm crit} + 0.2$



$g = g_{\rm crit}$

PT-symmetric O(*N*) anharmonic oscillator

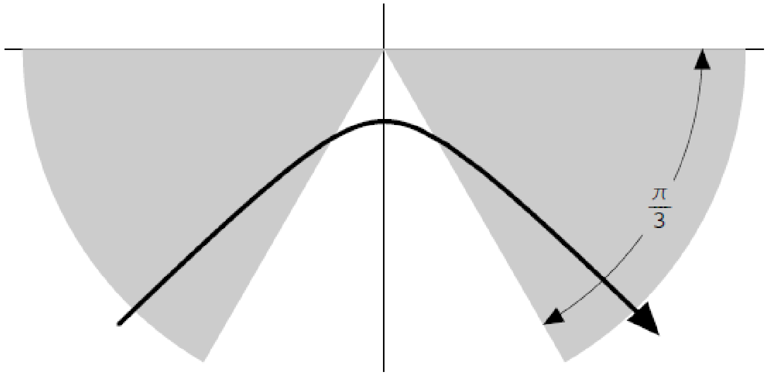
In polar coordinates:

$$-\frac{1}{N^2}\psi''(x)+\left(\frac{1}{4x^2}+\frac{1}{2}x^2-\frac{g}{4}x^4\right)\psi(x)=\mathcal{E}\psi(x)$$

Transform this Hamiltonian to one that is isospectral...

Pair of exactly isospectral Hamiltonians

$$H = \frac{1}{2m}p^2 - gx^4$$



$$\tilde{H} = \frac{\tilde{p}^2}{2m} - \hbar\sqrt{\frac{2g}{m}}\,z + 4gz^4$$

anomaly

CMB, D. C. Brody, J.-H. Chen, H. F. Jones , K. A. Milton, and M. C. Ogilvie
Physical Review D **74, 025016 (2006) [arXiv: hep-th/0605066]**

The isospectral potential valid on the real-*s* axis:

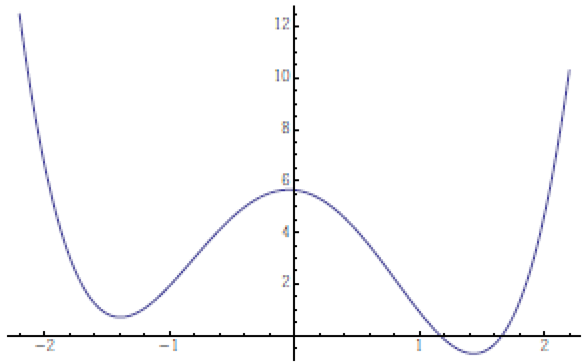
$$V(s) = -\frac{s}{2} + \frac{1}{4g}\left(\frac{1}{2}s^2 - 1\right)^2$$

Critical coupling: $g_{\rm crit} = (2/3)^{3/2} \approx 0.544331$

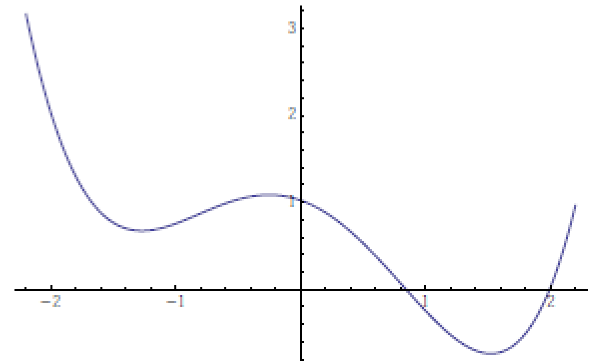
Universal function: $-\chi''(t) + \gamma t\left(1 - t^2\right)\chi(t) = 0$

$$\gamma = \frac{9}{16}6^{1/4}N^2(-G)^{5/4}$$

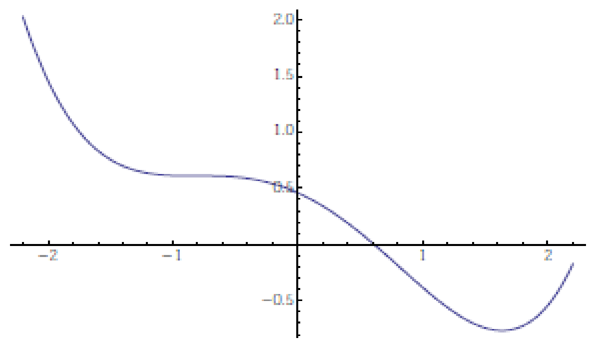
As you approach criticality:



$g = g_{\text{crit}} - 0.5$ where $g_{\text{crit}} = 0.544331$



$g = g_{\text{crit}} - 0.3$ where g_{crit}



$g = g_{\text{crit}}$

G measures departure from critical coupling...

Zero dimensions: $G \sim N^{-1/3}$

One dimension: $G \sim N^{-4/5}$