

# Symmetries and broken symmetries in non equilibrium systems

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**New Directions in Theoretical Physics**

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# Outline

Irreversibility and time reversal symmetry

Density fluctuations in non-equilibrium systems

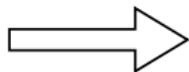
Second law and current fluctuations

Phase transitions in equilibrium and non-equilibrium systems

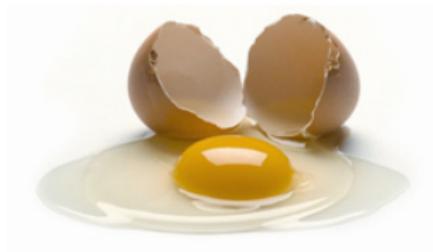
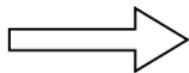
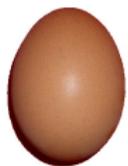
# Irreversibility in daily life



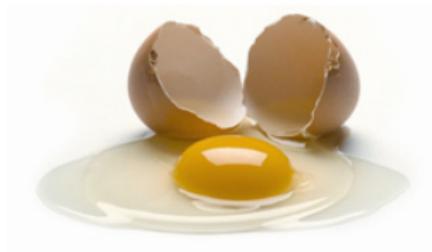
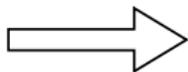
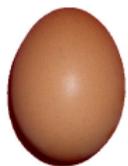
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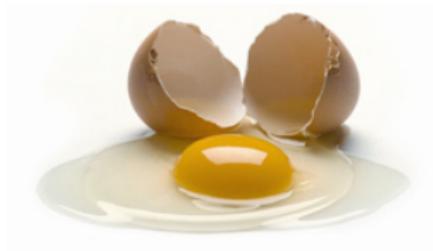
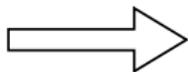
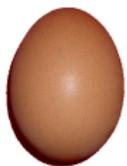
# Irreversibility in daily life



**Poincaré:**

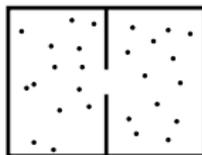
**"All this Maxwell and Boltzmann have explained ... "**

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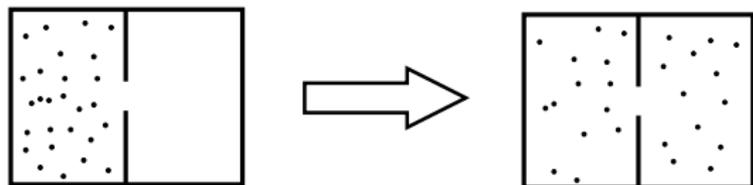


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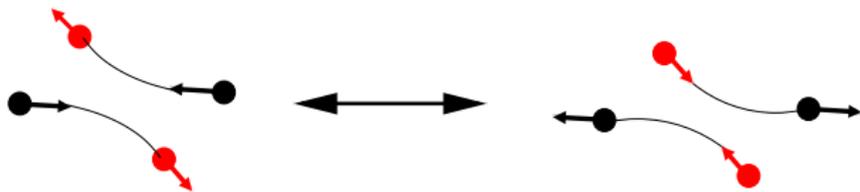
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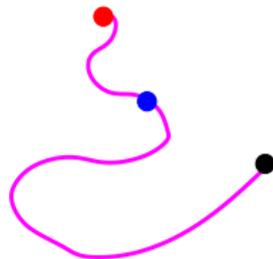
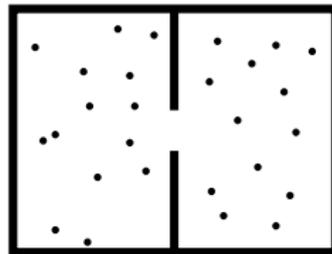
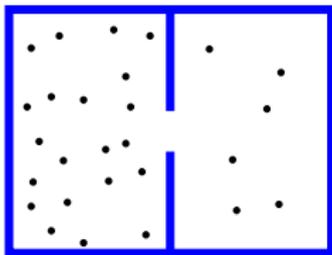
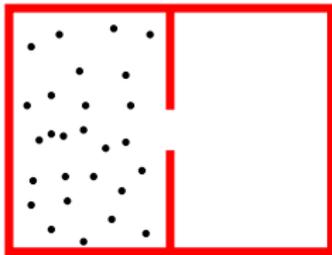


# Time reversal symmetry and irreversibility



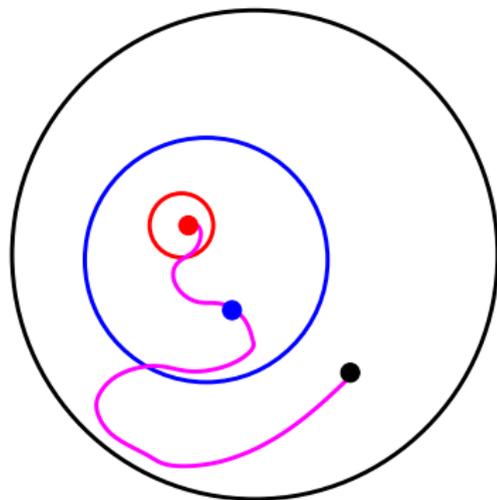
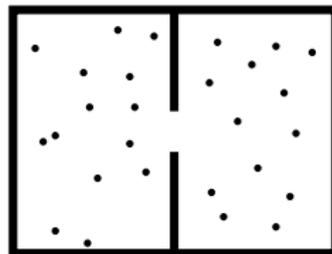
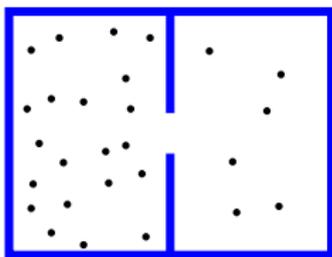
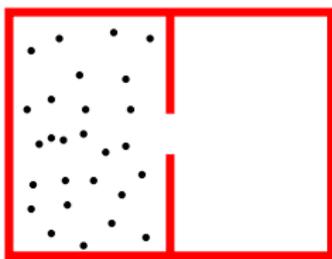
*At the microscopic scale, reversible dynamics*





In phase space

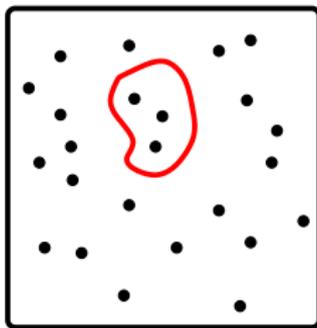




In phase space

$$\text{black volume} = 2^N \times \text{red volume}$$

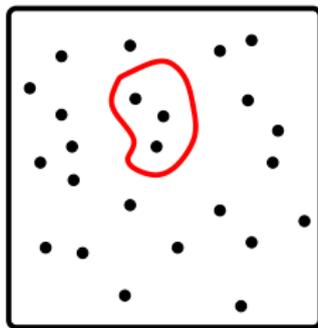
# Density fluctuations



$n$  particles in the volume  $v$

Time to see a 1% fluctuation  
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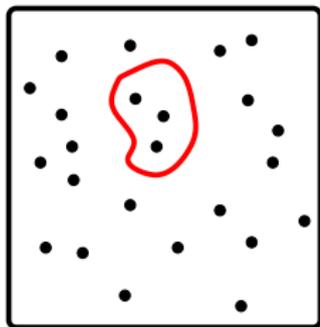
Chandrasekhar 1943:

Radius $a$	Time
$10^{-5}$ cm	$10^{-11}$ second
$2.5 \times 10^{-5}$ cm	1 second
$3. \times 10^{-5}$ cm	$10^6$ seconds
$5.10^{-5}$ cm	$10^{68}$ seconds
1 cm	$10^{10^{14}}$ seconds

# Fluctuation dissipation relation

Starting from  $S = k \log \Omega$

Einstein



$n$  particles in volume  $v$

$$\langle n^2 \rangle - \langle n \rangle^2 = kT v \rho \kappa$$

$\kappa$  is the compressibility,  
 $\rho$  the density in the reservoir  
et  $T$  la température.

Variance of the fluctuation =  $k \times$  Response coefficient

Boltzmann Constant  $k \sim 10^{-23} \text{ J/K}$

Time reversal symmetry of fluctuations at equilibrium

Probability of time dependent fluctuations

Onsager Machlup 1953

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Given a fluctuation at  $t = 0$

$$\rho(\mathbf{x}, 0) = \rho_{\text{equilibrium}}(\mathbf{x}) + \phi(\mathbf{x}, 0)$$

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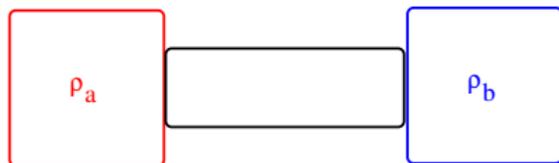
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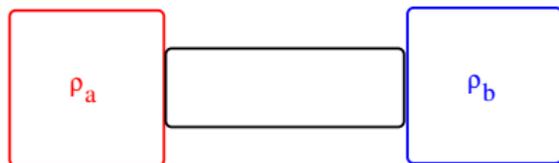
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$$\text{Equilibrium} \Rightarrow \boxed{\phi(\mathbf{x}, t) = \phi(\mathbf{x}, -t)}$$

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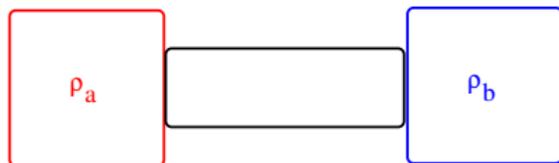
$$\rho(x, 0) = \rho_{\text{steady state}}(x) + \phi(x, 0)$$

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$t > 0$

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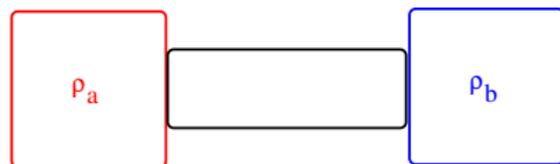
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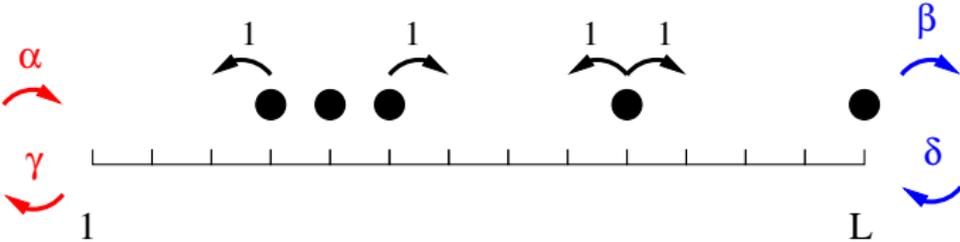
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# Exclusion processes

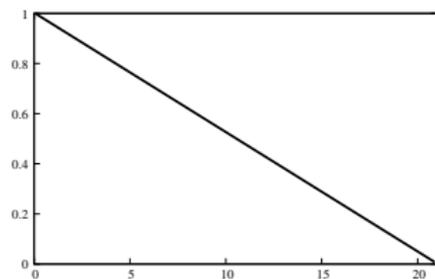
SSEP (Symmetric simple exclusion process)



$$\rho_a = \frac{\alpha}{\alpha + \gamma},$$

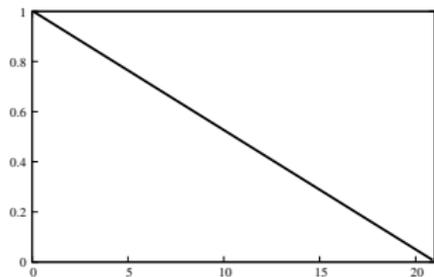
$$\rho_b = \frac{\delta}{\beta + \delta}$$

## Density profiles

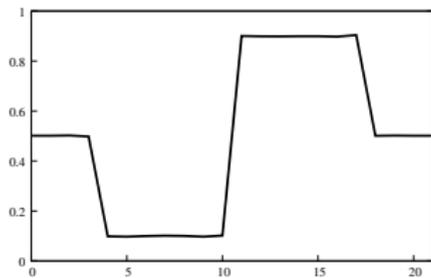


Steady state

## Density profiles

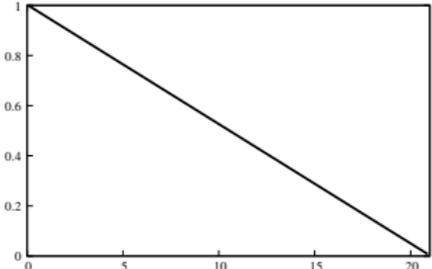


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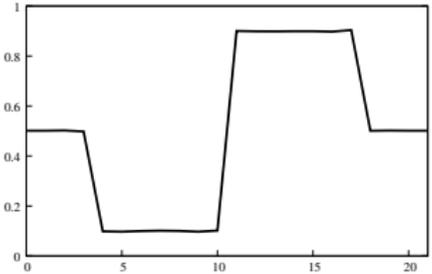


Fluctuation

# Density profiles

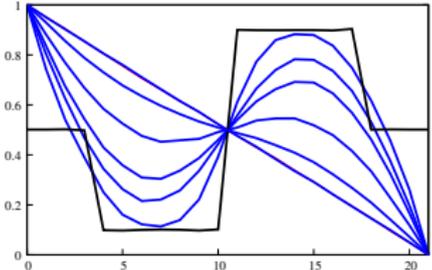


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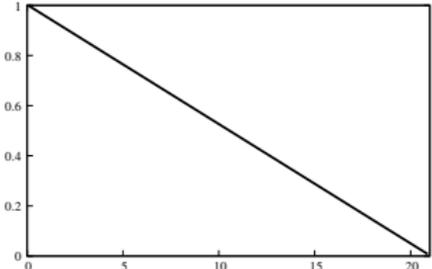


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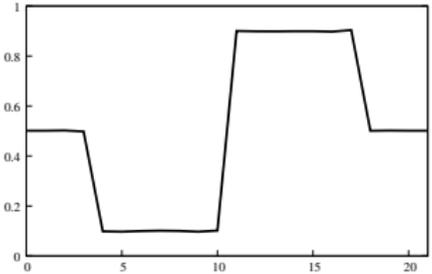
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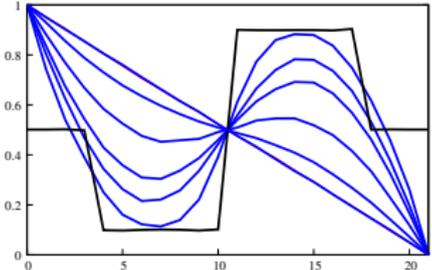


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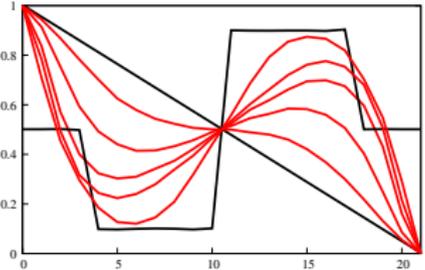


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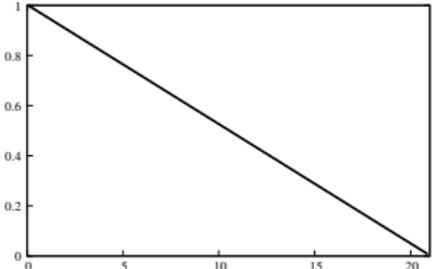
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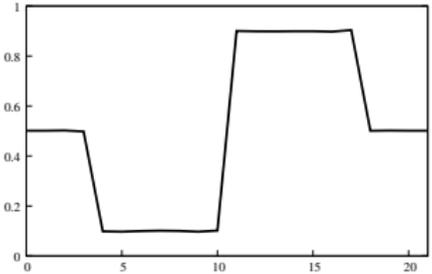
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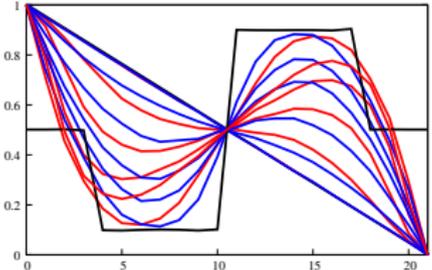
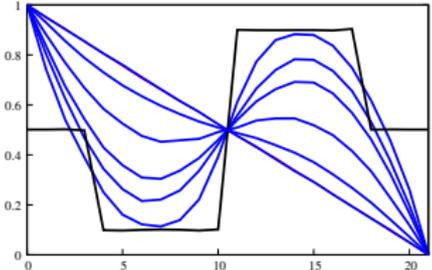
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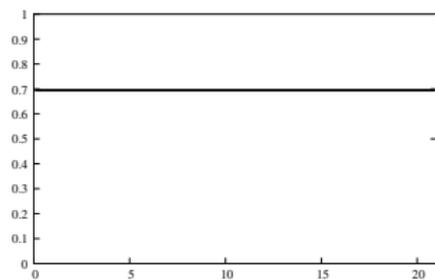
Fluctuation

How does a fluctuation relax?

Both

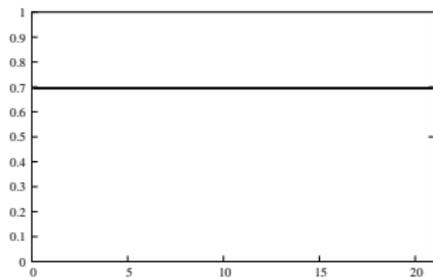


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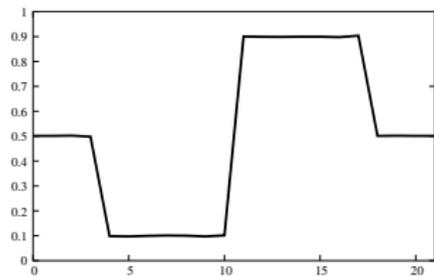


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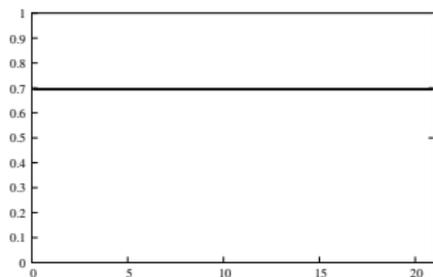


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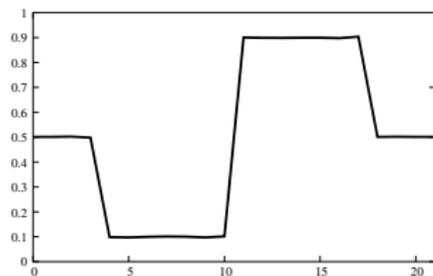


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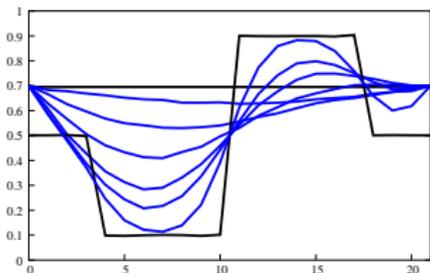


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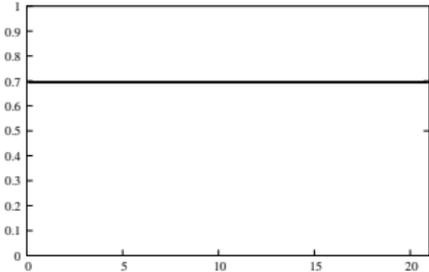


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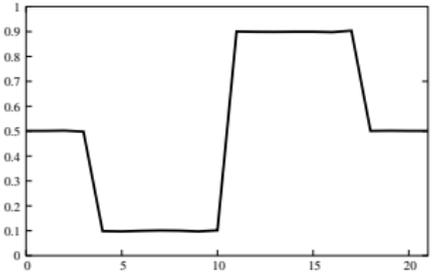
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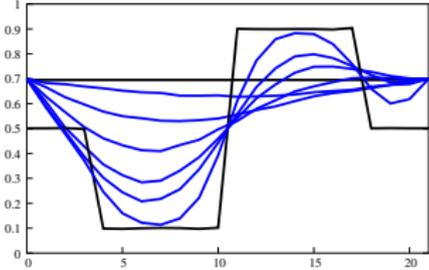


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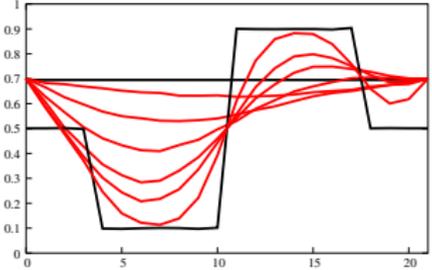


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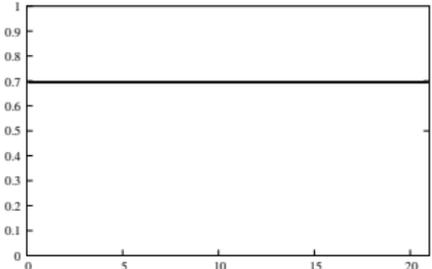
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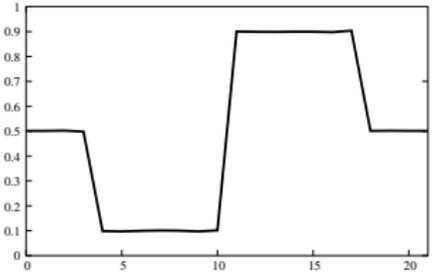
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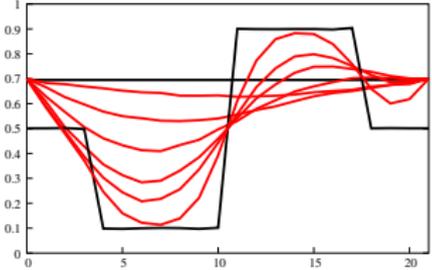


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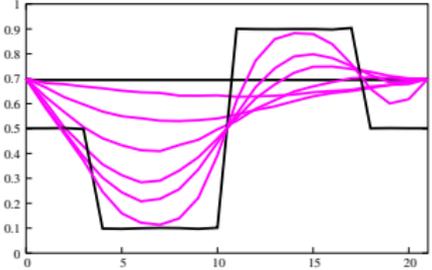


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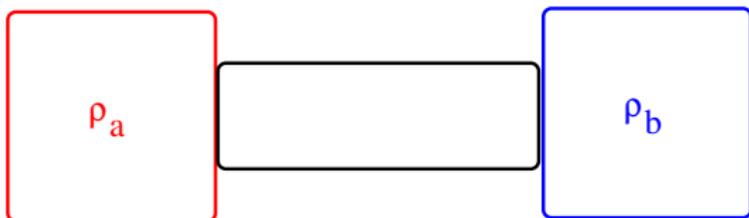
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# Large diffusive systems



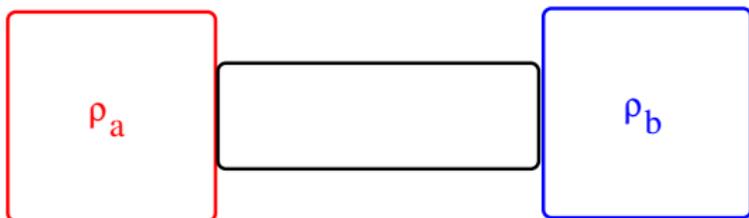
A diffusive system

► For  $\rho_a - \rho_b$  small:  $\frac{\langle Q_t \rangle}{t} = \frac{(\rho_a - \rho_b) D(\rho)}{L}$  Fick's law

►  $\rho_a = \rho_b = \rho$  :  $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

One has a theory for  $\phi(x, t)$  and  $\phi(x, t)$

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# Macroscopic fluctuation theory

## Onsager Machlup theory for non equilibrium diffusive systems

Kipnis Olla Varadhan 89  
Spohn 91

⋮  
Bertini De Sole Gabrielli  
Jona-Lasinio Landim 2001 →

Evolution of a profile  $\rho(x, t)$  for  $0 \leq t \leq T$

$$\text{Pro}(\{\rho(x, t), j(x, t)\}) \\ \exp \left[ -L \int_0^{T/L^2} dt \int_0^1 dx \frac{[j(x, t) + \rho'(x, t)D(\rho(x, t))]^2}{2\sigma(\rho(x, t))} \right]$$

▶  $\frac{d\rho}{dt} = -\frac{dj}{dx}$  (conservation law)

▶  $\rho(0, t) = \rho_a$  ;  $\rho(1, t) = \rho_b$

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$$j(x, t) = -\rho'(x, t)D(\rho(x, t)) + \frac{1}{\sqrt{L}}\eta(x, t) \quad \text{Fick's law} + \text{noise}$$

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$$j(x, t) = -\rho'(x, t)D(\rho(x, t)) + \frac{1}{\sqrt{L}}\eta(x, t) \quad \text{Fick's law} + \text{noise}$$

$$\langle \eta(x, t)\eta(x', t') \rangle = 2\sigma(\rho(x, t))\delta(x - x')\delta(t - t')$$

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad ; \quad \rho(0, t) = \rho_a \quad ; \quad \rho(1, t) = \rho_b$$

$$\text{Pro}(\{\rho(x, t), j(x, t)\}) \sim \exp[-\text{Action}] =$$

$$\exp \left[ -L \int_0^{T/L^2} dt \int_0^1 dx \frac{[j(x, t) + \rho'(x, t)D(\rho(x, t))]^2}{2\sigma(\rho(x, t))} \right]$$

How does a fluctuation  $\rho(x, 0)$  relax?

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$$\frac{d\rho}{dt} = (D(\rho(x, t))\rho(x, t))'$$

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How does a fluctuation  $\rho(x, 0)$  relax? Action = 0  $\Rightarrow$

$$\frac{d\rho}{dt} = (D(\rho(x, t))\rho(x, t))'$$

How is a fluctuation  $\rho(x, 0)$  produced?

Minimize the Action with  $\rho(x, -\infty) = \rho_{\text{steady state}}(x)$

# Large deviation functional

$$\text{Pro}(\{\rho(\vec{x})\}) \sim \exp \left[ -L^d \mathcal{F}(\{\rho(\vec{x})\}) \right]$$

## Equilibrium

1.  $\mathcal{F}$  local
2.  $\mathcal{F} = T^{-1} \int d\vec{x} f(\rho(\vec{x}))$  free energy
3. No phase transition in one dimension (short range interactions)

## Non-equilibrium

1.  $\mathcal{F}$  non local
2. Weak long range correlations (SSEP)

$$\langle (\rho(x) - \rho^*(x))(\rho(y) - \rho^*(y)) \rangle \simeq \frac{1}{L} G(x, y)$$

3. Phase transitions in one dimension

# Large deviation function for the SSEP

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

**Equilibrium**  $\rho_a = \rho_b = F$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[ (1 - \rho(x)) \log \frac{1 - \rho(x)}{1 - F} + \rho(x) \log \frac{\rho(x)}{F} \right]$$

**Non-equilibrium** ( $\rho_a \neq \rho_b$ )

D. Lebowitz Speer 2001-2002

Bertini De Sole Gabrielli Jona-Lasinio Landim 2002

$$\mathcal{F} = \sup_{F(x)} \int_0^1 dx \left[ (1 - \rho(x)) \log \frac{1 - \rho(x)}{1 - F(x)} + \rho(x) \log \frac{\rho(x)}{F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right]$$

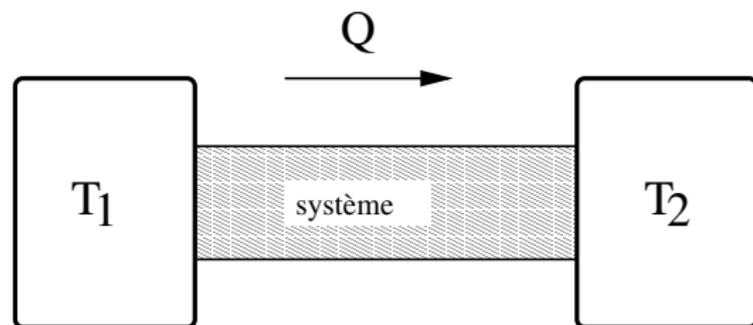
with  $F(x)$  monotone,  $F(0) = \rho_a$  and  $F(1) = \rho_b$ .

## Second law

**Clausius:** "Heat generally cannot flow spontaneously from a material at lower temperature to a material at higher temperature"

## Second law

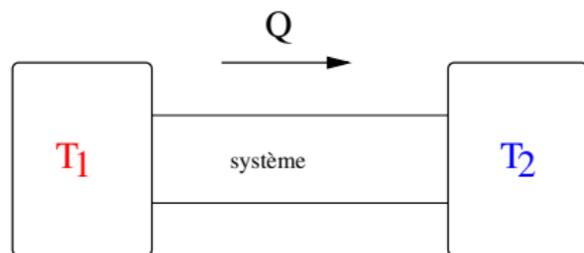
**Clausius:** "Heat generally cannot flow spontaneously from a material at lower temperature to a material at higher temperature"



Energy flows from the hot to the cold

$$Q > 0 \text{ si } T_1 > T_2$$

## Currents: fluctuation dissipation relation



$Q$  = energy transferred during time  $t$

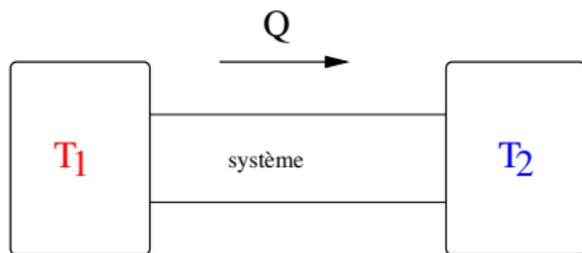
- ▶ Current fluctuation at equilibrium ( $T_1 = T_2 = T$ )

$$\langle Q^2 \rangle = k \times (2DT^2) \times t$$

- ▶ Response to a small temperature difference

$$\langle Q \rangle = D \times (T_1 - T_2) \times t$$

# Fluctuation theorem (large deviations)



Evans Cohen Morris 1994

Gallavotti Cohen 1995

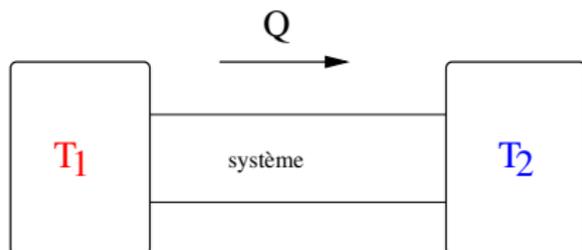
⋮

Lebowitz Spohn 1999

In the long time limit

$$\frac{\text{Pro}(Q)}{\text{Pro}(-Q)} \sim \exp \left[ Q \left( \frac{1}{kT_2} - \frac{1}{kT_1} \right) \right]$$

# Fluctuation theorem (large deviations)



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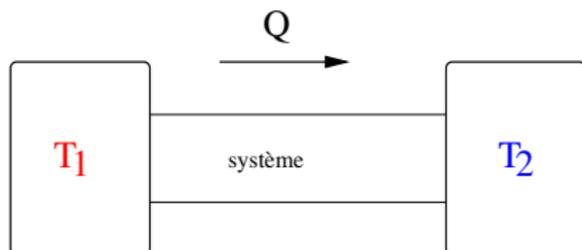
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$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\text{Pro}(Q = jt)}{\text{Pro}(Q = -jt)} = j \left( \frac{1}{kT_2} - \frac{1}{kT_1} \right)$$

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In the long time limit

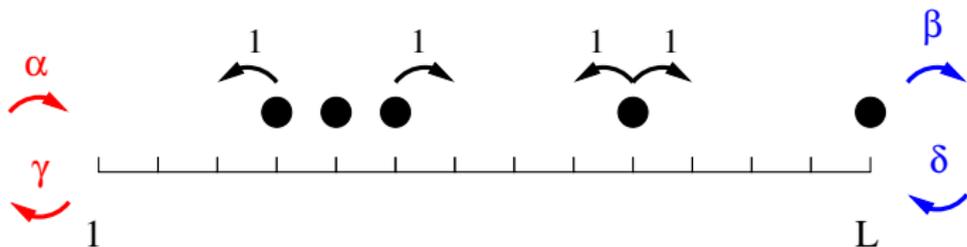
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- ▶ Probabilistic character of second law
- ▶ Universal relation
- ▶ Can be understood in terms of detailed balance

# Exclusion processes

SSEP (Symmetric simple exclusion process)



$$\rho_a = \frac{\alpha}{\alpha + \gamma},$$

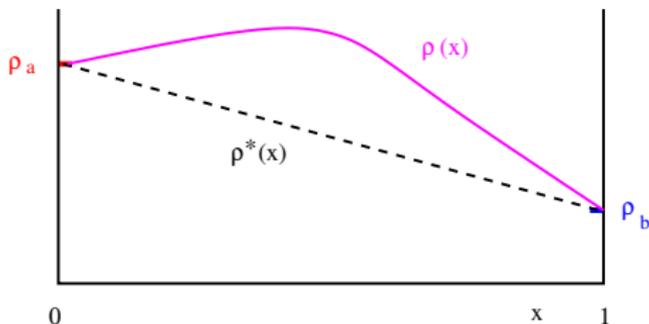
$$\rho_b = \frac{\delta}{\beta + \delta}$$

For large  $t$  :  $\text{Pro}(Q_t = jt) \sim \exp[-tF(j)]$

# Variational principle

Bodineau D. 2004

$$F(j) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \frac{[j L + \rho'(x) D(\rho(x))]^2}{2\sigma(\rho(x))}$$



Satisfies the fluctuation theorem

$$F(j) - F(-j) = j \int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

Gallavotti Cohen 1995  
Evans Searls 1994  
.... Kurchan 1998  
Lebowitz Spohn 1999

# Phase transitions

At equilibrium at temperature  $T$ :

$$\text{Pro}(\mathcal{C}) = \frac{1}{Z} \exp \left[ -\frac{E(\mathcal{C})}{kT} \right]$$

Boltzmann Gibbs

# Phase transitions

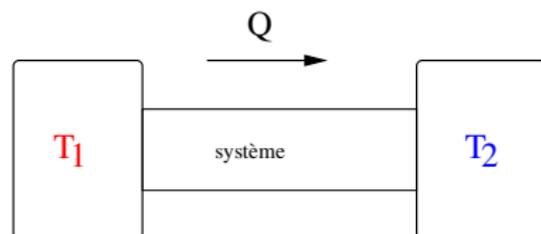
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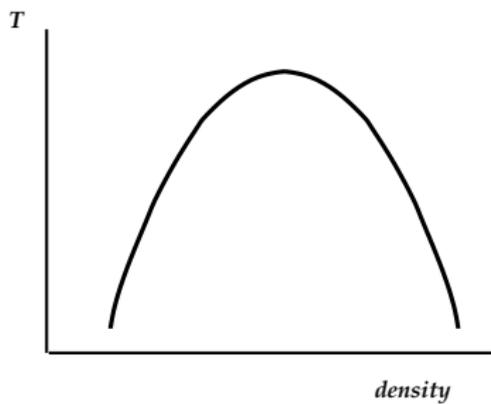
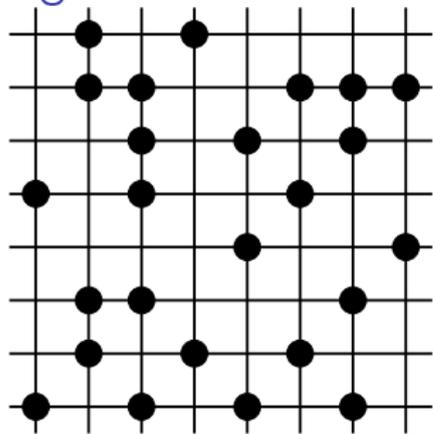
Boltzmann Gibbs

Out of equilibrium:

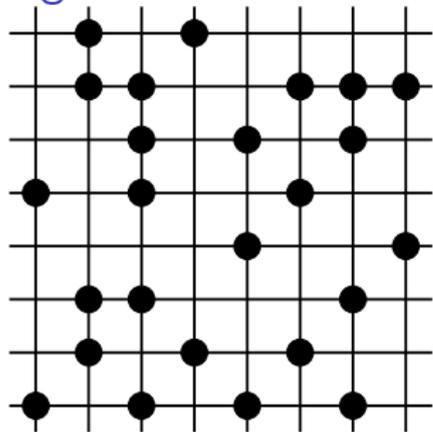
$$\text{Pro}(\mathcal{C}) = ?$$



# Ising model and broken symmetry



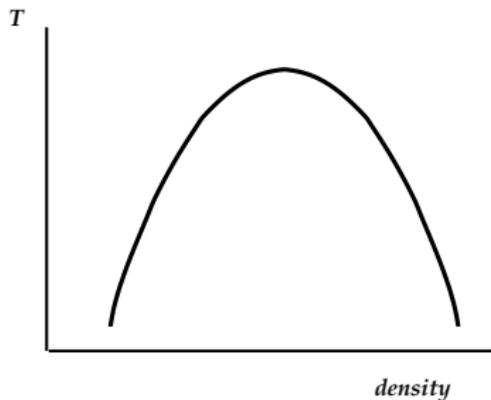
## Ising model and broken symmetry



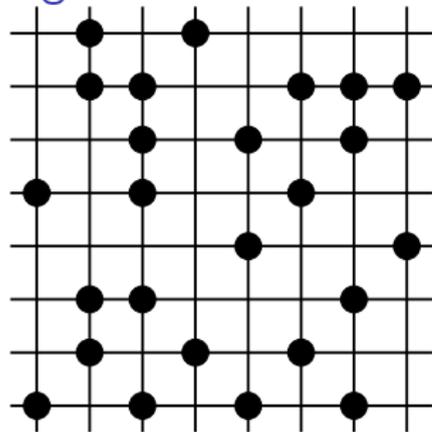
Energy = number of pairs of occupied neighbors on the lattice

$$\text{Pro}(C) = \frac{1}{Z} \exp \left[ -\frac{E(C)}{kT} \right]$$

$$E(C) = - \sum_{i \sim j} \eta_i \eta_j \quad \eta_i = 0 \text{ ou } 1$$



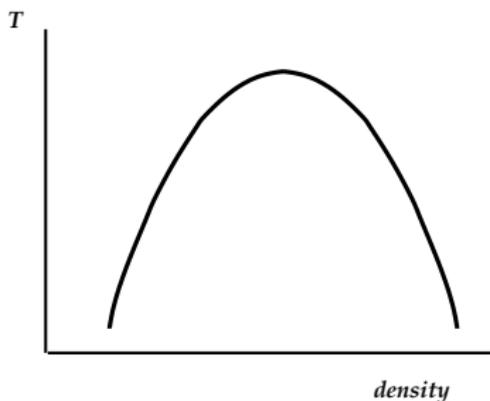
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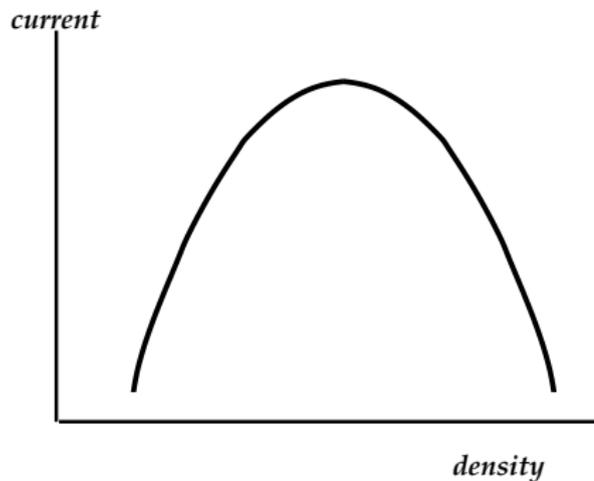
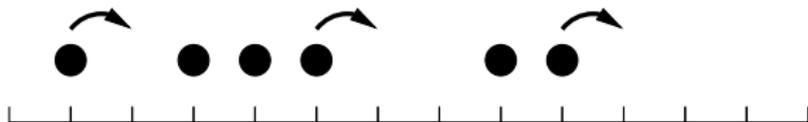


$$\rho_{\text{liquid}} - \rho_{\text{gas}} = \left[ 1 - \left( \sinh \frac{1}{2kT} \right)^{-4} \right]^{\frac{1}{8}}$$

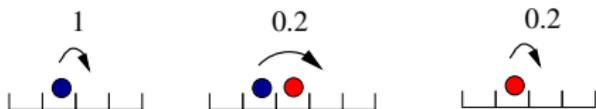
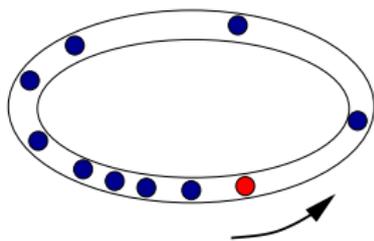
Onsager 1944 Yang 1952

# Traffic

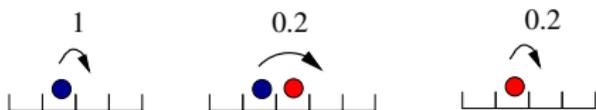
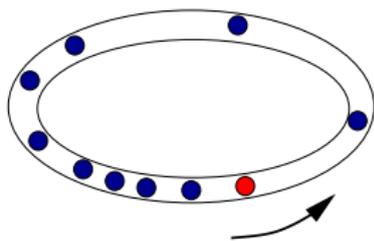
Asymmetric exclusion



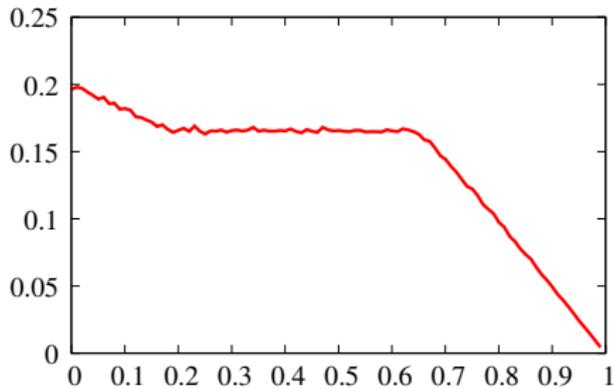
# Phase transition : Traffic



# Phase transition : Traffic

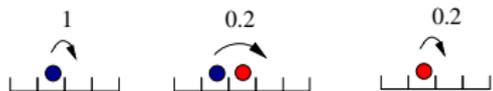
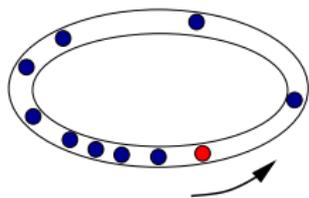


velocity

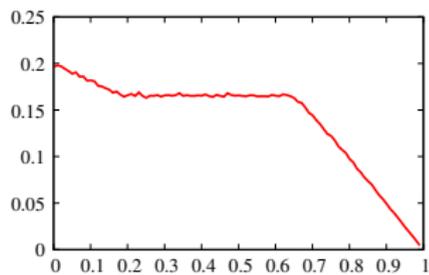


density

# Phase transition : Traffic



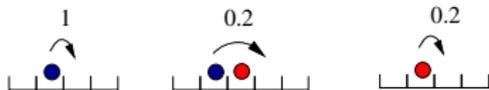
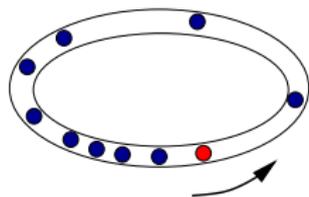
velocity



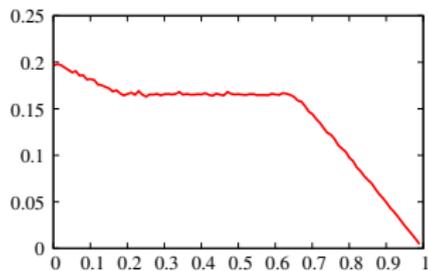
density

# Phase transition in one dimension

## Non equilibrium

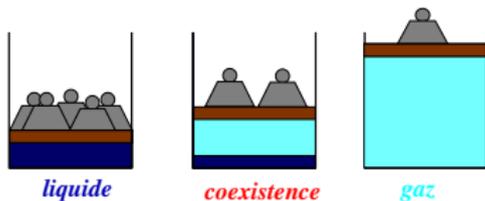


velocity

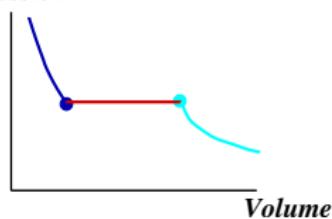


density

## Equilibrium



Pression



## Open questions

Stochastic thermodynamics for small systems

Driven systems and the KPZ (Kardar Parisi Zhang) equation

Mechanical (non diffusive) systems

Non steady state situations

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