Cluster Algebras and SYM Theory

Marcus Spradlin
Brown University

Higgs Centre for Theoretical Physics ■ 03 March 2020

Based on work with
Nima Arkani-Hamed, Thomas Lam, Anastasia Volovich
Luke Lippstreu, Jorge Mago, Anders Schreiber
Igor Prlina, Stefan Stanojevic

Introduction

In recent years we've seen remarkable progress on the problem of unlocking the hidden mathematical structure of quantum field theories.

Other talks may touch on applications to other field theories, but the focus of my talk (and several others) will be the particularly special

planar $\mathcal{N} = 4$ supersymmatric Yang-Mills theory

or SYM theory for short.

Why Mathematicians May Like SYM Theory

Think of SYM theory as an “encyclopedia” filled with collections of functions with remarkable properties and interrelationships.

In fact, the properties are so remarkable that the functions of SYM theory “barely exist”...

... to the extent that many aspire to find some purely mathematical problem to which these functions are the solution.

Introducing the Cast of Characters

Entries in this encyclopedia of functions are indexed by a pair of integers

$n \geq 6$	labels the number of particles ($n = 3, 4, 5$ are special cases)
$k \in \{0, 1, \ldots, n-4\}$	labels the "helicity sector" (called MHV, NMHV, ... N^kMHV)

Each n-particle function $\mathscr{A}_{n,k}$ depends on several continuous variables.

One of these variables, λ, is special and usually used to study series expansions

$$\mathscr{A}_{n,k}(\lambda) = \sum_{L=0}^{\infty} \lambda^L \mathscr{A}_{n,k}^{(L)}$$

(which are believed to have a non-zero and finite radius of convergence) where L is called the loop order.

Introducing the Cast of Characters

What is $\mathscr{A}_{n,k}^{(L)}$ a function of?

Using "momentum twistor" variables [Hodges] and the "dual conformal invariance" [Drummond, Henn, Korchemsky, Sokatchev] of SYM theory, we know that it is a function on the configuration space

$$\mathrm{Conf}_n(\mathbb{P}^3) \simeq \mathrm{Gr}(4,n)/(\mathbb{C}^*)^{n-1}$$

(the quotient group acts by independently rescaling columns).

The fact that this space has the structure of a cluster Poisson variety [Gekhtman, Shapiro, Vainshtein] apparently underlies the connection between amplitudes in SYM theory and cluster algebras [Golden, Goncharov, MS, Vergu, Volovich].

Introducing the Cast of Characters

In order to make contact with some of the other talks, let me clarify that to each amplitude $\mathscr{A}_{n,k}^{(L)}$ there is an associated, canonically defined integrand $\mathscr{I}_{n,k}^{(L)}$ related by

$$\mathscr{A}_{n,k}^{(L)} = \int_{\mathscr{C}_{n,k}^{(L)}} \mathscr{I}_{n,k}^{(L)}$$

where $\mathscr{C}_{n,k}^{(L)}$ is a contour in the configuration space of L lines in $\mathbb{P}^3$.

- Integrands are always rational functions on $\mathrm{Conf}_{L\ \mathrm{lines}}(\mathbb{P}^3) \times \mathrm{Conf}_n(\mathbb{P}^3)$,
- while (for $L > 0$) amplitudes are transcendental, multi-valued functions on $\mathrm{Conf}_n(\mathbb{P}^3)$.

Amplitude Singularity Theory

Following ideas that go back to Heisenberg, a goal of the "S-matrix program" is to be able to determine amplitudes in quantum field theory based on a few physical principles and a thorough knowledge of their analytic structure.

In the "old" approach this was hampered in part by the difficulty of identifying a suitable domain on which amplitudes could be expected to actually be analytic.

In generic field theories, this is manifested in perturbation theory by L-loop amplitudes typically having singularities that approach closer to the real axis (in some putative "physical domain") as L increases.

But in SYM theory there is a chance for Heisenberg's goal to be realized.

Amplitude Singularity Theory

The poles of $\mathscr{A}_{n,k}^{(L)}$ are completely understood; they occur on the subvariety of $\mathrm{Conf}_n(\mathbb{P}^3)$ given by

$$\prod_{i,j} \langle i\, i{+}1\, j\, j{+}1 \rangle = 0$$

(the bracket denotes a Plücker coordinate on $\mathrm{Gr}(4,n)$).

More interesting, and less trivial, are the branch points.

Theorem: For $n = 6, 7$, $\mathscr{A}_{n,k}^{(L)}$ can have branch points only on the subvariety of $\mathrm{Conf}_n(\mathbb{P}^3)$ given by $\prod_i a_i = 0$, where the product runs over the cluster variables of the $\mathrm{Gr}(4,n)$ cluster algebra.

I'll sketch the proof shortly; for now let me note that historically, evidence for this theorem (and the much stronger statement, still a conjecture, that the cluster variables provide a complete symbol alphabet for these amplitudes), was slowly collected over several years of very difficult calculations.

Del Duca, Duhr, Smirnov, Goncharov, Spradlin, Vergu, Volovich, Caron-Huot, Dixon, Drummond, Henn, He, von Hippel, Pennington, Harrington, Dulat, McLeod, Papathanasiou

Amplitude Singularity Theory

I'll sketch the proof shortly; for now let me note that historically, evidence for this theorem (and the much stronger statement, still a conjecture, that the cluster variables provide a complete symbol alphabet for these amplitudes), was slowly collected over several years of very difficult calculations.

[Some amplitudes belong to a class of functions called generalized polylogarithms. A symbol letter a of such a function is an algebraic function on $\mathrm{Conf}_n(\mathbb{P}^3)$ signifying the presence of a branch cut from $a = 0$ to $a = \infty$.]

Amplitude Singularity Theory

What happens for $n > 7$? Two new features:

- math: $\mathrm{Gr}(4, n)$ has infinitely many cluster variables
- physics: it is known that amplitudes have symbol letters that are algebraic functions of Plücker coordinates, and hence are not cluster coordinates of $\mathrm{Gr}(4, n)$

I will come back to (and other talks also discuss) the second point.

Regarding the first point: unlike for $n = 6, 7$, it could have happened that the subvariety of $\mathrm{Conf}_n(\mathbb{P}^3)$ on which $\mathscr{A}_{n,k}^{(L)}$ has branch points becomes more complicated as L increases, perhaps without bound.

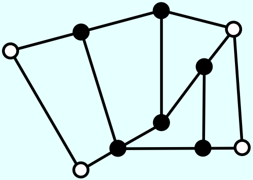
This would be a significant complication; fortunately we know it is not true.

The Landau Equations

Landau (1959) provided a criterion for determining the singularity locus of amplitudes in quantum field theory.

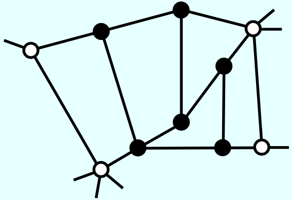
A (planar) Landau graph is a planar graph with

1. a complex Feynman parameter α_j assigned to each edge j,
2. a momentum vector $q_j \in \mathbb{C}^4$ assigned to each directed edge j (with $q_j \to -q_j$ under reversal of the edge orientation,
3. and momentum conservation $\sum_j q_j = 0$ imposed at each vertex,
4. except at certain privileged vertices called terminals



The Landau Equations

We can impose momentum conservation at the terminals by attaching some “external edges” that carry momentum into or out of the diagram.



The Landau Equations

An L-loop amplitude can have singularities only when the external momenta are such that the Landau equations

$$\alpha_j q_j^2 = 0 \text{ for each edge } j \text{, and}$$
$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$

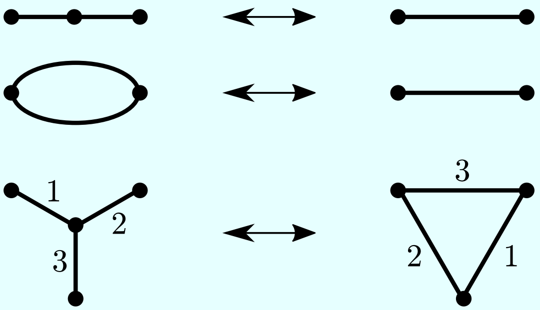
admit solutions for $\{\alpha_j, q_j\}$.

$\boxed{\text{The solution set of the Landau equations is preserved under the graph moves familiar from circuit theory.}}$

[Dennen, Prlina, Spradlin, Stanojevic, Volovich]

The Landau Equations

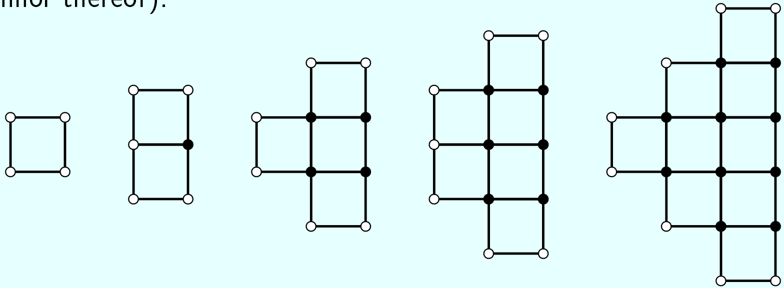
In massless theories (including SYM theory), the (locus of solutions of the) Landau equations is invariant under



Planar Graph Reduction

The problem of studying the reducibility of m-terminal graphs under the basic circuit operations is well studied in the mathematical literature.

The key result, for our purposes, comes from Gitler, who proved that any 2-connected m-terminal plane graph, with all terminals lying on a common face (which we take to be the "outer" face), can be reduced to what we call the m-terminal ziggurat graph, (or a minor thereof).



The Kinematic Domain

It follows that

> Theorem: For each n there is a finitely generated codimension-one subvariety $\mathscr{S}_n \subset \mathrm{Conf}_n(\mathbb{P}^3)$ (the Landau singularity locus of the n-terminal ziggurat graph) with the property that for all k and L, $\mathscr{A}_{n,k}^{(L)}$ has branch point singularities only on $\mathscr{S}_n \subset \mathrm{Conf}_n(\mathbb{P}^3)$.

In words: the (asymptotic) complexity of the singularity locus of amplitudes in SYM theory is determined only by n, and does not grow with loop order.

(There are accidental cancellations for very small k and/or L.)

The Kinematic Domain

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[Note: This analysis holds for all amplitudes; it is not restricted to those of polylogarithmic type.]

The Kinematic Domain and Positivity

So far we have only been able to explicitly compute $\mathscr{S}_n \subset \mathrm{Conf}_n(\mathbb{P}^3)$ for $n = 6$, and it is consistent with the $\mathrm{Gr}(4,6)$ symbol alphabet, but all evidence available to date is consistent with the

> Conjecture: $\mathscr{S}_n$ has empty intersection with the positive configuration space $\mathrm{Gr}_+(4,n)/T$.

In words: amplitudes do not have singularities, at any loop order, in the positive domain ($\langle i\,j\,k\,l\rangle > 0$ for $i < j < k < l$).

Introduction and Motivation

Because we are interested in a thorough knowledge of the analytic structure of amplitudes, and because their singularities can only occur on the boundary (or outside) of $\mathrm{Gr}_+(4,n)/T$, we are particularly keen to understand its boundary structure.

That is tantamount to identifying a suitable closure or compactification of $\mathrm{Gr}_+(4,n)/T$.

Moreover, motivated by similar problems that arise in physics (the open string moduli space), we are motivated to seek polytopal realizations of these compactifications, which exhibit all of the combinatorics of their boundaries (of arbitrary codimension).

Example: Six Particles

Using variables

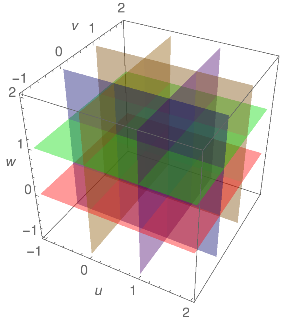
$$u = \frac{\langle 1234\rangle\langle 1456\rangle}{\langle 1245\rangle\langle 1346\rangle} \quad v = \frac{\langle 2345\rangle\langle 1256\rangle}{\langle 2356\rangle\langle 1245\rangle} \quad w = \frac{\langle 3456\rangle\langle 1236\rangle}{\langle 1346\rangle\langle 2356\rangle}$$

one finds that the singularity locus of six-particle amplitudes is

$$\mathscr{S}_6 = \bigcup_{s\in S_6} \{s=0\}, \qquad S_6 = \{u, v, w, 1-u, 1-v, 1-w, \frac{1}{u}, \frac{1}{v}, \frac{1}{w}\}$$

Example: Six Particles

In the (u, v, w) coordinate system, the six-particle positive domain $\mathrm{Gr}_+(4,6)/T$ is the interior of the unit cube. Amplitudes have no singularities inside this domain at any finite loop order in perturbation theory.



Question: Does this picture accurately portray the boundary structure of the closure $\overline{\mathrm{Gr}_+(4,6)/T}$?

Boundary Example

Let me illustrate the meaning of this question by an example.

Consider a toy model of a two-dimensional “kinematic space” parameterized by variables x, y that take values in the interior of the unit square: $x, y \in (0, 1)$.

Now let’s ask the question: is the point $(1, 1)$ really just a point, in the natural closure of this space?

Well, it depends what we mean by “natural”, and that is dictated by the class of functions that we find ourselves interested in.

For example, if we only encounter polynomials in x and y, then indeed the natural closure of the interior of the square is just a square, and $(1, 1)$ is really a point.

Boundary Example

But suppose the functions under study depend on quantities such as

$$\{u_1, \ldots, u_5\} = \left\{ x, \quad y, \quad \frac{1-y}{1-xy}, \quad 1-xy, \quad \frac{1-x}{1-xy} \right\}$$

Then if we approach the point $(1,1)$ by taking $t \to 0$ along the curve

$$x(t) = 1 - \alpha t \qquad y(t) = 1 - \beta t$$

where $\alpha, \beta > 0$, we find that

$$\{u_1, \ldots, u_5\} \to \left\{ 1, \quad 1, \quad \frac{\beta}{\alpha+\beta}, \quad 0, \quad \frac{\alpha}{\alpha+\beta} \right\}$$

Then it is evident that $(1,1)$ is not a single point, but a line segment parameterized by $u_3 = 1 - u_5 \in (0,1)$ as α, β range over all positive numbers.

The (x, y) coordinate system is not adequate to resolve this structure.

Boundary Example

In this case the interior of the unit square in (x, y) space is naturally mapped into the interior of a pentagon. Each edge of the pentagon is labeled by the unique "u" variable that vanishes on that edge.

Where does this "contrived" example come from? Consider

$$Z = \begin{pmatrix} 1 & 1 & 1 & & 0 \\ 0 & xy & y & 1 & 1 \end{pmatrix}$$

For $x, y \in (0,1)$ the minors of this matrix satisfy

$$\langle i\, j \rangle > 0 \qquad \forall i < j$$

If we read each column of this matrix as a homogeneous coordinate in $\mathbb{P}^1$, we see that this corresponds to a configuration of five ordered points

$$0 < xy < y < 1 < \infty$$

on the real axis, modulo conformal invariance (SL(2) acting from the left).

Boundary Example

Where does this "contrived" example come from? Consider

$$Z = \begin{pmatrix} 1 & 1 & 1 & & 0 \\ 0 & xy & y & 1 & 1 \end{pmatrix}$$

So this is a configuration of five open string vertex operators, the variables

$$\{u_1, u_2, u_3, u_4, u_5\} = \left\{ \frac{\langle 13\rangle\langle 45\rangle}{\langle 14\rangle\langle 35\rangle}, \frac{\langle 12\rangle\langle 45\rangle}{\langle 13\rangle\langle 25\rangle}, \frac{\langle 25\rangle\langle 34\rangle}{\langle 24\rangle\langle 35\rangle}, \frac{\langle 15\rangle\langle 24\rangle}{\langle 14\rangle\langle 25\rangle}, \frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle} \right\}$$

are conformally invariant cross-ratios, and the "compactification" described above — wherein the point $(1,1)$ was blown up into a line segment — is just the familiar Deligne-Mumford compactification of this moduli space.

Polytopes for SYM Theory

Moving back to SYM theory: As emphasized, it should be the amplitudes themselves that tell us the appropriate compactification of the positive domain on which they are actually defined.

As a shortcut, to sidestep that seemingly difficult problem, we will take a cue from string theory and investigate natural compactifications of $\mathrm{Gr}_+(4,n)/T$ that generalize the well-known Deligne-Mumford compactification of the string moduli space $\mathrm{Gr}_+(2,n)/T$.

Polytopes for SYM Theory

For $k > 2$ the $\mathrm{Gr}_+(k,n)/T$ problem is more challenging than $\mathrm{Gr}_+(2,n)/T$ because in the latter each codimension-one boundary corresponds to a collision between adjacent points, but in the former codimension-one boundaries of the moduli space can be realized in more complicated ways, such as three adjacent "points" becoming collinear, etc.

[An equivalent problem, arising from a variety of different motivations (generalized bi-adjoint ϕ^3 theory; amplituhedra; ...?), has recently been considered by many authors, including Cachazo, Early, Guevara, Mizera, Drummond, Foster, Gürdoğan, Kalousios, Rojas, Borges, Umbert, Zhang, He, Ren, Henke, Papathanasiou, Łukowski, Parisi, Williams, Moerman and will likely be discussed in several other talks.]

Construction of Polytopes

We use a construction due to N. Arkani-Hamed, S. He and T. Lam (equivalent, but dual, to the construction of Speyer and Williams (2005) using tropical geometry):

given a (cluster) chart $\vec{\mathbf{x}} = \{x_1, \ldots, x_m\}$ where $m = 3(n-5)$ and each x ranges over $(0, \infty)$ in the interior of $\mathrm{Gr}_+(4,n)/T$, [for example, x, y in the toy model considered above]

and given some collection of variables $a_1(\vec{\mathbf{x}}), \ldots, a_k(\vec{\mathbf{x}})$ (typically $k \gg m$) [for example,

$$x, \quad y, \quad 1-x, \quad 1-y, \quad 1-xy$$

in the toy model considered above]

we compute the Minkowski sum of the Newton polytopes associated to the a's.

Construction of Polytopes

The (dual of the) Speyer-Williams fan is the polytope obtained by taking the Minkowski sum over all Plücker coordinates...

... but one can make other choices. For example, keeping only Plücker coordinates of the form $\langle i\, i{+}1\, j\, j{+}1 \rangle$ and $\langle i\, j{-}1\, j\, j{+}1 \rangle$ gives, for $n = 7, 8$, polytopes we call $\mathscr{C}^\dagger(4, n)$, having f-vectors

$(1, 595, 1918, 2373, 1393, 385, 42, 1)$

$(1, 49000, 249306, 536960, 635176, 447284, 189564, 46312, 5782, 274, 1)$

These polytopes are apparently sufficient to encapsulate all (currently known) properties of 7- and 8-particle amplitudes, in a manner that will hopefully be explained in more detail by Drummond and Papathanasiou.

Notably, Drummond, Foster, Gürdoğan and Kalousios showed that all currently-known symbol letters of 8-particle amplitudes can be extracted from $\mathscr{C}^\dagger(4, 8)$!

Interlude: Scattering Diagrams

For the physicists:

Cluster variables in a rank d cluster algebra are parameterized by $\mathbf{g}$-vectors; elements of $\mathbb{Z}^d$.

To every cluster $a_1, \ldots, a_d$ is associated the cone in $\mathbb{R}^d$ generated by the corresponding $\mathbf{g}_1, \ldots, \mathbf{g}_d$.

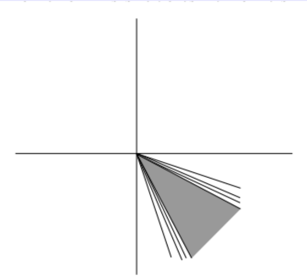
Cluster cones are non-overlapping and the union of cluster cones — called the cluster fan — covers all of $\mathbb{R}^d$ if the algebra is finite, while otherwise there are "gaps".

Every integer lattice point inside the cluster fan is naturally associated to a cluster monomial — a product of powers of cluster variables from a common cluster.

Lattice points in the gaps are associated to basis elements of the cluster algebra that are not cluster monomials.

Interlude: Scattering Diagrams

Mathematicians use (what they call) scattering diagrams to depict the **g**-vectors and cluster cones associated to an algebra; here's a rank 2 example. To each integer point in this diagram is associated some basis element; those in the white regions are cluster monomials while those in the grey "gap" are not cluster monomials.



Back to the polytopes: For six (seven) particles, the 9 (42) normal directions to the polytopes are generated by $\mathbf{g}$-vectors of the associated $\mathrm{Gr}(4, n)$ cluster algebras. (These are the rays of the Speyer-Williams fan.)

For eight particles, we find that 272 (out of 274) normal vectors are $\mathbf{g}$-vectors, and therefore correspond to cluster coordinates, many of which are known to appear in eight-point amplitudes, others of which might at higher loop order than any calculations have been done — remember we anticipate that there will exist some finite polytope that will encapsulate all-loop order predictions.

Exceptional Rays

But let's look at one of the two remaining normal rays of $\mathscr{C}^\dagger(4,8)$, generated by

$$\mathbf{v} = (-1,1,0,1,0,-1,0,-1,1)$$

This is not a $\mathbf{g}$-vector of the $\mathrm{Gr}(4,8)$ cluster algebra, in fact it is outside the cluster fan — hold on a second, how can one know this? — We know this because it happens to be one of the examples studied recently by Chang, Duan, Fraser and Li (CDFL).

Exceptional Rays

Using the algorithm explained in CDFL, one can (in principle) compute a basis element associated to each integer point along the ray generated by $\mathbf{v}$. One finds:

$$\mathscr{B}(1\mathbf{v}) = A$$
$$\mathscr{B}(2\mathbf{v}) = A^2 - B$$
$$\mathscr{B}(3\mathbf{v}) = A^3 - 2AB$$

where

$$A = \langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle$$
$$B = \langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

Conjecture: these are generated by

$$\sum_{s\geq 0} t^s \mathscr{B}(s\mathbf{v}) = \frac{1}{1 - At + Bt^2}$$

which, in particular, is a rational function of t. (Hard to test further because the computational complexity is $\mathscr{O}((4s)!)$.)

This motivates a purely mathematical question:

Let $\mathscr{A}$ be a cluster algebra of rank d, let $\mathbb{Z}^d$ be a lattice that parameterizes bases of $\mathscr{A}$, and let $\mathscr{B}(v)$ be the basis element associated to the lattice point $\mathbf{v} \in \mathbb{Z}^d$.

For $\mathbf{v} \in \mathbb{Z}^d$ define

$$f_{\mathbf{v}}(t) = \sum_{s \geq 0} \mathscr{B}(s\mathbf{v}) t^s$$

Question: under what conditions is $f_{\mathbf{v}}(t)$ a rational function of t?

If $\mathbf{v}$ lies in the cluster fan of $\mathscr{A}$ then it obviously is, since $\mathscr{B}(s\mathbf{v}) = \mathscr{B}(\mathbf{v})^s$ so the series is geometric: $f_{\mathbf{v}}(t) = 1/(1 - t\mathscr{B}(\mathbf{v}))$.

It's easy to check that $f_{\mathbf{v}}(t)$ is always rational for surface algebras, using any of the three types of bases considered by Thurston (bangles, bands, or bracelets).

Algebraic Directions in General Cluster Algebras

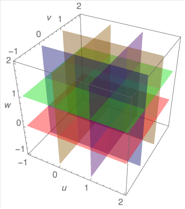
Question: under what conditions is $f_{\mathbf{v}}(t)$ a rational function of t?

But it seems to not be true in general, even at rank 2.

Of course, for amplitudes we are primarily interested not in completely general cluster algebras, but in the Grassmannian algebras.

Conclusion

- n particle scattering amplitudes in SYM theory are multi-valued functions on $\mathrm{Conf}_n(\mathbb{P}^3) \cong \mathrm{Gr}(4,n)/(\mathbb{C}^*)^{n-1}$, the configuration space of n points in $\mathbb{P}^3$.
- for each n there is a codimension subvariety $\mathscr{S}_n$; n-point amplitudes have singularities everywhere on, and only on, $\mathscr{S}_n$.
- a central conjecture is that the positive configuration space $\mathrm{Gr}_+(4,n)/T$ lies entirely inside one of the chambers of $\mathscr{S}_n$.
- more detailed information about amplitudes — for example symbol alphabets and "cluster adjacency" — is apparently encoded in combinatoric and cluster algebraic properties of this configuration space, in particular its boundaries.



- Because nothing involves the loop order L, these statements should hold to all finite loop order, and (we hope) non-perturbatively.

Conclusion

To do:

- Understand which compactification (or, in the dual language, which tropical fan) "most faithfully" exhibits the properties of amplitudes, and why.
- Harness this knowledge to learn more about amplitudes!
- Is there some canonical way in which these spaces beg to have certain functions naturally associated to them (in a manner analogous to the way that positive geometries have naturally associated canonical forms), such that those functions turn out to precisely be the amplitudes of SYM theory?