From matroid subdivisions of hypersimplices to generalized Feynman diagrams

Nick Early

Perimeter Institute for Theoretical Physics

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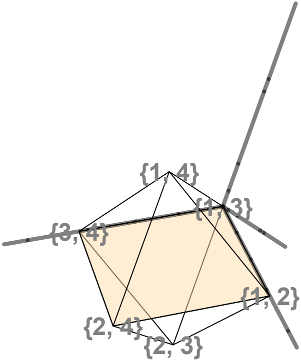

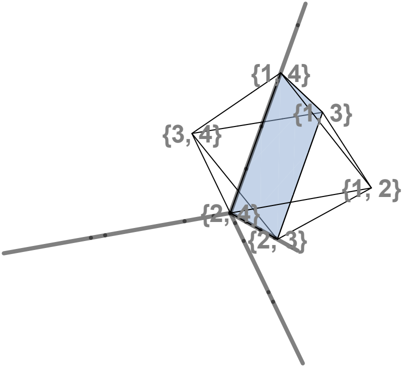

Blades on the $\mathcal{U}$niversal $\mathcal{O}$ctahedron.

The unreasonable effectiveness of mathematics.

- Background. I'll tell two parallel stories, both starting in 2012-2013, which ultimately joined forces in 2018/2019: my (math) Ph.D. thesis on symmetries and invariants for subdivisions of hypersimplices, and the CHY method for computing biadjoint scattering amplitudes.

Objectives

- Objective (0): interpret the biadjoint scalar scattering amplitude $m^{(k=2)}(\mathbb{I}_n, \mathbb{I}_n)$ in terms of matroid subdivisions of a convex polytope known as the (second) hypersimplex, $\Delta_{2,n}$.
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- Objective (3): show how these arrangements select a natural set of planar functions on the kinematic space for all n and k. These define a basis, the *planar basis*.
- *Remark:* These specialize for $k = 2$ to the planar basis introduced by [CHY2013] and denoted X_{ij} in [ABHY2017].

A model for $m^{(k)}(\mathbb{I}_n, \mathbb{I}_n)$

- The kinematic space $\mathcal{K}_{k,n}$ lives in the space of symmetric tensors in $\mathbb{R}^{\binom{n}{k}}$:

$$\mathcal{K}_{k,n} = \left\{ (s) \in \mathbb{R}^{\binom{n}{k}} : \sum_{J : J \ni a} s_J = 0 \text{ for each } a = 1, \ldots, n \right\}.$$

- Goal: to construct $m^{(k)}(\mathbb{I}_n, \mathbb{I}_n)$ inside $\mathcal{K}_{k,n}$ for $m^{(k)}(\mathbb{I}_n, \mathbb{I}_n)$ using exactly two bits of information: one convex polytope, the hypersimplex

$$\Delta_{k,n} = \text{convex hull} \left\{ x \in \{0,1\}^n : \sum x_i = k \right\}$$

and the facets of the simplex[1]

$$\{x \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

[1] More precisely, we need its normal fan.

Some history

- [CHY, 2013]. Scattering equations compute $k = 2$ biadjoint scalar scattering amplitudes: finite sum over critical points of a potential function in the moduli space of n points on the torus quotient of the Grassmannian $X(2, n) = G(2, n)/\left(\mathbb{C}^*\right)^n$,

$$\mathcal{S} = \sum_{1 \le i < j \le n} s_{ij} \log\left(\det\begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix}\right),$$

  where (s_{ij}) are symmetric matrices, with entries Mandelstam invariants; for well-definedness one assumes

$$\sum_{j \neq i} s_{ij} = 0 \text{ for all } i.$$

- One writes, schematically

$$m^{(2)}(\mathbb{I}_n, \mathbb{I}_n) = \int_{X(2,n)} \delta\left(\frac{\partial \mathcal{S}}{\partial y}\right) PT(12\cdots n)^2,$$

  where the $SL(2)$-measure is implicit.

Motivation for the generalized biadjoint scalar theory

- Key observation: duality for Grassmannians implies that CHY holds for both moduli spaces:

$$X(2,n) \longleftarrow \cdots \longrightarrow X(n-2,n),$$

where we identify $X(k,n) = G(k,n)/(\mathbb{C}^*)^n$.
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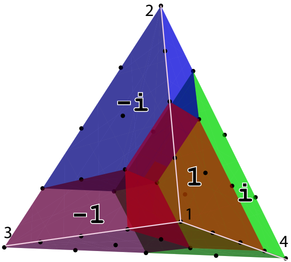

Basic ingredients of the generalized biadjoint scalar theory, (here for k=3)

- [CEGM March, 2019]. We generalized the CHY method, from n points on the Riemann sphere, $X(2, n)$, to the configuration space $X(k, n)$ of n points in $\mathbb{CP}^{k-1}$.
- This filled in the gap $X(2, n) \Leftrightarrow X(n-2, n)$...
- Along the way we discovered that some of our formulas appeared in the math literature, in the context of the tropical Grassmannian (and its positive part)!
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- Then the connection to matroid subdivisions quickly fell into place.
- Now define a (potential) function $\mathcal{S} : X(3,n) \to \mathbb{C}$:

$$\mathcal{S} = \sum_{1\leq a<b<c\leq n} s_{abc} \log\left(|abc|\right)$$

Introducing $m^{(3)}(\mathbb{I}_n, \mathbb{I}_n)$

- Generalized Mandelstam invariants s_{abc} are linear functions on the space of (symmetric) tensors, $\mathbb{R}^{\binom{n}{3}}$.
- **Rem.** The s_{abc} satisfy $\sum_{\{(a<b)\neq t\}} s_{abt} = 0$ for each t if and only if $\mathcal{S}$ is well-defined on $X(3,n)$.
- [CEGM2019]. The k-Parke Taylor factor (in the natural cyclic order) is

$$\mathrm{PT}^{(3)}(1,2,\ldots,n) := \frac{1}{|123||234|\cdots|n12|}$$

and for n-cycles α, β, the generalized biadjoint scalar amplitude is

$$m_n^{(3)}(\alpha|\beta) = \sum_{I=1}^{\mathcal{N}_n^{(3)}} \left(\frac{1}{\det'\Phi^{(3)}}\mathrm{PT}^{(3)}(\alpha)\mathrm{PT}^{(3)}(\beta)\right)\Bigg|_{x_a=x_a^{(I)},y_a=y_a^{(I)}}$$

The generalized biadjoint scattering amplitude is computed by summing over the critical points of $\mathcal{S}$, for a given choice of kinematics (s_{abc}). Here $\Phi^{(3)}$ is the Hessian matrix for $\mathcal{S}$.

Initial computations

- [CEGM, April 2019]. We solved the scattering equations directly, choosing kinematics s_{abc} to be large prime numbers, to obtain $m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$.

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- Remark: by changing the integrand we also found a non-planar Feynman diagram (not in $m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$),
$$\frac{1}{s_{123}s_{345}s_{561}s_{246}}.$$

New features of $m^{(k\geq 3)}(\mathbb{I}_n, \mathbb{I}_n)$

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$$m^{(3)}(\mathbb{I}_6, \mathbb{I}_6) = \underbrace{\frac{1}{s_{123}s_{456}t_{1236}t_{3456}}}_{\text{4 compatible here}} + \underbrace{\frac{R_{12,34,56} + R_{12,56,34}}{t_{1234}t_{3456}t_{1256}R_{12,34,56}R_{12,56,34}}}_{\text{but 5 here!}} + (46\text{ more})$$

Here

$$t_{1234} = s_{123} + s_{134} + s_{124} + s_{234}, \;\; R_{12,34,56} = t_{1234} + s_{125} + s_{126}.$$

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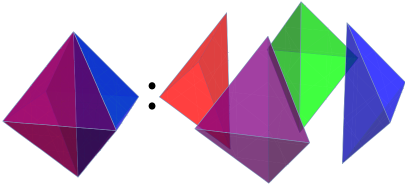
Rest of the talk will be mostly combinatorial...

Background and connections 1

- [Mizera, June 2017] and [ABHY, Nov 2017]: identification of $m^{(2)}(\mathbb{I}_n, \mathbb{I}_n)$ with an associahedron in the $(k = 2)$ kinematic space.
- Circa December, 2017 and beyond. After ABHY's treatment of kinematic space and pole compatibility on the associahedron: the story connecting amplitudes to matroid subdivisions started to become coalesce...

The $\mathcal{U}$niversal $\mathcal{O}$ctahedron determines pole compatibility for $\Delta_{2,n}$ (and beyond)



Main example: the four tetrahedra now share a common edge connecting e_{13} to e_{24}, which is not a *root* $e_i - e_j$; this is bad!

- Octahedron: $\Delta_{2,4} = \text{convex hull}\{e_i + e_j : 1 \leq i < j \leq 4\}$. All edges are parallel to roots $e_i - e_j$.

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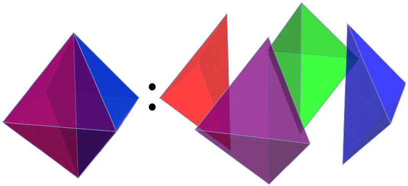

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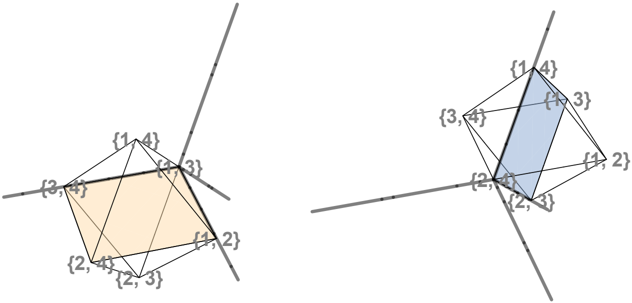
- Octahedron: $\Delta_{2,4} = \text{convex hull}\{e_i + e_j : 1 \le i < j \le 4\}$. All edges are parallel to roots $e_i - e_j$.
- Three ways to split the octahedron into two half-pyramids $\Leftrightarrow \{\frac{1}{s_{12}}, \frac{1}{s_{23}}, \frac{1}{s_{13}}\}\ldots$
- But only $\frac{1}{s_{12}}$ and $\frac{1}{s_{23}}$ appear in $m^{(2)}((1234),(1234))$.

What is a matroid subdivision?



Fact: A subdivision of $\Delta_{k,n}$ is (1) matroidal and (2) compatible with the cyclic order $(12\cdots n)$, if and only if on any octahedral face it coincides with one of these two pictures, replacing $(1 < 2 < 3 < 4)$ with any $(a < b < c < d)$.

The construction using the gray rays of *blade arrangements* is new [E,Oct2019] and will be explained.

First clue: momentum conservation on the octahedron

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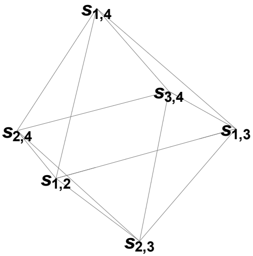

So, sum s_{ij} over any triangular face and get zero.

Second clue

- For more particles $n \geq 5$, momentum conservation translates to averaging over the facets $x_j = 1$ of the 2nd hypersimplices

$$\Delta_{2,n} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 2 \right\}.$$

- Second clue: the Mandelstam variables are reflection invariant; similarly, there is an invariant hyperplane:

$$(s_J = s_{J^c}) \Leftrightarrow \left\{ x \in \Delta_{2,n} : \sum_{j \in J} x_j = 1 = \sum_{j \in J^c} x_j \right\}.$$

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- Each such invariant hyperplane divides $\Delta_{2,n}$ into a pair of convex polytopes.
- Third clue: namely, the matroid polytopes $\sum_{j \in J} x_j \geq 1$ and $\sum_{j \in J} x_j \leq 1$!
- Fourth clue: compatibility for poles of Feynman diagrams...

Compatible poles

- Following usual notation, Mandelstam invariants are sums of coordinate functions on the kinematic space: $s_J = \sum_{(i<j)\in J} s_{ij}$ for J a subset of $[n]$, and $s_J = s_{J^c}$ by momentum conservation.
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- For example:

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- For example:

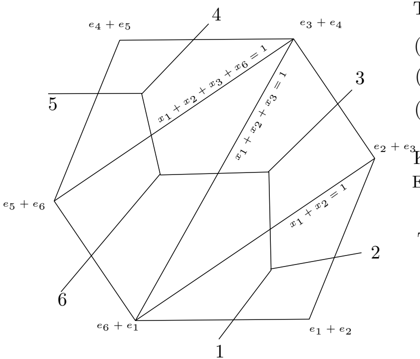
$$m^{(2)}(\mathbb{I}_5, \mathbb{I}_5) = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{234}}$$

- Now it's easy to see that summands are in bijection with the five *finest* matroid subdivisions of $\Delta_{2,5}$ *compatible with the cyclic order* (12345).

Matroid subdivisions and Feynman diagrams: bijection

$\Delta_{2,6} = \{x \in [0,1]^6 : x_1 + x_2 + \cdots + x_6 = 2\}$

$\Downarrow$ (for $\Delta_{k,n}$ all k projection see [Postnikov 2018])



Three 2-splits of $\Delta_{2,6}$:

(1) $x_1 + x_2 = 1 \Leftrightarrow s_{12}$

(2) $x_1 + x_2 + x_3 = 1 \Leftrightarrow s_{123}$

(3) $x_1 + x_2 + x_3 + x_6 = 1 \Leftrightarrow s_{1236}$

Key insight: these 2-splits are pairwise compatible!
E.g. $(\{1,2,3\} \cap \{4,5\} = \emptyset)$, satisfying compatibility.

Three compatible hyperplanes $\Leftrightarrow$ poles in $\frac{1}{s_{12}s_{123}s_{1236}}$

Now easy to conclude: given a planar order $(12\cdots n)$, **finest positroidal subdivisions of $\Delta_{2,n}$ are in bijection with Feynman diagrams with legs ordered** $(12\cdots n)$**.**

In fact, the whole amplitude $m^{(2)}(id, id)$ is a sum over all finest positroidal subdivisions of $\Delta_{2,n}$.

Can the identification be made explicit? $\Delta_{k,n} = \{[0,1]^n : \sum_{i=1}^{n} x_i = k\}$?

Parametrizing $m^{(k)}(\mathbb{I}_n, \mathbb{I}_n)$ for any k with positroidal subdivisions

- Before introducing the planar basis and weakly separated collections, we'll need some general results...
- **Defn.** The kinematic space $\mathcal{K}_{k,n}$ is a codimension n subspace of $\mathbb{R}^{\binom{n}{k}}$,

$$\mathcal{K}_{k,n} = \left\{ (s) \in \mathbb{R}^{\binom{n}{k}} : \sum_{J : J \ni a} s_J = 0 \text{ for each } a = 1, \ldots, n \right\}.$$

Hypersimplices, matroid polytopes & subdivisions

- A hypersimplex is an integral cross-section of a unit cube:

$$\Delta_{k,n} = \left\{x \in [0,1]^n : \sum_{i=1}^{n} x_i = k\right\},\ 1 \leq k \leq n-1.$$

$$= \text{convex hull}\left\{e_J : J \in \binom{[n]}{k}\right\},$$

  where $e_J = e_{j_1} + \cdots + e_{j_k}$ and J runs over all k-element subsets of $[n] = \{1, \ldots, n\}$.
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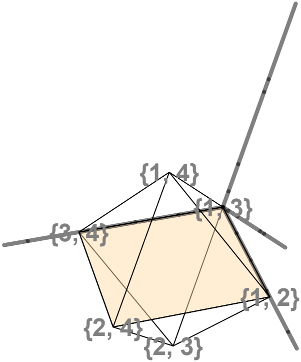
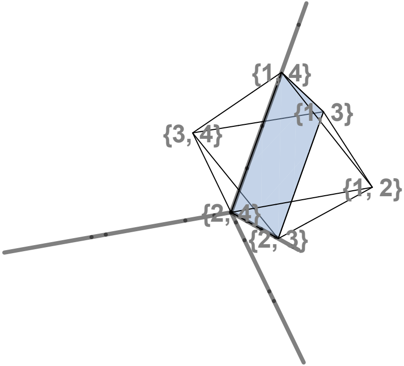
- For example, $\Delta_{1,4}, \Delta_{3,4}$ are tetrahedra, while $\Delta_{2,4}$ is an octahedron.
- A *matroid polytope* is... a subpolytope P of (some) $\Delta_{k,n}$ such that every edge of P is parallel to an edge of $\Delta_{i,j}$: $\{e_i - e_j : i \neq j\}$.
- A *matroid subdivision* is a decomposition of $\Delta_{k,n}$ into matroid polytopes intersecting only on their common facets.

2-splits of $\Delta_{2,n}$

- **Defn/Example.** A *2-split* (of $\Delta_{2,n}$) is a decomposition $\Pi_1 \cup \Pi_2 = \Delta_{2,n}$ into matroid polytopes sharing a common facet $\Pi_1 \cap \Pi_2$.
- For $\Delta_{2,n}$ these look like $\sum_{j \in J} x_j = 1$ with $2 \leq |J| \leq n-2$. The common facet is a Cartesian product of simplices of dimensions $|J| - 1$ and $|J^c| - 1$.
- $\hookrightarrow$ 2-splits of $\Delta_{2,n}$ are well-understood mathematically, and familiar from $m^{(2)}(\mathbb{I}_n, \mathbb{I}_n)$.

The two subdivisions of an octahedron





The two nontrivial blade arrangements on the octahedron $\Delta_{2,4}$. Edges of the octahedron are in the directions $e_i - e_j$. Same for the pairs of square pyramids.

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Compatible 2-splits of $\Delta_{2,n}$

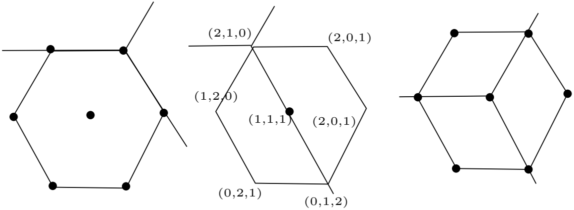
- There's a well-known compatibility rule for 2-splits of the second hypersimplex $\Delta_{2,n}$...
- Maximal cells of the subdivision of $\Delta_{2,n}$ induced by the pair of hyperplanes $\sum_{i\in J_1} x_i = 1$ and $\sum_{i\in J_2} x_i = 1$ are *matroid* polytopes if and only if at least one intersection is empty: $J_1 \cap J_2,\ J_1 \cap J_2^c,\ J_1^c \cap J_2,\ J_1^c \cap J_2^c$.
- The compatibility rule for pairs of matroid subdivisions of $\Delta_{k,n}$ involves checking a condition on each little octahedral facet of $\Delta_{k,n}$!

Blades on a hexagon

- Recap: for $\Delta_{2,n}$, poles are 2-splits and Feynman diagrams are superpositions of 2-splits.
- New for $k \geq 3$ subdivisions:

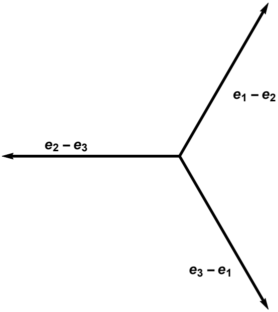
Blades on a hexagon

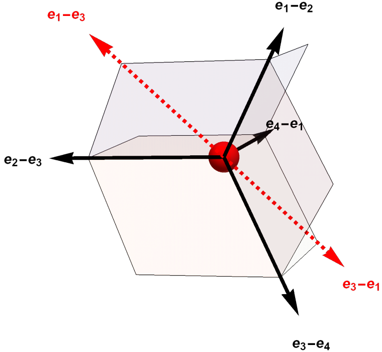
- Recap: for $\Delta_{2,n}$, poles are 2-splits and Feynman diagrams are superpositions of 2-splits.
- New for $k \geq 3$ subdivisions: poles correspond to splittings of $\Delta_{k,n}$ into more than 2 chambers!
- [E,Oct2019] Introduced a new method to induce splits:



1-split, 2-split, 3-split: induced by gluing a single *blade* $((1,2,3))$ to a vertex of a hexagon.

Blades in higher dimensions





Blades $((1,2,3))$ and $((1,2,3,4))$. Left: bends of the function $h(x) = \min\{x_2 + 2x_3, x_3 + 2x_1, x_1 + 2x_2\}$. Definition due to my Ph.D. co-adviser, A. Ocneanu.

Blade Definition

- Definition [Ocneanu]. Fix an integer $n \geq 3$. The blade $((1, 2, \ldots, n))$ is the union of the boundaries of n polyhedral cones:

$$((1, 2, \ldots, n)) = \bigcup_{j=1}^{n} \partial \left\{ \sum_{i \neq j} t_i(e_i - e_{i+1}) : t_i \geq 0 \right\}.$$

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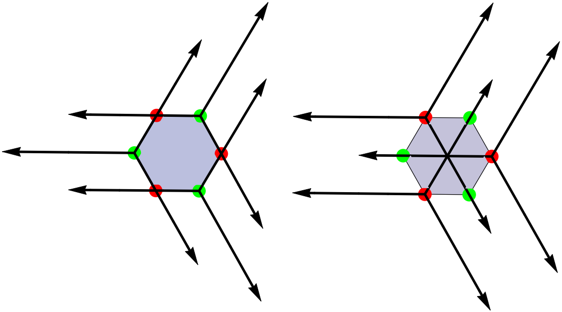
- Some related notions:
- **Prop**[E,2019]. This is a particular kind of *tropical hypersurface*. It is also the ($n-2$ skeleton of) the *normal fan* to the simplex $x_1 \leq \cdots \leq x_n \leq x_1 + 1$.

Main constructions

- **Definition.** A *blade arrangement* is a superposition of several copies of a blade on a hypersimplex $\Delta_{k,n}$.
- **Definition**. A *matroidal* blade arrangement is an arrangement of blades on some of the $\binom{n}{k}$ vertices $\sum_{j=1}^{k} e_{i_j}$ of a *hypersimplex* such that every maximal cell is *matroidal*: i.e., every edge of every maximal cell is in a root direction $e_i - e_j$.
- **Prop**[E, Oct2019]. Any matroid subdivision that is induced by a matroidal blade arrangement is *positroidal:* locally it intersects the octahedron in one of the splits induced by one of the two (equatorial) squares, $x_1 + x_2 = 1$ or $x_1 + x_4 = 1$.
- Idea of proof: compute explicitly the internal facet inequalities. They should be of the form $x_i + x_{i+1} + \cdots + x_{i+j} \geq a_{ij}$, where the index sets are cyclic modulo n.

Three blades arranged on $\Delta_{3,6}$ (seen projected into the plane)



Two arrangements of the blade $((1, 2, 3))$ on the vertices of a hexagon. Blade arrangement on left induces the trivial subdivision. Blade arrangement on right induces a 6-chamber subdivision.

These are projections of matroidal blade arrangements on $\Delta_{3,6}$.

- **Definition.** (due to Leclerc-Zelevinsky; we rephrase for our purposes). A pair of k-element subsets $J_1, J_2 \in \binom{[n]}{k}$ is weakly separated[2] if $e_{J_1} - e_{J_2}$ avoids $(\cdots + e_a - e_b + e_c - e_d + \cdots)$ for the cyclic pattern $a < b < c < d$.
- Example $(k, n) = (3, 6)$. Then

[2] compare in what follows to the classical Steinmann relations

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- Example $(k, n) = (3, 6)$. Then $e_{134} - e_{245} = e_1 - e_2 + e_3 - e_5$, so $\{134, 245\}$ is *not* weakly separated.

[2] compare in what follows to the classical Steinmann relations

Main combinatorial result

- **Thm**[E,Oct.2019]. An arrangement of the blade $((1,2,\ldots,n))$ on the vertices $e_{J_1},\ldots,e_{J_N}\in\Delta_{k,n}$ induces a matroid subdivision of $\Delta_{k,n}$ if and only if the collection $\{J_1,\ldots,J_N\}$ is *weakly separated*.
- Comments:

Main combinatorial result

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- Comments: This is actually really strong. For generic matroid subdivisions of $\Delta_{k,n}$ for large k and n we would have a large computational task to determine their compatibility.
- Can our construction can be used to leverage larger computations for generic matroidal subdivisions?
- We conclude with some illustrations...

Numbers of finest positroidal subdivisions $\Delta_{k,n}$ *induced by blade arrangements*

The table below uses weakly separated collections to enumerate *subsets* of the set of finest positroidal subdivisions of $\Delta_{k,n}$.
Prop. These map to arrays of Feynman diagrams from [CGUZ]. Notably,

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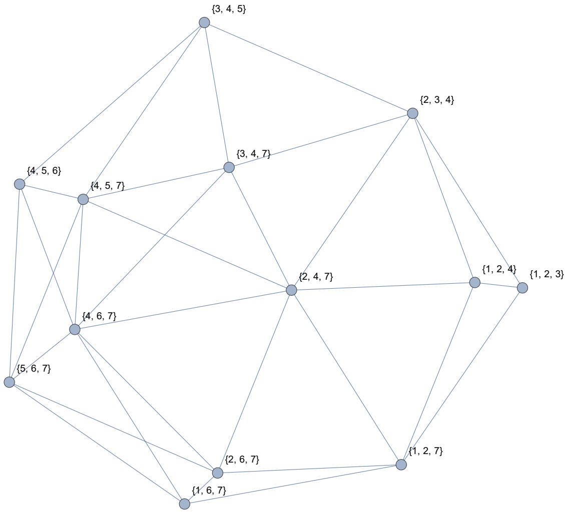
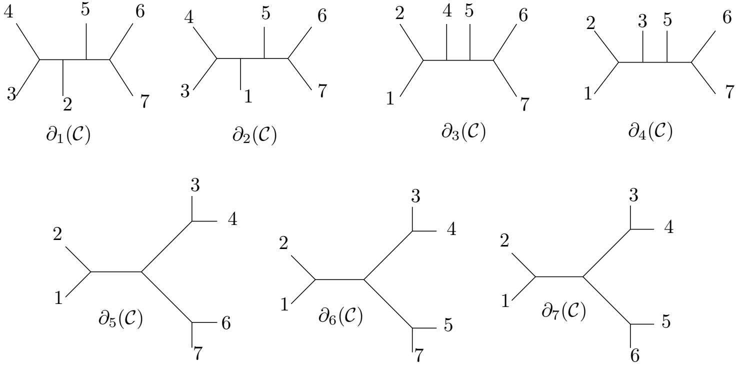
Prop. These map to arrays of Feynman diagrams from [CGUZ]. Notably, all generalized Feynman diagrams here have *exactly* $(k-1)(n-k-1)$ poles. This is *not* true for positroidal subdivisions in general!

$n \setminus k$	2	3	4	5	6	7	8	9	10
4	2								
5	5	5							
6	14	34	14						
7	42	259	259	42					
8	132	2136	5470	2136	132				
9	429	18600	122361	122361	18600	429			
10	1430	168565	2889186	7589732	2889186	168565	1430		
11	4862	1574298	71084299			71084299	1574298	4862	
12	16796	15051702						15051702	16796



Embedding of the blade arrangement $\{124, 247, 267, 347, 457, 467\}$ on the 1-skeleton of $\Delta_{3,7}$.

Blade arrangements induce collections of Feynman diagrams



The seven boundaries of the blades labeled by the weakly separated collection $\{124, 247, 267, 347, 457, 467\}$. Each tree encodes

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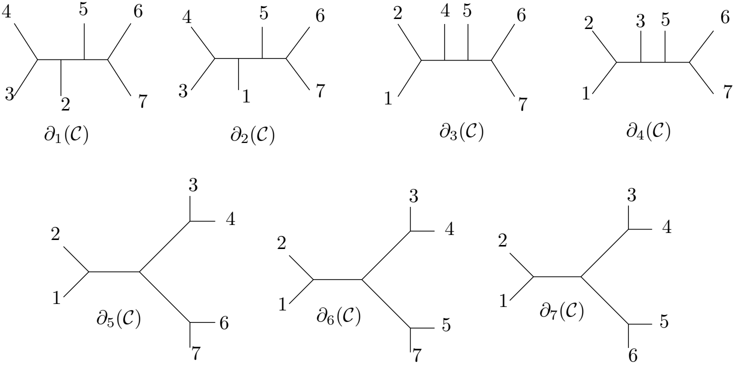

The seven boundaries of the blades labeled by the weakly separated collection $\{124, 247, 267, 347, 457, 467\}$. Each tree encodes a matroid subdivision of a face of $\Delta_{3,7}$, i.e., a copy of $\Delta_{2,6}$. The boundary operator can be defined directly on sets of k-tuples (not obvious!).

Blades induce positroidal multi-splits

- An essential question: which matroid subdivisions are induced by matroidal blade arrangements?

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- **Theorem**[E, Oct2019] The blade $((1,2,\ldots,n))_{e_J}$ induces a multi-split positroidal subdivision of $\Delta_{k,n}$, where the maximal cells are nested matroids. The number of maximal cells in the subdivision equals the number of cyclically consecutive intervals in the labels in J.

Towards an all-k planar basis

Let $V_0^n \subset \mathbb{R}^n$ be the hyperplane $x_1 + \cdots + x_n = 0$.

Defn. Let $h : V_0^n \to \mathbb{R}$ be the piece-wise linear function $h(x) = \min\{L_1(x), \ldots, L_n(x)\}$, where

$$L_j = x_{j+1} + 2x_{j+2} + \cdots (n-1)x_{j-1}.$$

Prop.[E,Oct2019].

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Prop.[E,Oct2019]. The blade $((1,2,\ldots,n))$ equals the bend locus of the function $h(x)$. That is,

$$((1,2,\ldots,n)) = \{x \in V_0^n : (L_i(x) = L_j(x)) \leq L_\ell(x) \text{ for all } \ell \neq i,j\}.$$

Height functions

Defn.[E,Dec 2019]. At each vertex $e_J (= \sum_{j\in J} e_j) \in \Delta_{k,n}$, we'll glue a copy of $((1,2,\ldots,n))$ and define a linear form on $\mathcal{K}_{k,n}$:

[3] *frozen* elements are zero: $\eta_{i,i+1,\ldots,i+(k-1)} = 0$

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$$\rho_J(x) = h(x-e_J), \text{ and } \quad \eta_J = -\frac{1}{n}\sum_{e_I\in\Delta_{k,n}} \rho_J(e_I)s_I.$$

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[E, Dec2019]. The set $\{\eta_J : J \text{ is nonfrozen}\}$ is a basis[3], the *planar basis*, for the space of linear forms on the kinematic space $\mathcal{K}_{k,n}$. These objects η_J have some useful properties which we discuss now...

[3] *frozen* elements are zero: $\eta_{i,i+1,\ldots,i+(k-1)} = 0$

Why linear forms η_J?

Warm up, $k = 2$. On the kinematic space $\mathcal{K}_{2,6}$

$$\eta_{24} = \frac{1}{4}\left(3s_{12} + 2s_{13} + s_{14} + s_{23} + 3s_{34}\right)$$

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Of course this all works beautifully for $k \geq 3$: e.g., (3,6):

$$\eta_{135} = \frac{1}{6}(3s_{123} + 2s_{124} + s_{125} + 6s_{126} + \cdots + s_{356} + 6s_{456})$$

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&\equiv s_{123}+s_{126}+s_{136}+s_{234}+s_{235}+s_{236}.
\end{aligned}$$

This is one of the new poles ("$R_{16,23,45}$") in $m^{(3)}(\mathbb{I}_6,\mathbb{I}_6)$!

Inverse transformation

- Nice "cubical" rule for expanding s_J as a sum of η_J's ($k=2$ case familiar):

$$s_{25} = -(\eta_{14} - \eta_{15} - \eta_{24} + \eta_{25}).$$

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- There is a generalization to $k \geq 3$:

$$\begin{aligned} -s_{235} &= \eta_{235} - \eta_{234} - \eta_{135} + \eta_{134} \\ -s_{246} &= \eta_{246} - \eta_{146} - \eta_{236} + \eta_{136} - \eta_{245} + \eta_{145} + \eta_{235} - \eta_{135}. \end{aligned}$$

Planar basis: explicit inversion formula

- [E,Dec2019] Given a nonfrozen vertex $e_J \in \Delta_{k,n}$ with $t(\geq 2)$ cyclic intervals, with cyclic initial points say $j_1, \ldots, j_t$, consider the t-dimensional cube

$$C_J = \left\{J_L = \{j_1 - \ell_1, \ldots, j_t - \ell_t\} : L = (\ell_1, \ldots, \ell_t) \in \{0,1\}^t\right\}.$$

Then the following "cubical" relation among linear functionals holds identically on $\mathbb{R}^{\binom{n}{k}}$, as well as on the subspace $\mathcal{K}_{k,n}$:

$$\sum_{L \in C_J} (-1)^{L \cdot L} \eta_{J_L} = -s_J,$$

where $L \cdot L$ is the number of 1's in the $0/1$ vector L.

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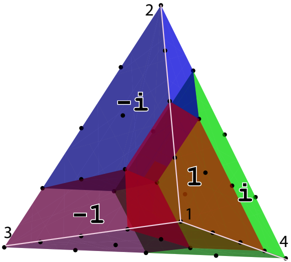
- In the planar basis, $m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$ has the expression

$$\begin{aligned} m^{(3)}(\mathbb{I}_6, \mathbb{I}_6) &= \frac{1}{\eta_{125}\eta_{134}\eta_{135}\eta_{145}} + \frac{1}{\eta_{124}\eta_{125}\eta_{134}\eta_{145}} \\ &+ \frac{1}{\eta_{136}\eta_{145}\eta_{146}\left(-\eta_{135} + \eta_{136} + \eta_{145} + \eta_{235}\right)} \\ &+ \frac{\eta_{136} + \eta_{145} + \eta_{235}}{\eta_{135}\eta_{136}\eta_{145}\eta_{235}\left(-\eta_{135} + \eta_{136} + \eta_{145} + \eta_{235}\right)} + 44 \text{ more.} \end{aligned}$$

Thank you!



q-plate in dimension 3

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