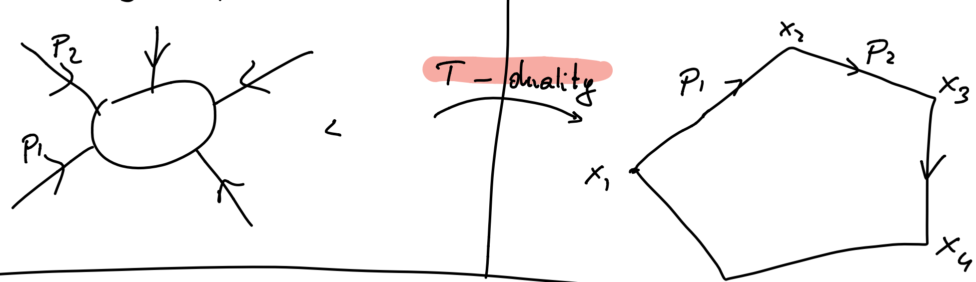
From amplituhedron to positive tropical Graßmannian via hypersimplex.

with Matteo Parisi and Lauren Williams, 2002.06164.

(1) Motivation

In planar $\mathcal{N}=4$ SYM: two sets of variables:

on-shell space $(\lambda^a, \tilde{\lambda}^{\dot{a}})$	momentum twistors: Z^A
scattering amplitudes	polygonal Wilson loops



$N^k MHV$ n-point amplitudes

[Damgaard, Ferro, TL, Parisi]	[Arkani-Hamed, Trnka]
Momentum amplituhedron: $\mathcal{M}^{(4)}_{n,k}$	Amplituhedron: $\mathcal{A}^{(4)}_{n,k}$
$\Phi_{\Lambda,\tilde{\Lambda}}: G_+(k+2,n) \to G(k+2,k+4) \times G(n-k-2,n-k)$	$\Phi_Z: G_+(k,n) \to G(k,k+4)$
$\Phi_{\Lambda,\tilde{\Lambda}}(c) = (c^\perp \Lambda, c\tilde{\Lambda})$	$\Phi_Z(c) = cZ$

Amplitude encoded in a logarithmic differential form (triangulate)

$\omega_{n,k} = \sum_{S_\sigma} \omega^{S_\sigma}$	$\tilde{\omega}_{n,k} = \sum_{S_\tau} \omega^{S_\tau}$
S_σ - a cells in $G_+(k+2,n)$ with $\dim = 2n-4$	S_τ - a cell in $G_+(k,n)$ with $\dim = 4k$

These cells are related by T-duality map:

$$\boxed{\tau(i) = \hat{\sigma}(i) = \sigma(i-2)}$$

Conjecture: The collection of cells $\{S_\sigma\}$ triangulates $\mathcal{M}^{(4)}_{n,k}$ iff the collection $\{S_{\hat{\sigma}}\}$ triangulates $A^{(4)}_{n,k}$.

- **Open question**: How to find all possible triangulations of $\mathcal{M}^{(4)}_{n,k} / \mathcal{A}^{(4)}_{n,k}$?

② **Toy example: $m=2$**

We have an additional space we want to study:

$$\begin{array}{ccccc}
G_+(k+1,n) & \xleftrightarrow{\text{identity}} & G_+(k+1,n) & \xrightarrow{\text{T-duality}} & G_+(k,n) \\
\downarrow \mu & & \downarrow \Phi_{\Lambda,\tilde{\Lambda}} & & \downarrow \Phi_Z \\
\Delta_{k+1,n} & & \mathcal{M}^{(2)}_{n,k} & & \mathcal{A}^{(2)}_{n,k}
\end{array}$$

hypersimplex

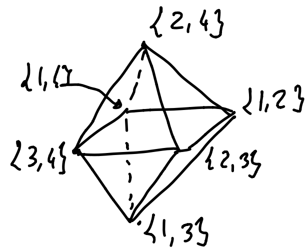
Hypersimplex is the image of $G_+(k+1,n)$ through the moment map μ:

$$\mu(c) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(c)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(c)|^2}$$

where $[n] = \{1, 2, \ldots, n\}$, $e_I = e_{i_1} + \ldots + e_{i_k}$ and $p_I(c)$ is a Plücker coordinates for matrix c.

Example: $k=1,\ n=4$

$$\Delta_{2,4} = \operatorname{conv}\{ \underset{\{1,2\}}{(1,1,0,0)}, \underset{\{1,3\}}{(1,0,1,0)}, \underset{\{1,4\}}{(1,0,0,1)}, \underset{\{2,3\}}{(0,1,1,0)}, \underset{\{2,4\}}{(0,1,0,1)}, \underset{\{3,4\}}{(0,0,1,1)} \}$$



Dimension of $\Delta_{k+1,n}$ is $n-1$ since $x_1+x_2+\dots+x_n=k$.

- We consider **positroid dissections** of the hypersimplex: take any positroid cell S_σ of $G_+(k+1,n)$ for which $\mu(S_\sigma)$ has dimension $n-1$. We call such images **positroid polytopes** Γ_σ

Example: take $S_\sigma \in G_+(2,4)$ such that $P_{\{1,2\}}(c)=0$.
Then $\Gamma_\sigma = \operatorname{conv}\{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.

Definition: A positroid dissection of $\Delta_{k+1,n}$ is a collection of cells $\{S_\sigma\}$ such that

- $\dim \Gamma_\sigma = n-1$
- $\Gamma_\sigma \cap \Gamma_{\sigma'} = \phi$
- $\bigcup_\sigma \overline{\Gamma_\sigma} = \Delta_{k+1,n}$

for amplituhedron $\mathcal{A}^{(2)}_{n,k}$

- $\dim \Phi_Z(S_\sigma) = 2k$
- $\Phi_Z(S_\sigma) \cap \Phi_Z(S_{\sigma'}) = \phi$
- $\bigcup_\sigma \overline{\Phi_Z(S_\sigma)} = \mathcal{A}^{(2)}_{n,k}$

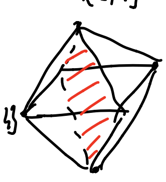
If additionally we have that μ is injective for all S_σ then we call it a <u>positroid triangulation</u>

<u>Claim</u>: The collection $\{S_\sigma\}^{\text{in } G_+(k+1,n)}$ gives a triangulation (dissection) of $\Delta_{k+1,n}$ ($\mathcal{A}^{(2)}_{n,k}$) iff the collection $\{S_{\hat\sigma}\}$ in $G_+(k,n)$ gives a triangulation (dissection) of $\mathcal{A}^{(2)}_{n,k}$, where

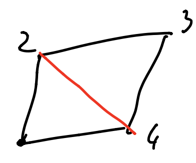
$$\hat\sigma(i) = \sigma(i-1)$$

<u>Example</u>: $k=1$, $n=4$

hypersimplex $\Delta_{2,4}$



amplituhedron $\mathcal{A}^{(2)}_{4,1}$



- Three types of triangulations/dissections
  - triangulations/dissections
  - <u>good</u> triangulations/dissections: codimension one boundaries are properly aligned
  - <u>regular</u> triangulations/dissections: come from a height function.

First example of triangulation which is not a good triang:

$$\Delta_{3,6} \quad \text{or} \quad \mathcal{A}^{(2)}_{6,2}.$$

First example of good triangulation which is not regular:

$$\Delta_{3,9} \quad \text{or} \quad \mathcal{A}^{(2)}_{9,2} \qquad \underline{\text{or}} \qquad \Delta_{4,8} \quad \text{or} \quad \mathcal{A}^{(2)}_{8,3}.$$

- Relation to <u>positive tropical Graßmannian.</u>
  - take a real-valued function:
    $$\{I\} \to P_I \qquad , I \in \binom{[n]}{k}$$
  - consider points $(e_I, P_I) \in \Delta_{k,n} \times \mathbb{R}$ and take their convex hull
  - take the lower faces and project them back to $\Delta_{k,n}$ - this gives a subdivision of $\Delta_{k,n}$
  - subdivisions obtained in this way are called <u>regular</u>

<u>Claim</u>: A collection $\{S_\sigma\}$ of positroid cells of $G_+(k,n)$ gives a regular triangulation of $\Delta_{k,n}$ iff this triangulation comes from a height function with $P = (P_I)$ being a positive tropical Plücker vector from a maximal cone of $\mathrm{Trop}_+ G(k,n)$.

<u>Corollary</u>: The number of regular positroidal triangulations of the hypersimplex $\Delta_{k,n}$ equals the number of maximal cones in the positive tropical Graßmannian $\mathrm{Trop}_+ G(k,n)$.