Progress in $\mathcal{N} = 4$ amplitude calculations

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Introduction

Feynman integrals are interesting for their connection to physics and also mathematics (number-theory and algebraic-geometry). Mathematical tools like twistor space, cluster algebras, polylogarithms, symbols, coproducts, cosmic Galois group, homology, have been useful and there are certainly more connections to be made.
Quantum Field Theories can be pretty complicated, but a very simple one I will be mostly focusing on, is the $\mathcal{N} = 4$ supersymmetric gauge theory. A lot of what I'll be saying applies in other cases as well.

Feynman integrals

Starting from any oriented graph G (in fact any multigraph) we can build an integral by multiplying terms associated to edges and vertices. To each edge $e \in G$ we associate a term $\frac{1}{p_e^2 - m_e^2}$ called *propagator*, where m_e is called *mass* and p_e is called *momentum*. The square p_e^2 is calculated with a quadratic form with signature $(1, d-1)$ (the Minkowski metric).
Momentum is conserved at vertices. Usually there are contributions arising from vertices and the propagators may also have a nontrivial numerator (depending on the type of particle). The generic form of a Feynman integral is

$$\int (d^d k)^L \frac{N}{\prod_{e \in G}(p_e^2 - m_e^2)}, \tag{1}$$

where the integral is over the momenta which satisfy the conservation relations.

Example: box integral

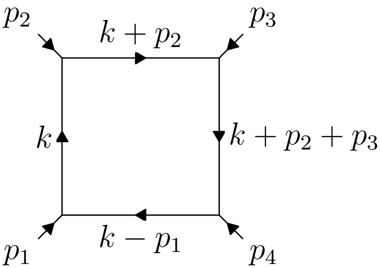

Figure: The box integral.

This graph has associated the following integral (we set all the masses to zero)

$$I_\Box = \int d^4k \frac{1}{k^2(k+p_2)^2(k+p_2+p_3)^2(k-p_1)^2}. \tag{2}$$

Polylogarithms

Definition (leading singularity, Cachazo)

Given an integral, one can compute its *leading singularities* by taking residues in as many denominators as possible. If the residues are all ± 1, then the integral is *well-normalized*.

The numerator factor is essential for the integral to be well-normalized.

Definition (multiple polylogarithms)

For positive integers $n_1, \ldots, n_k$ a *multiple polylogarithm* is the analytic continuation of the power series

$$\mathrm{Li}_{n_1,\ldots,n_p}(x_1,\ldots,x_p) = \sum_{0<k_1<\cdots<k_p} \frac{x_1^{k_1}\cdots x_p^{k_p}}{k_1^{n_1}\cdots k_p^{n_p}}. \tag{3}$$

Polylogarithms

In some cases, there are more integration variables than propagators. After computing the residues in all the propagators, one obtains a Jacobian factor which can itself be singular so we can compute extra residues. The process stops when there are no more residues or there is no dependence on the integration variables k.

Conjecture

Well-normalized Feynman integrals can be expressed as a linear combination of (multiple) polylogarithms with rational coefficients. The arguments of the multiple polylogarithms are algebraic functions of the external momenta.

Dual conformal symmetry

The box integral can be written in dual space (Broadhurst) as

$$I_{\square} = \int \frac{d^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}, \tag{4}$$

where $x_{ij} = x_i - x_j$ and $x_{10} = k$, $x_{20} = k + p_2$, $x_{30} = k + p_2 + p_3$, $x_{40} = k - p_1$.

Definition (conformal group)

The *conformal group* is the group of space-time coordinate transformations which preserve the metric upto a multiplicative factor.

The box integral has a (dual) conformal symmetry in the dual variables x.

Theorem ((Drummond, Henn, Korchemsky, Sokatchev), (Arkani-Hamed, Bourjaily, Cachazo, Trnka))

The $\mathcal{N} = 4$ planar integrands have a dual superconformal symmetry.

Momentum twistors

A dual vector $x \in \mathbb{R}^4$ can be complexified and its components placed in a 2×2 matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}. \tag{5}$$

We may use the quadratic form $\det X = X_{11} X_{22} - X_{12} X_{21}$ as the Minkowski metric.[1] This quadratic form can be polarized to a scalar product

$$(X, Y) = \frac{1}{2} \det(X) \operatorname{tr}(X^{-1} Y) = \frac{1}{2} \det(Y) \operatorname{tr}(X Y^{-1}). \tag{6}$$

There is an "inversion" conformal transformation $X \to X^{-1}$. This transformation together with translation $X \to X + A$ and Lorentz transformations $X \to BXC$ with $\det B \det C = 1$ generates the dual conformal group acting as $X \to (AX + B)(CX + D)^{-1}$ where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}(4). \tag{7}$$

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[1] If the components $X_{ij} \in \mathbb{R}$, then the metric has split signature $(2, 2)$.

Grassmannian

Consider the space of 4×2 matrices $\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$, modulo the right action by an invertible 2×2 matrix. The conformal group $\mathrm{PGL}(4)$ acts linearly on this space, with $X = UV^{-1}$, when V is invertible. This space is just the Grassmannian $G(2,4)$. Upon projectivisation we obtain $\mathbb{G}(1,3)$, the space of projective lines in $\mathbb{P}^3$. This $\mathbb{P}^3$ is called *dual twistor space*. It was introduced by Hodges, following similar constructions of Penrose for usual (not dual!) conformal symmetry.

dual space	twistor space
point X	line L_X
$(X, Y) = 0$	intersecting lines L_X, L_Y

Table: Correspondence

Box integral leading singularities

To the four external dual points correspond four lines, which we take to be skew and such that they don't all lie on a quadric. Given these four lines, we want to find the lines which intersect all of them (leading singularity locus).

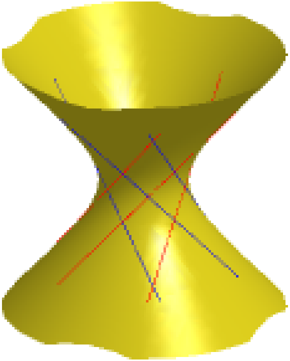


Figure: Quadric with two rulings.

Quadrics

Properties of non-singular quadrics in $\mathbb{P}^3$:

1. Three skew lines determine a non-singular quadric Q.
2. Each quadric is rules by two families of lines.
3. Each line in one family intersects all the lines in the other.
4. Through each point on Q passes one line from the first family and one line from the other.
5. A non-singular quadric in $\mathbb{P}^3$ is the image of the Segre map

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \tag{8}$$

$$([s_0 : s_1], [t_0 : t_1]) \mapsto (s_0 t_0, s_0 t_1, s_1 t_0, s_1 t_1). \tag{9}$$

Equivalently, the quadric is the locus in $\mathbb{P}^3$ with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ where $x_0 x_3 = x_1 x_2$.

Leading singularities of the box

We construct the leading singularity locus as follows (Hodges):

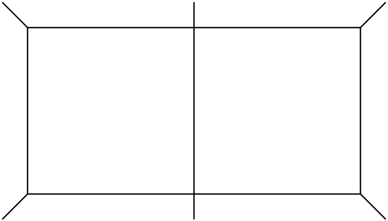
1. From three of the lines L_1, L_2 and L_3 construct a quadric Q. These lines are in one of the rulings of Q
2. The intersection $Q \cap L_4$ is two points (Bézout's theorem)
3. Through each of these points passes a line in the ruling opposite to the one determined by L_1, L_2 and L_3. Hence, it intersects all of them.

There are two transversals ℓ_1 and ℓ_2 to the lines L_1, L_2, L_3 and L_4. On ℓ_1 and ℓ_2 there are four points of intersection so we can form two cross-ratios z_1, z_2. The result for the normalized[2] box integral is

$$NI_{\square} = 2\,\mathrm{Li}_2(z_1) - 2\,\mathrm{Li}_2(z_2) - \log(z_1 z_2) \log \frac{1 - z_1}{1 - z_2}. \qquad (10)$$

[2] The numerator vanishes when the lines L_i lie on a single quadric.

Two-loop train-track



Consider the two-loop train-track integral (Caron-Huot, Larsen). What is the leading singularity in this case? It turns out it is a genus one curve. After taking all the possible residues we are left with a holomorphic one-form so no further residues are possible.

Q: How to normalize the integral?

A: One could think of normalizing the integral so that the integral of this holomorphic one-form along a homology cycle is one. But which homology cycle?

Q: What other more complicated geometries appear?

A: (Possibly singular) Calabi-Yau manifolds.

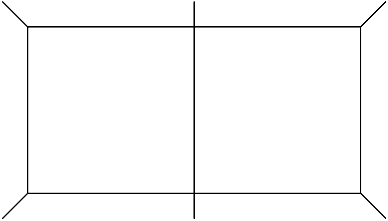
Twistor construction for the two-loop train-track

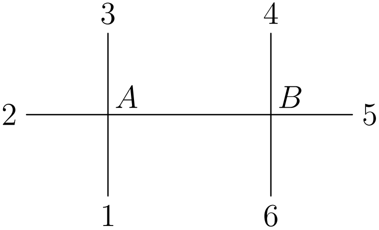
What is the leading singularity locus (and the holomorphic one-form) in twistor language? We can build a quadric Q_l from the left three lines and a quadric Q_r from the right three lines. These quadrics intersect in a curve $C = Q_l \cap Q_r$. Given a point $p \in C$, through p passes a line intersecting the three defining lines of Q_l and a line intersecting the three defining lines of Q_r.
The holomorphic one-form can be found by taking two Poincaré residues

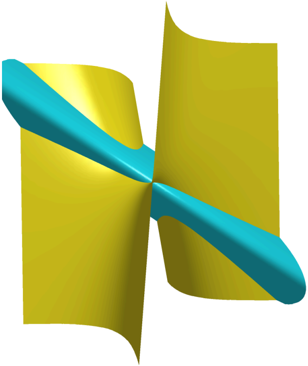
$$\omega_C = \operatorname{Res}_{Q_l} \operatorname{Res}_{Q_r} \frac{\omega_{\mathbb{P}^3}}{Q_l Q_r}, \tag{11}$$

where $\omega_{\mathbb{P}^3} = x_0 \, dx_1 \wedge dx_2 \wedge dx_3 - x_1 \, dx_0 dx_2 dx_3 + \cdots$ is the $\mathrm{PGL}(4)$-invariant weight four form on $\mathbb{P}^3$.

Geometry of the leading singularity genus one curve







Comparing genus one curves

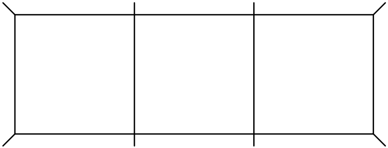
The curve C can be characterized by the complex structure modulus τ or by the j-invariant. The computation of τ involves integrals, while j can be defined algebraically.
The curve C is the intersection of a pencil of quadrics $\mu_0 Q_l + \mu_1 Q_r$. A member of this pencil becomes singular at four points.[3] From these four points in $\mathbb{P}^1$ with coordinates $[\mu_0 : \mu_1]$ we can build a cross-ratio λ. Then the j-invariant is

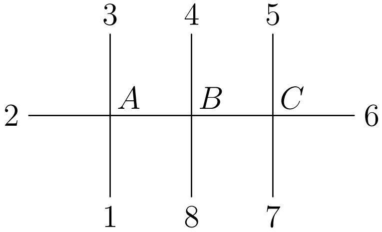
$$j = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}. \tag{12}$$

The j-invariant can also be calculated by doing the integrals using Feynman parametrization. This calculation looks very different but the j-invariants agree.

[3] A quadric in $\mathbb{P}^3$ can be thought as a 4×4 matrix which becomes singular when its determinant vanishes. This determinant is of degree four in λ_0, λ_1.

Three-loop train-track





Consider next the three-loop train-track diagram. Its leading singularity locus has been studied by Bourjaily, He, McLeod, von Hippel, Wilhelm by some laborious procedure (using Feynman parametrization and involving computer calculations using `Macaulay2`).

We can instead do this analysis in momentum twistor space.

Three-loop train-track twistor geometry

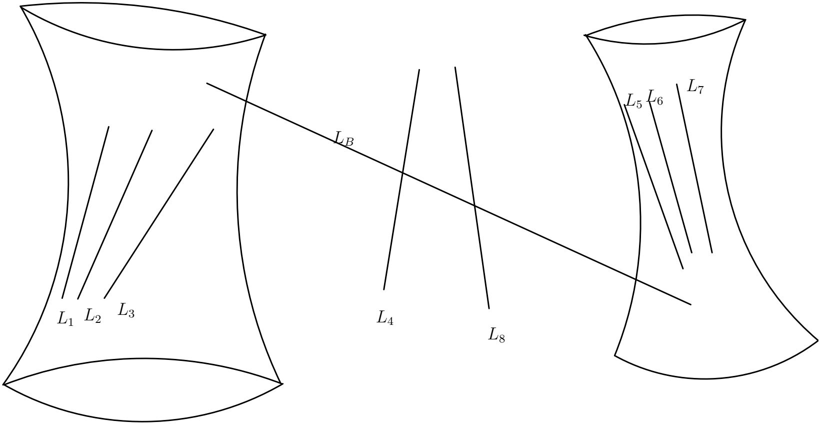


Figure: K3 twistor geometry.

Three-loop train-track twistor geometry

Here are the steps of the geometric construction:

1. A point on L_4 and another on L_8 define a line L_B.
2. The lines L_1, L_2 and L_3 determine a quadric Q_l.
3. The lines L_5, L_6 and L_7 determine a quadric Q_r.
4. The line L_B generically intersects[4] Q_l in two points and Q_r in two points (Bézout).
5. The condition that the line L_B is tangent to Q_l is an equation of bidegree $2, 2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (which is a genus one curve).
6. The K3 surface is then a branched cover over $\mathbb{P}^1 \times \mathbb{P}^1$.

[4] We take the line L_B not to be contained in Q_l or Q_r.

Leading singularity as a branched cover

The leading singularity locus is a four-fold cover over a generic point in $\mathbb{P}^1 \times \mathbb{P}^1$ (two intersections with Q_l and two intersections with Q_r). It is a double cover over the genus one curve C_l (corresponding to a tangent to Q_l and two intersections with Q_r). It is also a double cover over C_r (tangent to Q_r and two intersections with Q_l). Finally, there is no branching over the eight intersection points of $C_l \cap C_r$.
This is an analog of the construction of a genus one curve as a double cover branched over four points on $\mathbb{P}^1$. From these four points we can compute a cross-ratio and a j-invariant. What is the analog for K3?

Euler characteristic

We use surgery. We have

- four copies of the points $\mathbb{P}^1 \times \mathbb{P}^1 - C_l \cup C_r$
- two copies of the points $C_l \cup C_r - C_l \cap C_l$
- one copy of the points $C_l \cap C_r$

We also know that

- $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = \chi(\mathbb{P}^1)^2$.
- $\chi(\mathbb{P}^1) = 2$ since $\mathbb{P}^1$ is a two-sphere.
- $\chi(C_l) = \chi(C_r) = 0$ since C_l and C_r are tori.
- $\chi(pt) = 1$.
- inclusion-exclusion $\chi(C_l \cup C_r) = \chi(C_l) + \chi(C_r) - \chi(C_l \cap C_r)$.

Euler characteristic

Then,

$$\begin{aligned}\chi(S) = 4(\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(C_l \cup C_r))+ \\ 2(\chi(C_l \cup C_r) - \chi(C_l \cap C_r)) + \chi(C_l \cap C_r) = \\ 4\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(C_l \cup C_r) - \chi(C_l \cap C_r) = \\ 4 \times 2 \times 2 - 2 \times (-8) - 8 = 24.\end{aligned} \tag{13}$$

The Hodge diamond of K3 is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{14}$$

Holomorphic two-form

The holomorphic two-form is

$$\omega_{K3} = \frac{\omega_{\mathbb{P}^1}\omega_{\mathbb{P}^1}}{\sqrt{C_l}\sqrt{C_r}}. \tag{15}$$

The measure $\omega_{\mathbb{P}^1}\omega_{\mathbb{P}^1}$ has weight $2, 2$ while C_l and C_r have bidegree $2, 2$ each. More accurately, we introduce new coordinates y_l and y_r of bidegree $1, 1$ with properties $y_l^2 = C_l$ and $y_r^2 = C_r$ so the holomorphic two-form is

$$\omega_{K3} = \frac{\omega_{\mathbb{P}^1}\omega_{\mathbb{P}^1}}{y_l y_r}. \tag{16}$$

Embedding in toric varieties

We can define K3 as a codimension one algebraic variety in a space defined by the equivalences

$$(\underbrace{a_1, a_2}_{\mathbb{P}^1}, \underbrace{b_1, b_2}_{\mathbb{P}^1}, y_1, y_2) \sim (sa_1, sa_2, tb_1, tb_2, sty_1, sty_2). \tag{17}$$

This is a toric variety we denote by

$$\mathbb{P}\left(\begin{smallmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{smallmatrix}\right) \tag{18}$$

Then, in this manifold we impose two equations of degrees $2, 2$ each. The resulting manifold is denoted by

$$\mathbb{P}\left(\begin{smallmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{smallmatrix}\right) \quad \left[\begin{array}{c|c} 2 & 2 \\ 2 & 2 \end{array}\right] \tag{19}$$

Toric CICY

The general framework for doing these calculations is that of toric CICY's, as formulated by Batyrev & Borisov.

- From the data of the embedding space build a (reflexive) lattice polytope Δ^* with dual Δ.
- The vertices of Δ^* can be partitioned in some way (nef partition) and $\Delta^* = \mathrm{conv}(\nabla_1, \dots, \nabla_r)$.
- One can build dual lattice polytopes Δ_i.
- The points of Δ_i are the Newton polytopes for the equations defining the embedding.

Mirror symmetry is built in.[5] What role does mirror symmetry play? (Some studies by Bloch, Kerr, Vanhove for the case of sunrise integral).

[5] Swap Δ_i with ∇_i and convex hull with Minkowski sum.

The three-fold

One can also build the three-fold (corresponding to a four-loop train-track) as a toric CICY.

$$\mathbb{P}\begin{pmatrix} 1&1&0&0&1&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&1 \\ 0&0&0&0&0&0&0&1&1&1 \end{pmatrix} \quad \left[\begin{array}{c|c|c} 2&0&1 \\ 2&0&1 \\ 0&2&1 \\ 0&2&1 \end{array}\right]^{12,28}_{-32} \tag{20}$$

We have $h_{11} = 12$, $h_{2,1} = 28$ and $\chi = -32$. This result was obtained using the `nef.x` computer program by Kreutzer et al. The complete intersections become high codimension which makes them hard to analyze.

Genus one in superspace

So far, we have not made use of the dual *super*-conformal symmetry. In this case the dual super-conformal symmetry is $\mathrm{PGL}(4|4)$ and has a simple action on the super-twistor space $\mathbb{P}^{3|4}$ with homogeneous coordinates

$$\mathcal{Z} = [Z_0 : Z_1 : Z_2 : Z_3 \mid \chi_1 : \chi_2 : \chi_3 : \chi_4], \tag{21}$$

where the χ_i coordinates are nilpotent $\chi_i^2 = 0$, $\chi_i \chi_j = -\chi_j \chi_i$. The space $\mathbb{P}^{3|4}$ is special in that it has a weight zero $3|4$-form

$$\omega_{\mathbb{P}^{3|4}} = \omega_{\mathbb{P}^3}\, d\chi_1 \cdots d\chi_4, \tag{22}$$

where $d\chi_i$ has weight -1. This means that the $3|4$-form $\omega_{\mathbb{P}^{3|4}}$ is *canonically* normalized. These spaces are sometimes called super-Calabi-Yau.

Supersymmetric Dirac delta functions

We define[6]

$$\delta^{4|4}(\mathcal{Z}) = \delta(Z_0)\cdots\delta(Z_3)\chi_1\cdots\chi_4. \tag{23}$$

Then the version on $\mathbb{P}^{3|4}$ is

$$\delta^{3|4}_{\mathbb{P}^{3|4}}(\mathcal{Z};\mathcal{Y}) = \int_{\mathbb{P}^1} \frac{\omega_{\mathbb{P}^1}(\alpha)}{\alpha_0\alpha_1}\delta^{4|4}(\alpha_0\mathcal{Z}+\alpha_1\mathcal{Y}). \tag{24}$$

In the same way we can define $\delta^{1|4}_{\mathbb{P}^{3|4}}(L_1;L_2)$ of two lines to intersect and $\delta^{2|4}_{\mathbb{P}^{3|4}}(L;\mathcal{Z})$ of the point $\mathcal{Z}$ to lie on the line L. Finally we can define $\delta^{1|8}_{\mathbb{P}^{3|4}}(\mathcal{Z};\mathcal{Q}_l)$ of the point $\mathcal{Z}$ to lie on the superquadric $\mathcal{Q}_l$.

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[6] Such supersymmetric delta functions were considered by Mason & Skinner in the context of perturbation theory in twistor space. We could take $\delta(Z_i)$ to be $(0,1)$-currents in the sense of de Rham.

SUSY analogs of Poincaré residues

The expression for the holomorphic one-form is now characterized by the equality[7]

$$\int_C \omega_C^{0|12}(Z) f(Z) = \int \omega_{\mathbb{P}^{3|4}}(\mathcal{Z}) \delta^{1|8}_{\mathbb{P}^{3|4}}(\mathcal{Z}; \mathcal{Q}_l) \delta^{1|8}_{\mathbb{P}^{3|4}}(\mathcal{Z}; \mathcal{Q}_r) f(Z), \quad (25)$$

for all f.

Hence, it is possible to define a holomorphic one-form ω_C of degree 12 in the odd variables of the external points. This form is *canonically* normalized, but its coefficient is nilpotent! The degree 12 corresponds to a sector of the ten-point amplitude called N^3MHV.

[7] There is no need to specify a contour if we take f, $\delta^{1|8}_{\mathbb{P}^{3|4}}$ to be $(0,1)$-forms.

Disentangle even and odd

In $\omega_C^{0|12}$, the dependence on Z is entangled with that on the odd variables χ in combinations of weight $0|4$ called R-invariants. Can they be disentangled? Use the idea of Lagrange interpolation which appeared in (Kosower, Roiban, CV).

The End