

Helicity Distributions and Orbital Angular Momentum at Small x

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Based on work done with Dan Pitonyak and Matt Sievert (2015-2018) and with Florian Cougoulic (2019-2020).

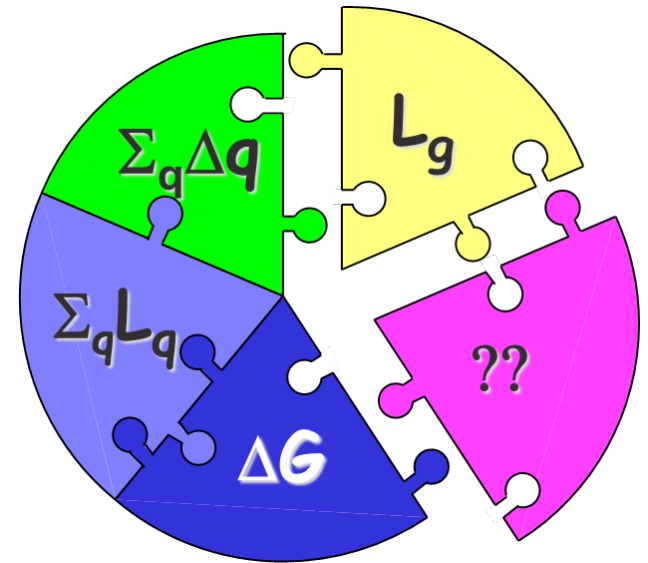
Also see talks by Daniel Adamiak (JAM-small x) and Yossathorn Tawabutr (helicity evolution at large- N_c & N_f and at LLA) in Breakout Room 11 later today. See talk by Gabriel Santiago on STSA in $p^\uparrow + A$ collisions in the same session.

Outline

- Goal: understanding the proton spin coming from helicities and OAM of small- x quarks and gluons.
- Quark Helicity (“simplify-evolve-solve” prescription):
 - Quark helicity distribution at small x
 - Small- x evolution equations for quark helicity
 - Small- x asymptotics of quark helicity
- Gluon Helicity:
 - Gluon helicity distribution at small x
 - Small- x evolution equations for gluon helicity
 - Small- x asymptotics of gluon helicity TMDs
- Quark and Gluon OAM at small x : results.
- Valence quark transversity at small x : results.
- Conclusions and Outlook

Introduction and goals

Proton Spin Puzzle



- Helicity sum rule (Jaffe-Manohar form):

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta\Sigma(x, Q^2)$$

$$S_g(Q^2) = \int_0^1 dx \Delta G(x, Q^2)$$

- The helicity parton distributions are

$$\Delta f(x, Q^2) \equiv f^+(x, Q^2) - f^-(x, Q^2)$$

with the net quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

- L_q and L_g are the quark and gluon orbital angular momenta

Our goal

- The goal is to constrain theoretically the amount of proton spin and OAM coming from small x .
- Any existing and future experiment probes the helicity distributions and OAM down to some $x_{\min}$.

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta\Sigma(x, Q^2)$$


$$S_g(Q^2) = \int_0^1 dx \Delta G(x, Q^2)$$


$$L_{q+\bar{q}}(Q^2) = \int_0^1 dx L_{q+\bar{q}}(x, Q^2)$$


$$L_G(Q^2) = \int_0^1 dx L_G(x, Q^2)$$


- At very small x (for the proton), saturation sets in: that region likely carries a negligible amount of proton spin. But what happens at larger (but still small) x ?

Our goal

- Ultimately the aim is to make predictions for helicity distributions at EIC.
- If the predictions are not too far off, one could extrapolate the theory curves down to $x=0$, getting a (hopefully) good estimate for quark and gluon spin coming from small x .
- For OAM the story is more complicated, but perhaps we may be able to constrain small- x OAM this way too.

Quark Helicity at Small x (flavor-singlet case)

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],
arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph],
arXiv:1703.05809 [hep-ph], arXiv:1808.09010 [hep-ph]

Quark Helicity TMD

- We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

$$g_{1L}^q(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{1}{2} \sum_{S_L} S_L \int d^2r dr^- e^{ik \cdot r} \langle p, S_L | \bar{\psi}(0) \mathcal{U}[0, r] \frac{\gamma^+ \gamma^5}{2} \psi(r) | p, S_L \rangle_{r^+=0}$$

- At small- x , in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in $A^-=0$ gauge for the + moving proton)

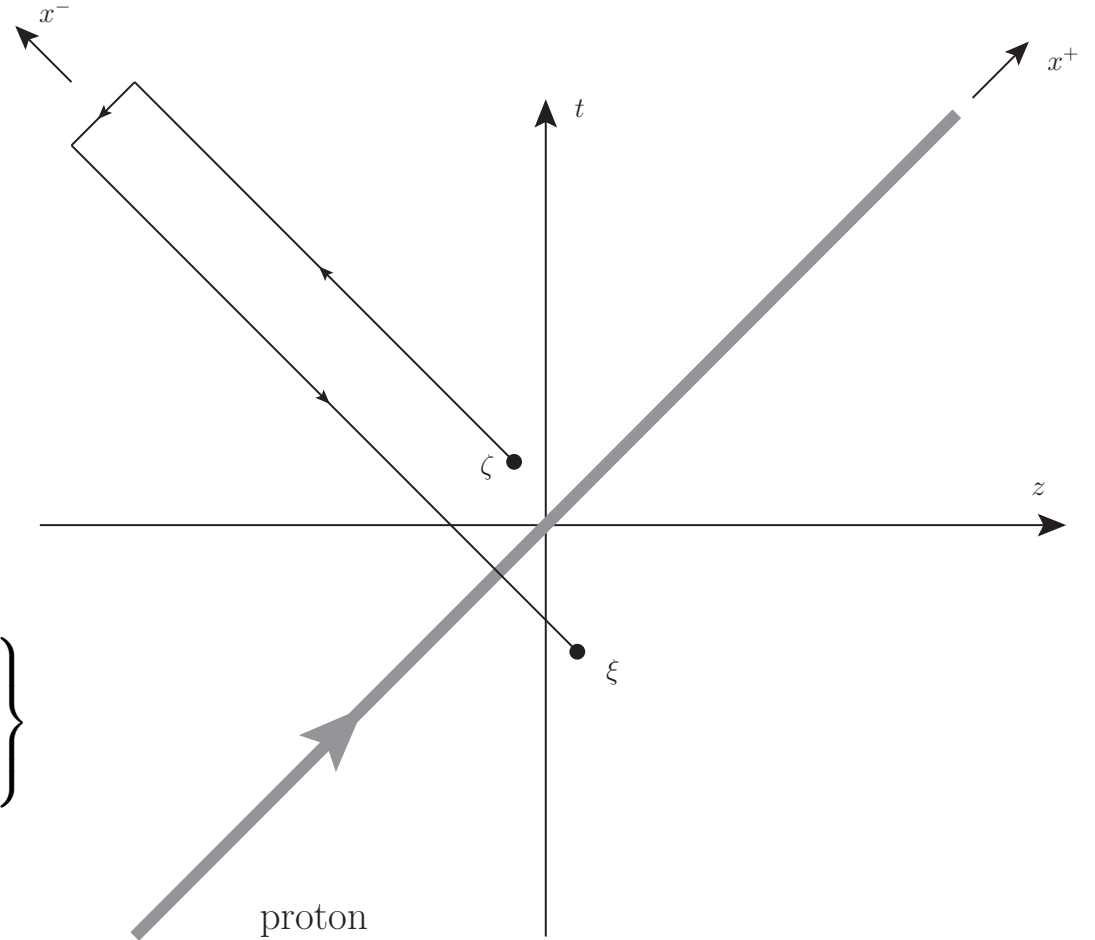
$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta d\zeta^- d^2\xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-, \infty] V_{\underline{\zeta}}[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

where the fundamental light-cone Wilson line is

$$V_{\underline{x}}[b^-, a^-] = \text{P exp} \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, \underline{x}) \right\}$$

Quark Helicity TMD

Space-time representation of the quark helicity TMD definition.



$$V_{\underline{x}}[b^-, a^-] = \text{P exp} \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, \underline{x}) \right\}$$

$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta d\zeta^- d^2\xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-, \infty] V_{\underline{\zeta}}[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

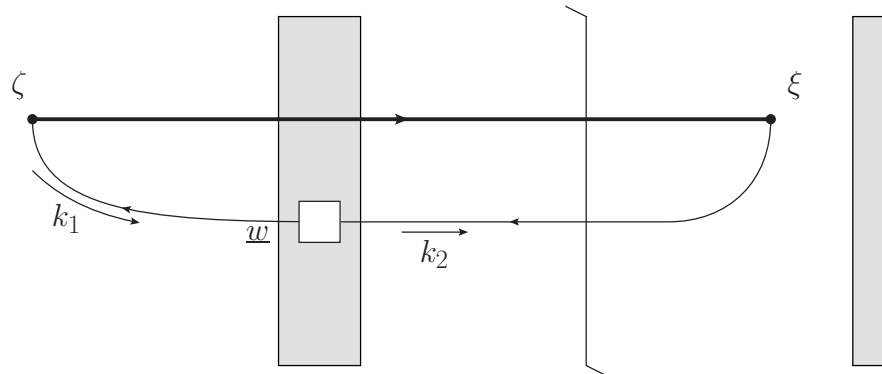
Quark Helicity TMD at Small x

- Only one diagram contributes, giving

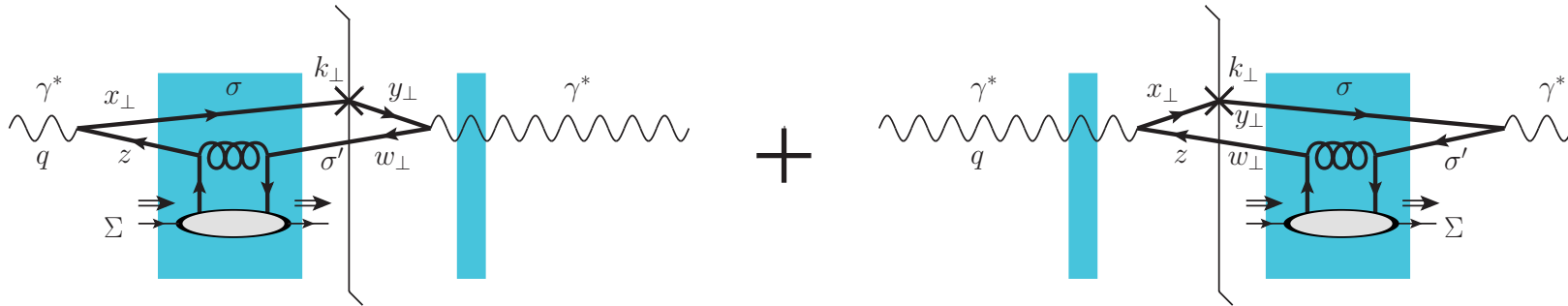
$$g_{1L}^q(x, k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta d^2w d^2y e^{-i\underline{k} \cdot (\underline{\zeta} - \underline{y})} \int_{\Lambda^2/s}^1 \frac{dz}{z} \frac{\underline{\zeta} - \underline{w}}{|\underline{\zeta} - \underline{w}|^2} \cdot \frac{\underline{y} - \underline{w}}{|\underline{y} - \underline{w}|^2} G_{\underline{w}, \underline{\zeta}}(zs)$$

where $G_{w\zeta}$ is the polarized dipole amplitude (defined on the next slide).

- Here s is the cms energy squared, Λ is some IR cutoff, underlining denotes transverse vectors, z = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark



Quark Helicity Observables at Small x



- One can show that the g_1 structure function and quark helicity PDF (Δq) and TMD at small-x can be expressed in terms of the polarized dipole amplitude (flavor singlet case):

$$g_1^S(x, Q^2) = \frac{N_c N_f}{2 \pi^2 \alpha_{EM}} \int_{z_i}^1 \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda \sigma \sigma'} |\psi_{\lambda \sigma \sigma'}^T|^2_{(x_{01}^2, z)} + \sum_{\sigma \sigma'} |\psi_{\sigma \sigma'}^L|^2_{(x_{01}^2, z)} \right] G(x_{01}^2, z),$$

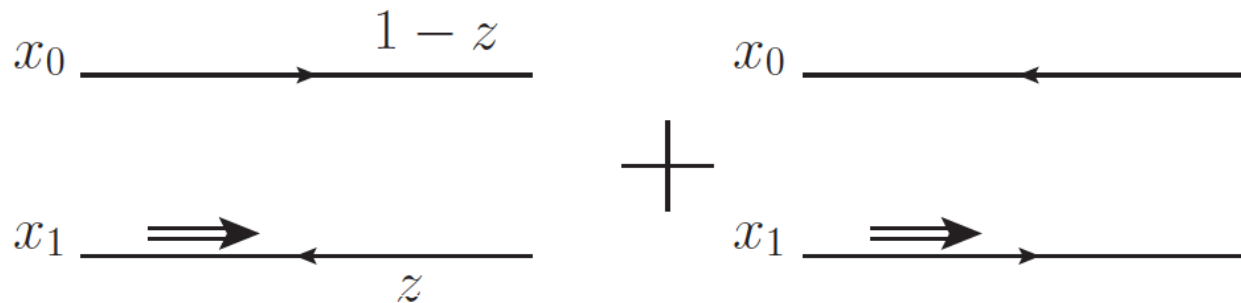
$$\Delta q^S(x, Q^2) = \frac{N_c N_f}{2 \pi^3} \int_{z_i}^1 \frac{dz}{z} \int_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2} G(x_{01}^2, z), = \Delta \Sigma(x, Q^2)$$

$$g_{1L}^S(x, k_T^2) = \frac{8 N_c N_f}{(2\pi)^6} \int_{z_i}^1 \frac{dz}{z} \int d^2 x_{01} d^2 x_{0'1} e^{-i \underline{k} \cdot (\underline{x}_{01} - \underline{x}_{0'1})} \frac{\underline{x}_{01} \cdot \underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2} G(x_{01}^2, z)$$

- Here s is cms energy squared, $z_i = \Lambda^2/s$, $G(x_{01}^2, z) \equiv \int d^2 b G_{10}(z)$

Polarized Dipole

- All flavor-singlet small- x helicity observables depend on one object, “polarized dipole amplitude”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \text{Re} \left\langle\left\langle \text{Tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol \dagger} \right] + \text{Tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^\dagger \right] \right\rangle\right\rangle(z)$$

unpolarized quark
polarized quark: eikonal propagation,
non-eikonal spin-dependent interaction

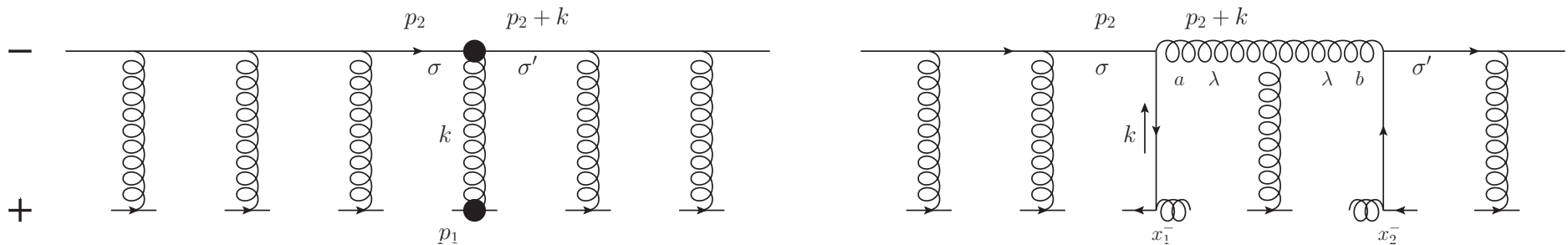
$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{\infty} dx^+ A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\left\langle \mathcal{O} \right\rangle\right\rangle(z) \equiv z s \left\langle \mathcal{O} \right\rangle(z)$$

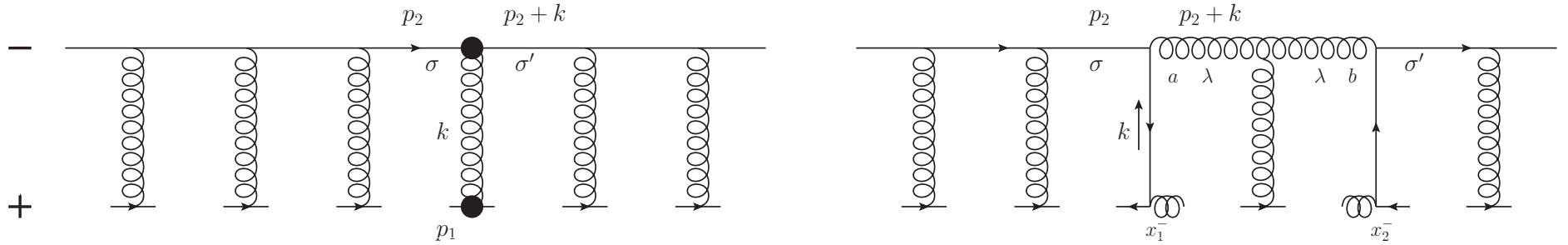
Polarized fundamental “Wilson line”

- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized “Wilson line” V^{pol} , which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.

Polarized fundamental “Wilson line”



- In the end one arrives at (KPS '17; YK, Sievert, '18; cf. Chirilli '18)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int_{-\infty}^{\infty} dx^- V_{\underline{x}}[+\infty, x^-] F^{12}(x^-, \underline{x}) V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[+\infty, x_2^-] t^b \psi_{\beta}(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5 \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty].$$

- We have employed an adjoint light-cone Wilson line
$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int_{a^-}^{b^-} dx^- \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$
- Note the simple physical meaning of the first term:

$$-\vec{\mu} \cdot \vec{B} = -\mu_z B_z = \mu_z F^{12}$$

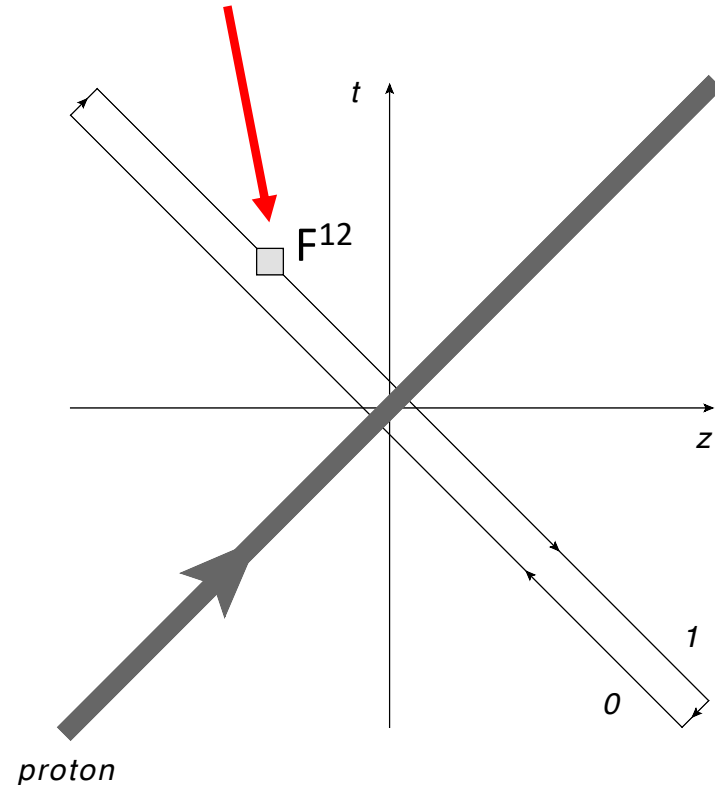
Polarized Dipole Amplitude

- The polarized dipole amplitude is then defined by

$$G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] (-ig) \underline{\nabla} \times \tilde{\underline{A}}(x^-, \underline{x}) V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) \sim F^{12}$$

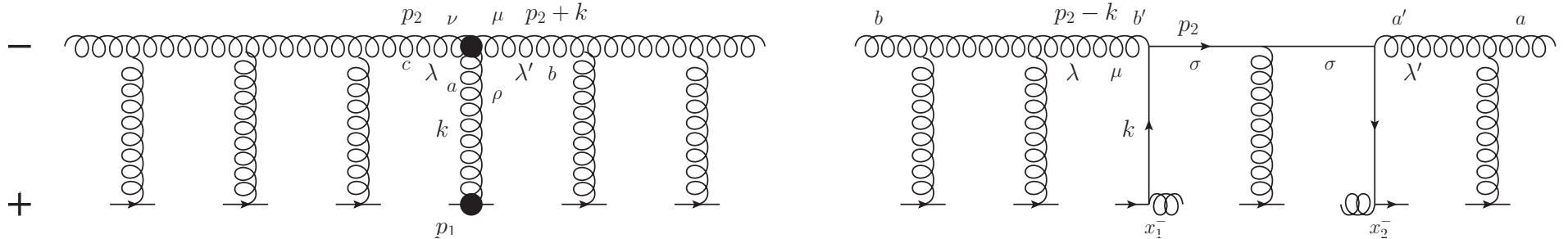
with the standard light-cone
Wilson line

$$V_{\underline{x}}[b^-, a^-] = \text{P exp} \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, \underline{x}) \right\}$$



Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.

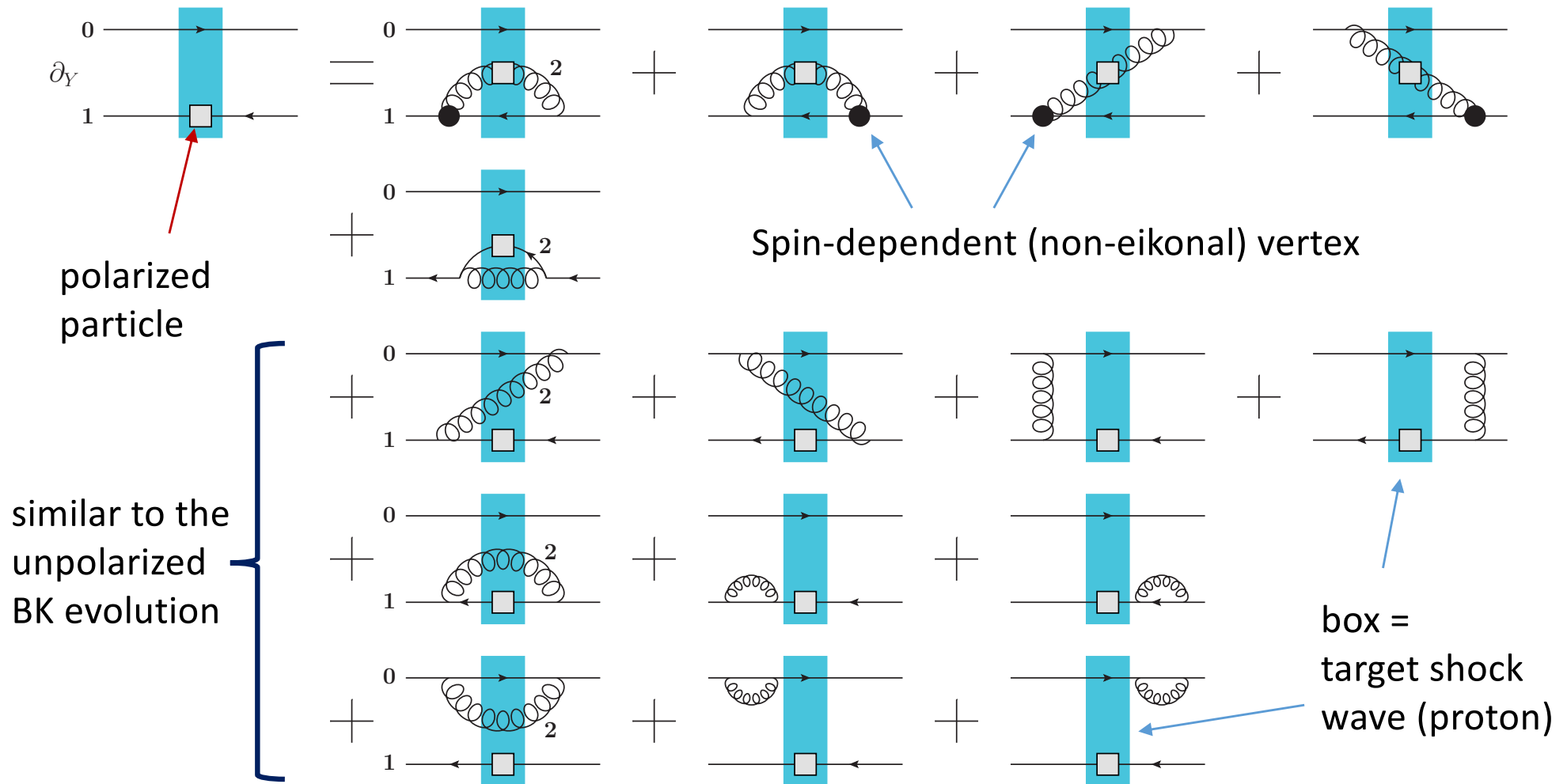


- The calculation is similar to the quark scattering case. It yields (YK, Sievert, '18; cf. Chirilli '18)

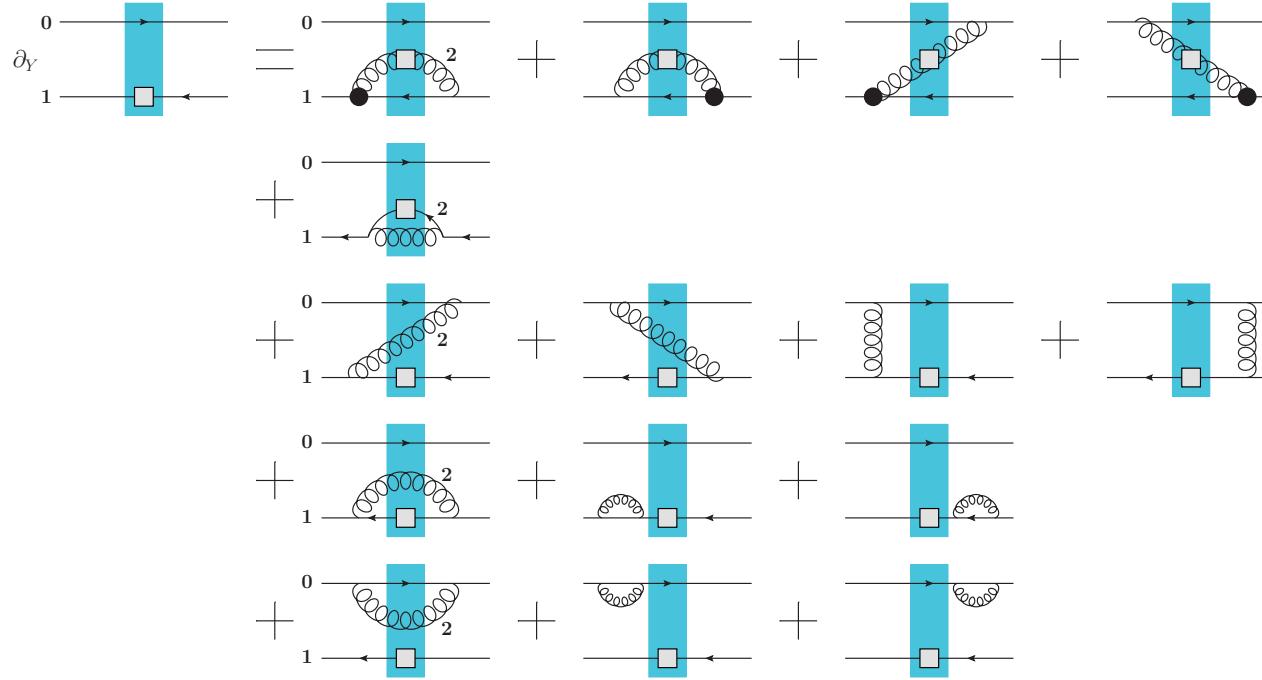
$$\begin{aligned}
 (U_{\underline{x}}^{pol})^{ab} &= \frac{2i g p_1^+}{s} \int_{-\infty}^{+\infty} dx^- \left(U_{\underline{x}}[+\infty, x^-] \mathcal{F}^{12}(x^+ = 0, x^-, \underline{x}) U_{\underline{x}}[x^-, -\infty] \right)^{ab} \\
 &- \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- U_{\underline{x}}^{aa'}[+\infty, x_2^-] \bar{\psi}(x_2^-, \underline{x}) t^{a'} V_{\underline{x}}[x_2^-, x_1^-] \frac{1}{2} \gamma^+ \gamma_5 t^{b'} \psi(x_1^-, \underline{x}) U_{\underline{x}}^{b'b}[x_1^-, -\infty] - \text{c.c.}
 \end{aligned}$$

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \dots \rangle\rangle = \frac{1}{z s} \langle \dots \rangle$$

$$\rho'^2 = \frac{1}{z' s}$$

$$\frac{1}{N_c} \langle\langle \text{tr} [V_0^{unp} V_1^{pol \dagger}] \rangle\rangle (z) = \frac{1}{N_c} \langle\langle \text{tr} [V_0^{unp} V_1^{pol \dagger}] \rangle\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int_{z_i}^z \frac{dz'}{z'} \int_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \left\{ \theta(x_{10} - x_{21}) \frac{2}{N_c} \langle\langle \text{tr} [t^b V_0^{unp} t^a V_1^{unp \dagger}] U_2^{pol ba} \rangle\rangle (z') \right.$$

Equation does not close!

$$+ \theta(x_{10}^2 z - x_{21}^2 z') \frac{1}{N_c} \langle\langle \text{tr} [t^b V_0^{unp} t^a V_2^{pol \dagger}] U_1^{unp ba} \rangle\rangle (z')$$

$$+ \theta(x_{10} - x_{21}) \frac{1}{N_c} \left[\langle\langle \text{tr} [V_0^{unp} V_2^{unp \dagger}] \text{tr} [V_2^{unp} V_1^{pol \dagger}] \rangle\rangle (z') - N_c \langle\langle \text{tr} [V_0^{unp} V_1^{pol \dagger}] \rangle\rangle_{18} (z') \right] \Big\}$$

Polarized Dipole Evolution in the Large- N_c Limit

In the large- N_c limit the equations close, leading to a system of 2 equations:

The diagrammatic equations represent the evolution of the dipole Green's function $G_{10}(z)$ and the dipole splitting function $\Gamma_{02,21}(z')$ in the large- N_c limit. The first equation shows the evolution of $G_{10}(z)$ with terms involving $\Gamma_{02,21}(z)$, $S_{02}(z)$, $G_{21}(z)$, $S_{02}(z)$, and $\Gamma_{01,21}(z)$. The second equation shows the evolution of $\Gamma_{02,21}(z')$ with terms involving $\Gamma_{03,32}(z')$, $S_{03}(z')$, $G_{32}(z')$, $S_{03}(z')$, and $\Gamma_{02,32}(z')$.

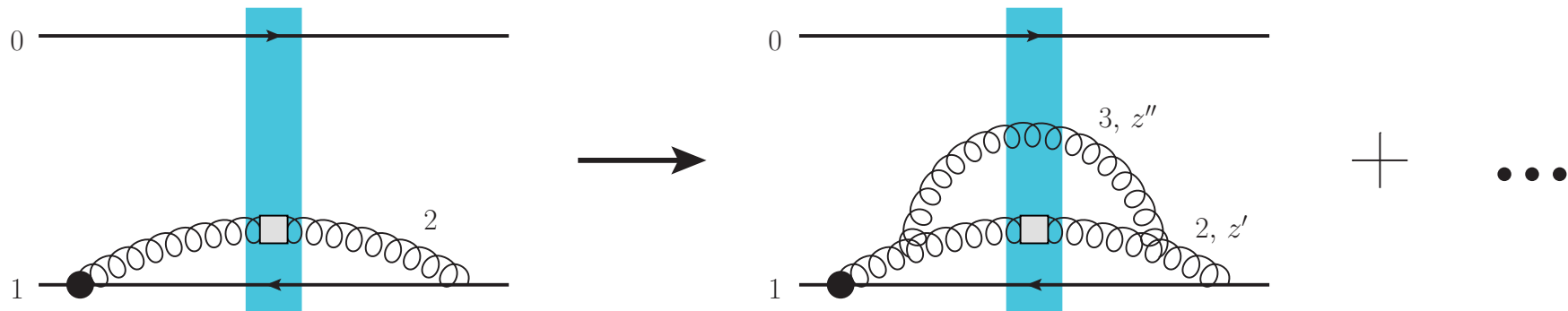
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^z \frac{dz'}{z'} \int_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} [2 \Gamma_{02,21}(z') S_{21}(z') + 2 G_{21}(z') S_{02}(z') + G_{12}(z') S_{02}(z') - \Gamma_{01,21}(z')]$$

$$\Gamma_{02,21}(z') = \Gamma_{02,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^{z'} \frac{dz''}{z''} \int_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} [2 \Gamma_{03,32}(z'') S_{23}(z'') + 2 G_{32}(z'') S_{03}(z'') + G_{23}(z'') S_{03}(z'') - \Gamma_{02,32}(z'')]$$

S = found from BK/JIMWLK, it is LLA

“Neighbor” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ‘know’ about another dipole:



$$x_{21}^2 z' \gg x_{32}^2 z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02, 21}(z')$

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ln(1/x)$$

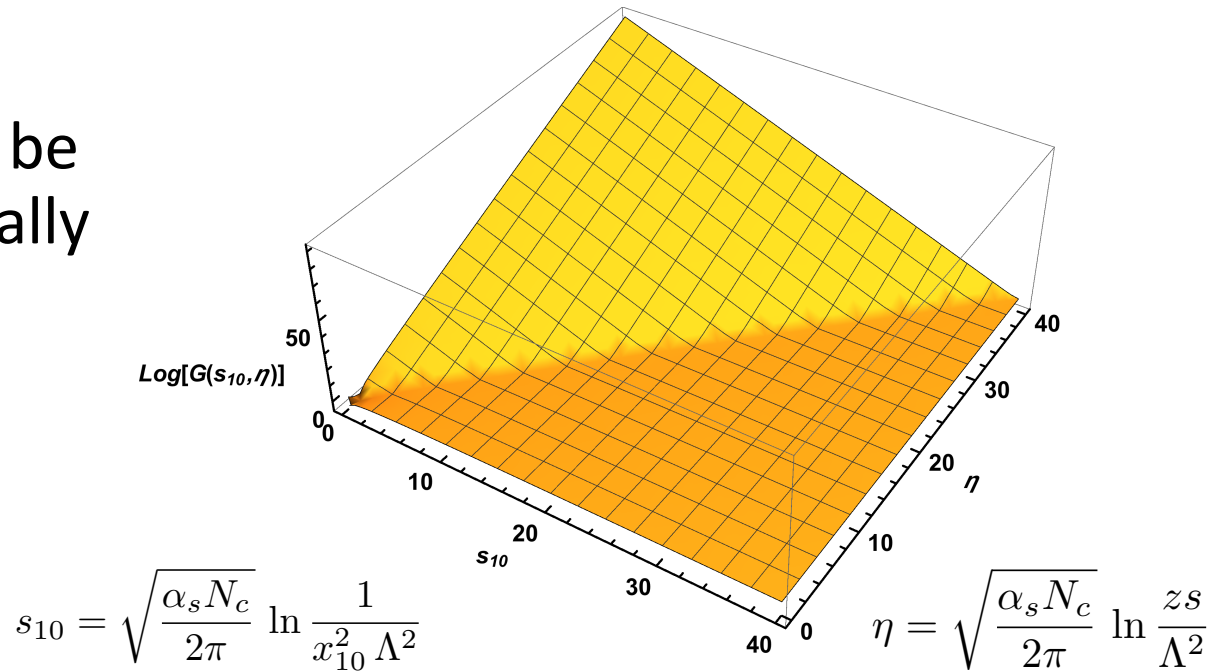
- Helicity evolution resummation parameter is double-logarithmic (DLA):

$$\alpha_s \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.
- This was known before: Kirschner and Lipatov '83; Kirschner '84; Bartels, Ermolaev, Ryskin '95, '96; Griffiths and Ross '99; Itakura et al '03; Bartels and Lublinsky '03.

Quark Helicity at Small x

- These equations can be solved both numerically and analytically.
(KPS '16-'17)



- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

The Value of the Intercept

- If we take

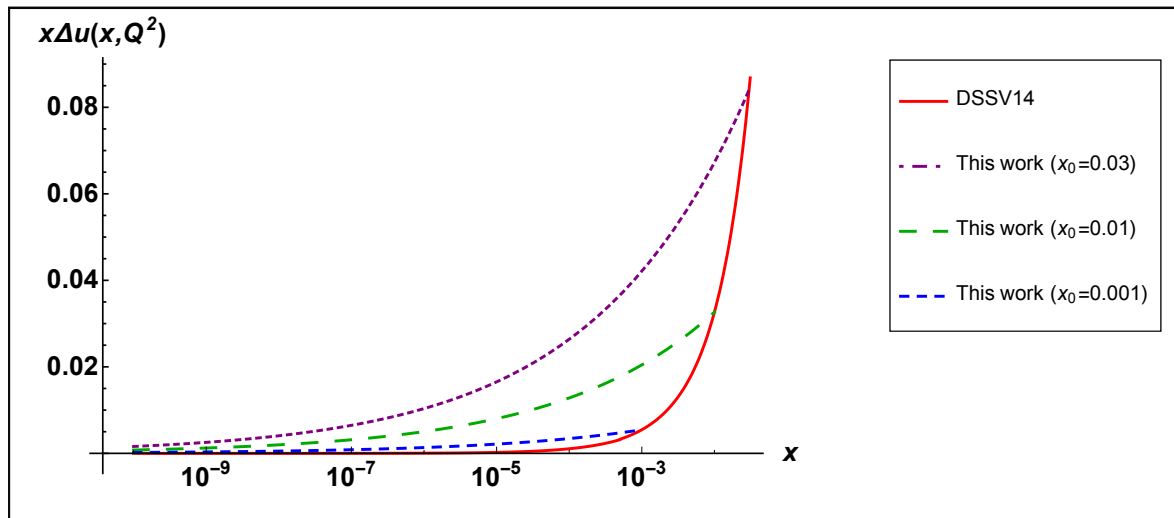
$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

and plug in $\alpha_s = 0.25$ we get $\alpha_h^q = 0.80$.

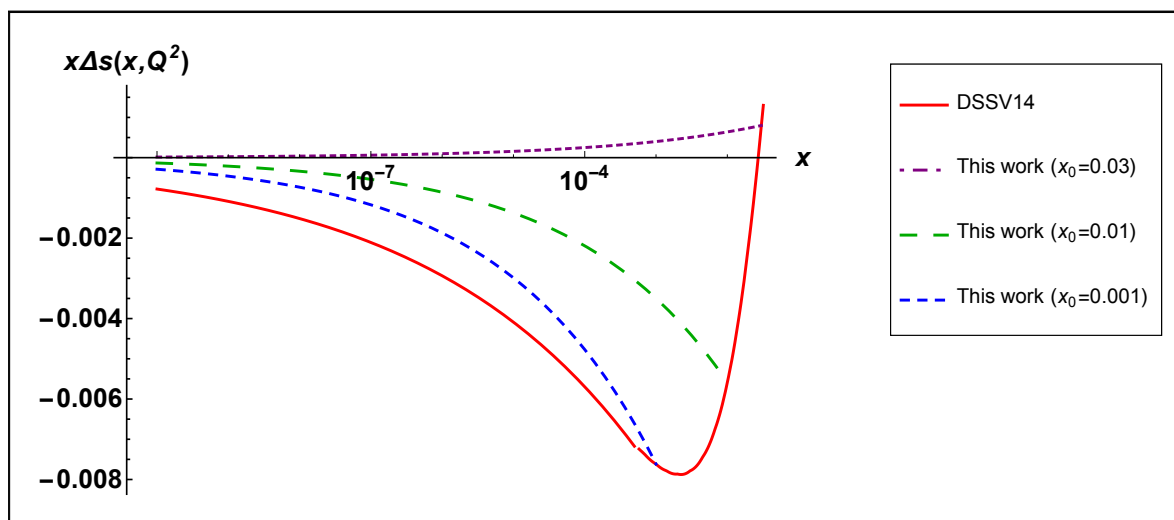
- Compare this to DSSV '09 which has $\alpha_h^q = 0.836$.
- Beyond large- N_c : the evolution equations also close in the large- N_c & N_f limit. See the talk by Yossathorn (Josh) Tawabutr for their solution.

Impact of our $\Delta\Sigma$ on the proton spin

- We have attached a $\Delta\tilde{\Sigma}(x, Q^2) = N x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



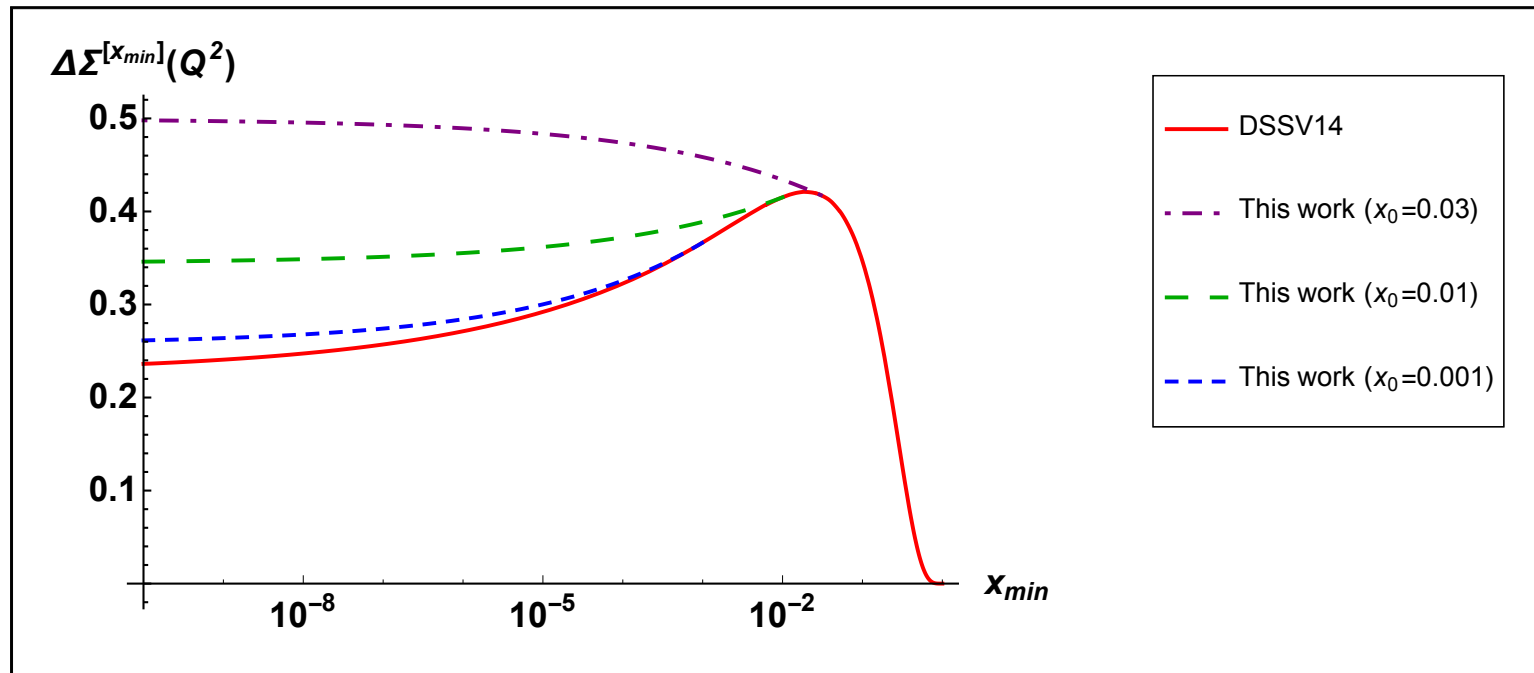
“ballpark”
phenomenology (KPS 2017)



See the talk by D. Adamiak
for proper phenomenology
using JAM.

Impact of our $\Delta\Sigma$ on the proton spin

- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^1 dx \Delta\Sigma(x, Q^2)$ we plot it for $x_0=0.03, 0.01, 0.001$:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

$$\alpha = A^+, \quad \beta = F^{12}$$

Helicity JIMWLK

$$\langle \mathcal{O}_{\alpha,\beta,\psi,\bar{\psi}} \rangle_Y = \frac{\int \mathcal{D}\alpha \mathcal{D}\beta \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{O}_{\alpha,\beta,\psi,\bar{\psi}} \mathcal{W}_Y[\alpha, \beta, \psi, \bar{\psi}]}{\int \mathcal{D}\alpha \mathcal{D}\beta \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{W}_Y[\alpha, \beta, \psi, \bar{\psi}]}$$

- To go beyond the large- N_c and large- $N_c \& N_f$ limits need to write a helicity analogue of JIMWLK evolution.
- This has been done recently (F. Cougoulic, YK, arXiv:1910.04268 [hep-ph], arXiv:2005.14688 [hep-ph]):

$$W_\tau[\alpha, \beta, \psi, \bar{\psi}] = W_\tau^{(0)}[\alpha, \beta, \psi, \bar{\psi}] + \int d^3\tau' \mathcal{K}_h[\tau, \tau'] \cdot W_{\tau'}[\alpha, \beta, \psi, \bar{\psi}]$$

with $\tau \equiv \{z, z X_\perp^2, z Y_\perp^2\}$ and the kernel

$$\mathcal{K}_h[\tau, \tau'] = \frac{\alpha_s}{\pi^2} \int d^2w_\perp \frac{\underline{X}' \cdot \underline{Y}'}{X'^2 Y'^2} \theta^{(3)}(\tau - \tau') \theta\left(z' - \frac{\Lambda^2}{s}\right) \theta\left(X'^2 - \frac{1}{z' s}\right) \theta\left(Y'^2 - \frac{1}{z' s}\right)$$

Life-time ordered!

$$\times \left\{ U_w^{ba} D_{x,a,<}^+ D_{y,b,>}^+ - \frac{1}{2} (D_{x,a,<}^+ D_{y,a,<}^+ + D_{x,a,>}^+ D_{y,a,>}^+) \right.$$

Regular JIMWLK

$$+ \frac{1}{2} U_w^{pol,ba} (D_{x,a,<}^+ D_{y,b,>}^\perp + D_{x,a,<}^\perp D_{y,b,>}^+)$$

Polarized gluon emissions

$$+ \left(\frac{1}{2} \gamma^5 \gamma^- \right)_{\beta\alpha} \frac{1}{2} \left((V_w^{pol})_{ij} D_{x,j,\alpha,<}^{\bar{\psi}} D_{y,i,\beta,>}^\psi + (V_w^{pol\dagger})_{ij} D_{x,j,\alpha,>}^{\bar{\psi}} D_{y,i,\beta,<}^\psi \right)$$

Polarized quark emissions

Helicity McLerran-Venugopalan Model

- The initial conditions for helicity JIMWLK are given by the helicity MV model, with the weight functional (YK, F. Cougoulic, 2005.14688 [hep-ph])

Standard MV

$$\mathcal{W}^{(0)}[\alpha, \beta, \psi, \bar{\psi}] \propto \exp \left\{ - \int d^2 x_{\perp} dx^- \operatorname{tr} \left[(\nabla_{\perp}^2 \alpha)^2 \frac{\mu_+^2 + \mu_-^2}{8\mu_+^2 \mu_-^2} + (\langle p^+ \rangle \beta)^2 \frac{\mu_+^2 + \mu_-^2}{2\mu_+^2 \mu_-^2} + (\nabla_{\perp}^2 \alpha) \langle p^+ \rangle \beta \frac{\mu_+^2 - \mu_-^2}{2\mu_+^2 \mu_-^2} \right] \right\} \\ \times \exp \left\{ \int d^2 x_{\perp} dx^- \langle p^+ \rangle \left[\frac{\nu_+^2 + \nu_-^2}{\nu_+^2 \nu_-^2} \bar{\psi} \frac{1}{2} \gamma^+ \nabla_{\perp}^2 \psi - \frac{\nu_+^2 - \nu_-^2}{\nu_+^2 \nu_-^2} \bar{\psi} \frac{1}{2} \gamma^+ \gamma^5 \nabla_{\perp}^2 \psi \right] \right\}.$$

- Here $\alpha = A^+$, $\beta = F^{12}$
- The parameters $\mu_{\pm}$, $\nu_{\pm}$ are $\sim$ “saturation scales” of helicity-plus and helicity-minus partons.

Gluon Helicity at Small x

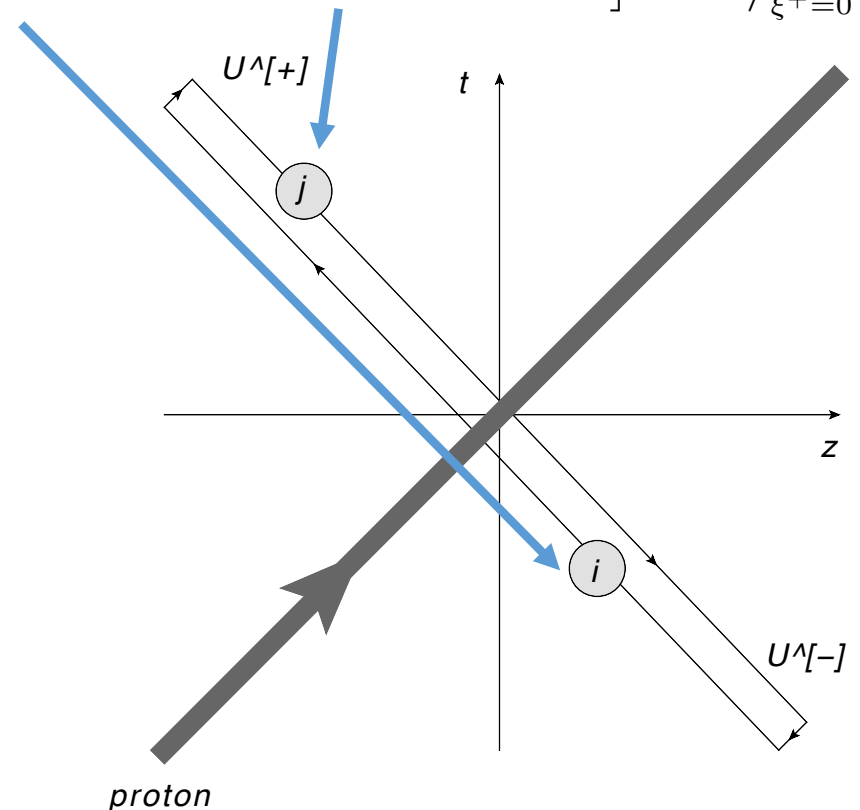
Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.
- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x, k_T^2) = \frac{-2i S_L}{x P^+} \int \frac{d\xi^- d^2\xi}{(2\pi)^3} e^{ixP^+ \xi^- - i\vec{k} \cdot \vec{\xi}} \left\langle P, S_L \left| \epsilon_T^{ij} \text{tr} \left[F^{+i}(0) \mathcal{U}^{[+] \dagger}[0, \xi] F^{+j}(\xi) \mathcal{U}^{[-]}[\xi, 0] \right] \right| P, S_L \right\rangle_{\xi^+=0}$$

- Here $\mathcal{U}^{[+]}$ and $\mathcal{U}^{[-]}$ are future and past Wilson line staples (hence the name 'dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

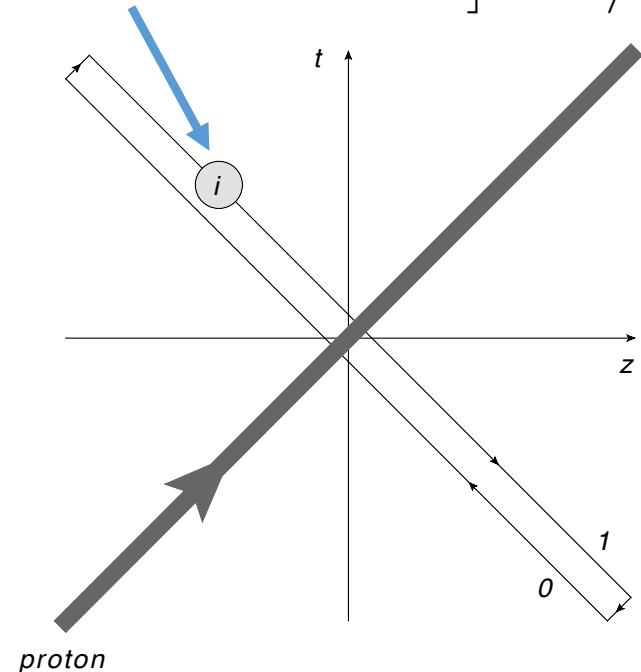
- At small x , the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G \text{ dip}}(x, k_T^2) = \frac{8i N_c S_L}{g^2 (2\pi)^3} \int d^2 x_{10} e^{i \underline{k} \cdot \underline{x}_{10}} k_\perp^i \epsilon_T^{ij} \left[\int d^2 b_{10} G_{10}^j(zs = \frac{Q^2}{x}) \right]$$

- Here we obtain a new operator, which is a transverse vector (written here in $A^-=0$ gauge, cf. Hatta et al, 2016):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \tilde{A}^i(x^-, \underline{x}) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- Note that $k_\perp^i \epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-, \underline{x})$ -- different from the polarized dipole amplitude!



Dipole TMD vs dipole amplitude

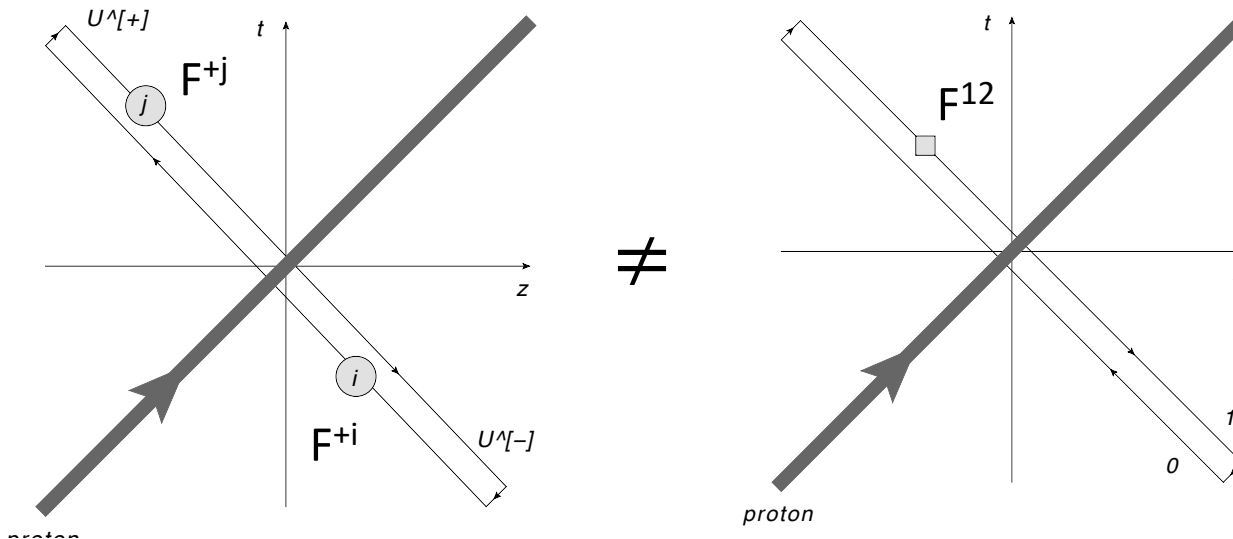
- Note that the operator for the dipole gluon helicity TMD

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \tilde{A}^i(x^-, \underline{x}) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

is different from the polarized dipole amplitude

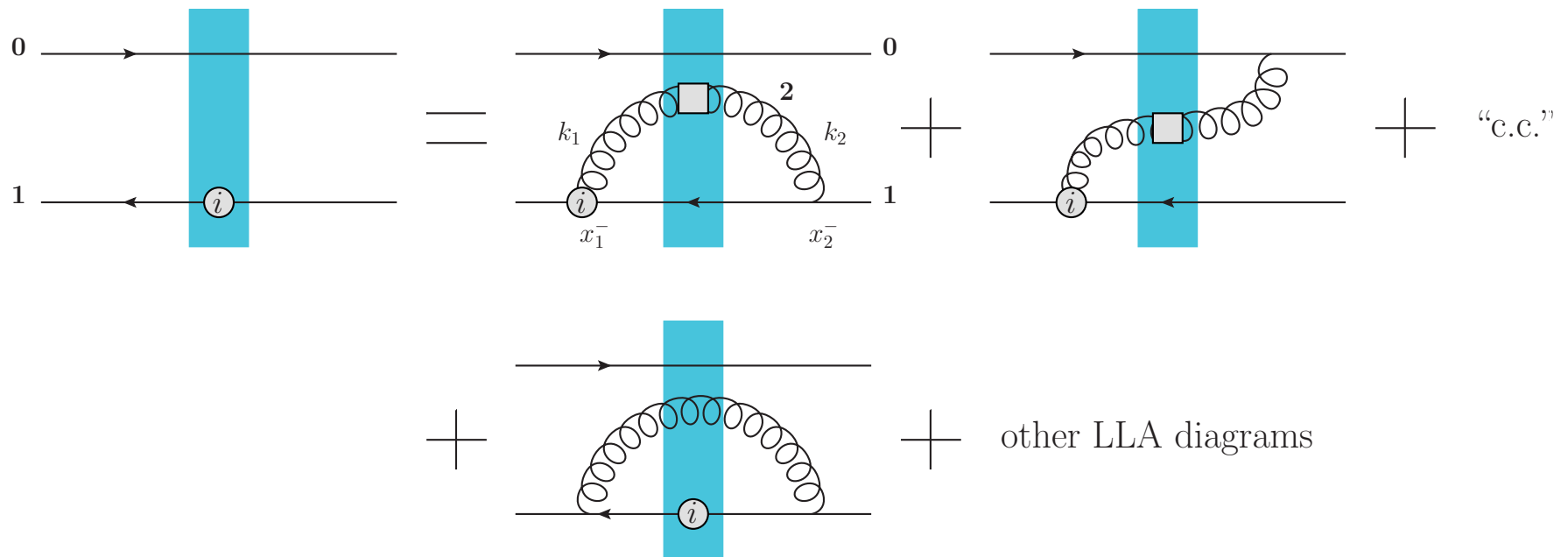
$$G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \underline{\nabla} \times \tilde{\underline{A}}(x^-, \underline{x}) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) \sim F^{12}$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMDs.) This is different from the unpolarized gluon TMD case.



Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



Large- N_c Evolution: Equations

- This results in the following evolution equations:

$$\begin{aligned}
 G_{10}^i(zs) &= G_{10}^{i(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2 x_2 \ln \frac{1}{x_{21} \Lambda} \frac{\epsilon_T^{ij}(x_{21})_{\perp}^j}{x_{21}^2} \left[\Gamma_{20,21}^{gen}(z's) + G_{21}(z's) \right] \\
 &\quad - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2 x_2 \ln \frac{1}{x_{21} \Lambda} \frac{\epsilon_T^{ij}(x_{20})_{\perp}^j}{x_{20}^2} \left[\Gamma_{20,21}^{gen}(z's) + \Gamma_{21,20}^{gen}(z's) \right] \\
 &\quad + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z's}}^{\frac{x_{10}^2}{x_{21}^2}} \frac{dx_{21}^2}{x_{21}^2} \left[G_{12}(z's) - \Gamma_{10,21}^i(z's) \right] \\
 \Gamma_{10,21}^i(z's) &= G_{10}^{i(0)}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \ln \frac{1}{x_{31} \Lambda} \frac{\epsilon_T^{ij}(x_{31})_{\perp}^j}{x_{31}^2} \left[\Gamma_{30,31}^{gen}(z''s) + G_{31}(z''s) \right] \\
 &\quad - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \ln \frac{1}{x_{31} \Lambda} \frac{\epsilon_T^{ij}(x_{30})_{\perp}^j}{x_{30}^2} \left[\Gamma_{30,31}^{gen}(z''s) + \Gamma_{31,30}^{gen}(z''s) \right] \\
 &\quad + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int_{\frac{1}{z''s}}^{\min \left[x_{10}^2, x_{21}^2 \frac{z'}{z''} \right]} \frac{dx_{31}^2}{x_{31}^2} \left[G_{13}(z''s) - \Gamma_{10,31}^i(z''s) \right].
 \end{aligned}$$

Large- N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small- x gluon helicity intercept

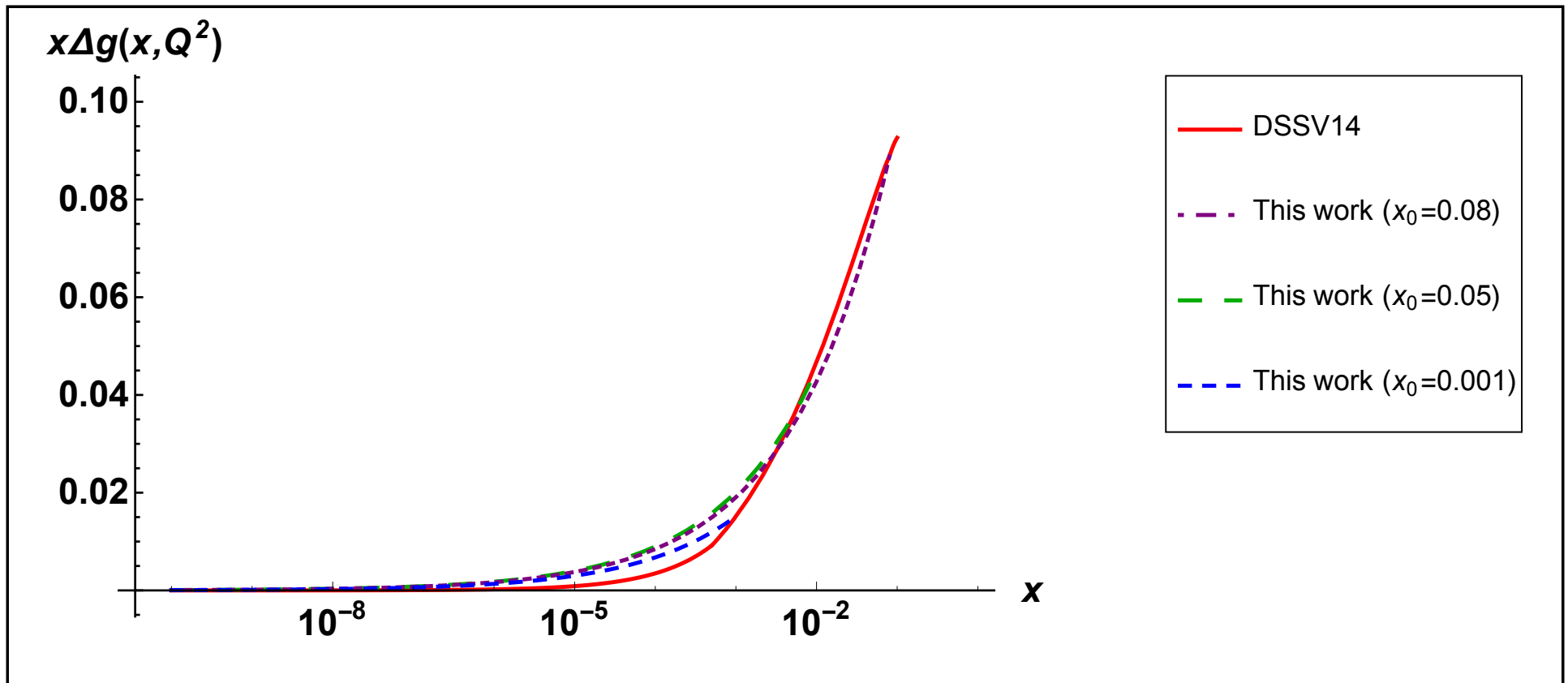
$$\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- We obtain the small- x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Impact of our ΔG on the proton spin

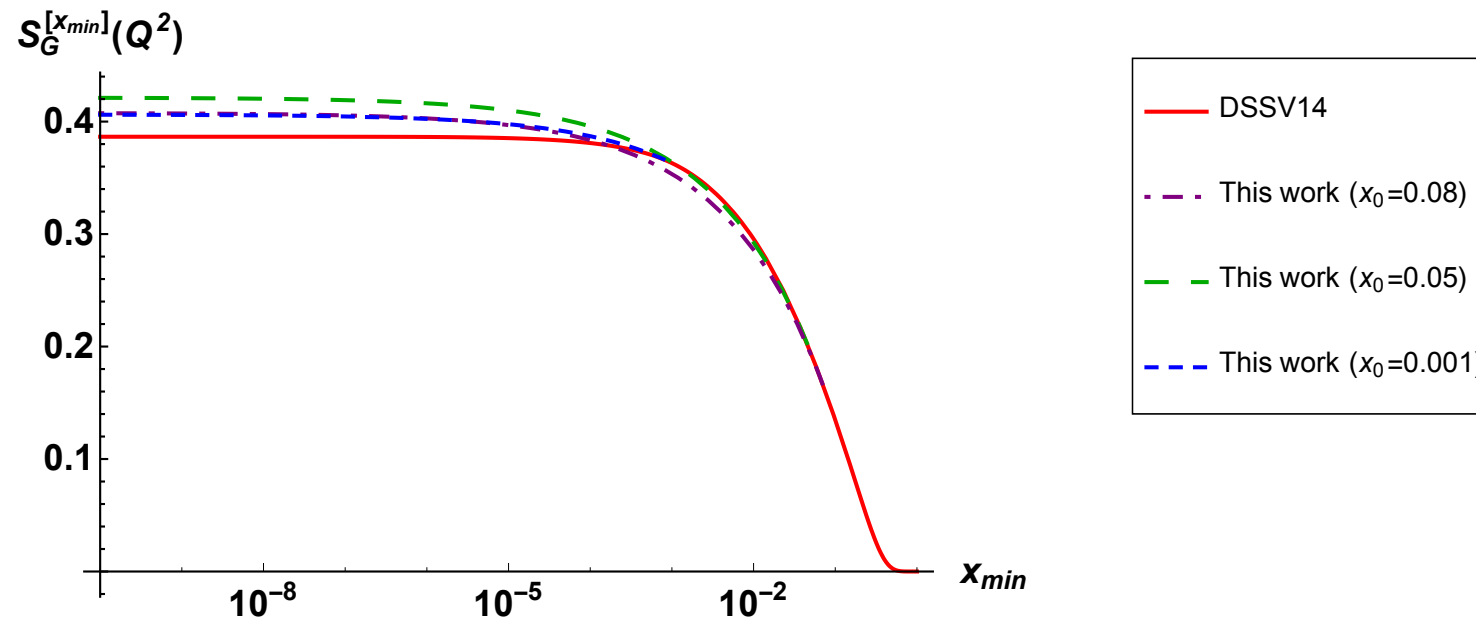
- We have attached a $\Delta\tilde{G}(x, Q^2) = N x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



“ballpark”
phenomenology

Impact of our ΔG on the proton spin

- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^1 dx \Delta G(x, Q^2)$ we plot it for $x_0=0.08, 0.05, 0.001$:



- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.

Musings on gluon helicity

- Small- x evolution for helicity mixes two operators (in the DLA):

$$\vec{\mu} \cdot \vec{B} \sim F^{12} \quad \text{and} \quad \bar{\psi} \frac{1}{2} \gamma^+ \gamma^5 \psi \sim \Delta q \sim \Delta \Sigma(x, Q^2)$$

- See for instance the polarized fundamental Wilson line:

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int_{-\infty}^{\infty} dx^- V_{\underline{x}}[+\infty, x^-] F^{12}(x^-, \underline{x}) V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[+\infty, x_2^-] t^b \psi_{\beta}(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5 \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty].$$

- This is different from DGLAP evolution in Q^2 , which mixes

$$\Delta G \sim \epsilon^{ij} F^{+i} F^{+j} \quad \text{and} \quad \bar{\psi} \frac{1}{2} \gamma^+ \gamma^5 \psi \sim \Delta q \sim \Delta \Sigma(x, Q^2)$$

- This may be related to the difficulties in defining the gluon helicity PDF.
Any thoughts from the experts?
- Certainly, $\vec{\mu} \cdot \vec{B} \sim F^{12}$ is a local operator and has no collinear limit...

Musings on gluon helicity

- At LLA (subleading to DLA), the two gluon operators

$$\vec{\mu} \cdot \vec{B} \sim F^{12} \quad \text{and} \quad \Delta G \sim \epsilon^{ij} F^{+i} F^{+j}$$

mix with each other (YK, Tarasov, Tawabutr, in preparation).

- The above operators involved in small-x evolution,

$$\vec{\mu} \cdot \vec{B} \sim F^{12} \quad \text{and} \quad \bar{\psi} \frac{1}{2} \gamma^+ \gamma^5 \psi \sim \Delta q \sim \Delta \Sigma(x, Q^2)$$

were independently confirmed by G. Chirilli (2018+in preparation).

- The DLA evolution presented above has been independently cross-checked and confirmed by A. Tarasov in the gluon sector (in preparation).

Quark and Gluon OAM at Small x

YK, [1901.07453](#) [hep-ph]

Quark OAM: small-x asymptotics

- We get (cf. Y. Hatta & D.-J. Yang, 2018 for the equality, not for power of x)

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

where the flavor-singlet quark helicity is

$$\Delta\Sigma(x, Q^2) = \sum_f [\Delta q^f(x, Q^2) + \Delta \bar{q}^f(x, Q^2)] \sim \left(\frac{1}{x}\right)^{\alpha_h^q} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}} \approx \left(\frac{1}{x}\right)^{2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that this is not a complete cancellation, the contribution to the proton spin is

$$\frac{1}{2} \Delta\Sigma(x, Q^2) + L_{q+\bar{q}}(x, Q^2) = -\frac{1}{2} \Delta\Sigma(x, Q^2)$$

Gluon OAM: small-x asymptotics

- We arrive at the following relation

$$L_G(x, Q^2) = \left(\frac{\alpha_h^q}{4} \ln \frac{Q^2}{\Lambda^2} \right) \Delta G(x, Q^2)$$

where

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- We conclude that

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}} \sim \left(\frac{1}{x} \right)^{1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that with the DLA accuracy we could also simply conclude that

$$|L_G| \ll |\Delta G|$$

Valence Quark Transversity at Small x

Yu.K., M. Sievert, [arXiv:1808.10354](https://arxiv.org/abs/1808.10354) [hep-ph]

Small-x Asymptotics of Quark Transversity

- Solution of the transversity evolution equation is straightforward.
- The resulting small-x asymptotics is (cf. Kirschner et al, 1996)

$$h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2 \sqrt{\frac{\alpha_s C_F}{\pi}}$$

- Note the suppression by x^2 compared to the unpolarized quark TMDs.
- For $\alpha_s = 0.3$ we get

$$h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp NS}(x, k_T^2) \sim x^{0.243}$$

- This certainly satisfies the Soffer bound, but is not likely to produce much tensor charge from small x .

$$\delta q(Q^2) = \int_0^1 dx h_1(x, Q^2)$$

Conclusions and Outlook

- At large N_c we have obtained the following small- x asymptotics:

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}},$$

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Future helicity and OAM work will involve solving the large- N_c & N_f equations + including running coupling corrections + LLA corrections (Y. Tawabutr's talk) + phenomenology to constrain the spin+OAM coming from small- x quarks and gluons (D. Adamiak's talk).
- EIC should be able to measure helicity TMDs with high precision and down to fairly small x . We may also be able to learn something about OAM.

Conclusions and Outlook

- The same formalism can be applied to any TMD in order to determine its small- x asymptotics.
- For transversity we got

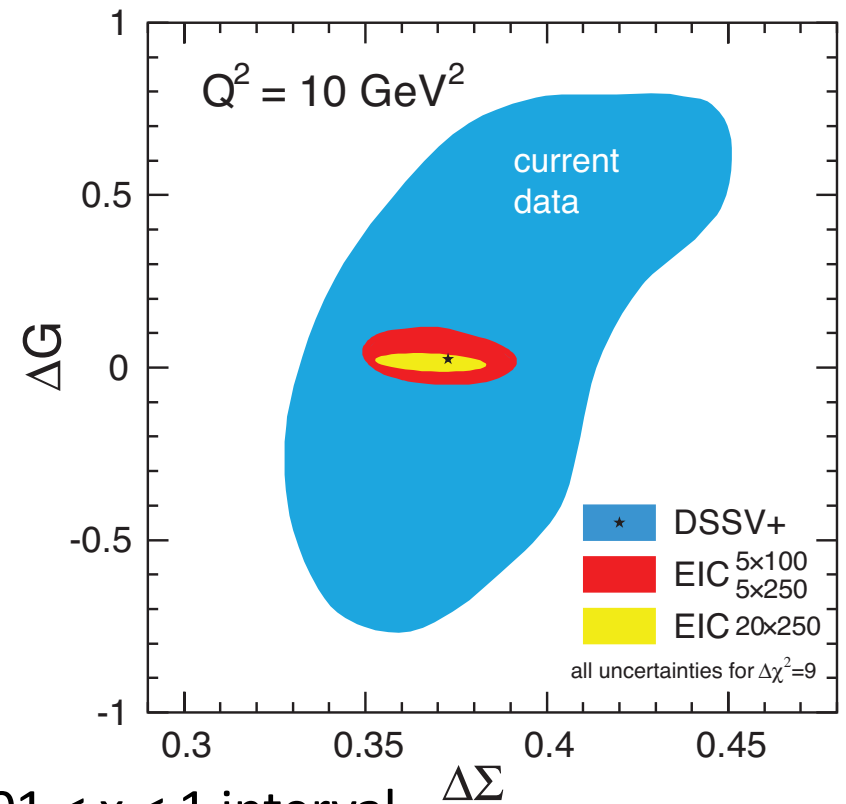
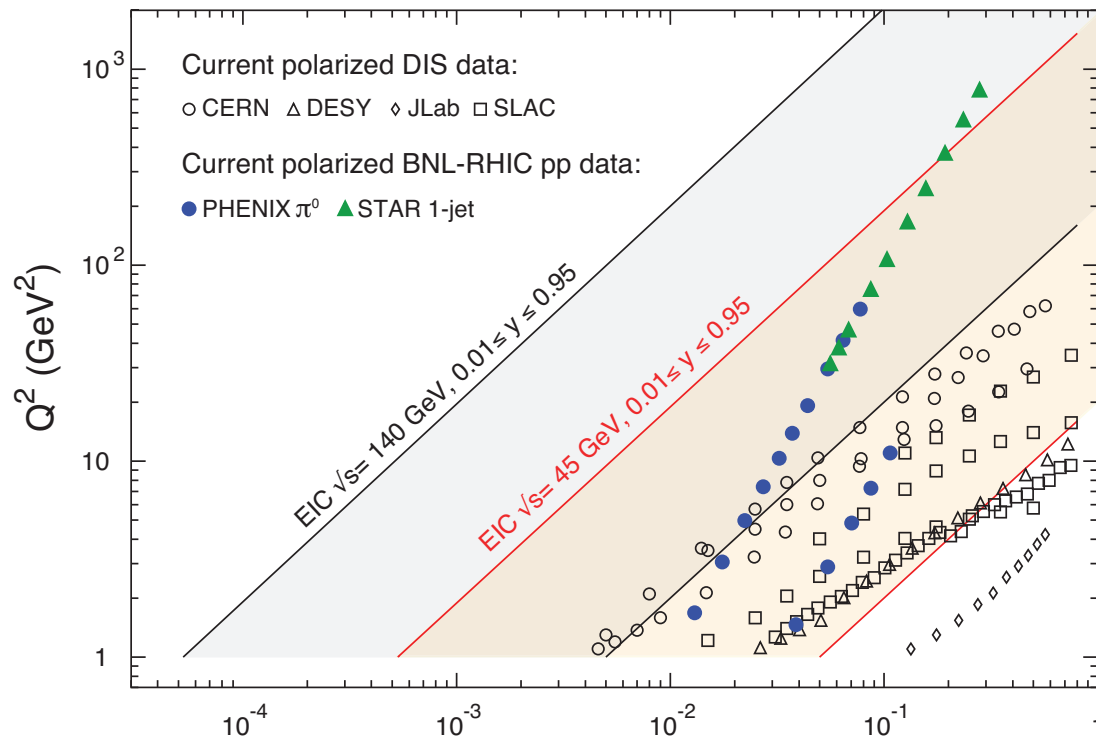
$$h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2\sqrt{\frac{\alpha_s C_F}{\pi}}$$

- Small- x asymptotics of the Sivers function will be obtained shortly by our method (G. Santiago+YK, in preparation).

Backup Slides

EIC & Spin Puzzle

- Parton helicity distributions are sensitive to low- x physics.
- EIC would have an unprecedented low- x reach for a polarized DIS experiment, allowing to pinpoint the values of quark and gluon contributions to proton's spin:



- ΔG and $\Delta \Sigma$ are integrated over x in the $0.001 < x < 1$ interval.

Proton Spin Puzzle $S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta\Sigma(x, Q^2)$

- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$\Delta\Sigma \approx 0.1 \div 0.2$$

- It appeared quarks do not carry all of the proton spin (which would have corresponded to $\Delta\Sigma = 1$).

- Missing spin can be
 - Carried by gluons
 - In the orbital angular momenta of quarks and gluons
 - At small x (both helicity and OAM):

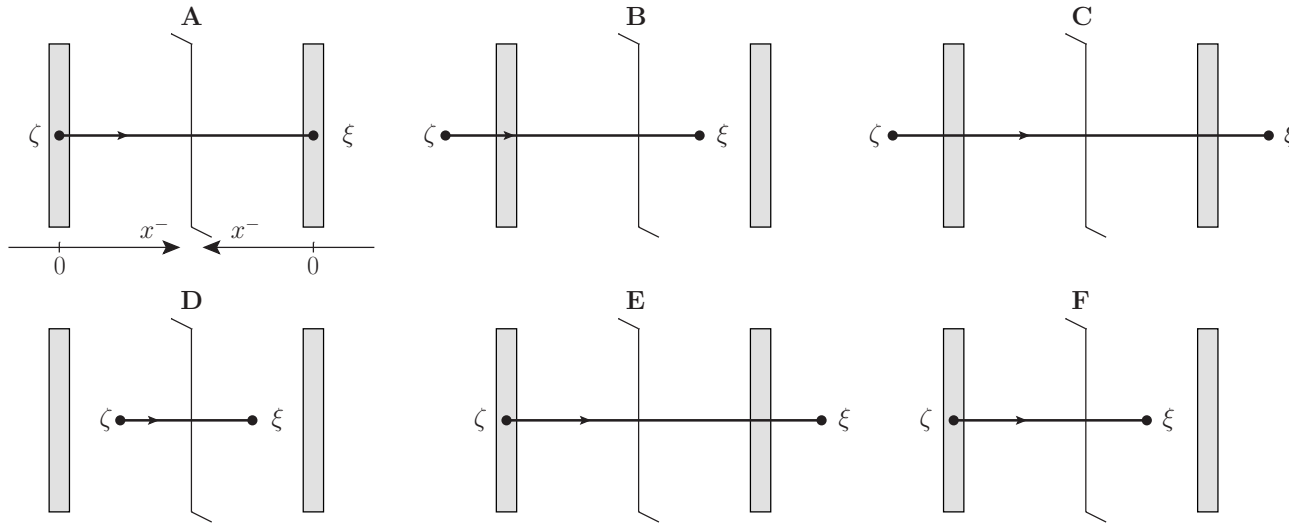
$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta\Sigma(x, Q^2) \quad S_g(Q^2) = \int_0^1 dx \Delta G(x, Q^2)$$

Can't integrate down to zero, use $x_{\min}$ instead!

- Or all of the above!

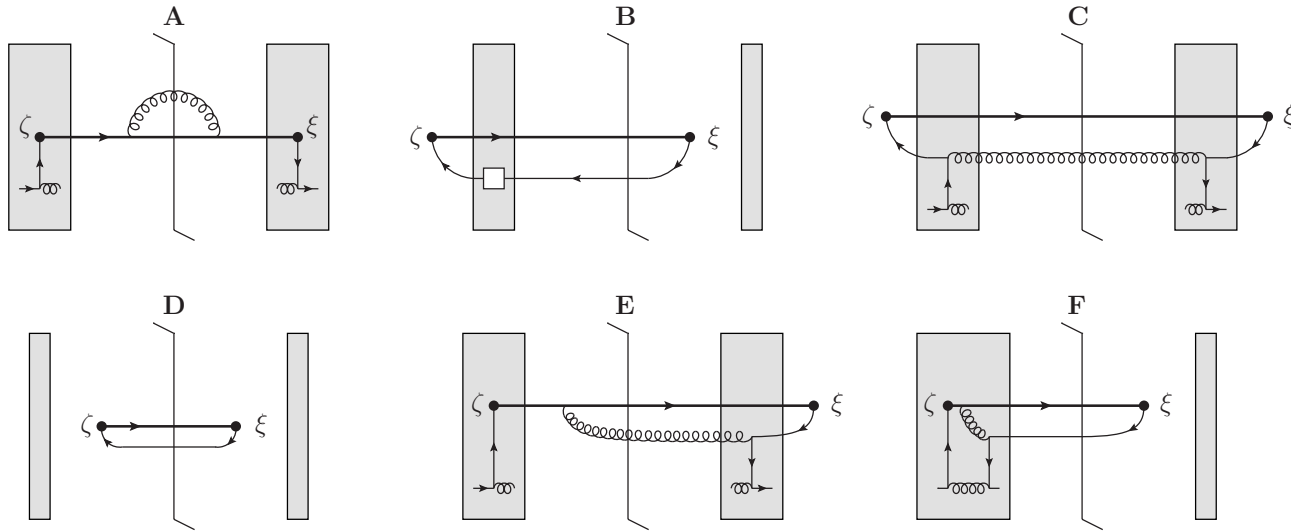
Quark Helicity TMD at Small x

- At high energy/small-x the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:



$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \sum_X \int d^2\zeta d\zeta^- d^2\xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_\xi[\xi^-, \infty] | X \right\rangle \left\langle X | V_\zeta[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

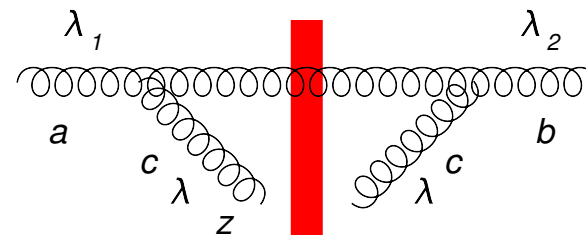
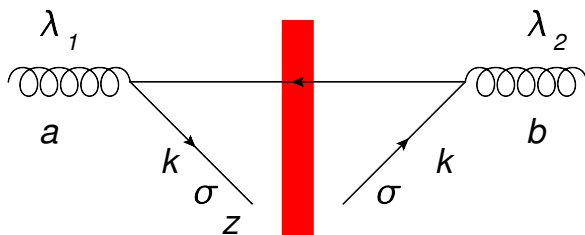
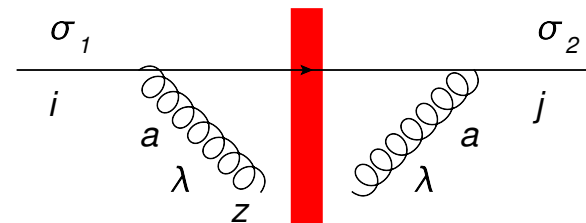
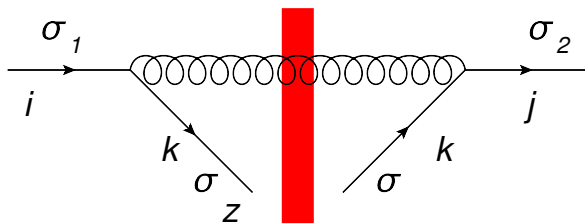
Quark Helicity TMD at Small x



- Diagram D does not transfer spin information from the target. Diagram C is canceled as we move t-channel quarks across the cut.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other at the leading (DLA) order (Ward identity).
- We are left with the diagram B.

Helicity Evolution Ingredients

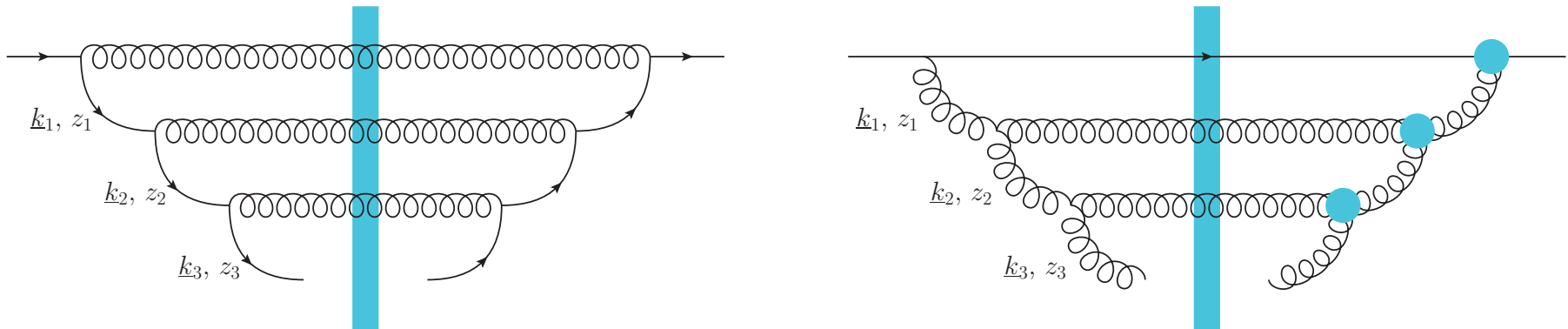
- Unlike the unpolarized evolution, in one step of helicity evolution we may emit a soft gluon or a soft quark (all in $A^+=0$ LC gauge of the projectile):



- When emitting gluons, one gluon is eikonal, while another one is soft, but non-eikonal, as is needed to transfer polarization down the cascade/ladder.

Helicity Evolution: Ladders

- To get an idea of how the helicity evolution works let us try iterating the splitting kernels by considering ladder diagrams (circles denote non-eikonal gluon vertices):

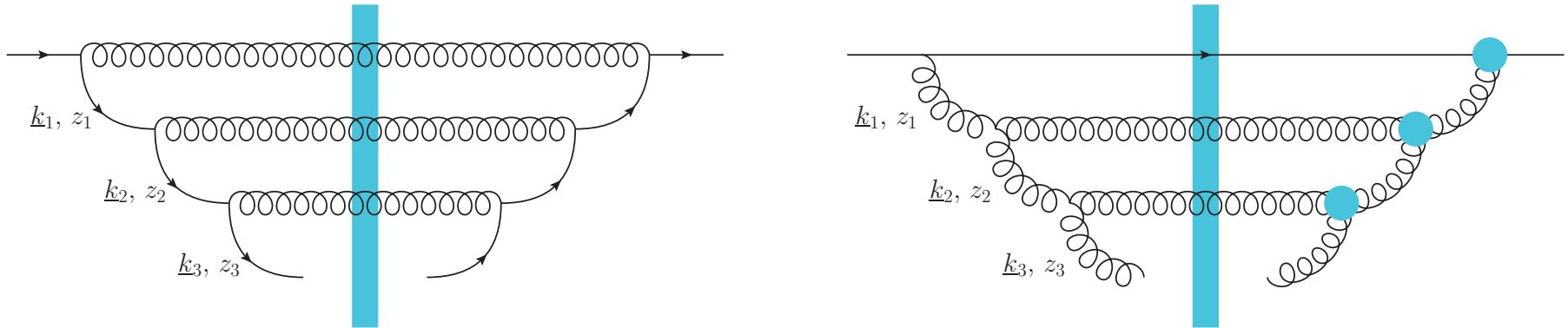


- To get the leading-energy asymptotics we need to order the longitudinal momentum fractions of the quarks and gluons (just like in the unpolarized evolution case) $1 \gg z_1 \gg z_2 \gg z_3 \gg \dots$

obtaining a nested integral

$$\alpha_s^3 \int_{z_i}^1 \frac{dz_1}{z_1} \int_{z_i}^{z_1} \frac{dz_2}{z_2} \int_{z_i}^{z_2} \frac{dz_3}{z_3} z_3 \otimes \frac{1}{z_3 s} \sim \frac{1}{s} \alpha_s^3 \ln^3 s$$

Helicity Evolution: Ladders



- However, these are not all the logs of energy one can get here. Transverse momentum (or distance) integrals have UV and IR divergences, which lead to logs of energy as well.
- If we order transverse momenta / distances as (Sudakov- β ordering)

$$\frac{k_1^2}{z_1} \ll \frac{k_2^2}{z_2} \ll \frac{k_3^2}{z_3} \ll \dots$$

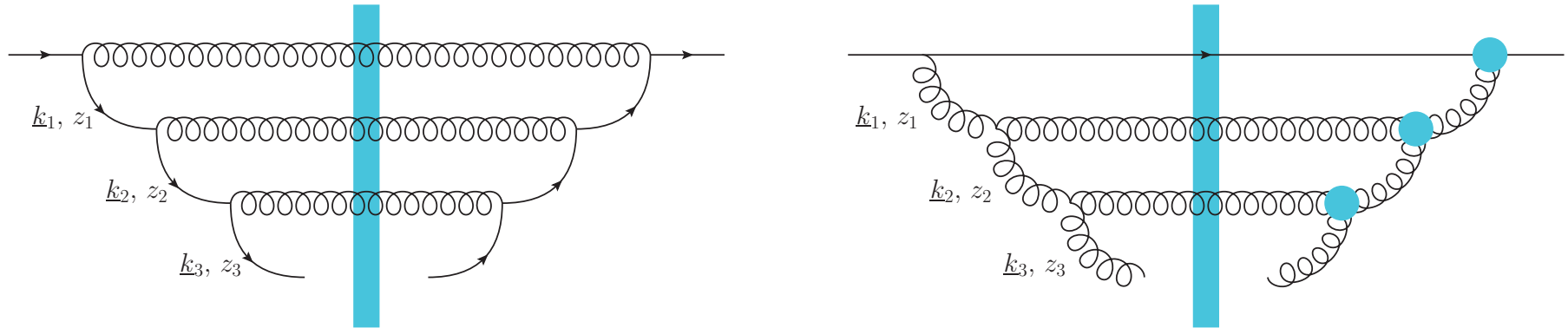
$$z_1 x_1^2 \gg z_2 x_2^2 \gg z_3 x_3^2 \gg \dots$$

we would get integrals like

also generating logs of energy.

$$\int_{1/(z_n s)}^{x_{n-1,\perp}^2 z_{n-1}/z_n} \frac{dx_{n,\perp}^2}{x_{n,\perp}^2}$$

Helicity Evolution: Ladders



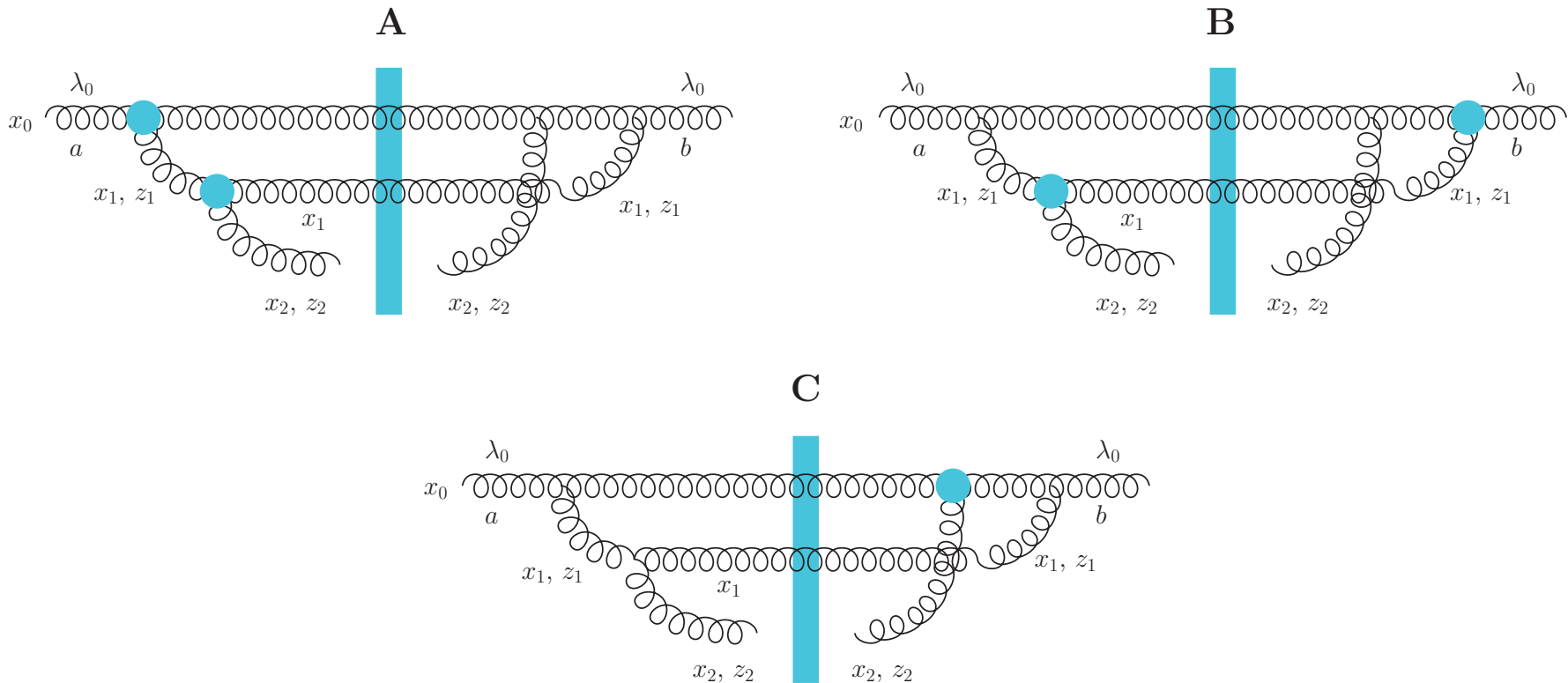
- To summarize, the above ladder diagrams are parametrically of the order

$$\frac{1}{s} \alpha_s^3 \ln^6 s$$

- Note two features:
 - $1/s$ suppression due to non-eikonal exchange
 - two logs of energy per each power of the coupling!

Non-Ladder Diagrams

- Ladder diagrams are not the whole story. The non-ladder diagrams below are also leading-order (that is, DLA).



- Non-ladder soft quark emissions cancel for flavor-singlet observables we are primarily interested in. Non-ladder soft gluons do not cancel.

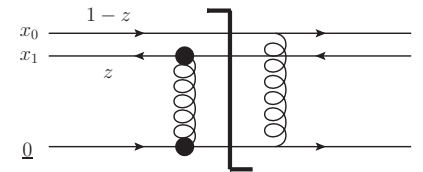
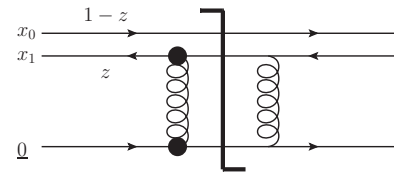
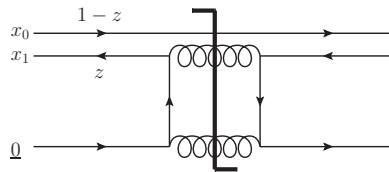
Large- N_c Evolution

- In the strict DLA limit ($S=1$) and at large N_c we get (here Γ is an auxiliary function we call the ‘neighbor dipole amplitude’) (KPS ‘15)

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} [\Gamma(x_{10}^2, x_{21}^2, z') + 3 G(x_{21}^2, z')]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int_{\frac{1}{z'' s}}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{dx_{32}^2}{x_{32}^2} [\Gamma(x_{10}^2, x_{32}^2, z'') + 3 G(x_{32}^2, z'')]$$

- The initial conditions are given by the Born-level graphs



$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2 \ln(zs x_{10}^2) \right]$$

Large- N_c Evolution: Equations

- Here

$$\Gamma_{20,21}^{gen}(z's) = \theta(x_{20} - x_{21}) \Gamma_{20,21}(z's) + \theta(x_{21} - x_{20}) G_{20}(z's)$$

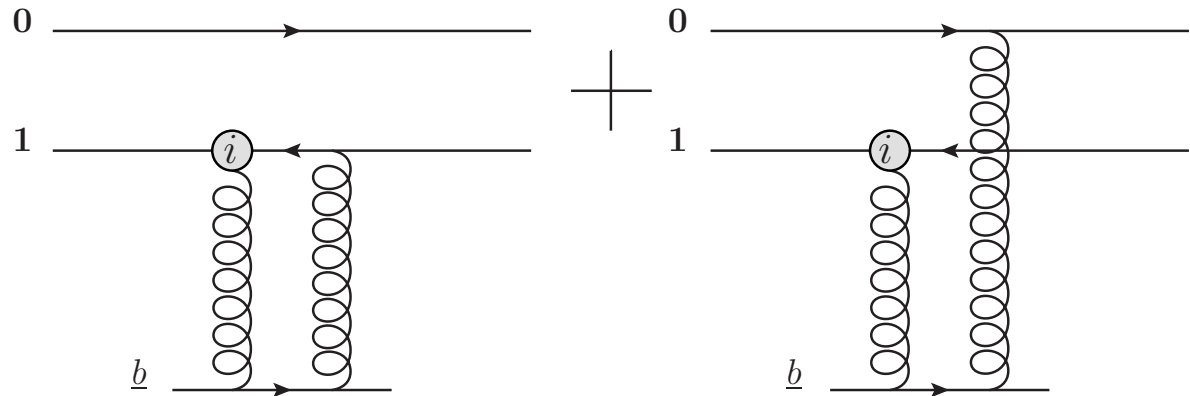
is an object which we know from the quark helicity evolution, as the latter gives us G and Γ .

- Note that our evolution equations mix the gluon (G^i) and quark (G) small- x helicity evolution operators:

$$\begin{aligned} G_{10}^i(zs) = & G_{10}^{i(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{21})_\perp^j}{x_{21}^2} \left[\Gamma_{20,21}^{gen}(z's) + G_{21}(z's) \right] \\ & - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{20})_\perp^j}{x_{20}^2} \left[\Gamma_{20,21}^{gen}(z's) + \Gamma_{21,20}^{gen}(z's) \right] \\ & + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G_{12}(z's) - \Gamma_{10,21}^i(z's) \right] \end{aligned}$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2 b_{10} G_{10}^{i(0)}(zs) = \int d^2 b_{10} \Gamma_{10,21}^{i(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c} \pi \epsilon^{ij} x_{10}^j \ln \frac{1}{x_{10} \Lambda}$$

- Note that these initial conditions have no $\ln s$, unlike the initial conditions for the quark evolution:

$$\int d^2 b_{10} G_{10}^{(0)}(zs) = \int d^2 b_{10} \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c} \pi \ln(zs x_{10}^2)$$

Small-x Evolution at large N_c

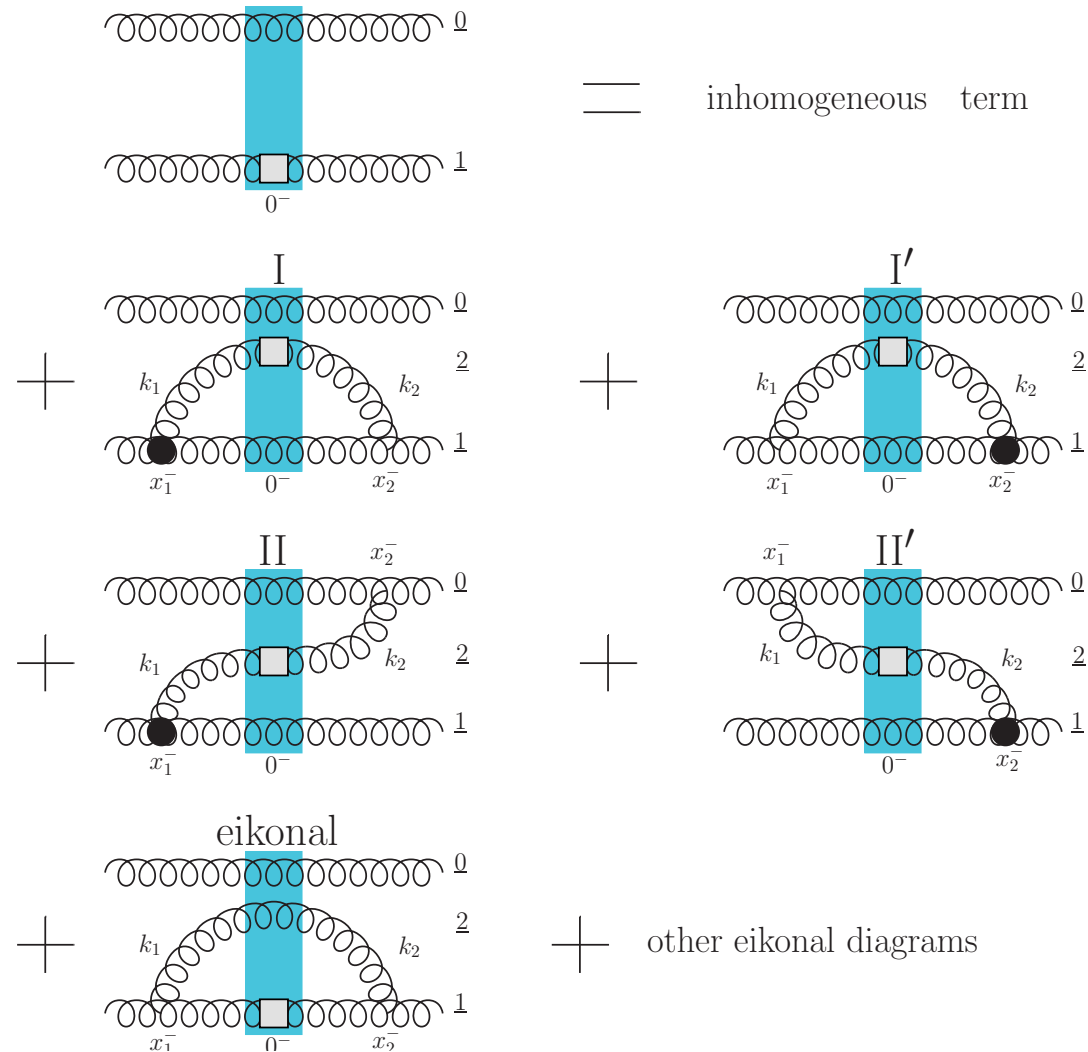
- At large N_c the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large N_c the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4 G_{10}(z)$$

(Note that the factor is 4, not 2 like in the unpolarized dipole case.)

Small-x Evolution at large N_c

- We need to sum the following diagrams (box denotes the polarized “Wilson lines”):



Quark OAM: Definition

- We begin by writing the (Jaffe-Manohar) quark OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp db^- d^2 k_\perp dk^+}{(2\pi)^3} (\underline{b} \times \underline{k})_z W(k, b)$$

with the quark SIDIS Wigner distribution

$$W^{q,SIDIS}(k, b) = 2 \sum_X \int d^2 r dr^- e^{ik \cdot r} \left\langle \bar{\psi}_\alpha \left(b - \frac{1}{2} r \right) V_{\underline{b} - \frac{1}{2} \underline{r}} \left[b^- - \frac{1}{2} r^-, \infty \right] |X\rangle \left(\frac{1}{2} \gamma^+ \right)_{\alpha\beta} \right. \\ \left. \times \langle X | V_{\underline{b} + \frac{1}{2} \underline{r}} \left[\infty, b^- + \frac{1}{2} r^- \right] \psi_\beta \left(b + \frac{1}{2} r \right) \right\rangle$$

- Here, and above, the angle brackets denote "CGC averaging" in the (polarized) proton target:

$$\langle \hat{\mathcal{O}}(b, r) \rangle = \frac{1}{2P^+} \int \frac{d^2 \Delta d\Delta^+}{(2\pi)^3} e^{ib \cdot \Delta} \left\langle P + \frac{\Delta}{2} \left| \hat{\mathcal{O}}(0, r) \right| P - \frac{\Delta}{2} \right\rangle$$

Quark OAM: small-x simplifications

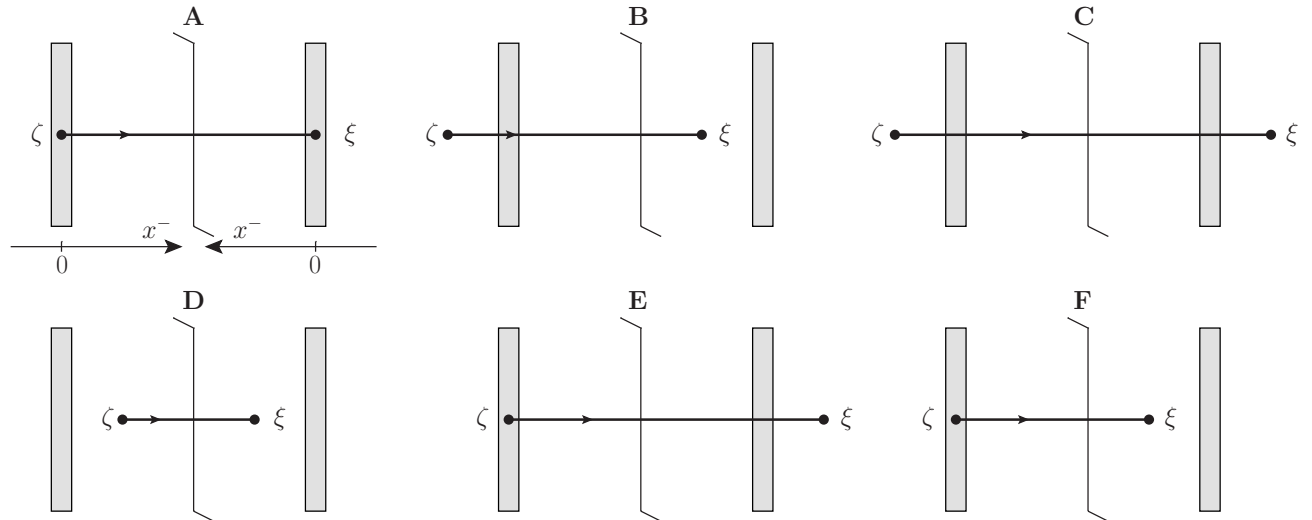
- The resulting quark OAM “PDF” is

$$L_q(x, Q^2) = \frac{2P^+}{(2\pi)^3} \sum_X \int d^2 k_\perp d^2 \zeta d\zeta^- d^2 \xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left(\frac{\zeta + \xi}{2} \times \underline{k} \right) \left\langle \bar{\psi}_\alpha(\xi) V_\xi[\xi^-, \infty] | X \right\rangle \left(\frac{1}{2} \gamma^+ \right)_{\alpha\beta} \\ \times \left\langle X | V_\zeta[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

- This can be compared to quark helicity,

$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2 \zeta d\zeta^- d^2 \xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_\xi[\xi^-, \infty] V_\zeta[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

- The operators are different, but the structure is similar. The quark OAM can be evaluated in the same way as the quark helicity operator: only diagram B survives.



Quark OAM: small-x expression

- After some algebra we arrive at the following small-x expression for quark OAM:

$$L_{q+\bar{q}}(x, Q^2) = \frac{8N_c}{(2\pi)^5} \int d^2k_\perp d^2x_{10} d^2x_1 e^{i\vec{k} \cdot \vec{x}_{10}} \frac{\underline{x}_{10}}{x_{10}^2} \times \frac{\underline{k}}{k^2} \underline{x}_1 \times \underline{k} \int_{\Lambda^2/s}^1 \frac{dz}{z} G_{10}(zs) - \sum_f [\Delta q^f(x, Q^2) + \Delta \bar{q}^f(x, Q^2)]$$

- The result is written in terms of the polarized dipole amplitude $G_{10}(z)$. It seems we are done, right?
- This is almost correct. The remaining minor technicality is that the above quark OAM depends on the “first moment” of the polarized dipole amplitude

$$I^k(\underline{x}_{10}, zs) = \int d^2x_1 x_1^k G_{10}(zs)$$

while all our earlier results for the quark helicity were derived for the “zeroth moment”, the impact-parameter integrated polarized dipole amplitude

$$G(x_{10}^2, zs) = \int d^2x_1 G_{10}(zs)$$

Gluon OAM: definition

- The gluon OAM story is similar. We start with the Wigner distribution definition

$$L_z = \int \frac{d^2 b_\perp db^- d^2 k_\perp dk^+}{(2\pi)^3} (\underline{b} \times \underline{k})_z W(k, b)$$

with the dipole Wigner distribution for gluons

$$W^{G \text{ dip}}(k, b) = \frac{4}{xP^+} \int d\xi^- d^2 \xi_\perp e^{ixP^+ \xi^- - i\underline{k} \cdot \underline{\xi}} \\ \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] F^{+i}(b + \tfrac{1}{2}\xi) \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

- We obtain the following expression for the gluon OAM “PDF” (cf. Hatta et al, 2016)

$$L_G(x, Q^2) = \frac{4}{(2\pi)^3 x} \int d^2 b_\perp db^- d^2 k_\perp d\xi^- d^2 \xi_\perp (\underline{b} \times \underline{k}) e^{ixP^+ \xi^- - i\underline{k} \cdot \underline{\xi}} \\ \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] F^{+i}(b + \tfrac{1}{2}\xi) \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

Gluon OAM: small-x expression

- Gluon OAM at small x can (similarly to the quark OAM) be rewritten in terms of the “moment” of the polarized dipole amplitude G_{10}^i for the gluon helicity TMD. This object is different from the polarized amplitude for the quark.

- We get

$$L_G(x, Q^2) = -\frac{8iN_c}{g^2 (2\pi)^3} \int d^2 x_{10} d^2 k_{\perp} e^{i\vec{k} \cdot \vec{x}_{10}} (\vec{k} \cdot \vec{x}_{10}) G_5 \left(x_{10}^2, zs = \frac{Q^2}{x} \right)$$

where

$$\int d^2 x_1 x_1^j \nabla_{10}^i G_{10}^i(zs) = x_{10}^j G_4(x_{10}^2, zs) + \epsilon^{jk} x_{10}^k G_5(x_{10}^2, zs)$$

- We write down and solve the equations for G_5 .