Andrea Simonelli,
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In collaboration with M. Boglione

Factorization of $e^+e^- \to HX$ cross section, differential in z_h, P_T and T in the 2-jet limit

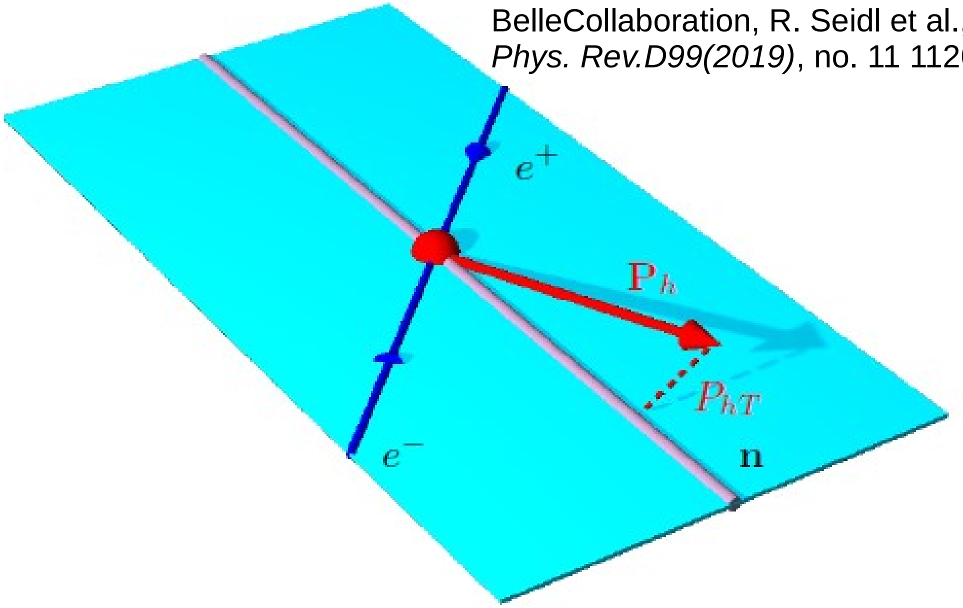
The BELLE Cross Section

Cross section of $e^+e^- \to HX$, differential in:

z_h	T	P_T
Fractional energy of the detected hadron:	Topology of the final state:	Transverse Momentum of the detected hadron with respect to the thrust axis.
$z_h = \frac{2P \cdot q}{q^2} = \frac{2E_H}{Q}$	$T = \mathrm{Max}\frac{\sum_i \lvert\vec{p}_i \cdot \vec{n}\rvert}{\sum_i \lvert\vec{p}_i\rvert}$	$P_T << Q$
$Q = 10.58 \text{ GeV}$	$0.5 \leq T \leq 1$	$P_T \leq 2.5 \text{ GeV}$

Assumptions:

- Thrust axis = jet axis
- 2-jet limit $T \sim 1$
- Spinless hadron (charged pions $\pi^\pm$)



BelleCollaboration, R. Seidl et al., *Phys. Rev.D99(2019)*, no. 11 112006

The BELLE Cross Section

Naive structure (single hadron): $\sigma \sim \widehat{\sigma} \otimes D_1$

Actual structure, resulting from (CSS) factorization procedure:

$$\frac{d\sigma}{dz_h\, dT\, dP_T^2} = \pi \sum_j \int_{z_h}^{1} \frac{dz}{z} \, \frac{d\widehat{\sigma}_j}{dz_h/z\, dT} \, D_{1,\,\pi^\pm/j}(z,\, P_T) \left[1 + \begin{array}{c}\text{power}\\ \text{suppressed}\\ \text{terms}\end{array} \right]$$

Partonic Cross Section, totally predicted by pQCD (**NLO**)

Unpolarized TMD FF, includes:
- Perturbative contributions (**NLL**)
- Non-Perturbative contributions (**pheno**menological models)

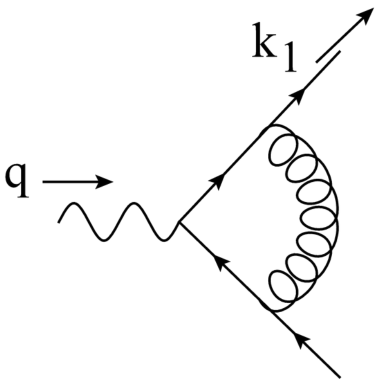
Suppressed corrections of order P_T^2/Q^2 and M_π^2/Q^2

Still not the final version:
- Subtraction and Renormalization mechanism (Partonic Cross Section)
- Rapidity cut-offs

Partonic Cross Section at NLO

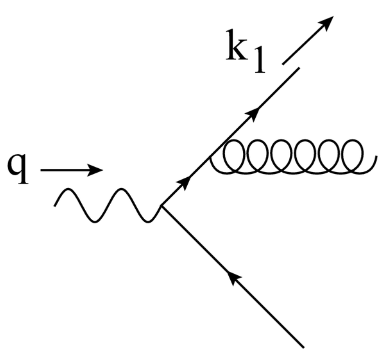
Lets consider a fragmenting quark of flavor f:

- Virtual Emission



$\longrightarrow$ LO $\times V^{[1]}(\epsilon)\;\delta(1-T)\,\delta(1-z)$

- Real Emission



$\longrightarrow$ 2-jet topology

Gluon collinear to fragmenting quark

$\propto J^{[1]}_{q/q}(\epsilon;\,1-T,\,z)$

Soft Gluon $\quad\propto S^{[1]}(\epsilon;\,1-T)\,\delta(1-z)$

Gluon collinear to antiquark (backward)

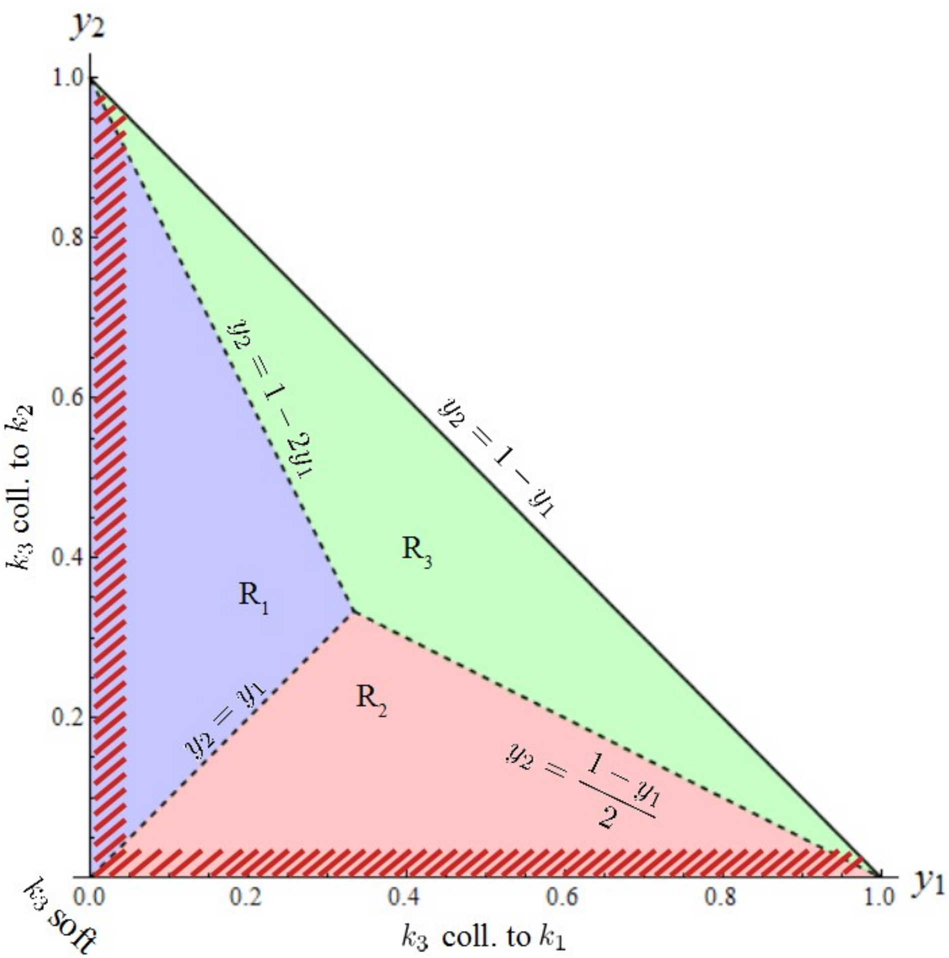
$\propto J^{[1]}_{B}(\epsilon;\,1-T)\,\delta(1-z)$

Partonic Cross Section at NLO

Topology cut-off:
$\tau = 1 - T \leq \tau_{\mathrm{MAX}}$

2-jet limit

$\tau_{\mathrm{MAX}} \to 0$

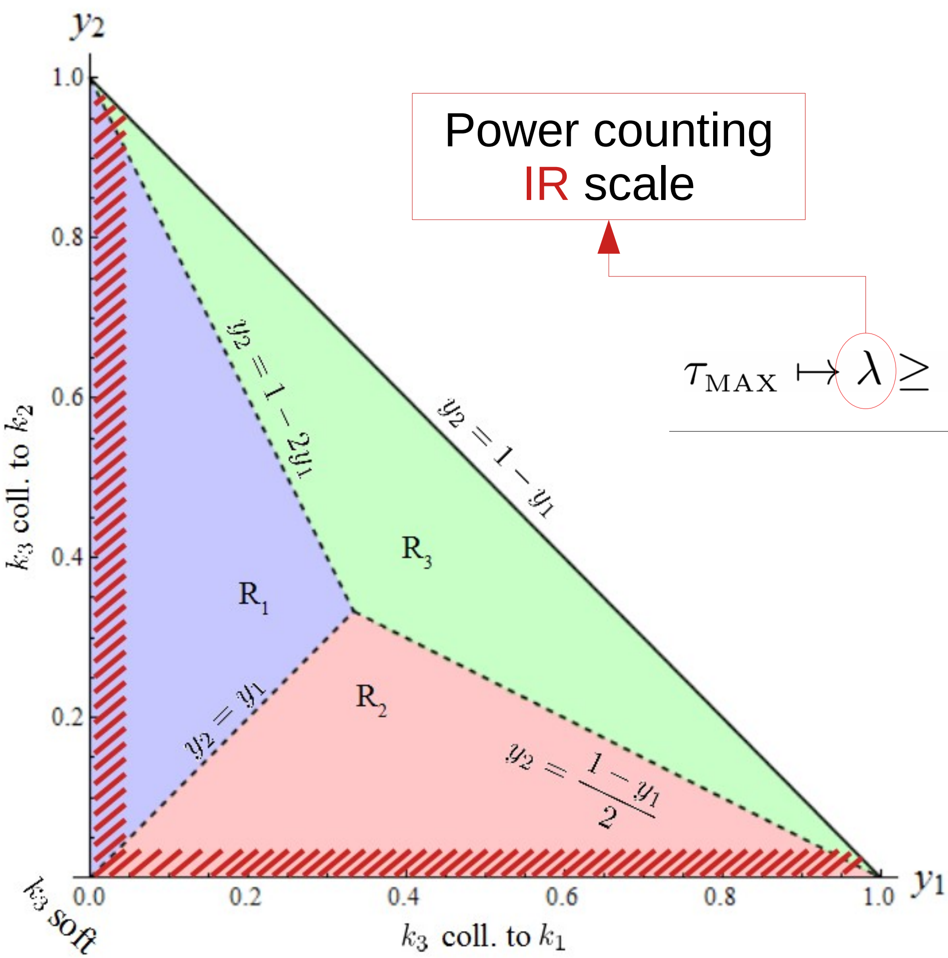


Partonic Cross Section at NLO

Topology cut-off:

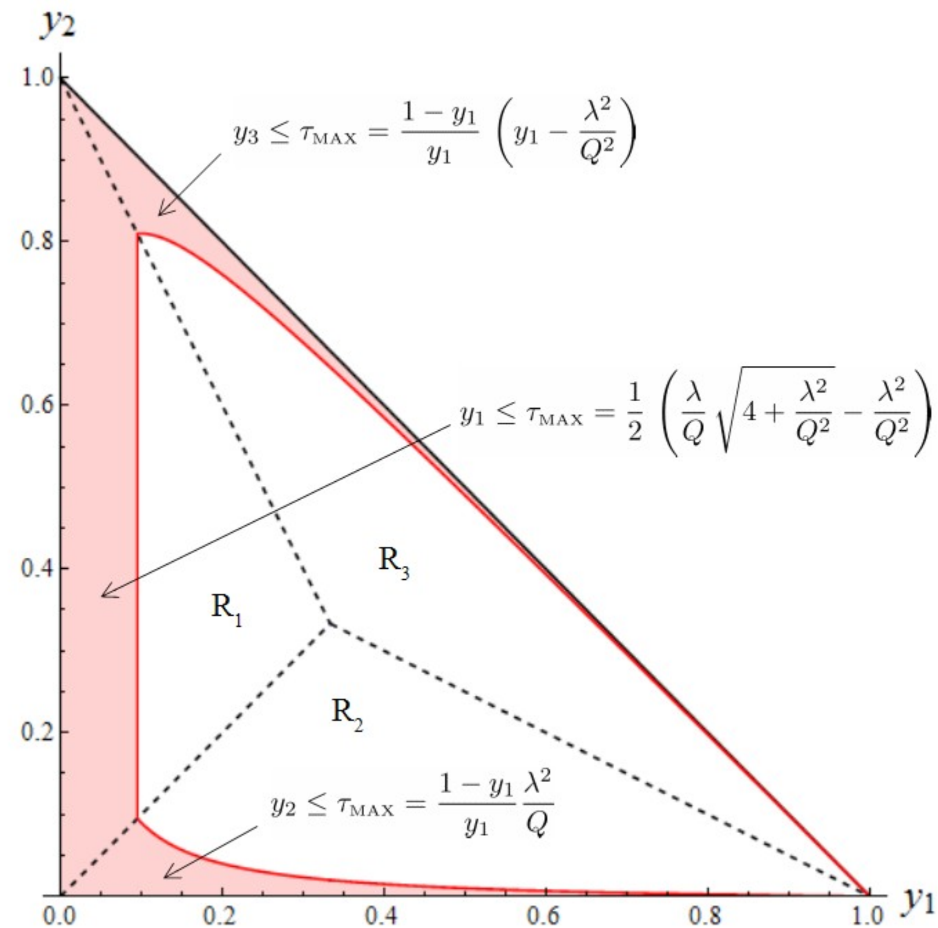
$\tau = 1 - T \leq \tau_{\mathrm{MAX}}$

2-jet limit $\longleftrightarrow$ $\tau_{\mathrm{MAX}} \to 0$ $\longleftrightarrow$ $\lambda \to 0$



Power counting
IR scale

$\tau_{\mathrm{MAX}} \mapsto \lambda \geq k_T$



Partonic Cross Section at NLO

$$\frac{d\widehat{\sigma}_f^{[1]}}{dz\,dT} = \overbrace{\sigma_B}^{\frac{4\pi\alpha^2}{3Q^2}} z\, N_C\, e_f^2 \Big[\delta(1-z)\Big(\delta(\tau)\,\overbrace{V^{[1]}(\epsilon)}^{\text{Virtual}} + \overbrace{S^{[1]}(\epsilon;\,\tau)}^{\text{Soft}} + \overbrace{J_B^{[1]}(\epsilon;\,\tau)}^{\text{Backward}}\Big) +$$

$$+ \underbrace{J_{q/q}^{[1],\,(\lambda)}(\epsilon;\,\tau,\,z)}_{\text{Coll. to thrust axis}}\Big]$$

Partonic Cross Section at NLO

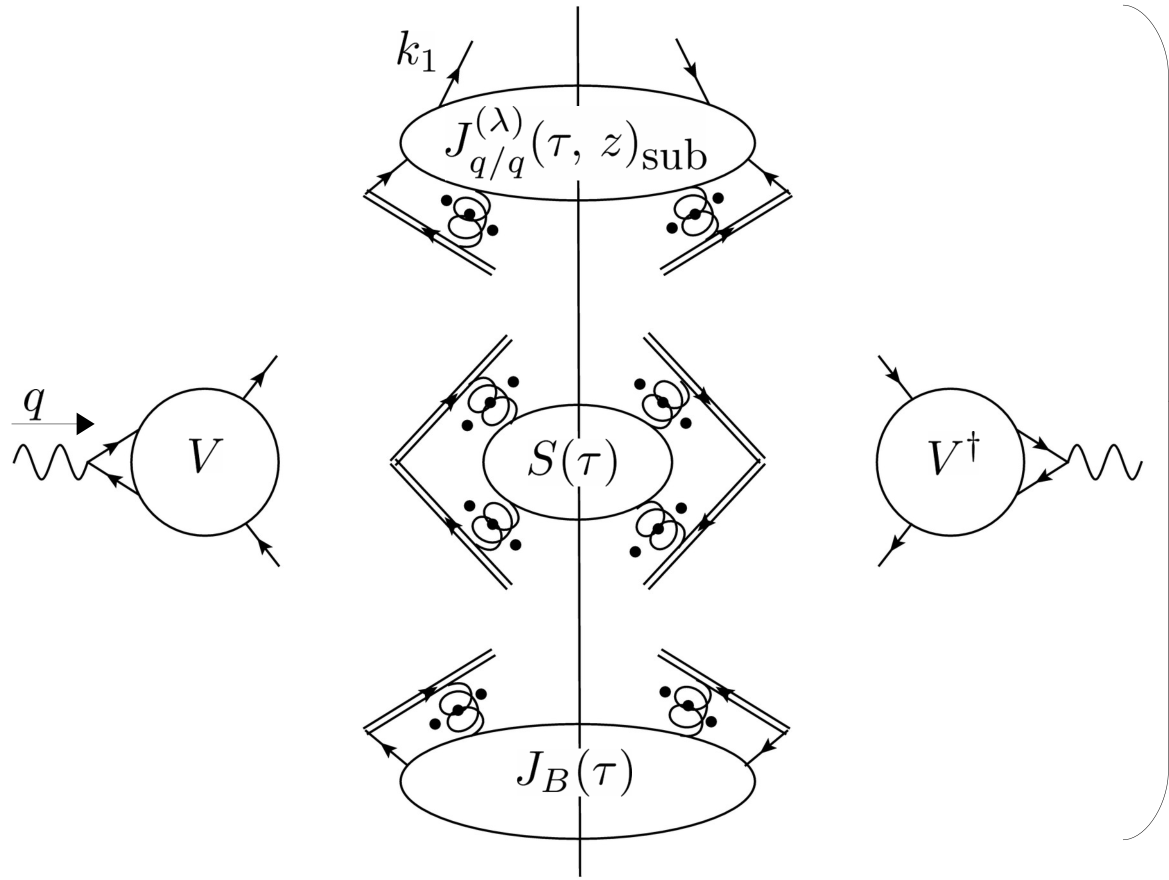
$$\frac{d\widehat{\sigma}_f^{[1]}}{dz\,dT} = \overbrace{\sigma_B}^{\frac{4\pi\alpha^2}{3Q^2}} z\, N_C\, e_f^2 \Big[\delta(1-z)\Big(\delta(\tau)\,\overbrace{V^{[1]}(\epsilon)}^{\text{Virtual}} + \overbrace{S^{[1]}(\epsilon;\,\tau)}^{\text{Soft}} + \overbrace{J_B^{[1]}(\epsilon;\,\tau)}^{\text{Backward}}\Big) + \\ + \underbrace{J_{q/q}^{[1],\,(\lambda)}(\epsilon;\,\tau,\,z)}_{\text{Coll. to thrust axis}} \underbrace{- z\,\widetilde{D}_{q/q}^{[1],\,(\lambda)}(\epsilon;\,z,\,\zeta)\,\delta(\tau)}_{\text{Subtraction Term}}\Big]$$

Subtraction Term:

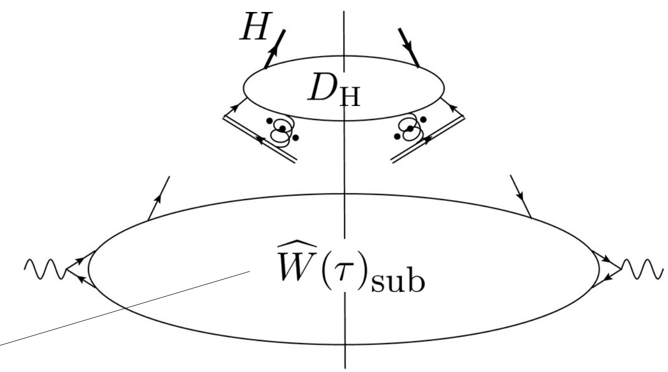
- Cancellation of all poles (collinear, UV)
- Renormalization
- Cancellation of double counting (overlapping between hard and collinear momentum region)

Generalization to ALL ORDERS

Partonic Cross Section



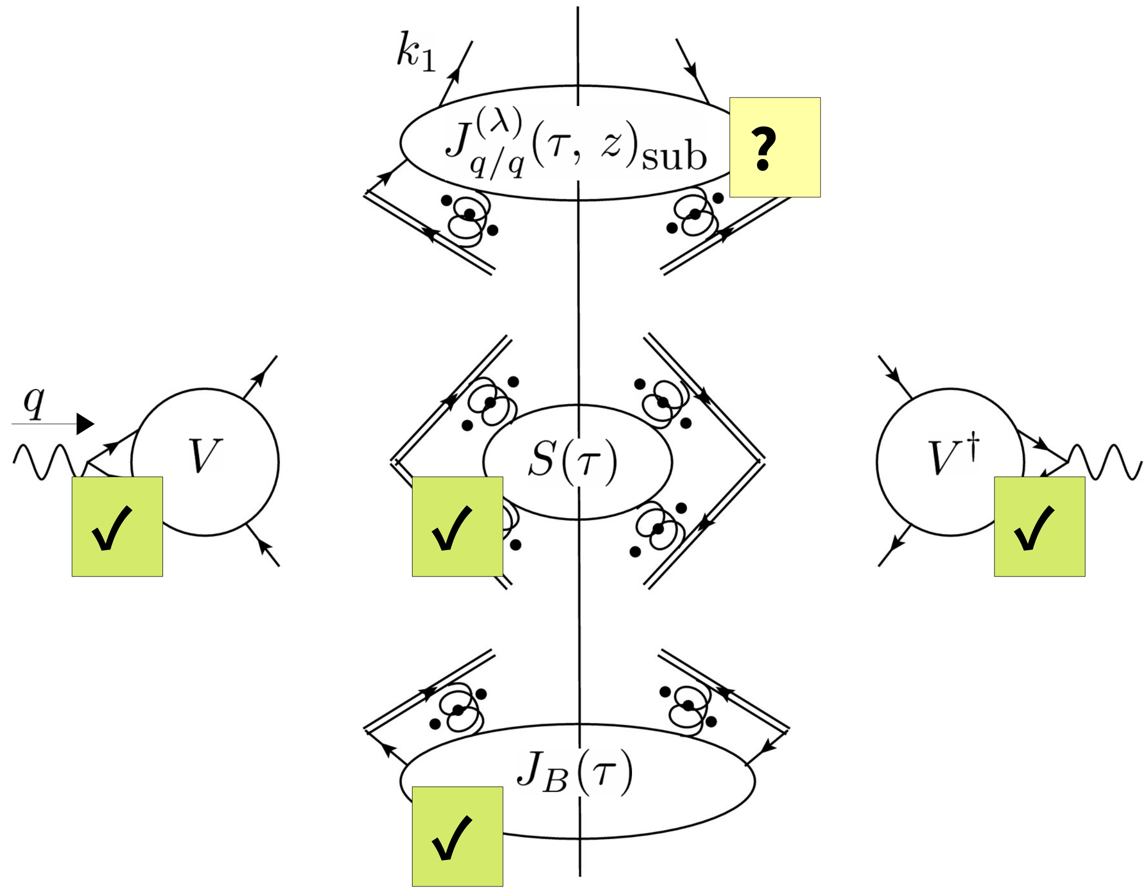
Recall: $\sigma \sim \widehat{\sigma} \otimes D_1$



The partonic final state tensor is totally predicted by pQCD and it encodes soft, backward radiation etc...

$$\frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = \sigma_B\, z\, N_C\, e_f^2\, \underbrace{|V|^2}_{\text{Virtual}} \left(\overbrace{J^{(\lambda)}_{q/q}(\, z,\, \zeta)\Big|_{\rm sub}}^{\text{Coll. to thrust axis}} \otimes \underbrace{J_B}_{\text{Backward}} \otimes \overbrace{S}^{\text{Soft}} \right)(\tau)$$

Partonic Cross Section



Next Step:
RESUMMATION

$$\frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = \sigma_B\, z\, N_C\, e_f^2\, \underbrace{|V|^2}_{\text{Virtual}} \left(\overbrace{J^{(\lambda)}_{q/q}(z,\, \zeta)\Big|_{\text{sub}}}^{\text{Coll. to thrust axis}} \otimes \underbrace{J_B}_{\text{Backward}} \otimes \overbrace{S}^{\text{Soft}} \right)(\tau)$$

Evolution Equations

Four energy scales: Q, μ, ζ and λ :

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial \log \mu} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = -\gamma_D\left(\alpha_S(\mu),\, \zeta/\mu^2\right) \\
&\frac{\partial}{\partial \log \sqrt{\zeta}} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = \frac{1}{2}\, \widehat{K}\left(\alpha_S(\mu),\, \mu^2/\lambda^2\right) \\
&\frac{\partial}{\partial \log \lambda} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda)}{dz\, dT} = G\left(\alpha_S(\mu),\, \mu^2/Q^2,\, \zeta/\mu^2,\, \mu^2/\lambda^2\right)
\end{aligned}
\right.
$$

$\Downarrow$

$$
\begin{aligned}
\frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = \left.\frac{d\widehat{\sigma}_f}{dz\, dT}\right|_{\text{ref.}} &\exp\left\{ \int_\mu^Q \frac{d\mu'}{\mu'}\, \gamma_D\left(\alpha_S(\mu'),\, \zeta/(\mu')^2\right) \right\} \times \\
\times \exp&\left\{ \frac{1}{4}\, \widehat{K}\left(\alpha_S(Q),\, 1\right) \log \frac{\zeta}{Q^2} - \int_\lambda^Q \frac{d\lambda'}{\lambda'}\, G\left(\alpha_S(Q),\, 1,\, \zeta/Q^2,\, Q^2/(\lambda')^2\right) \right\}
\end{aligned}
$$

Evolution Equations

Four energy scales: $Q, \cancel{\mu}, \cancel{\zeta}$ and λ :

$$\frac{\partial}{\partial \log \mu} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = -\gamma_D\left(\alpha_S(\mu),\, \zeta/\mu^2\right)$$

RG-invariance of the whole cross section

$$\frac{\partial}{\partial \log \sqrt{\zeta}} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda,\, \zeta)}{dz\, dT} = \frac{1}{2}\, \widehat{K}\left(\alpha_S(\mu),\, \mu^2/\lambda^2\right)$$

$$\frac{\partial}{\partial \log \lambda} \log \frac{d\widehat{\sigma}_f(\mu,\, \lambda)}{dz\, dT} = G\left(\alpha_S(\mu),\, \mu^2/Q^2,\, \zeta/\mu^2,\, \mu^2/\lambda^2\right)$$

Correct **subtraction mechanism** only if $\zeta = \lambda^2$

$$\frac{d\widehat{\sigma}_f(\cancel{\mu},\, \lambda,\, \cancel{\zeta})}{dz\, dT} = \left.\frac{d\widehat{\sigma}_f}{dz\, dT}\right|_{\text{ref.}} \exp\left\{\int_\mu^Q \frac{d\mu'}{\mu'}\, \gamma_D\left(\alpha_S(\mu'),\, \zeta/(\mu')^2\right)\right\} \times$$

$$\times \exp\left\{\frac{1}{4}\, \widehat{K}\left(\alpha_S(Q),\, 1\right) \log \frac{\overset{\lambda^2}{\cancel{\zeta}}}{Q^2} - \int_\lambda^Q \frac{d\lambda'}{\lambda'}\, G\left(\alpha_S(Q),\, 1,\, \overset{\lambda^2}{\cancel{\zeta}}/Q^2,\, Q^2/(\lambda')^2\right)\right\}$$

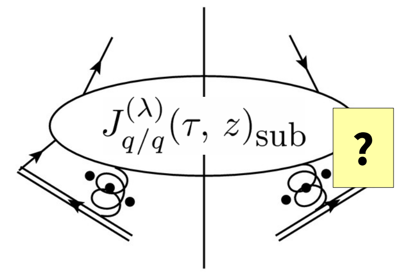
Evolution Equations

$$\frac{d\widehat{\sigma}_f}{dz\,dT} = \left.\frac{d\widehat{\sigma}_f}{dz\,dT}\right|_{\text{ref.}} \exp\left\{\frac{1}{4}\,\widehat{K}\left(\alpha_S(Q)\right)\,\log\frac{\lambda^2}{Q^2} - \int_\lambda^Q \frac{d\lambda'}{\lambda'}\,G\left(\alpha_S(Q),\,Q^2/(\lambda')^2\right)\right\}$$

1-loop

$$\exp\left\{-\frac{\alpha_S(Q)}{4\pi}\,3C_F\log^2\frac{\lambda^2}{Q^2} + \mathcal{O}\left(\alpha_S(Q)^2\right)\right\}$$

- Expected suppression as $\lambda \to 0$.
- Not a rigorous resummation



NLO

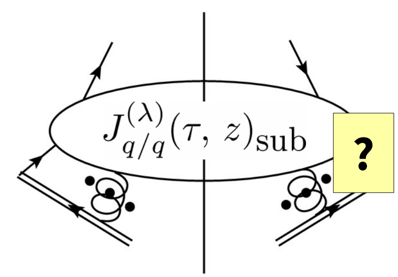
$$\sigma_B\,e_f^2\,N_C\Bigg(\delta(1-z)\,\delta(\tau)+$$
$$+\frac{\alpha_S(Q)}{4\pi}\,2\,C_F\left\{\delta(1-z)\left[\delta(\tau)\left(-\frac{9}{2}+\frac{\pi^2}{3}\right)-\frac{3}{2}\left(\frac{1}{\tau}\right)_+ - 4\left(\frac{\log\tau}{\tau}\right)_+\right]+\right.$$
$$\left.+\,2\left[-\frac{z}{1-z}\,\log z - \log(1-z) + \left(\frac{\log(1-z)}{1-z}\right)_+\right]\delta(\tau)\right\} + \mathcal{O}\left(\alpha_S(Q)^2\right)\Bigg)$$

Partonic Cross Section at NLO (ref. scale)

$$\left.\frac{d\widehat{\sigma}_f}{dz\,dT}\right|_{\text{ref.}} \overset{\text{NLO}}{=} \sigma_B\, e_f^2\, N_C \Bigg(\delta(1-z)\,\delta(\tau) + \\ + \frac{\alpha_S(Q)}{4\pi}\, 2\, C_F \Bigg\{ \delta(1-z) \left[\delta(\tau) \left(-\frac{9}{2} + \frac{\pi^2}{3} \right) - \frac{3}{2} \left(\frac{1}{\tau} \right)_+ - 4 \left(\frac{\log\tau}{\tau} \right)_+ \right] + \\ + 2 \left[-\frac{z}{1-z}\log z - \log(1-z) + \left(\frac{\log(1-z)}{1-z} \right)_+ \right] \delta(\tau) \Bigg\} + \mathcal{O}\left(\alpha_S(Q)^2\right) \Bigg)$$

Written in terms of τ-distributions → Pheno requires functions

Solution: RESUMMATION in both z and τ



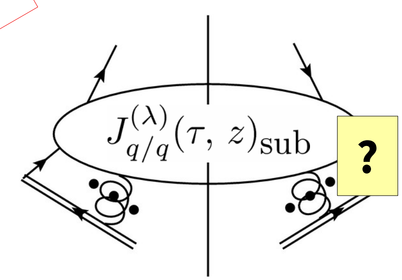
Partonic Cross Section at NLO (ref. scale)

$$\frac{d\widehat{\sigma}_f}{dz\,dT}\Big|_{\text{ref.}} \overset{\text{NLO}}{=} \sigma_B\, e_f^2\, N_C \Bigg(\delta(1-z)\,\delta(\tau) + \\ + \frac{\alpha_S(Q)}{4\pi}\, 2\, C_F \Bigg\{ \delta(1-z) \Bigg[\delta(\tau)\left(-\frac{9}{2}+\frac{\pi^2}{3}\right) - \frac{3}{2}\left(\frac{1}{\tau}\right)_+ - 4\left(\frac{\log\tau}{\tau}\right)_+ \Bigg] + \\ + 2\left[-\frac{z}{1-z}\log z - \log(1-z) + \left(\frac{\log(1-z)}{1-z}\right)_+ \right]\delta(\tau) \Bigg\} + \mathcal{O}\left(\alpha_S(Q)^2\right) \Bigg)$$

Written in terms of τ-distributions → Pheno requires functions

Solution: RESUMMATION in both

DIFFICULT TASK



Shortcut: neglect $\tau = 0$

Partonic Cross Section at NLO

$$\frac{d\widehat{\sigma}_f}{dz\,dT} = \left[-\sigma_B e_f^2 N_C \frac{\alpha_S(Q)}{4\pi} C_F \delta(1-z)\left[\frac{3+8\log\tau}{\tau}\right] + \mathcal{O}\left(\alpha_S(Q)^2\right)\right] e^{-\frac{\alpha_S(Q)}{4\pi}\,3C_F\,\log^2\frac{\lambda^2}{Q^2}+\mathcal{O}\left(\alpha_S(Q)^2\right)}$$

- The limit $T=1$ cannot be reached
- The z-dependence is compromised, especially at large T (not drastic misbehavior up to $T \leq 0.9$)

→ Pheno in the range: $0.7 \leq T \leq 0.9$

Partonic Cross Section at NLO

$$\frac{d\widehat{\sigma}_f}{dz\,dT} = \left[-\sigma_B e_f^2 N_C \frac{\alpha_S(Q)}{4\pi} C_F \delta(1-z)\left[\frac{3+8\log\tau}{\tau}\right] + \mathcal{O}\left(\alpha_S(Q)^2\right)\right] e^{-\frac{\alpha_S(Q)}{4\pi}\,3C_F\log^2\frac{\lambda^2}{Q^2}+\mathcal{O}\left(\alpha_S(Q)^2\right)}$$

- The limit $T=1$ cannot be reached
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Pheno in the range: $0.7 \leq T \leq 0.9$

Almost finished...we still have to fix λ! Remember: $\boxed{k_T \leq \lambda}$

But k_T is naturally constrained by kinematics: $\boxed{k_T \leq \sqrt{\tau}Q}$

Partonic Cross Section at NLO

$$\frac{d\widehat{\sigma}_f}{dz\,dT} = \left[-\sigma_B e_f^2 N_C \frac{\alpha_S(Q)}{4\pi} C_F \delta(1-z)\left[\frac{3+8\log\tau}{\tau}\right] + \mathcal{O}\left(\alpha_S(Q)^2\right)\right] e^{-\frac{\alpha_S(Q)}{4\pi}\,3C_F(\log\tau)^2 + \mathcal{O}\left(\alpha_S(Q)^2\right)}$$

- The limit $T = 1$ cannot be reached
- The z-dependence is compromised, especially at large T (not drastic misbehavior up to $T \leq 0.9$)

Pheno in the range: $0.7 \leq T \leq 0.9$

Almost finished...we still have to fix λ! Remember: $\boxed{k_T \leq \lambda}$

But k_T is naturally constrained by kinematics: $\boxed{k_T \leq \sqrt{\tau}Q}$

$\boxed{\lambda = \sqrt{\tau}Q}$

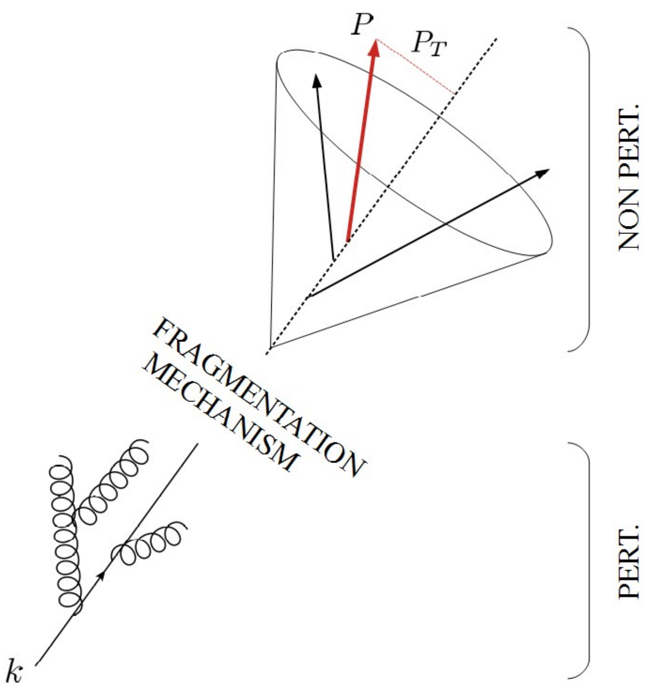
Now we are ready for phenomenology!

Final Results

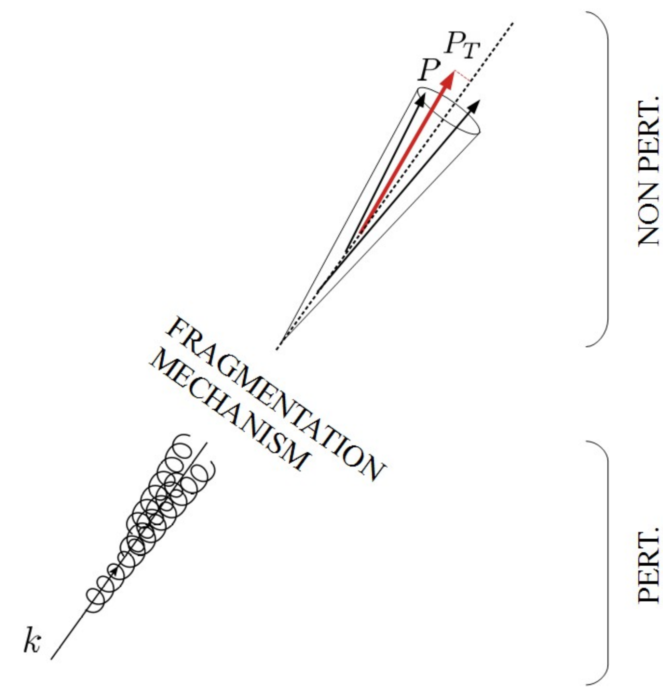
$$\frac{d\sigma}{dz_h\,dT\,dP_T^2} = \pi \sum_f \int_{z_h}^{1} \frac{dz}{z}\, \overbrace{\frac{d\widehat{\sigma}_f}{dz_h/z\,dT}}^{\textbf{NLO}}\, \overbrace{D_{1,\,\pi^\pm/f}(z,\, P_T,\, Q,\, \underbrace{(1-T)\,Q^2})}^{\textbf{NLL}}$$

Only fermions,
the fragmenting gluon
is suppressed by $\mathcal{O}(1-T)$

The TMD FF acquires a
dependence on thrust
through its rapidity cut-off.



$$\xrightarrow[T\,\to\,1]{\text{2-jet limit}}$$



Final Results

$$\frac{d\sigma}{dz_h\, dT\, dP_T^2} = \pi \sum_f \int_{z_h}^1 \frac{dz}{z} \overbrace{\frac{d\widehat{\sigma}_f}{dz_h/z\, dT}}^{\textbf{NLO}} \overbrace{D_{1,\,\pi^\pm/f}(z,\, P_T,\, Q,\, (1-T)\,Q^2)}^{\textbf{NLL}}$$

Fourier Transform of:

Collinear FFs, **NNFF10NLO**

NNPDFCollaboration,
V. Bertone, S. Carrazza, N. P. Hartland,
E. R. Nocera, and J. Rojo,
A,Eur. Phys. J. C77(2017), no. 8 516

$$\begin{aligned}\widetilde{D}_{1,\,\pi^\pm/f}(z,\, b_T;\, Q,\, \tau Q^2) &= \frac{1}{z^2} \sum_k \left[\boxed{d_{\pi^\pm/k}} \otimes \mathcal{C}_{k/f}\right](\mu_b) \times \\ &\times \exp\left\{\frac{1}{4}\,\widetilde{K}\,\log\frac{\tau Q^2}{\mu_b^2} + \int_{\mu_b}^{Q} \frac{d\mu'}{\mu'}\left[\gamma_D - \frac{1}{4}\,\gamma_K\,\log\frac{\tau Q^2}{\mu'^2}\right]\right\} \times \\ &\times \underbrace{(M_D)_{f,\,\pi^\pm}(z,\, b_T)}\,\exp\left\{-\frac{1}{4}\,\underbrace{g_K(b_T)}\,\log\left(\tau\,\frac{z_h^2\, Q^2}{M_H^2}\right)\right\}\end{aligned}$$

Computed up to NLL in pQCD

Non-Perturbative content

The very soul of the specific TMD FF

Universal, independent of the TMD definition used

Final Results

Non-perturbative functions:

$g_K(b_T) = a\, b_T^2$ $\longrightarrow$ **Quadratic behavior** (common choice)

From past extractions: $0.01\ \mathrm{GeV}^2 \leq a \leq 0.1\ \mathrm{GeV}^2$

Our choice: $a = 0.05\ \mathrm{GeV}^2$

$$(M_D)_{f,\,\pi^\pm}(z,\, b_T) \equiv M_D(b_T) = \frac{2^{2-p}}{\Gamma(p-1)}\,(b_T\, m)^{p-1}\, K_{p-1}(b_T\, m)$$

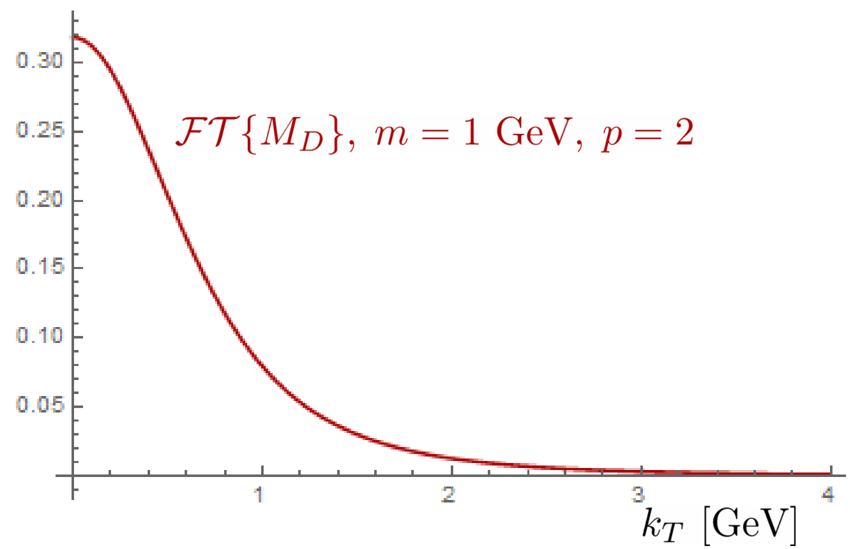
Power-law model

$$\mathcal{FT}\{M_D\} = \frac{\Gamma(p)}{\pi\,\Gamma(p-1)}\,\frac{m^{2(p-1)}}{(k_T^2 + m^2)^p}$$

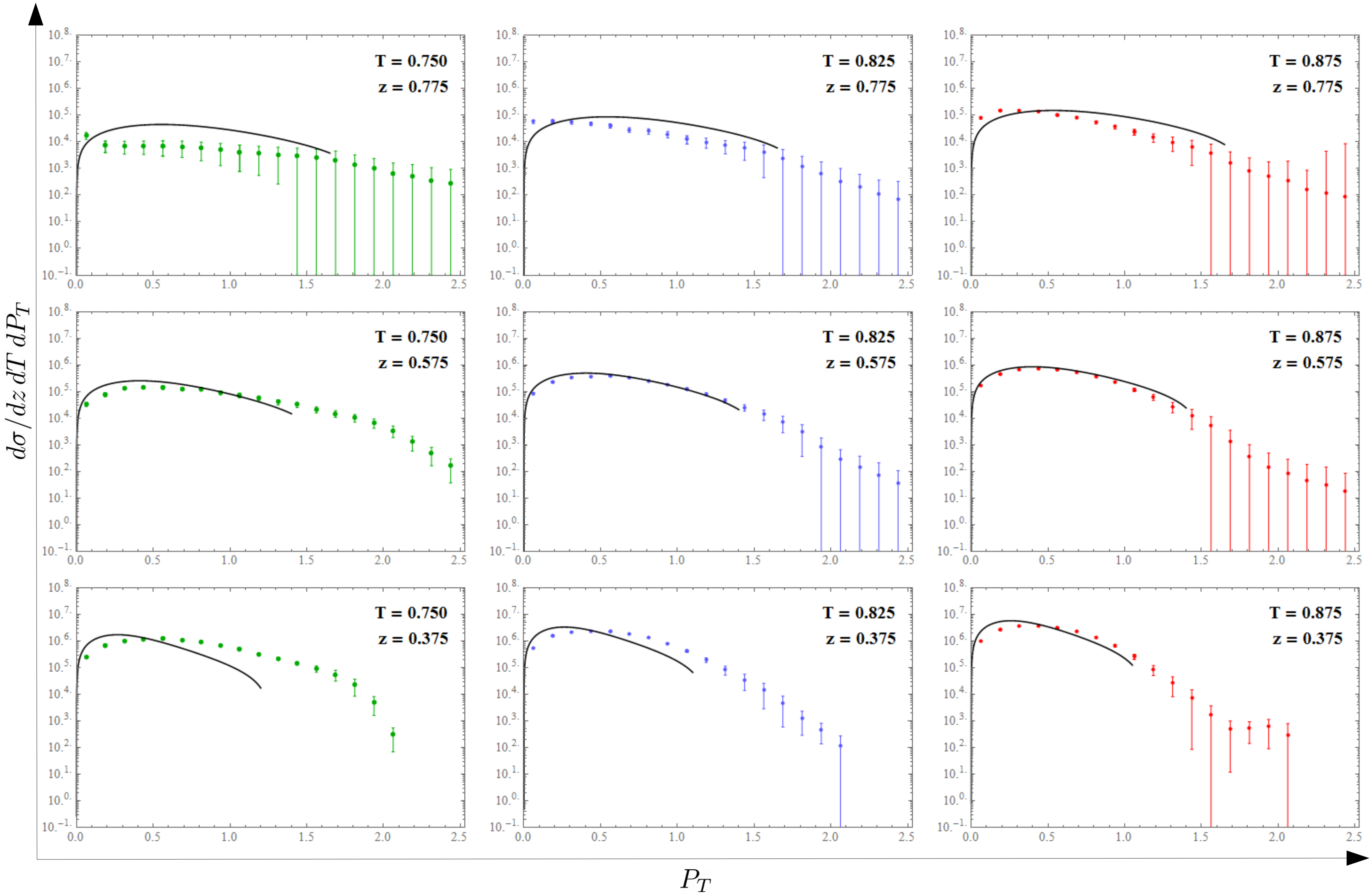
Common sense:

$m = 1\ \mathrm{GeV}$ generic hadronic mass

$p = 2$ propagator squared

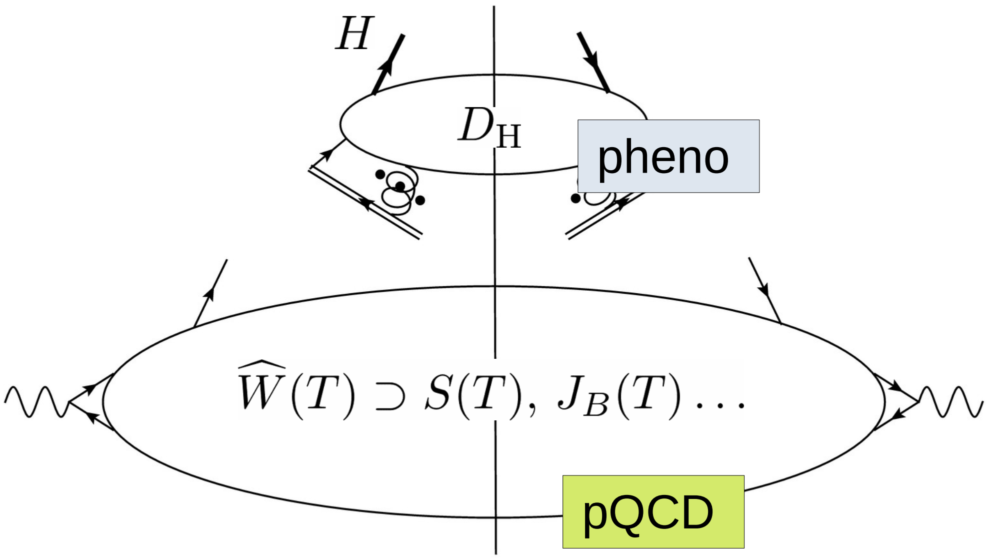


Final Result: Comparison with BELLE data (no fit)

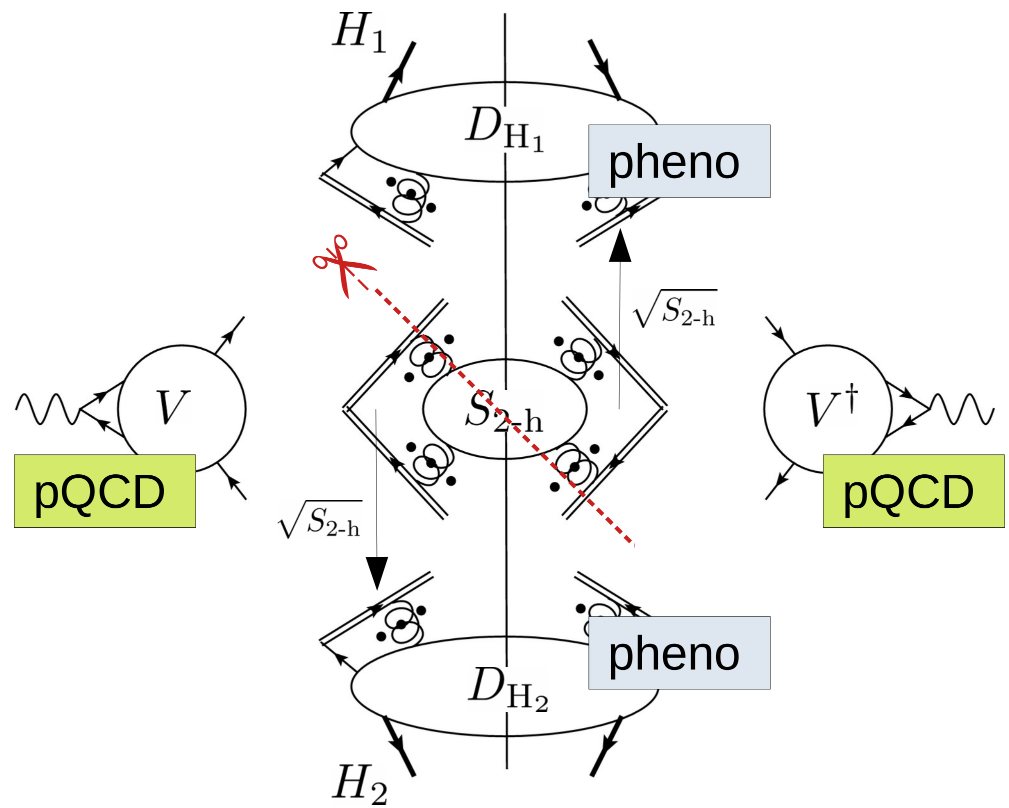


Single- vs Double-Hadron

$$e^+e^- \to H\,X \quad (T \sim 1)$$



$$e^+e^- \to H_1\,H_2\,X$$



Different TMDs !

$$\widetilde{D}^{\text{sqrt}}_{H_1/f} = \widetilde{D}_{1,\,H/f}\,\sqrt{M_S}$$

Soft Model

- Long-distance behavior of the 2-h Soft Factor $S_{\text{2-h}}$.
- Pivotal role of Soft Factor, see talk by A. Vladimirov, 09/12.

Conclusions and Future Remarks

M. Boglione and A. Simonelli arXiv:2007.13674

M. Boglione and A. Simonelli arXiv:2011.07366

- We have factorized (CSS) the cross section of $e^+e^- \to H\,X$, differential in z_h, P_T and T.

- We have obtained a very good agreement with BELLE data (only three parameters fixed to sensible values – no fit)

- The TMD FFs are defined differently from the usual square root definition:

$$\underbrace{\widetilde{D}^{\text{sqrt}}_{H_1/f}}_{\text{Pheno of past 20 years}} = \underbrace{\widetilde{D}_{1,\,H/f}}_{\text{Pheno on BELLE data}} \boxed{\sqrt{M_S}}$$

Remaining unknown

- Promising applications in SIDIS and $e^+e^- \to H_1\,H_2\,X$, with back-to-back hadrons, 3-h class processes, e.g. $\ell\,h \to \ell'\,J_1\,J_2\,X$ (see talk by F. Del Castillo, 08/12 Room 8), Λ-production in e^+e^- (single- and double-hadron), etc...

THANK YOU FOR YOUR ATTENTION!

Andrea Simonelli,
Università degli Studi di Torino

In collaboration with M. Boglione

BACKUP SLIDES

Definition of TMDs: Building Blocks

DEFINITIONS	RAPIDITY INTERVAL
Unsubtracted TMD FF: $\widetilde{D}^{(0),\,\text{unsub}}_{1,\,H/f}(z,\,b_T;\,\mu,\,y_P,\,-\infty) = \frac{1}{z}\sum_X \langle P\,(H),\,X;\,\text{out}\vert\overline{\psi}_f(-x/2)\,W_q(-x/2,\infty;\,n_1(y_1)\,)^\dagger\vert 0\rangle \langle 0\vert W_q(x/2,\infty;\,w_-\,)\,\psi_f(x/2)\vert P\,(H),\,X;\,\text{out}\rangle\,\vert_{\text{NO S.I.}}$	$-\infty \leq y \leq y_P\ (\sim +\infty)$
2-h Soft Factor: $\widetilde{\mathbb{S}}^{(0)}_{\text{2-h}}(b_T;\,\mu,\,y_1-y_2) = \frac{\text{Tr}_C}{N_C}\,\langle 0\vert W(-\vec{b}_T/2,\,\infty;\,n_1(y_1)\,)^\dagger\,W(\vec{b}_T/2,\,\infty;\,n_1(y_1)\,)\,W(\vec{b}_T/2,\,\infty;\,n_2(y_2)\,)^\dagger\,W(-\vec{b}_T/2,\,\infty;\,n_2(y_2)\,)\vert 0\rangle\,\vert_{\text{NO S.I.}}$	$y_2 \leq y \leq y_1$

Where:

- b_T is the variable Fourier conjugate to transverse momentum.
- (0) - label reminds that the quantities are *bare* (need for UV renormalization).

Definition of TMDs: Soft Subtraction

DEFINITIONS	RAPIDITY INTERVAL
■ Factorization Definition (see $e^+e^- \to H\,X$, $T \sim 1$) $$\widetilde{D}_{1,\,H/f}(z,\,b_T;\,\mu,\,y_P-y_1) = = Z_j(\mu,\,y_P-y_1) Z_2\left(\alpha_S(\mu)\right) \times \lim_{y_{u_2}\to-\infty} \frac{\widetilde{D}^{(0),\,\mathrm{unsub}}_{1,\,H/f}(z,\,b_T;\,\mu,\,y_P-y_{u_2})}{\mathbb{S}^{(0)}_{\text{2-h}}(b_T;\,\mu,\,y_1-y_{u_2})}$$ UV ct.	$y_1 \le y \le y_P \ (\sim +\infty)$
■ Square Root Definition (see $e^+e^- \to H_1\,H_2\,X$, back-to-back) $$\widetilde{D}^{\mathrm{sqrt}}_{H_1/f}(z,\,b_T;\,\mu,\,y_P-y_1) = = Z_j(\mu,\,y_P-y_1) Z_2\left(\alpha_S(\mu)\right) \times \lim_{\substack{y_{u_1}\to+\infty \\ y_{u_2}\to-\infty}} \widetilde{D}^{(0),\,\mathrm{unsub}}_{1,\,H/f}(z,\,b_T;\,\mu,\,y_P-y_{u_2}) \times \sqrt{\frac{\widetilde{\mathbb{S}}_{\text{2-h}}(b_T;\,\mu,\,y_{u_1}-y_1)}{\widetilde{\mathbb{S}}_{\text{2-h}}(b_T;\,\mu,\,y_{u_1}-y_{u_2})\,\widetilde{\mathbb{S}}_{\text{2-h}}(b_T;\,\mu,\,y_1-y_{u_2})}}$$ UV ct.	$y_1 \le y \le y_P \ (\sim +\infty)$

Single-Hadron Cross Section

$$\frac{d\sigma}{dz_h\, dT\, dP_T^2} = z_h \, \frac{\alpha^2}{4Q^4} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, L_{\mu\nu}(\theta) \, \frac{dW_H^{\mu\,\nu}(z_h,\, T,\, P_T)}{dP_T^2}$$

Leptonic Tensor (LO in QED):

$$L^{\mu\nu}(\theta) = l_1^{\mu}\, l_2^{\nu} + l_2^{\mu}\, l_1^{\nu} - g^{\mu\nu}\, l_1 \cdot l_2$$

Hadronic Tensor:

$$\begin{aligned} W_H^{\mu\,\nu}(z_h,\, T,\, P_T) &= 4\pi^3 \sum_X \delta^{(4)}\left(p_X + P - q\right) \times \\ &\times \langle 0|\, j^{\mu}(0) |\, P,\, X,\, \mathrm{out}\,\rangle_T\, {}_T\langle P,\, X,\, \mathrm{out}| j^{\nu}(0) |\, 0\rangle = \\ &= \frac{1}{4\pi} \sum_X \int d^4 z\, e^{iq\cdot z} \langle 0|\, j^{\mu}\left(z/2\right) |\, P,\, X,\, \mathrm{out}\,\rangle_T\, {}_T\langle P,\, X,\, \mathrm{out}| j^{\nu}\left(-z/2\right) |\, 0\rangle \end{aligned}$$

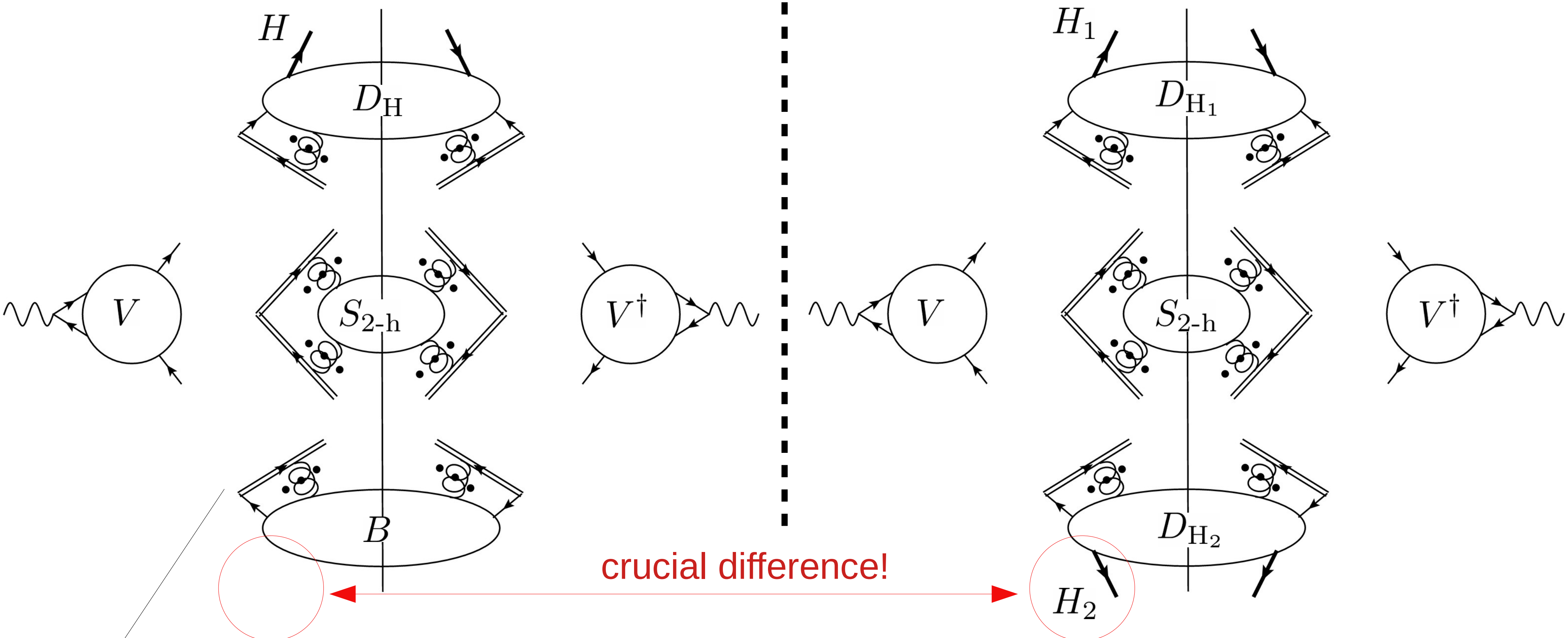
Normalization

J. Collins, *Foundations of perturbative QCD*.

CSS factorization (naive)

$e^+e^- \to H\,X \quad (T \sim 1)$

$e^+e^- \to H_1\,H_2\,X$



$\bar{q}$ plays the role of a hard real emission.

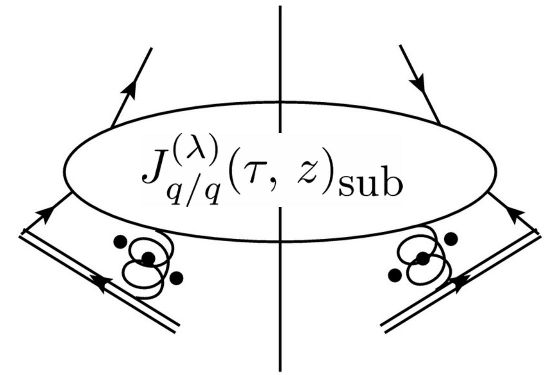
J. Collins, *Foundations of perturbative QCD*.

- B can be considered a “hard part”.
- $S_{2\text{-h}}$ turns out to be unity ($\mathbb{1}$).

REARRANGEMENT OF BLOBS in the SINGLE-HADRON case

Partonic Cross Section: Subtraction Mechanism

In the partonic cross section,
the contribution associated to the radiation collinear to the fragmenting quark is given by:

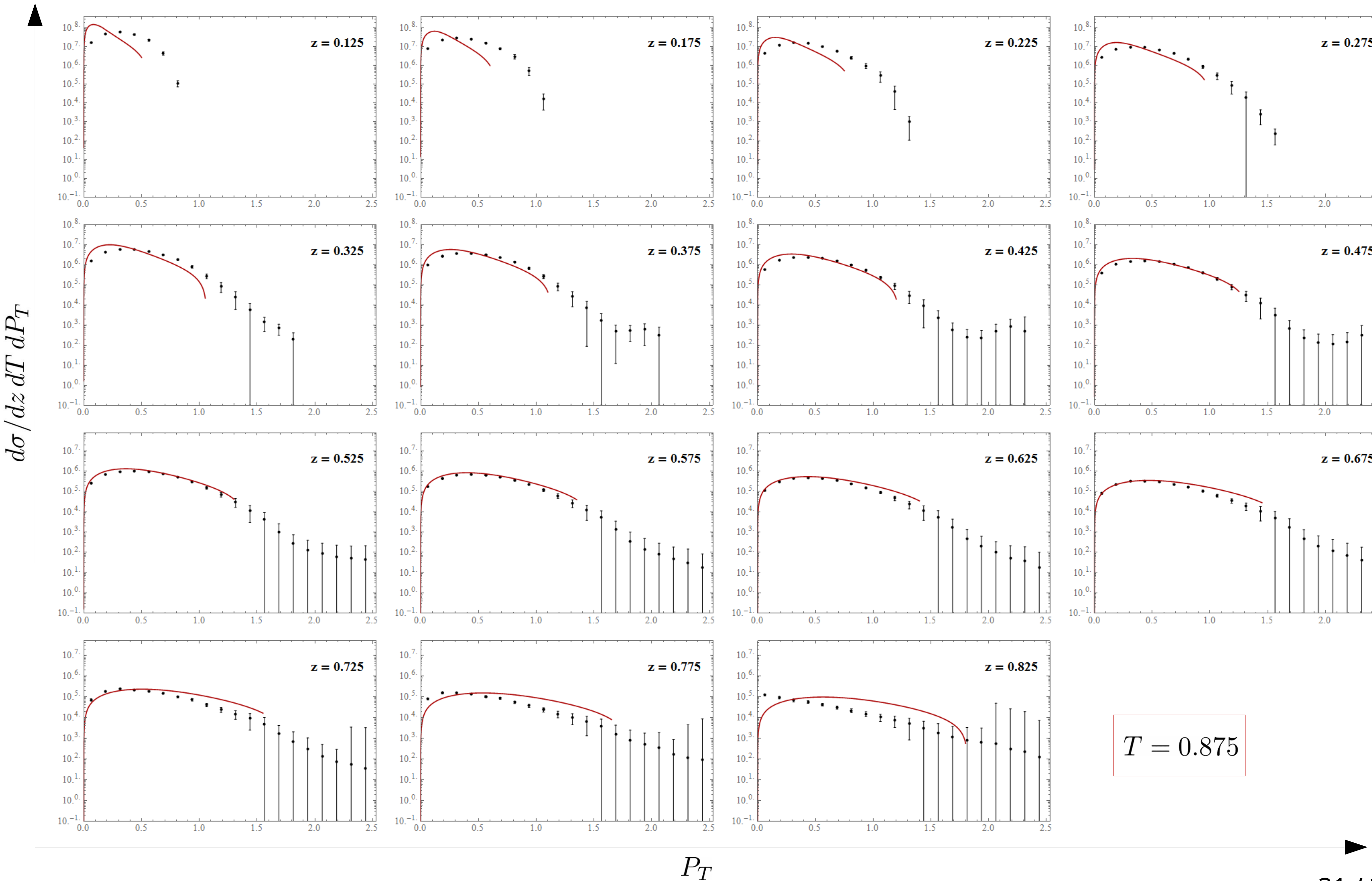


1 loop

	UNSUBTRACTED	SUBTRACTION TERM	
DEFINITIONS	$J^{[1],\,(\lambda)}_{q/q}(\epsilon;\, \tau,\, z)$	$-z\, \widetilde{D}^{[1],\,(\lambda)}_{q/q}(\epsilon;\, z,\, \zeta)\, \delta(\tau)$	
TRANSVERSE MOMENTUM INTERVAL	$0 \leq k_T \leq \lambda$	$0 \leq k_T \leq \lambda$	Matched
RAPIDITY INTERVAL	$\frac{1}{2} \log \frac{2(k^+)^2}{\lambda^2} \leq y \leq +\infty$	$\frac{1}{2} \log \frac{2(k^+)^2}{\zeta} \leq y \leq +\infty$	Matched only if $\zeta = \lambda^2$ $(\tau = 0)$

$\zeta = \lambda^2$ ensures a correct subtraction between "hard" and collinear momentum regions.

Final Result: Comparison with BELLE data (no fit)



Final Result: Comparison with BELLE data (no fit)