QCD Resummations in the Lattice Calculation of PDFs

Resummation, Evolution, Factorization 2020
December 7—11, 2020

YONG ZHAO

DEC. 7, 2020

BROOKHAVEN NATIONAL LABORATORY

TMD Collaboration

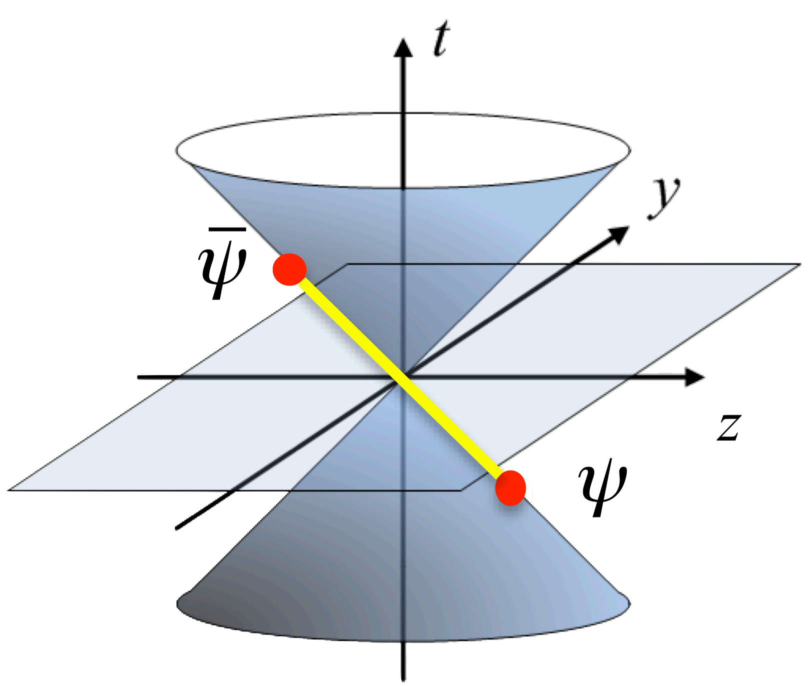
In collaboration with Xiang Gao (Tsinghua U. & BNL), Kyle Lee (LBNL) and Swagato Mukherjee (BNL), in preparation.

Outline

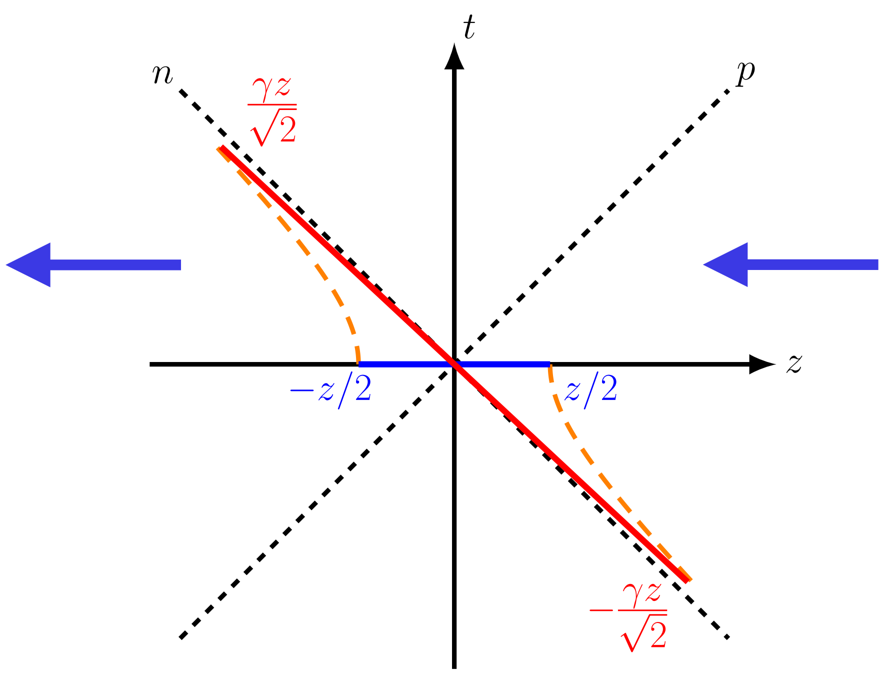
- Factorization formula for lattice calculation of PDFs
  - Large-momentum effective theory expansion
  - Short-distance expansion in coordinate space
- Threshold Resummation
  - Origin of threshold logarithms
  - Resummation formula in the Mellin moment space
- Implementation on lattice data

Large-Momentum Effective Theory (LaMET)

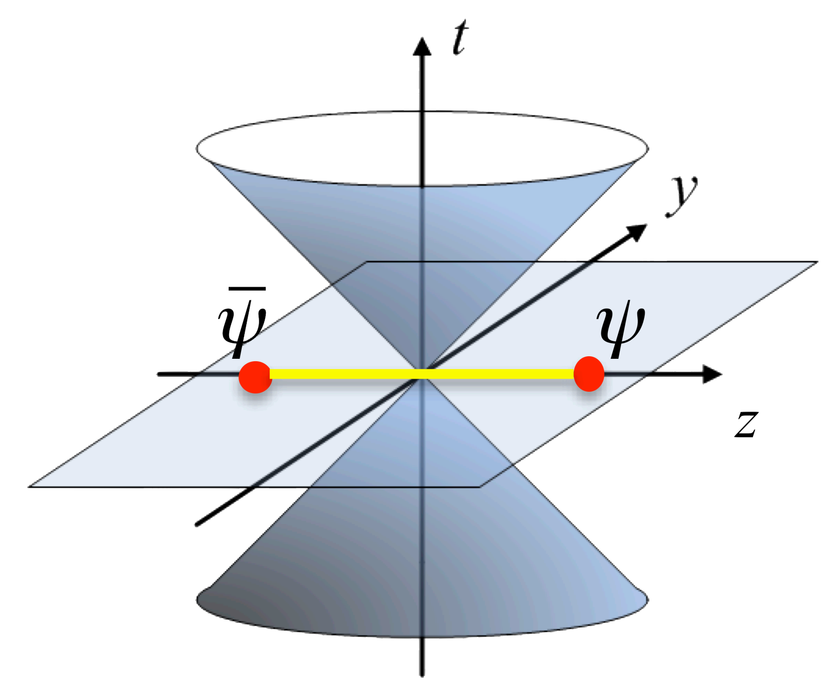
$z + ct = 0, \;\; z - ct \neq 0$



Related by Lorentz boost



$t = 0, \;\; z \neq 0$



PDF $f(x)$:
Cannot be calculated on the lattice

$$f(x) = \int \frac{dz^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(z^-) \times \frac{\gamma^+}{2} W[z^-,0] \psi(0) | P \rangle$$

- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{iz(xP^z)} \langle P | \bar{\psi}(z) \times \frac{\gamma^z}{2} W[z,0] \psi(0) | P \rangle$$

Large-momentum effective theory

- Large-momentum expansion:

$$f(x,\mu)=\int_{-\infty}^{\infty}\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{yP^z}\right)\tilde{f}(y,P^z,\mu)+\mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{(xP^z)^2},\frac{\Lambda^2_{\rm QCD}}{((1-x)P^z)^2}\right)$$

  - X. Ji, PRL 110 (2013); SCPMA57 (2014);
  - X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
  - Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).
  - X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.
  - X. Ji, YZ, et al., arXiv: 2008.03886.

- One-loop matching coefficient:

$$C^{(1)}\left(\xi,\frac{\mu}{yP^z}\right)=-\frac{\alpha_sC_F}{2\pi}\delta(1-\xi)\left[\frac{3}{2}\ln\frac{\mu^2}{4x^2P_z^2}+\frac{5}{2}\right]$$

$\xi=\dfrac{x}{y}$

$$-\frac{\alpha_sC_F}{2\pi}\begin{cases}\left(\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{\xi}{\xi-1}+1\right)_+ & \xi>1\\ \left(\dfrac{1+\xi^2}{1-\xi}\left[-\ln\dfrac{\mu^2}{y^2P_z^2}+\ln\left(4\xi(1-\xi)\right)-1\right]+1\right)_+ & 0<\xi<1\\ \left(-\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{-\xi}{1-\xi}-1\right)_+ & \xi<0\end{cases}$$

Large-momentum effective theory

- Large-momentum expansion:

$$f(x,\mu)=\int_{-\infty}^{\infty}\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{yP^z}\right)\tilde{f}(y,P^z,\mu)+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(xP^z)^2},\frac{\Lambda_{\rm QCD}^2}{((1-x)P^z)^2}\right)$$

  - X. Ji, PRL 110 (2013); SCPMA57 (2014);
  - X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
  - Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).
  - X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.
  - X. Ji, YZ, et al., arXiv: 2008.03886.

- One-loop matching coefficient

$$\ln\frac{\mu^2}{y^2P_z^2}=\ln\frac{\mu^2}{x^2P_z^2}+\ln\frac{x^2}{y^2}$$

$$C^{(1)}\left(\xi,\frac{\mu}{yP^z}\right)=-\frac{\alpha_sC_F}{2\pi}\delta(1-\xi)\left[\frac{3}{2}\ln\frac{\mu^2}{4x^2P_z^2}+\frac{5}{2}\right]$$

$$\xi=\frac{x}{y}$$

$$-\frac{\alpha_sC_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1\right)_+ & \\ \left(\frac{1+\xi^2}{1-\xi}\left[-\ln\frac{\mu^2}{y^2P_z^2}+\ln\left(4\xi(1-\xi)\right)-1\right]+1\right)_+ & 0<\xi<1\\ \left(-\frac{1+\xi^2}{1-\xi}\ln\frac{-\xi}{1-\xi}-1\right)_+ & \xi<0\end{cases}$$

Similar to DGLAP evolution:

$$\frac{dC(\xi,\mu/(xP^z))}{d\ln(xP^z)}=\frac{\alpha_sC_F}{\pi}\left[P_{qq}^{(0)}(\xi)-\frac{3}{2}\delta(1-\xi)\right]$$

X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.

Large-momentum effective theory

- Large-momentum expansion:

$$f(x,\mu)=\int_{-\infty}^{\infty}\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{yP^z}\right)\tilde{f}(y,P^z,\mu)+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(xP^z)^2},\frac{\Lambda_{\rm QCD}^2}{((1-x)P^z)^2}\right)$$

- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
- Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.
- X. Ji, YZ, et al., arXiv: 2008.03886.

- One-loop matching coefficient:

$$C^{(1)}\left(\xi,\frac{\mu}{yP^z}\right)=-\frac{\alpha_sC_F}{2\pi}\delta(1-\xi)\left[\frac{3}{2}\ln\frac{\mu^2}{4x^2P_z^2}+\frac{5}{2}\right]$$

$\xi=\frac{x}{y}$

$$-\frac{\alpha_sC_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1\right)_+ & \xi>1\\ \left(\frac{1+\xi^2}{1-\xi}\left[-\ln\frac{\mu^2}{y^2P_z^2}+\ln\left(4\xi(1-\xi)\right)-1\right]+1\right)_+ & 0<\xi<1\\ \left(-\frac{1+\xi^2}{1-\xi}\ln\frac{-\xi}{1-\xi}-1\right)_+ & \xi<0\end{cases}$$

Large-momentum effective theory

- Large-momentum expansion:

$$f(x,\mu)=\int_{-\infty}^{\infty}\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{yP^z}\right)\tilde{f}(y,P^z,\mu)+\mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{(xP^z)^2},\frac{\Lambda^2_{\rm QCD}}{((1-x)P^z)^2}\right)$$

- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
- Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.
- X. Ji, YZ, et al., arXiv: 2008.03886.

- One-loop matching coefficient:

$$C^{(1)}\left(\xi,\frac{\mu}{yP^z}\right)=-\frac{\alpha_s C_F}{2\pi}\delta(1-\xi)\left[\frac{3}{2}\ln\frac{\mu^2}{4x^2P_z^2}+\frac{5}{2}\right]$$

$$-\frac{\alpha_s C_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1\right)_+ & \\ \left(\frac{1+\xi^2}{1-\xi}\left[-\ln\frac{\mu^2}{y^2P_z^2}+\ln\left(4\xi(1-\xi)\right)-1\right]+1\right)_+ & 0<\xi<1\\ \left(-\frac{1+\xi^2}{1-\xi}\ln\frac{-\xi}{1-\xi}-1\right)_+ & \xi<0\end{cases}$$

$\xi=\frac{x}{y}$

$y\to 1,\ x\to 1\Rightarrow \xi=\frac{y}{x}\to 1$

Threshold logarithms:

$$C(\xi,\mu/(yP^z))\sim\frac{\alpha_s C_F}{2\pi}\left[\frac{2\ln|1-\xi|}{|1-\xi|}-\frac{2}{1-\xi}\ln\frac{\mu^2}{P_z^2}-\frac{2}{1-\xi}\right]_+$$

For small *x* logarithms, see Chirilli's talk.

Large-x behavior of the extracted PDF is sensitive to the large threshold logarithms.

The pseudo distribution approach

- Short distance expansion of the spatial correlator:

$$\tilde{h}(\lambda, z^2) = \frac{1}{2P^0}\langle P\,|\,\bar{\psi}(z)\gamma^0 W[z,0]\psi(0)\,|\,P\rangle$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017).

$\lambda = zP^z$

$$\tilde{h}(\lambda, z^2) = \int_{-1}^{1} d\alpha\, \mathscr{C}\left(\alpha, z^2\mu^2\right) \int_{-1}^{1} dx\, e^{ix\alpha\lambda} f(x,\mu) \;+\; \mathcal{O}\left(z^2\Lambda^2_{\rm QCD}\right)$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- X. Ji, J.-H. Zhang and YZ, Nucl.Phys.B 924 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017);
- A. V. Radyushkin, Phys.Lett.B 781 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).

- One-loop matching coefficient:

$$\begin{aligned}
\mathcal{C}^{(1)}(\alpha, z^2\mu^2) = {} & \delta(1-\alpha)\frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} + \frac{5}{2}\right] \\
& + \frac{\alpha_s C_F}{2\pi}\left\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+\left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right]\right. \\
& \left. - \left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)_+\right\}\theta(\alpha)\theta(1-\alpha)
\end{aligned}$$

The pseudo distribution approach

- Short distance expansion of the spatial correlator:

$$\tilde{h}(\lambda, z^2) = \frac{1}{2P^0}\langle P\,|\,\bar{\psi}(z)\gamma^0 W[z,0]\psi(0)\,|\,P\rangle$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017).

$\lambda = zP^z$

$$\tilde{h}(\lambda, z^2) = \int_{-1}^{1} d\alpha\ \mathscr{C}\left(\alpha, z^2\mu^2\right) \int_{-1}^{1} dx\ e^{ix\alpha\lambda} f(x,\mu)\ +\ \mathscr{O}\left(z^2\Lambda^2_{\mathrm{QCD}}\right)$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- X. Ji, J.-H. Zhang and YZ, Nucl.Phys.B 924 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017);
- A. V. Radyushkin, Phys.Lett.B 781 (2018);
- T. Izub

- One-loop matching coefficient:

$$\mathcal{C}^{(1)}(\alpha, z^2\mu^2) = \delta(1-\alpha)\frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} + \frac{5}{2}\right] + \frac{\alpha_s C_F}{2\pi}\left\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+\left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right] - \left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)_+\right\}\theta(\alpha)\theta(1-\alpha)$$

Similar to DGLAP evolution:

$$\frac{d\mathscr{C}(\alpha, \mu^2 z^2)}{d\ln z^2} = \frac{\alpha_s}{2\pi}\left[-P^{(0)}_{qq}(\alpha) - \frac{3}{2}\delta(1-\alpha)\right]$$

A. V. Radyushkin, Phys.Lett.B 781 (2018).

The pseudo distribution approach

- Short distance expansion of the spatial correlator:

$$\tilde{h}(\lambda, z^2) = \frac{1}{2P^0}\langle P\,|\,\bar{\psi}(z)\gamma^0 W[z,0]\psi(0)\,|\,P\rangle$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017).

$\lambda = zP^z$

$$\tilde{h}(\lambda, z^2) = \int_{-1}^{1} d\alpha\ \mathscr{C}\left(\alpha, z^2\mu^2\right) \int_{-1}^{1} dx\ e^{ix\alpha\lambda} f(x,\mu)\ +\ \mathcal{O}\left(z^2\Lambda^2_{\rm QCD}\right)$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- X. Ji, J.-H. Zhang and YZ, Nucl.Phys.B 924 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017);
- A. V. Radyushkin, Phys.Lett.B 781 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).

- One-loop matching coefficient:

$$\begin{aligned}\mathcal{C}^{(1)}(\alpha, z^2\mu^2) &= \delta(1-\alpha)\frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}+\frac{5}{2}\right] \\ &+\frac{\alpha_s C_F}{2\pi}\left\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+\left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}-1\right]\right. \\ &\left.-\left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ +2(1-\alpha)_+\right\}\theta(\alpha)\theta(1-\alpha)\end{aligned}$$

The pseudo distribution approach

- Short distance expansion of the spatial correlator:

$$\tilde{h}(\lambda, z^2) = \frac{1}{2P^0}\langle P|\bar{\psi}(z)\gamma^0 W[z,0]\psi(0)|P\rangle$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017).

$\lambda = zP^z$

$$\tilde{h}(\lambda, z^2) = \int_{-1}^{1} d\alpha\, \mathscr{C}\left(\alpha, z^2\mu^2\right) \int_{-1}^{1} dx\, e^{ix\alpha\lambda} f(x,\mu) + \mathcal{O}\left(z^2\Lambda_{\rm QCD}^2\right)$$

- A. V. Radyushkin, Phys.Rev.D 96 (2017);
- X. Ji, J.-H. Zhang and YZ, Nucl.Phys.B 924 (2017);
- K. Orginos et al., Phys.Rev.D 96 (2017);

$\alpha \to 1$

- One-loop matching coefficie[nt]

Threshold logarithms:

$$\mathscr{C}(\alpha, \mu^2 z^2) \sim \frac{\alpha_s C_F}{2\pi}\left[-\frac{4\ln(1-\alpha)}{1-\alpha} - \frac{2}{1-\alpha}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - \frac{2}{1-\alpha}\right]_+$$

$$\mathcal{C}^{(1)}(\alpha, z^2\mu^2) = \delta(1-\alpha)\frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}\right. + \frac{\alpha_s C_F}{2\pi}\left\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+\left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right] - \left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)_+\right\}\theta(\alpha)\theta(1-\alpha)$$

Threshold effect is easier to see in Mellin moment space.

Outline

- Factorization formula for lattice calculation of PDFs
  - Large-momentum effective theory expansion
  - Short-distance expansion in coordinate space

- Threshold Resummation
  - Origin of threshold logarithms
  - Resummation formula in the Mellin moment space

- Implementation on lattice data

Origin of threshold logarithms

- Deep inelastic scattering

$$\lim_{z\to 1} C_q\left(z, \frac{Q^2}{\mu^2}\right) \sim \frac{\alpha_s C_F}{2\pi}\left[\frac{2\ln(1-z)}{1-z} - \frac{3}{2(1-z)} + \frac{2}{1-z}\ln\frac{Q^2}{\mu^2}\right]_+$$

$$\frac{2}{1-z}\ln\frac{(1-z)Q^2}{\mu^2}$$

- Quasi-PDF

$$\lim_{\xi\to 1} C(\xi, \mu/(yP^z)) \sim \frac{\alpha_s C_F}{2\pi}\left[\frac{2\ln|1-\xi|}{|1-\xi|} - \frac{2}{1-\xi} - \frac{2}{1-\xi}\ln\frac{\mu^2}{P_z^2}\right]_+$$

$$\frac{2}{|1-\xi|}\ln\frac{|1-\xi|P_z^2}{\mu^2}$$

- Spatial correlator

$$\lim_{\alpha\to 1} \mathscr{C}(\alpha, \mu^2 z^2) \sim \frac{\alpha_s C_F}{2\pi}\left[-\frac{4\ln(1-\alpha)}{1-\alpha} - \frac{2}{1-\alpha} - \frac{2}{1-\alpha}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}\right]_+$$

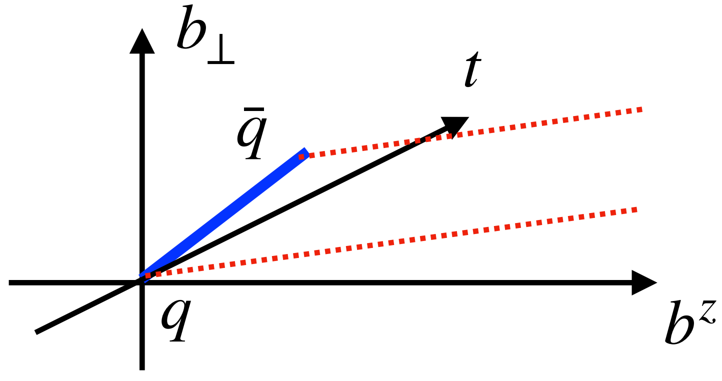
$$\frac{2}{1-\alpha}\ln\frac{4e^{-2\gamma_E}}{(1-\alpha)^2 z^2\mu^2}$$

Origin of threshold logarithms

- 3D momentum density:

$$\tilde{f}(\vec{k}, P^z) = \frac{1}{2P^0}\int \frac{d^3\vec{b}}{2\pi} e^{i\vec{k}\cdot\vec{b}} \langle P|\bar{\psi}(\vec{b})\gamma^0 W(\vec{b},0)\psi(0)|P\rangle$$

- Straight-line gauge link;
- Different from normal TMD distribution with a staple shaped gauge link.



X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.

- Relation to the spatial correlator:

$$\frac{1}{2P^0}\langle P|\bar{\psi}(\vec{b})\gamma^0 W(\vec{b},0)\psi(0)|P\rangle = \tilde{h}(P\cdot b, b^2)$$ A. V. Radyushkin, Phys.Rev.D 96 (2017).

$-P\cdot b = P^z b^z = \lambda,\ b^2 = -z^2$ uniquely determines the spatial correlator.

In the limit of $P^z \to \infty$, $b^z \to 0$, $P^z b^z$ finite,

$$\tilde{h}(\lambda = P^z b^z, b_\perp^2 = -z^2) = \int dx\, e^{-ix\lambda}\int d^2k_\perp\, e^{ik_\perp\cdot b_\perp} \lim_{P^z\to\infty} \tilde{f}(x, k_\perp, P^z)$$

"primordial TMD" distribution

A. V. Radyushkin, Phys.Rev.D 96 (2017).

- Relation to the quasi-PDF:

$$\tilde{f}\left(x = \frac{k^z}{P^z}, P^z\right) \equiv \int d^2k_\perp\, \tilde{f}(\vec{k}, P^z) = \int_{-\infty}^{\infty} dk^x \int_{-1}^{1} dx \lim_{P^z\to\infty} \tilde{f}(x, \sqrt{k_x^2 + (x-y)P_z^2}, P^z)$$

Origin of threshold logarithms

- Leading divergence in the one-loop diagram:

$(1-s)^{1-d}\delta^{(d-1)}\left(\vec{l}-\frac{\vec{k}-s\vec{p}}{1-s}\right)$

$$\tilde{f}(x,k_\perp,P^z) = -2ig^2C_F\frac{\partial}{\partial x}\int_0^1 ds\int\frac{dl^0d^{d-1}\vec{l}}{(2\pi)^d}\frac{l^0+p^z}{l^2(p-l)^2}e^{-i(\vec{p}-\vec{l})\cdot\vec{b}s}e^{-i\vec{l}\cdot\vec{b}}\delta^{(d-1)}((\vec{p}-\vec{l})s+\vec{l}-\vec{k})$$

$k_t^2=\frac{\vec{k}_\perp^2}{p_z^2}$

$$=\frac{2g^2C_F}{4(2\pi)^{d-1}}\int_0^1 ds\,\frac{(1-s)^{2-d}}{k_t^2}\left[\frac{k_t^2(-2s+x+1)+(x-s)^3}{\left(k_t^2+(s-x)^2\right)^{3/2}}-\frac{k_t^2(-2s+x+1)+(x-1)^3}{\left(k_t^2+(x-1)^2\right)^{3/2}}\right]_+$$

$k_t^2\ll 1$

$$=\frac{g^2C_F}{(2\pi)^{d-1}}\frac{1}{k_t^2}\left[\frac{1+x}{1-x}+2\epsilon\frac{\ln(1-x)}{1-x}+2\epsilon\frac{x}{1-x}\right]_+\theta(x)\theta(1-x)$$

Normal TMD has additional term $\delta(1-x)\ln(p_z^2/k_\perp^2)/k_\perp^2$.

Origin of threshold logarithms

- Integration over $k_\perp$:
  - Trivially reduces to the quasi-PDF if $k_t^2 = \vec{k}_\perp^2/p_z^2 \ll 1$ limit not taken;
  - Nontrivial check: $\tilde{f}\left(x = \frac{k^z}{P^z}, P^z\right) = \lim_{P^z\to 0} \int_{-\infty}^{\infty} dk^x \int_{-1}^{1} dx\, \tilde{f}(x, \sqrt{k_x^2 + (x-y)P_z^2}, P^z)$

$$\mu^{2\epsilon}\int_0^1 dy \int d^{d-3}k_\perp^x \frac{1}{(k_\perp^x)^2 + (x-y)^2 p_z^2}\left(\frac{1+y}{1-y} + 2\epsilon\frac{\ln(1-y)}{(1-y)} + 2\epsilon\frac{y}{1-y}\right)_+$$

$$\sim \frac{1+x}{1-x}\left[-\frac{1}{\epsilon} - \ln\frac{\mu^2}{p_z^2} + \ln(x(1-x))\right] + \frac{1-2x}{1-x}$$

Exactly reproduce the leading soft divergences in the quasi-PDF and spatial correlator!

- Fourier transform to λ and $b_\perp$ space:

$$\mu^{2\epsilon}\int dx\, e^{-ix\lambda}\int d^{d-2}k_\perp \frac{e^{-i\vec{k}\cdot\vec{b}_\perp}}{k_t^2}\left(\frac{1+x}{1-x} + 2\epsilon\frac{\ln(1-x)}{1-x} + 2\epsilon\frac{x}{1-x}\right)_+$$

$$\sim -\int dx\, e^{-ix\lambda}\left[\frac{1}{\epsilon} + \ln(\vec{b}_\perp^2 p_z^2 e^{2\gamma_E}/4)\right](\mu^2/p_z^2)^\epsilon\left(\frac{1+x}{1-x} + 2\epsilon\frac{\ln(1-x)}{1-x} + 2\epsilon\frac{x}{1-x}\right)_+$$

$$= \int dx\, e^{-ix\lambda}\left(-\left[\frac{1}{\epsilon_{IR}} + \ln(\vec{b}_\perp^2\mu^2 e^{2\gamma_E}/4)\right]\frac{1+x}{1-x} - \frac{2\ln(1-x)}{1-x} - \frac{2x}{1-x}\right)_+$$

All order form of resummation

- Operator product expansion of the spatial correlator:

$$\tilde{h}_n(\lambda, z^2) = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!} C_n(\mu^2 z^2) a_n(\mu) + \cdots \,, \qquad a_n(\mu) = \int_{-1}^{1} dx\ x^n f(x,\mu)$$

$$C_n(\mu^2 z^2) = 1 + \frac{\alpha_s(\mu) C_F}{2\pi}\left[\left(\frac{3+2n}{2+3n+n^2} + 2H_n\right)\ln\frac{\mu^2 z^2 e^{2\gamma_E}}{4} + \frac{5+2n}{2+3n+n^2} + 2(1-H_n)H_n - 2H_n^{(2)}\right],$$

- Large N limit:

$$z_0 = |z| e^{\gamma_E}/2 \qquad \lim_{N\to\infty} C_N\big(\alpha_s(1/z_0), 1\big) = 1 + \frac{\alpha_s(1/z_0) C_F}{2\pi}\left[-2(\ln N + \gamma_E)^2 + 2(\ln N + \gamma_E) - \frac{\pi^2}{3}\right]$$

  - Leading and subleading logarithms as a result of the soft divergence;
  - Threshold resummation suppresses the contribution.

All order form of resummation

- Resummation at NLL accuracy (all-order proof in preparation):

$$\ln C_N^{\rm NLL}\big(\alpha_s(1/z_0),1\big) = \int d\alpha\, \frac{\alpha^{N-1}-1}{1-\alpha}\left[\int_{1/z_0^2}^{\frac{1}{(1-\alpha)^2 z_0^2}} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s(1/((1-x)^2 z_0^2)))\right],$$

$$A(\alpha_s) = A^{(0)}\frac{\alpha_s}{2\pi} + A^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \cdots\,, \qquad B(\alpha_s) = B^{(0)}\frac{\alpha_s}{2\pi} + B^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \cdots\,,$$

$$A^{(0)} = -B^{(0)} = 2C_F\,, \qquad A^{(1)} = 2C_F\left[C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right) - \frac{10}{9}n_f T_F\right].$$

- Expanded to $O(\alpha_s^2)$,

$$\begin{aligned} C_N^{\rm NLL}\big(\alpha_s(1/z_0),1\big) = 1 &+ a_s\left[-A^{(0)}\ln^2(Ne^{\gamma_E}) - B^{(0)}\ln(Ne^{\gamma_E}) - A^{(0)}\frac{\pi^2}{6}\right] \\ &+ a_s^2\Big[\frac{\big(A^{(0)}\big)^2}{2}\ln^4(Ne^{\gamma_E}) + \frac{1}{3}A^{(0)}\left(3B^{(0)}+2\beta_0\right)\ln^3(Ne^{\gamma_E}) \\ &+ \left(-A^{(1)} + \frac{1}{2}\big(B^{(0)}\big)^2 + B^{(0)}\beta_0 + \big(A^{(0)}\big)^2\frac{\pi^2}{6}\right)\ln^2(Ne^{\gamma_E})\Big] + \mathcal{O}(a_s^2\ln N, a_s^3) \end{aligned}$$

$$a_s = \frac{\alpha_s}{2\pi}$$

Agrees with fixed-order calculation at NNLO. Li, Ma and Qiu, 2006.12370.

All order form of resummation

- Resummation at NLL accuracy (all-order proof in preparation):

$$\ln C_N^{\rm NLL}\big(\alpha_s(1/z_0),1\big) = \int d\alpha \, \frac{\alpha^{N-1}-1}{1-\alpha}\left[\int_{1/z_0^2}^{\frac{1}{(1-\alpha)^2 z_0^2}} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s(1/((1-x)^2 z_0^2)))\right] ,$$

$$A(\alpha_s) = A^{(0)}\frac{\alpha_s}{2\pi} + A^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 +$$

$$A^{(0)} = -B^{(0)} = 2C_F \, ,$$

Evolved to a UV fixed point. Recall the one-loop threshold logarithms:

$$\lim_{\alpha\to 1} \mathscr{C}(\alpha, \mu^2 z^2) \sim \frac{\alpha_s C_F}{2\pi}\left[\frac{2}{1-\alpha}\ln\frac{1}{(1-\alpha)^2 z_0^2 \mu^2}\right]_+$$

For arbitrary $\mu \sim 1/z_0$, change the lower limit of the integral to μ .

- Expanded to $O(\alpha_s^2)$,

$$C_N^{\rm NLL}\big(\alpha_s(1/z_0),1\big) = 1 + a_s\Big[-A^{(0)}\ln^2(Ne^{\gamma_E}) - B^{(0)}\ln(Ne^{\gamma_E}) - A^{(0)}\frac{\quad}{6}\Big]$$

$$a_s = \frac{\alpha_s}{2\pi}$$

$$+a_s^2\Big[\frac{\left(A^{(0)}\right)^2}{2}\ln^4(Ne^{\gamma_E}) + \frac{1}{3}A^{(0)}\left(3B^{(0)}+2\beta_0\right)\ln^3(Ne^{\gamma_E})$$

$$+\Big(-A^{(1)} + \frac{1}{2}\left(B^{(0)}\right)^2 + B^{(0)}\beta_0 + \left(A^{(0)}\right)^2\frac{\pi^2}{6}\Big)\ln^2(Ne^{\gamma_E})\Big] + \mathscr{O}(a_s^2\ln N, a_s^3)$$

Agrees with fixed-order calculation at NNLO. Li, Ma and Qiu, 2006.12370.

All order form of resummation

- Resummation at NLL accuracy (all-order proof in preparation):

$$\ln C_N^{\rm NLL}\left(\alpha_s(1/z_0),1\right) = \int d\alpha\, \frac{\alpha^{N-1}-1}{1-\alpha}\left[\int_{1/z_0^2}^{\frac{1}{(1-\alpha)^2 z_0^2}} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s(1/((1-x)^2 z_0^2)))\right],$$

$$A(\alpha_s) = A^{(0)}\frac{\alpha_s}{2\pi} + A^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \cdots, \qquad B(\alpha_s) = B^{(0)}\frac{\alpha_s}{2\pi} + B^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \cdots,$$

$$A^{(0)} = -B^{(0)} = 2C_F, \qquad A^{(1)} = 2C_F\left[C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right) - \frac{10}{9}n_f T_F\right].$$

- Expanded to $O(\alpha_s^2)$,

$$\begin{aligned} C_N^{\rm NLL}\left(\alpha_s(1/z_0),1\right) = 1 &+ a_s\left[-A^{(0)}\ln^2(Ne^{\gamma_E}) - B^{(0)}\ln(Ne^{\gamma_E}) - A^{(0)}\frac{\pi^2}{6}\right] \\ &+ a_s^2\Big[\frac{\left(A^{(0)}\right)^2}{2}\ln^4(Ne^{\gamma_E}) + \frac{1}{3}A^{(0)}\left(3B^{(0)}+2\beta_0\right)\ln^3(Ne^{\gamma_E}) \\ &+ \left(-A^{(1)} + \frac{1}{2}\left(B^{(0)}\right)^2 + B^{(0)}\beta_0 + \left(A^{(0)}\right)^2\frac{\pi^2}{6}\right)\ln^2(Ne^{\gamma_E})\Big] + \mathcal{O}(a_s^2\ln N, a_s^3) \end{aligned}$$

$$a_s = \frac{\alpha_s}{2\pi}$$

Agrees with fixed-order calculation at NNLO. Li, Ma and Qiu, 2006.12370.

All order form of resummation

- Resummation at NLL accuracy:

$$\ln C_N^{\rm NLL}\big(\alpha_s(1/z_0),1\big) = -\frac{\pi^2}{3}a_s C_F + \ln N' g_1(\lambda) + g_2(\lambda) + \mathcal{O}(\alpha_s^k \ln^{k-1} N')$$

$$\lambda = \beta_0 a_s \ln N' = \beta_0 a_s \ln(N e^{\gamma_E})$$

$$g_1(\lambda) = -\frac{A^{(0)}}{2\beta_0\lambda}\left[-2\lambda + (1+2\lambda)\ln(1+2\lambda)\right],$$

No Landau pole!

$$g_2(\lambda) = -\frac{A^{(0)}}{\beta_0}\frac{\beta_1}{4\beta_0^2}\ln^2(1+2\lambda) - \left(\frac{A^{(0)}}{\beta_0} - \frac{A^{(1)}}{\beta_1}\right)\frac{\beta_1}{2\beta_0^2}\left[-2\lambda + \ln(1+2\lambda)\right] - \frac{B^{(0)}}{2\beta_0}\ln(1+2\lambda)$$

- Inverse Mellin Transformation:

$$\mathscr{C}(\alpha, z^2\mu^2) \approx e^{-\frac{\pi^2}{3}a_s C_F}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\left(\frac{a_s(1/z_0)}{a_s(\mu)}\right)^{\frac{\gamma_N^{(0)}}{\beta_0}} \exp\left[\ln N' g_1(\lambda) + g_2(\lambda)\right]\alpha^{-N} dN\,.$$

Threshold resummation of the μ-independent part of the matching coefficient for the quasi-PDF is the same.

Outline

- Factorization formula for lattice calculation of PDFs
  - Large-momentum effective theory expansion
  - Short-distance expansion in coordinate space
- Threshold Resummation
  - Origin of threshold logarithms
  - Resummation formula in the Mellin moment space
- Implementation on lattice data

Fitting the moments:

- Resummed OPE formula at NLO+NLL accuracy:

$$\tilde{h}_n(\lambda, z^2) = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!} C_n(\mu^2 z^2) a_n(\mu) + \cdots \ ,$$

$$C_n^{\text{NLO+NLL}}(\mu^2 z^2) = C_n^{\text{NLL}}(\mu^2 z^2) - C_n^{\text{NLLEP}}(\mu^2 z^2) + C_n^{\text{NLO}}(\mu^2 z^2)$$

C_n^{NLLEP} is C_n^{NLL} expanded to $O(\alpha_s)$, so when n is small, the NLO Wilson coefficient dominates.

- Take DGLAP evolution into account:

$$C_n^{\text{NLOevo}}(\mu^2 z^2) = C_n^{\text{NLO}}(\alpha_s(1/z_0), 1) \left(\frac{\alpha_s(\mu)}{\alpha_s(1/z_0)} \right)^{\frac{-\gamma_N^{(0)}}{\beta_0}}$$

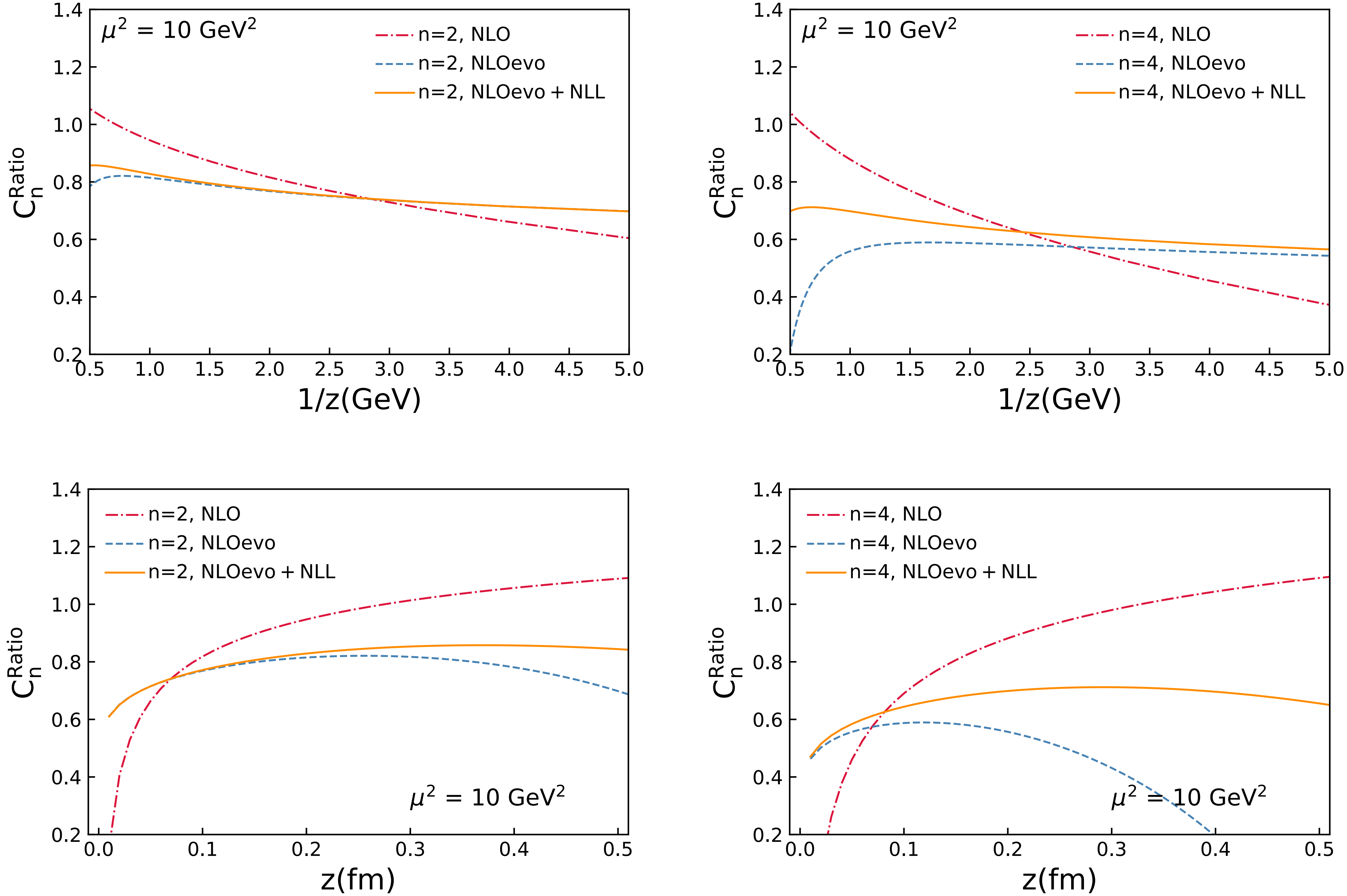
$$C_n^{\text{NLO+NLLevo}}(\mu^2 z^2) = C_n^{\text{NLO+NLL}}(\alpha_s(1/z_0), 1) \left(\frac{\alpha_s(\mu)}{\alpha_s(1/z_0)} \right)^{\frac{-\gamma_N^{(0)}}{\beta_0}}$$

Fitting the moments:

- Fitting the moments (least model assumption):

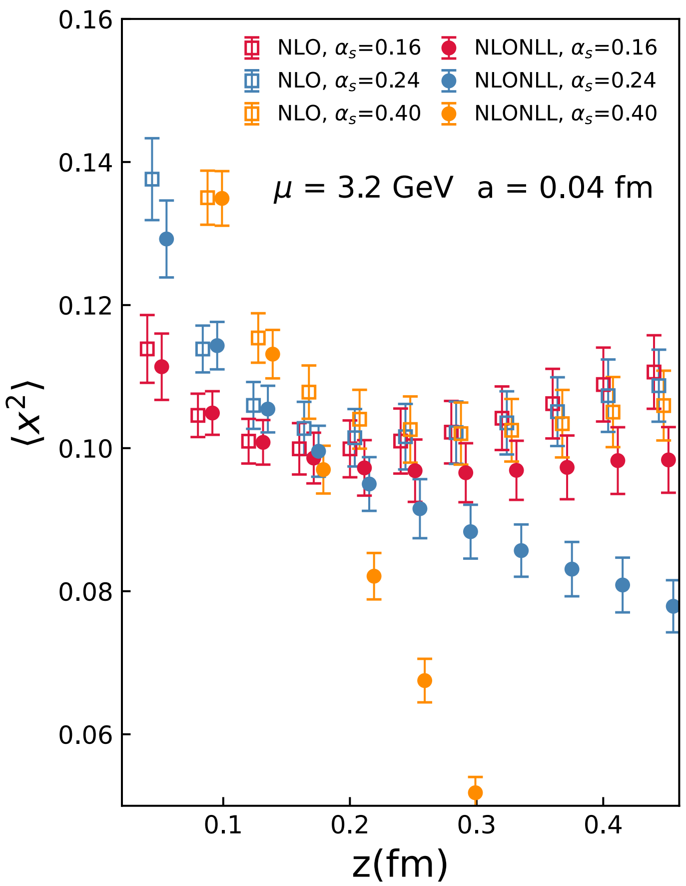
Comparison of the Wilson coefficients:

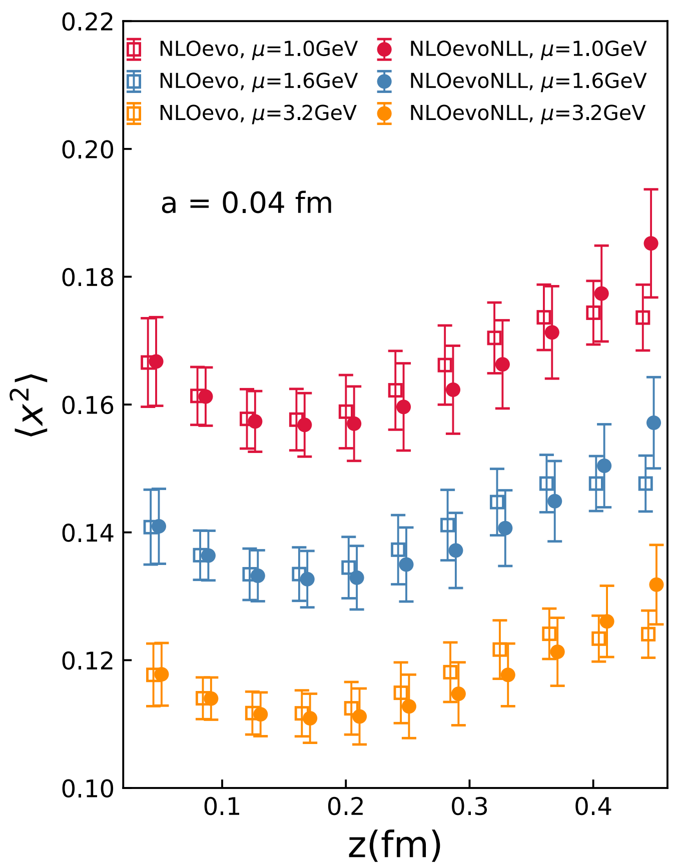
$$C_n^{\text{ratio}} = \frac{C_n(\mu^2 z^2)}{C_0(\mu^2 z^2)}$$



Fitting the moments:

- Fitting the moments (least model assumption):

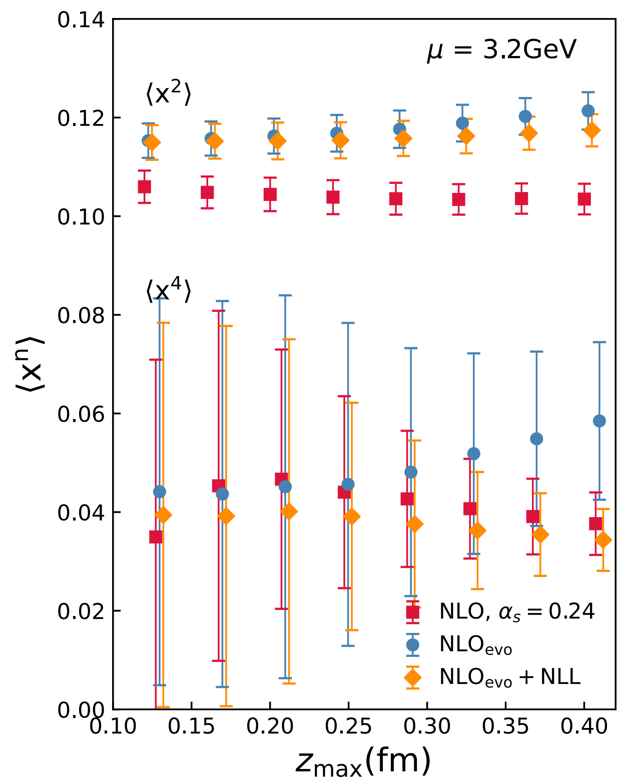


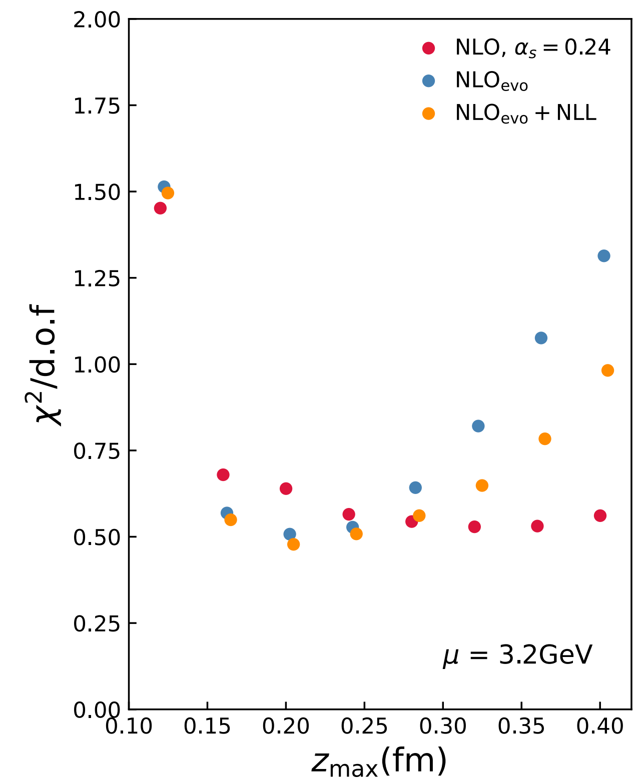


Lattice data for pion valence PDF: X. Gao, YZ et al., Phys.Rev.D 102 (2020)

Combined fit of 2nd and 4th moments for all $z \in [0.12, 0.40]$ fm. 👇

- Effect of threshold resummation is not significant within the uncertainty.;
- To probe higher moments requires large momentum with better precision.





👆 Fit of 2nd and 4th moments for each z:

- $0.97 \text{ GeV} < P^z < 2.42 \text{ GeV}$,
  $0.04 \text{ fm} < z < 0.48 \text{ fm}$, $z_{\text{max}} P^z_{\text{max}} = 5.9$;
- NLO or NLO+NLL cannot fit data with $\mu < 2$ GeV, or when $\alpha_s > 0.3$.

Fitting to a model of the PDF:

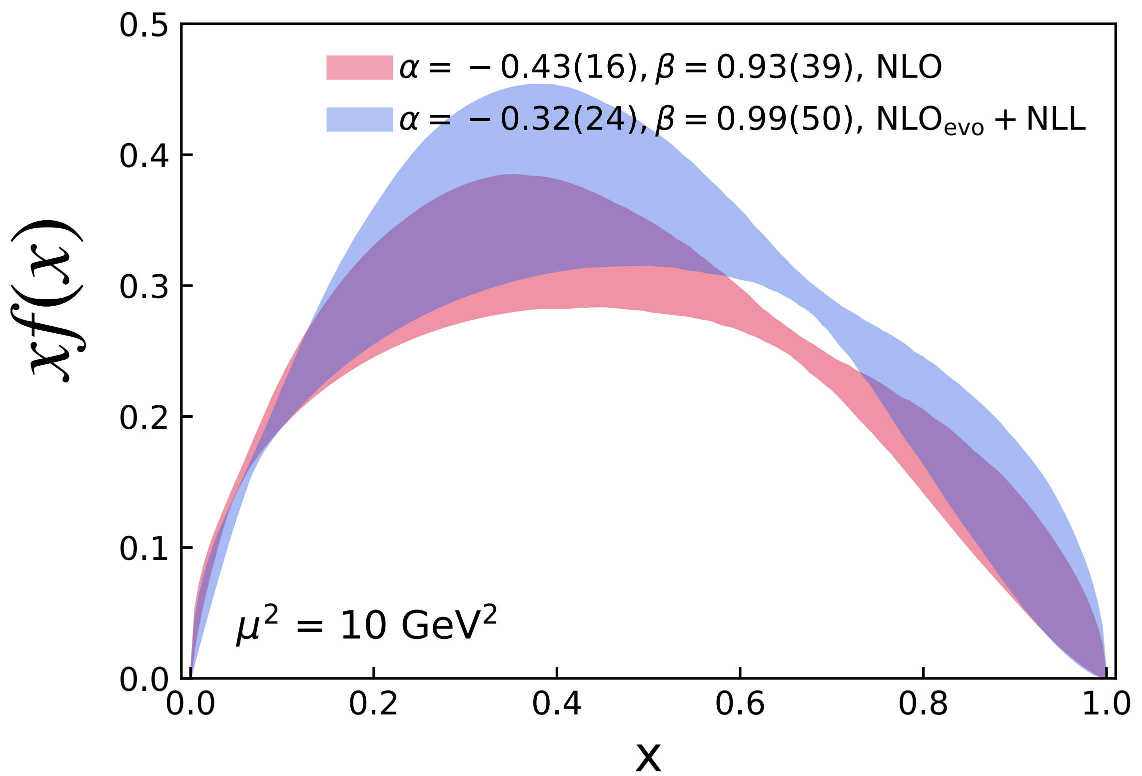
- Investigate effects of threshold resummation on the large-x behavior of the PDF, model-independence not considered.

Simple parametrization: $f(x) = \Gamma(\alpha+1)\Gamma(\beta+1)\dfrac{x^\alpha(1-x)^\beta}{\Gamma(\alpha+\beta+2)}$

- Strategy: $\tilde{h}_n(zP^z, z^2) = \sum_{n=0}^{N_{\max}} \dfrac{(-i\lambda)^n}{n!} C_n(\mu^2 z^2) a_n(\alpha,\beta)$

Fitting to lattice data for $z \in [0.12,\, 0.48]$ fm with N_{max}=20:

- No significant change to the large-x behavior of the fitted result;
- Main reason is because lattice data are not sensitive to large-x contributions.



Summary

- Precision lattice calculation of PDFs will require QCD evolution and resummation;
- The origin of threshold logarithms in the quasi-PDF and spatial correlators is identified, which can be resummed using standard techniques;
- DGLAP evolution effect is significant for current lattice data which uses large values of z;
- Current lattice data are only sensitive to the lowest moments or finite-x range of the PDF, so the effect of threshold resummation is not significant;
- Threshold resummation will be important for future calculations with larger hadron momenta to study the large-x behavior of the PDF.

Origin of threshold logarithms

- Leading divergence in the one-loop diagram:

$$\tilde{f}(x, k_\perp, P^z) = 2g^2 C_F \int \frac{d^{d-1}\vec{b}}{(2\pi)^{d-1}} e^{i\vec{k}\cdot\vec{b}} \int \frac{dl^0 d^{d-1}\vec{l}}{(2\pi)^d} \frac{l^0 b^z + \vec{l}\cdot\vec{b}}{l^2(p-l)^2} \int_0^1 ds\; e^{-i(\vec{p}-\vec{l})\cdot\vec{b}s} e^{-i\vec{l}\cdot\vec{b}}$$

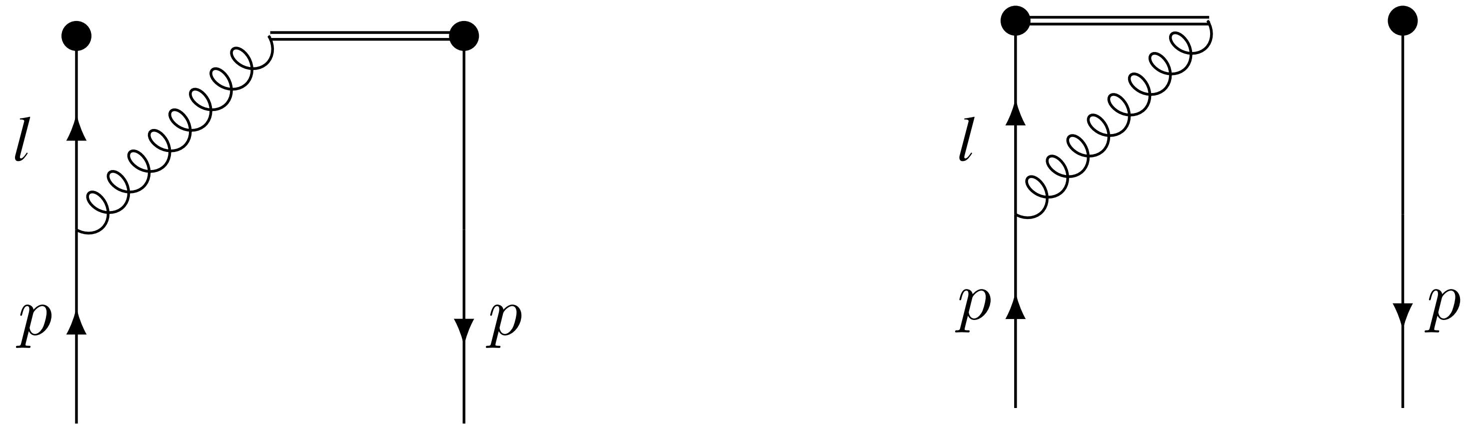
$$l^0 b^z + \vec{l}\cdot\vec{b} = (l^0 + p^z) b^z - (\vec{p} - \vec{l})\cdot\vec{b}$$

$$\tilde{f}(x, k_\perp, P^z) = 2g^2 C_F \int \frac{d^{d-1}\vec{b}}{(2\pi)^{d-1}} e^{i\vec{k}\cdot\vec{b}} \int \frac{dl^0 d^{d-1}\vec{l}}{(2\pi)^d} \frac{(l^0 + p^z) b^z}{l^2(p-l)^2} \int_0^1 ds\; e^{-i(\vec{p}-\vec{l})\cdot\vec{b}s} e^{-i\vec{l}\cdot\vec{b}}$$

$$b^z = \frac{\partial}{\partial i k^z} = -\frac{i}{p^z}\frac{\partial}{\partial x} \qquad + i2g^2 C_F \int \frac{d^{d-1}\vec{b}}{(2\pi)^{d-1}} e^{i\vec{k}\cdot\vec{b}} \int \frac{dl^0 d^{d-1}\vec{l}}{(2\pi)^d} \frac{1}{l^2(p-l)^2} \left[e^{-i\vec{l}\cdot\vec{b}} - e^{-i\vec{p}\cdot\vec{b}} \right]$$

Origin of threshold logarithms

- Leading divergence in the one-loop diagram:



$(1-s)^{1-d}\delta^{(d-1)}\left(\vec{l}-\frac{\vec{k}-s\vec{p}}{1-s}\right)$

$$\tilde{f}(x,k_\perp,P^z) = -2ig^2C_F\frac{\partial}{\partial x}\int_0^1 ds \int \frac{dl^0 d^{d-1}\vec{l}}{(2\pi)^d}\frac{l^0+p^z}{l^2(p-l)^2}e^{-i(\vec{p}-\vec{l})\cdot\vec{b}s}e^{-i\vec{l}\cdot\vec{b}}\delta^{(d-1)}((\vec{p}-\vec{l})s+\vec{l}-\vec{k})$$

$k_t^2 = \frac{\vec{k}_\perp^2}{p_z^2}$

$$= \frac{2g^2C_F}{4(2\pi)^{d-1}}\int_0^1 ds\,\frac{(1-s)^{2-d}}{k_t^2}\left[\frac{k_t^2(-2s+x+1)+(x-s)^3}{\left(k_t^2+(s-x)^2\right)^{3/2}}-\frac{k_t^2(-2s+x+1)+(x-1)^3}{\left(k_t^2+(x-1)^2\right)^{3/2}}\right]_+$$

$k_t^2 \ll 1$

$$= \frac{2g^2C_F}{4(2\pi)^{d-1}}\left(\int_0^1 ds\,\frac{(1-s)^{2-d}}{k_t^2}\left[\frac{x-s}{|s-x|}+\frac{1-x}{|1-x|}\right]+\int_0^1 ds\,(1-s)^{2-d}\delta(x-s)\frac{2(1-s)}{k_t^2}\right)_+$$

$$= \frac{g^2C_F}{(2\pi)^{d-1}}\frac{1}{k_t^2}\left[\frac{1+x}{1-x}+2\epsilon\frac{\ln(1-x)}{1-x}+2\epsilon\frac{x}{1-x}\right]_+\theta(x)\theta(1-x)$$

Normal TMD has additional term $\delta(1-x)\ln k_\perp^2/k_\perp^2$