Three-Reggeon ladders and four-loop amplitudes in the high-energy limit

Resummation, Evolution, Factorization 2020

G. Falcioni

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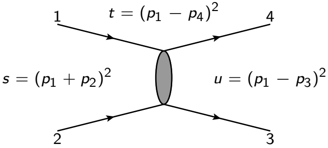
E. Gardi, C. Milloy and L. Vernazza

Amplitudes in the high-energy limit



High-energy limit

$$s \gg -t$$
$$u \simeq -s$$

Large logarithms in the amplitude

$$L = \underbrace{\frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)}_{\textbf{Signature}\text{ even}}$$

Expansion in loops ($a_s = \alpha_s/\pi$) and **towers** of logarithms

$$\mathcal{M} = \mathcal{M}^0 + a_s L\,\mathcal{M}^{(1,1)} + a_s\,\mathcal{M}^{(1,0)}$$
$$+ a_s^2 L^2\,\mathcal{M}^{(2,2)} + a_s^2 L\,\mathcal{M}^{(2,1)} + a_s^2\,\mathcal{M}^{(2,0)}$$

LL	NLL	NNLL
$a_s L\,\mathcal{M}^{(1,1)}$, $a_s^2 L^2\,\mathcal{M}^{(2,2)}$	$a_s\,\mathcal{M}^{(1,0)}$, $a_s^2 L\,\mathcal{M}^{(2,1)}$	$a_s^2\,\mathcal{M}^{(2,0)}$

Probing deep structures in QFT

Impressive progress in understanding the high-energy limit

- All orders at any multiplicities in planar $\mathcal{N} = 4$ sYM (Caron-Huot et al. 2019)
- QCD in full colour: 4 points to 13 loops at NLL (Caron-Huot, Gardi, Reichel, Vernazza 2020)

Crucial input for the bootstrap programme

- Planar $\mathcal{N} = 4$ (following Dixon, Drummond, Henn 2011)
- Infrared (IR) singularities to 3 loops in QCD (Almelid, Duhr, Gardi, McLeod, White 2018)

Focus here

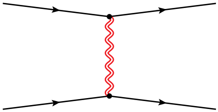
High-energy amplitudes: 4 points to 4 loops at NNLL.

- IR singularities at 4 loops in QCD
- Real part of amplitudes in non-planar $\mathcal{N} = 4$

Reggeon exchanges

Leading tower: one-Reggeon exchange in the t-channel
(Lipatov;Fadin,Kuraev,Lipatov 1976)

$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\frac{a_s C_A}{\epsilon} r_\Gamma}$$



Leading Logarithms of the amplitude

$$\mathcal{M}^{\mathrm{LL}} = \mathcal{M}^0\, e^{\frac{a_s C_A L}{\epsilon} r_\Gamma} \qquad r_\Gamma = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

Higher towers: compound states of multiple Reggeons.
$\mathcal{M}$ has Even $(+)$ and Odd $(-)$ terms under $s \leftrightarrow u$

- $\mathcal{M}^{(+)}$ **immaginary** with **even** number of Reggeons
- $\mathcal{M}^{(-)}$ Purely **real** with **odd** number of Reggeons

Real (odd) amplitudes

At NLL $\mathcal{M}^{(-)}$ is still given by a single-Reggeon exchange (Fadin, Kozlov, Reznichenko 2015).

Three-Reggeons states appear at NNLL

Observed in two-loop amplitudes (Del Duca, Glover 2001).
Recent investigations up to three loops

- IR singularities (Del Duca, G.F, Magnea, Vernazza 2013-14; Fadin 2016; Fadin, Lipatov 2017)
- Complete result (Caron-Huot, Gardi, Vernazza 2017)

New questions

- Can we compute the whole NNLL tower?
  Start looking into **Four-loop amplitudes** at NNLL

High-energy limit perspective

Does the three-Reggeon contribution exponentiate?

Beyond the high-energy limit

Directly relevant to bootstrap IR singularities

$$\mathcal{M} = \mathbf{Z}\ \mathcal{H} \longrightarrow \text{IR finite}$$

Constraint four-loop ansatz of $\mathbf{Z}$ (Becher, Neubert 2019)

From particles to Wilson lines

Wilson lines key to compute high-energy processes (from Korchemskaya, Korchemsky 1995; Balitsky 1996).

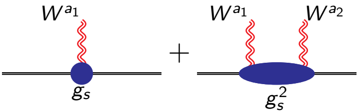
Following (Caron-Huot 2013)

$$U^{\eta}(z) = \mathcal{P}\exp\left[ig_s \mathbf{T}^a \int_{-\infty}^{+\infty} dx^+ \, A_+^a(x^+, x^- = 0, z)\right]$$
$$\equiv e^{ig_s \mathbf{T}^a W^a(z)}$$

- $\mathbf{T}^a$ group generator in the parton representation.
- $\eta = L$ (implicit) rapidity cutoff.

Scattering states $|\psi_i\rangle$ are expanded in Reggeon fields W^a

$|\psi_i\rangle \sim$



$+ \ldots \equiv \begin{pmatrix} W \\ W\,W \\ \ldots \end{pmatrix}$

$\frac{d}{dL}|\psi_i\rangle = -H|\psi_i\rangle$ Balitsky-JIMWLK Hamiltonian

- High-energy limit
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Scattering of Wilson lines

$$\frac{i}{2s}\frac{1}{Z_i\,Z_j}\mathcal{M} = \langle\psi_j|e^{-L\,H}|\psi_i\rangle \qquad Z_i \text{ collinear poles}$$

- To leading order, Reggeons are free fields

$$\langle W^a(q)|W^b(p)\rangle = \frac{i}{p^2}\delta^{ab}\delta^{2-2\epsilon}(p-q) + \mathcal{O}(g^2),$$

Structure of the hamiltonian at Leading Order in a_s

$$H = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \dots \\ 0 & H_{2\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Diagonal elements start $\mathcal{O}(a_s)$
Suppressed Off-diagonal elements $\mathcal{O}(a_s^2)$

The NNLL reduced amplitude

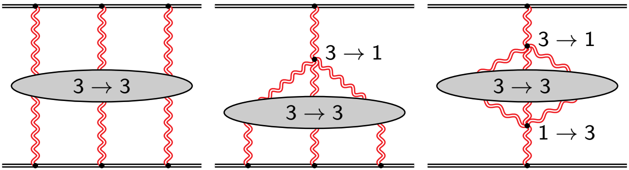
Three-Reggeon contribution to the **NNLL tower** involves

- Balitsky-JIMWLK evolution at Leading Order
- Projectile and target states at Leading Order

Reduced amplitude: subtract single-Reggeon exchange

$$\frac{i}{2s}\hat{\mathcal{M}} \equiv \langle\psi_j|e^{-(H-H_{1\to1})L}|\psi_i\rangle = \langle\psi_j|e^{-\hat{H}L}|\psi_i\rangle$$

To all loop orders, $\hat{\mathcal{M}}^{(-,\mathrm{NNLL})}$ is the sum of



$\langle j_3|\left(\hat{H}_{3\to3}\right)^k|i_3\rangle$ $\langle j_1|\hat{H}_{3\to1}\left(\hat{H}_{3\to3}\right)^k|i_3\rangle$ $\langle j_1|\hat{H}_{3\to1}\left(\hat{H}_{3\to3}\right)^k\hat{H}_{1\to3}|i_1\rangle$

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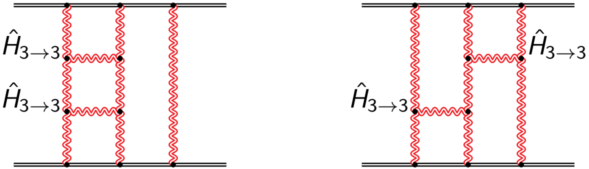
Three-Reggeon ladders

Two independent contributions

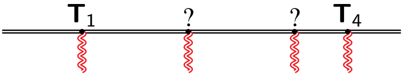


- Single kinematic scale t. s is generated by evolution All the integrals are massless 4-loop propagators: ✓ gamma functions or computed with FORCER (Ruijl, Ueda, Vermaseren 2020)
- Problem: factorise the colour structure in **universal** operators acting on the tree level amplitude, with

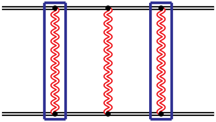
$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 \quad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 \quad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3$$

A colour puzzle

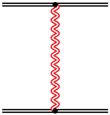
Outmost generators clearly associated to external particles



At lowest order there is no ambiguity

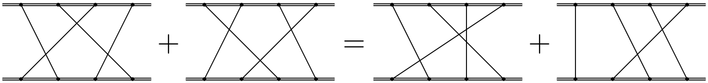


$$= \left[\frac{1}{2} \left(\mathbf{T}^2_{s-u} - \frac{\mathbf{T}^2_t}{2} \right) \right]^2 \cdot$$



where $\mathbf{T}^2_{s-u} = \frac{\mathbf{T}^2_s - \mathbf{T}^2_u}{2}$.

Reduce *entangled* configurations using identities such as



Three-Reggeon Ladders - Result

$$\langle j_3 | \hat{H}^2_{3\to 3} | i_3 \rangle = \frac{1}{144}\left[\frac{\mathbf{C}_{33}^{(4,-4)}}{\epsilon^4} + \frac{2f_\epsilon}{\epsilon}\mathbf{C}_{33}^{(4,-1)} + \mathcal{O}(\epsilon)\right]\mathcal{M}^{(0)}$$

- $f_\epsilon = \zeta_3 + \frac{3}{2}\epsilon\,\zeta_4$ (appearing in **every term** at NNLL!)
- Colour operators $\mathbf{T}_t^2$ and $\mathbf{T}_{s-u}^2$ acting on $\mathcal{M}^{(0)}$
- Contribution of quartic Casimir

Three-Reggeon Ladders - Result

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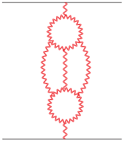
- Contribution of quartic Casimir

$$\begin{aligned}
\mathbf{C}_{33}^{(4,-4)} &= 6\left(17C_A\mathbf{T}_t^2 - 6C_A^2 - 6(\mathbf{T}_t^2)^2\right)\mathbf{C}_{33}^{(2)} \\
&\quad - \frac{3}{4}\mathbf{T}_{s-u}^2(\mathbf{T}_t^2)^2\mathbf{T}_{s-u}^2 + \frac{25}{144}C_A^4 + \frac{1}{3}\frac{d_{AA}}{N_A} - 3C_A\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right) \\
\mathbf{C}_{33}^{(4,-1)} &= 18\left(521C_A\mathbf{T}_t^2 - 300C_A^2 - 220(\mathbf{T}_t^2)^2\right)\mathbf{C}_{33}^{(2)} - 101\,\mathbf{C}_{33}^{(4,-4)} \\
\mathbf{C}_{33}^{(2)} &= \frac{1}{24}\left(\mathbf{T}_{s-u}^2 - \frac{C_A^2}{12}\right)
\end{aligned}$$

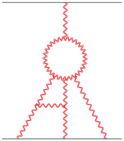
Transitions to single Reggeons

To all orders, terms with a single Reggeon are $\propto \mathcal{M}^{(0)}$

- Colour must flow through a single Reggeon



$$= \frac{1}{432}\left[-\left(\frac{C_A^4}{12}+\frac{d_{AA}}{N_A}\right)\frac{1}{\epsilon^4} + \left(\frac{101}{6}C_A^4 + 220\frac{d_{AA}}{N_A}\right)\frac{f_\epsilon}{\epsilon}\right]\mathcal{M}^{(0)}$$



$$= \frac{C_A}{144}\frac{d_{AR_i}}{N_{R_i}}\left[\frac{1}{\epsilon^4} - 208\frac{f_\epsilon}{\epsilon}\right]\mathcal{M}^{(0)}$$

Complete Reduced amplitude

- High-energy limit
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A four-loop amplitude (almost) fitting in one line

$$\hat{\mathcal{M}}^{(-,4,2)} = \frac{r_\Gamma^4 \pi^2}{144}\left[\frac{\mathbf{C}_{\mathcal{M}}^{(-4)}}{\epsilon^4} + \mathbf{C}_{\mathcal{M}}^{(-1)}\frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon)\right]\mathcal{M}^{(0)}$$

$$\mathbf{C}_{\mathcal{M}}^{(-4)} = \frac{\mathbf{C}_{33}^{(4,-4)}}{2} - \frac{C_A^4}{72} - \frac{1}{6}\frac{d_{AA}}{N_A} + \frac{1}{2}\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right)$$

$$\mathbf{C}_{\mathcal{M}}^{(-1)} = \mathbf{C}_{33}^{(4,-1)} + \frac{101\,C_A^4}{36} + \frac{110}{3}\frac{d_{AA}}{N_A} - 104\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right)$$

Result holds in every gauge theory. Next steps

- Extract the universal infrared singularities
- Compute the odd amplitude in $\mathcal{N} = 4$

Factorisation of $\hat{\mathcal{M}}$

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Divergences and finite parts

Isolate the universal IR divergences from the finite terms

$$\mathcal{H} = \tilde{\mathbf{Z}}^{-1}\, e^{-H_{1\to 1} L}\, \hat{\mathcal{M}},$$

$\tilde{\mathbf{Z}} = \mathcal{P}\exp[-\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\tilde{\mathbf{\Gamma}}]$. $\tilde{\mathbf{\Gamma}}$ soft anomalous dimension.

- $\tilde{\mathbf{\Gamma}} = \frac{\gamma_K}{2}\left[L\,\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2\right] + \mathbf{\Delta}$

Four-loop soft anomalous dimension at NNLL

$$\mathrm{Re}\left[\mathbf{\Delta}^{(4,2)}\right] = \zeta_2\zeta_3\mathbf{C}_\Delta$$

$$\mathbf{C}_\Delta = \frac{\mathbf{T}_t^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]}{4} + \frac{3}{4}[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 + \left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)$$

- Planar terms in $\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)$ cancel
- Manifestly non-planar, new quartic Casimir in $\tilde{\mathbf{\Gamma}}$

Finite parts in $\mathcal{N} = 4$

Finite parts are theory dependent as they involve

- Two-loop impact factors
- $H_{1\to1}$ to three loops

both determined in (Caron-Huot, Gardi, Vernazza 2017) using three-loop ampliudes (Henn, Mistlberger 2016).

Result

$$\text{Re}\left[\mathcal{H}^{(4,2)}_{\mathcal{N}=4}\right] = \left[\frac{C_A^4}{128}\zeta_3^2 + \frac{3}{16}\zeta_4\zeta_2\mathbf{C}_\Delta^{(4,2)}\right]\mathcal{M}^{(0)}$$

- Match large N_c limit
- New non-planar term: **proportional** to $\mathbf{\Delta}^{(4,2)}$

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Conclusions

- NNLL amplitudes feature exchanges of 3 Reggeons.
- Three-Reggeon contributions are predicted to all loops with LO Balitsky-JIMWLK equation!
- We computed three-Reggeon exchanges to 4 loops.
- Using this result we find
  - New constraints on the IR divergences in four-loop QCD amplitudes.
  - The real part of $2 \to 2$ amplitude in non-planar $\mathcal{N} = 4$ sYM at 4 loops.

Interestingly, the non-planar terms in the finite parts are proportional to the IR divergent contributions.

Outlook

Many directions for further investigation

On the high-energy side

- Closer look at the origin of planar and non-planar contributions
- All-order iteration of the Balitsky-JIMWLK hamiltonian.

  $\rightarrow$ Resummation of the entire NNLL tower.

And towards general kinematics

- Bootstrapping IR singularities to four loops

Thank you!