TMD PDFs and FFs:
N^3LO, Analytic Continuation, and Reciprocity

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Based on 1912.05778, 2006.10534, 2012.03256

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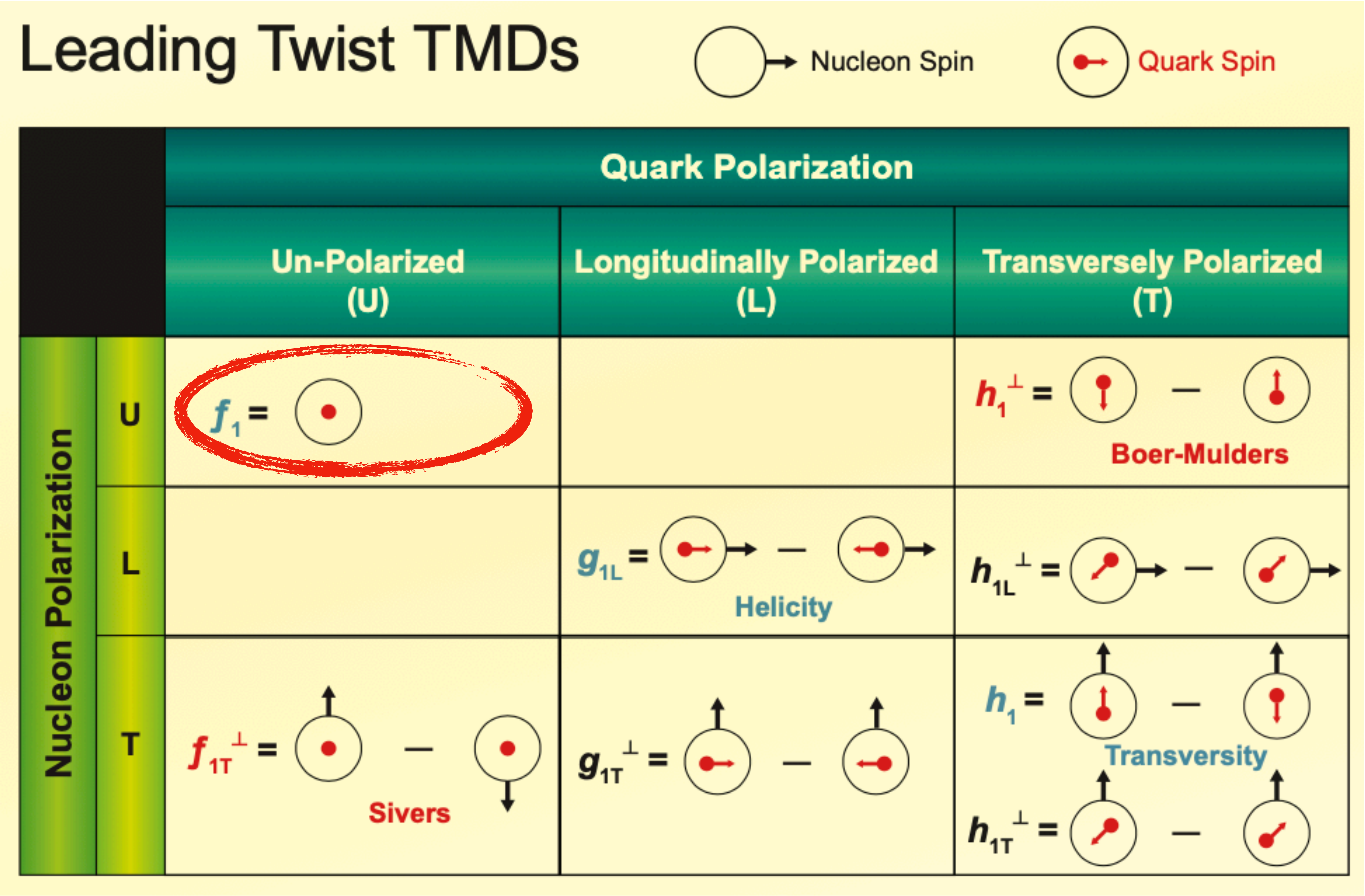
Transverse Momentum Dependent Distributions



Similar classification for gluon and for Fragmentations

- 3D imaging
- Orbital motion
- Spin-orbit correlations
- Gauge invariance and universality

[EIC white paper]

Operator definitions for unpolarized TMD PDFs

Operator defintion for gluon TMD PDFs

$$\mathcal{B}_{g/N}^{\text{bare},\mu\nu}(x, b_\perp) = -\, xP_+ \int \frac{db^-}{2\pi} e^{-ixb^- P^+} \langle N(P)|\mathcal{A}_{n\perp}^{a,\mu}(0, b^-, b_\perp)\mathcal{A}_{n\perp}^{a,\nu}(0)|N(P)\rangle$$

OPE onto perturbatively calculable matching coefficients and collinear PDFs at leading twist

$$\mathcal{B}_{g/N}^{\text{bare},\mu\nu}(x, b_\perp) = \sum_i \int_x^1 \frac{d\xi}{\xi} \mathcal{I}_{gi}^{\text{bare},\mu\nu}(\xi, b_\perp) \phi_{i/N}^{\text{bare}}(x/\xi) + \text{power corrections}$$

$$\mathcal{I}_{gi}^{\text{bare},\mu\nu}(\xi, b_\perp) = \frac{g_\perp^{\mu\nu}}{d-2} \mathcal{I}_{gi}^{\text{bare}}(\xi, b_T) + \left(\frac{g_\perp^{\mu\nu}}{d-2} + \frac{b_\perp^\mu b_\perp^\nu}{b_T^2}\right) \mathcal{I}'^{\text{bare}}_{gi}(\xi, b_T)$$

Unpolarized

Linearly polarized

Operator defintion for gluon TMD FFs

$$\mathcal{D}_{N/g}^{\text{bare},\mu\nu}(z, b_\perp) = -\, \frac{P_+}{z^2} \sum_X \int \frac{db^-}{2\pi} e^{iP^+ b^-/z} \langle 0|\mathcal{A}_{n\perp}^{a,\mu}(0, b^-, b_\perp)|N(P), X\rangle\langle N(P), X|\mathcal{A}_{n\perp}^{a,\nu}(0)|0\rangle$$

Status of perturbative corrections to unpolarized TMDs

- **NNLO**
  - Extraction from full QCD cross section
    - quark [Catani, Grazzini, 1106.4652]; gluon [Catani, Cieri, de Florian, Ferrera, Grazzini, 1209.0158]
  - Calculation from operator definition in SCET (rapidity regularization necessary)
    - quark, unpolarized gluon [Gehrmann, Lubbert, L.L. Yang, 1209.0682; 1403.6451]; [Echevarria, Scimemi, Vladimirov, 1604.07869][M.X. Luo, X. Wang, X. Xu, L.L. Yang, T.Z. Yang, HXZ, 1908.03831; M.X. Luo, X. T.Z. Yang, HXZ, Y.J. Zhu, 1909.13820]
    - Linearly polarized gluon [Gutierrez-Reyez, Leal-Gomez, Scimemi, Vladimirov, 1907.03780] [M.X. Luo, X. T.Z. Yang, HXZ, Y.J. Zhu, 1909.13820]
- **NNNLO**
  - quark [M.X. Luo, X. T.Z. Yang, HXZ, Y.J. Zhu, 1912.05778]
  - quark, unpolarized gluon [Ebert, Mistlberger, Vita, 2006.05329]
  - unpolarized gluon + all unpolarized FFs [M.X. Luo, X. T.Z. Yang, HXZ, Y.J. Zhu, 2012.03256; with H. Chen, 2006.10534]

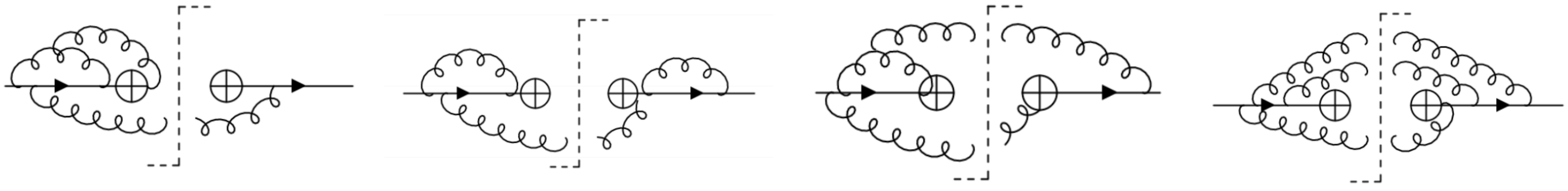
👉 See also Berhnard's talk

Strategy: Divide and Conquer

- Matching coefficients are independent of infrared regularization
- Replace non-perturbative hadron state by perturbative partonic state
- Insert a complete set of states in perturbative Fock space

$$\mathcal{B}_{q/N}^{\text{bare}}(x, b_\perp) = \int \frac{db_-}{2\pi} e^{-ixb^- P^+} \langle q(P)|\bar{\chi}_n(0, b^-, b_\perp) \sum_{X_n} |X_n\rangle\langle X_n| \frac{\bar{n}\cdot\gamma}{2}\chi_n(0)|q(P)\rangle$$

Ingredients@N3LO (plus renormalization of lower order terms)



(a) VVR (b) VV*R (c) VRR (d) RRR

[Y. Li, Neill, HXZ, 1604.00392] [Chiu, Jain, Neill, Rothstein, 1104.0881]

Rapidity divergence: $\int_0^\infty \frac{dk}{k}$

Exponential regularization: $\int \frac{d^3 p_i}{2E_i} \exp(-\tau b_0 E_i)$

Rapidity Renormalization $\ln \tau \to \ln \frac{1}{\nu}$

Generalized IBP systems and differential equations

- VVR and VV*R involves only a single phase space integral can be done in a straightforward way
- VRR + RRR more challenge: 40,000 integrals in total
- Treat real on-shell constraint as cut propagator (reverse unitarity) [Aanstasiou, Melnikov; hep-ph/0207004; Aanstasiou, Melnikov, Dixon, Petriello, hep-ph/0306192]

$$\delta(p^2) = \frac{1}{2\pi i}\left(\frac{1}{p^2 - i0} - \frac{1}{p^2 + i0}\right)$$

Chetyrkin, Tkachov, 1981

- Treating loop and phase space in the same IBP framework $\int d^d k \frac{1}{\partial k^\mu} f(k, \cdots) = 0$

$$0 = \int d^d q \, \frac{\partial}{\partial q^\mu}\left[e^{-b_0 \tau \frac{P\cdot K}{P^+}} F(\{\tilde{l}\})\right]$$

$$= \begin{cases} \int d^d q \, e^{-b_0 \tau \frac{P\cdot K}{P^+}} \left[-b_0 \tau \frac{P_\mu}{P^+} + \frac{\partial}{\partial q^\mu}\right] F(\{\tilde{l}\}), & q = K, \\ \int d^d q \, e^{-b_0 \tau \frac{P\cdot K}{P^+}} \frac{\partial}{\partial q^\mu} F(\{\tilde{l}\}), & q \neq K, \end{cases}$$

IBP equation can be generalized to include exponential regulator

[M.X. Luo, X. T.Z. Yang, HXZ, Y.J. Zhu, 1912.05778]

Differential equations

- Rapidity regularized integrals depend on two independent variables, z and τ
- Most conveniently solved by differential equation [Kotikov 1991; Gehrmann, Remiddi hep-ph/0008287; Henn, 1304.1806]

example: NNLO RR

$$J_1 = \int [dPS] \,,$$
$$J_2 = \int [dPS] \frac{1}{P\cdot k_1} \,,$$
$$J_3 = \int [dPS] \frac{1}{(P-K)^2} \,,$$
$$J_4 = \int [dPS] \frac{1}{\bar{n}\cdot k_1 P\cdot k_1} \,,$$
$$J_5 = \int [dPS] \frac{1}{\bar{n}\cdot k_1 P\cdot k_1 (P-K)^2} \,,$$
$$J_6 = \int [dPS] \frac{1}{\bar{n}\cdot k_1 P\cdot (K-k_1)(P-K)^2} \,,$$
$$J_7 = \int [dPS] \frac{1}{\bar{n}\cdot k_1 P\cdot k_1 P\cdot (K-k_1)} \,,$$
$$J_8 = \int [dPS] \frac{1}{\bar{n}\cdot k_1 P\cdot (K-k_1) K^2} \,,$$

$$\int [dPS] = \int [d^d K] \int [d^d k_1]$$

$$[d^d K] = e^{-2\tau P\cdot K} \delta(K^+ - (1-z)) \delta(-1 + K^2 - 2(1-z) K\cdot P) \,,$$
$$[d^d k_1] = d^d k_1 \delta_+(k_1^2) \delta_+((K-k_1)^2) \,.$$

$$\frac{\partial \vec{J}}{\partial \tau} = A(\tau, z) \vec{J} \,,$$
$$\frac{\partial \vec{J}}{\partial z} = B(\tau, z) \vec{J} \,,$$

Not solvable in terms of usual functions

But we don't need the exact τ dependence!

Expansion of DE in the rapidity regulator

$$J_i(z,\tau,\epsilon) \overset{\tau\to 0}{=} \sum_j \sum_n \sum_{k=0} J_i^{(j,n,k)}(z,\epsilon)\tau^{j+n\epsilon}\ln^k \tau$$

$$\frac{\partial \vec{J}}{\partial \tau} = A(\tau,z)\vec{J}\,,$$
$$\frac{\partial \vec{J}}{\partial z} = B(\tau,z)\vec{J}\,,$$

$$\frac{\partial J_i^{(j,n,k)}}{\partial z} = \sum_{a,b,c,d} C(z,\epsilon)_{ijnk,abcd} J_a^{(b,c,d)}$$

- For individual master integral, j can be negative ⇒ power divergence in τ ⇒ but cancel out in the full integrand
- For individual master integral, n can be non-zero. ⇒ rapidity logarithms from dimensional regularization ⇒ but cancel out in the full integrand

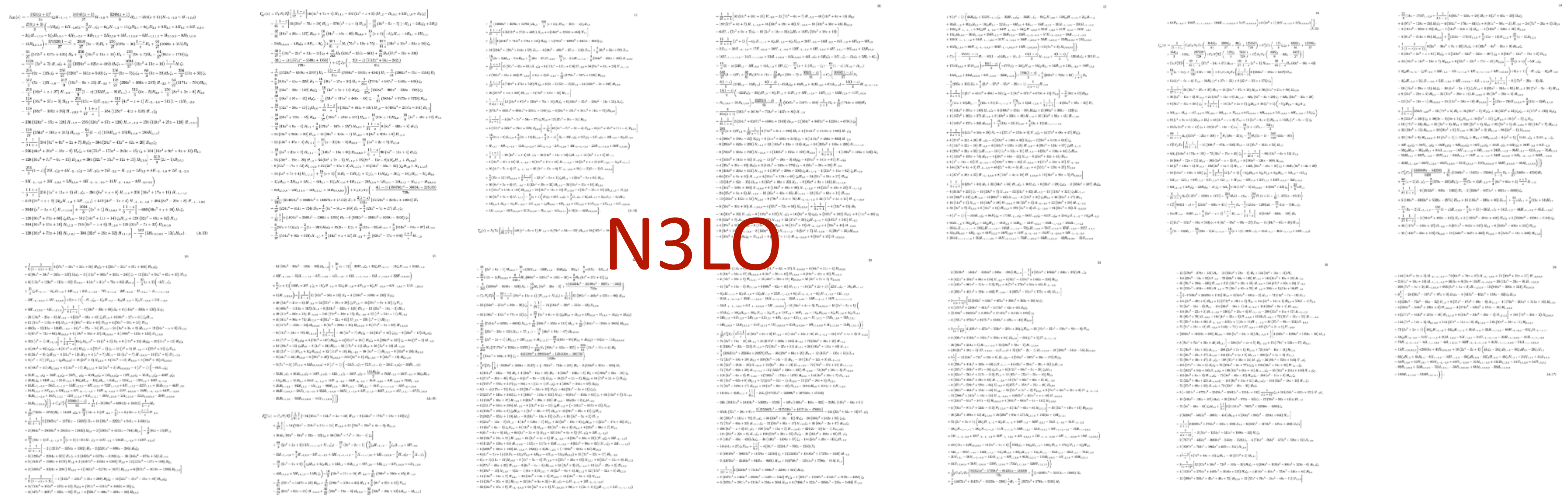
Analytic results for TMD PDFs@N3LO

- Individual integrals built from iterative integral with denominator drawn from

$$\{z, 1-z, 1+z, 2-z, z^2-z+1\}$$

Harmonic Polylogarithms
[Remiddi, Vermaseren, hep-ph/9905237]

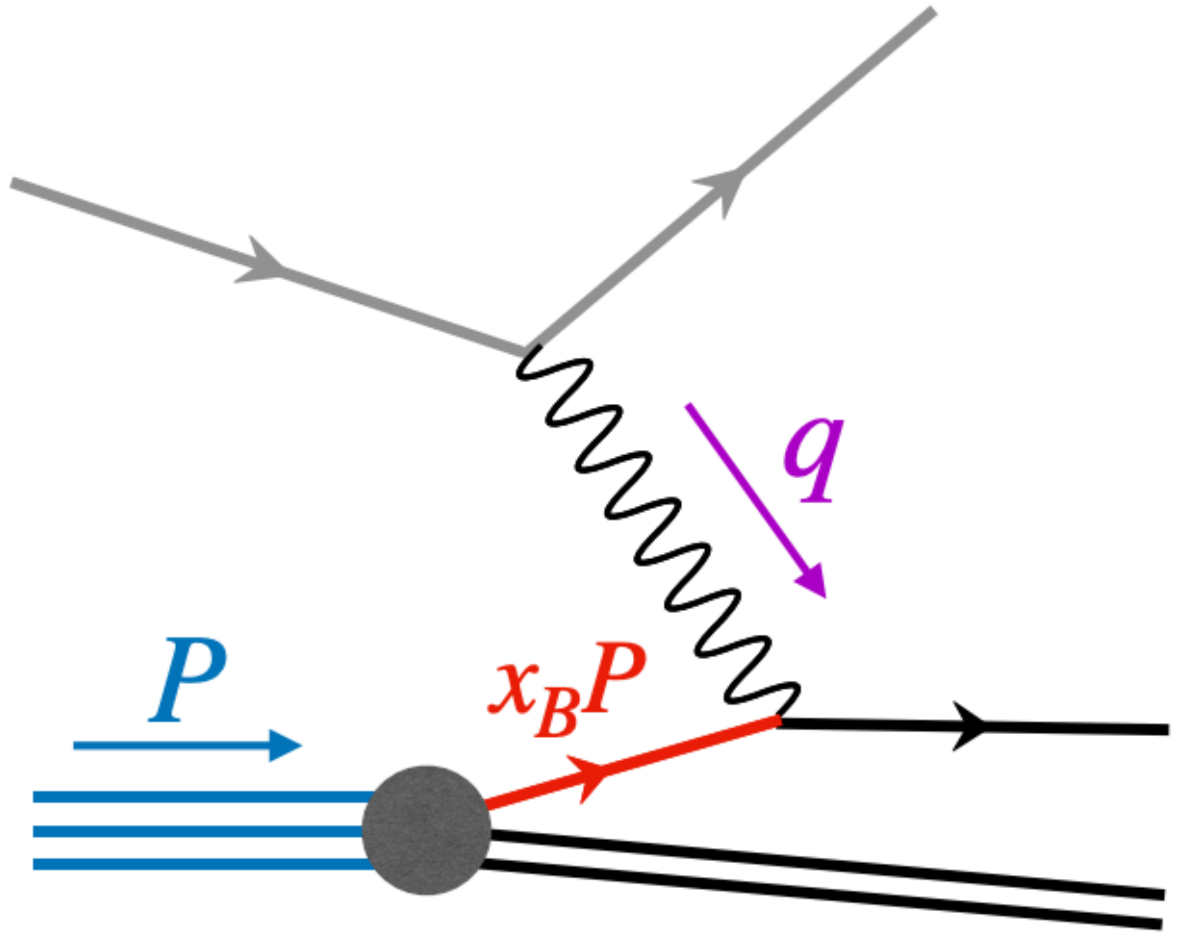
Cancel out between VRR and RRR



Full analytic results and numerical fitting for both unpolarized quark and gluon can be found in [1912.05778, 2012.03256]

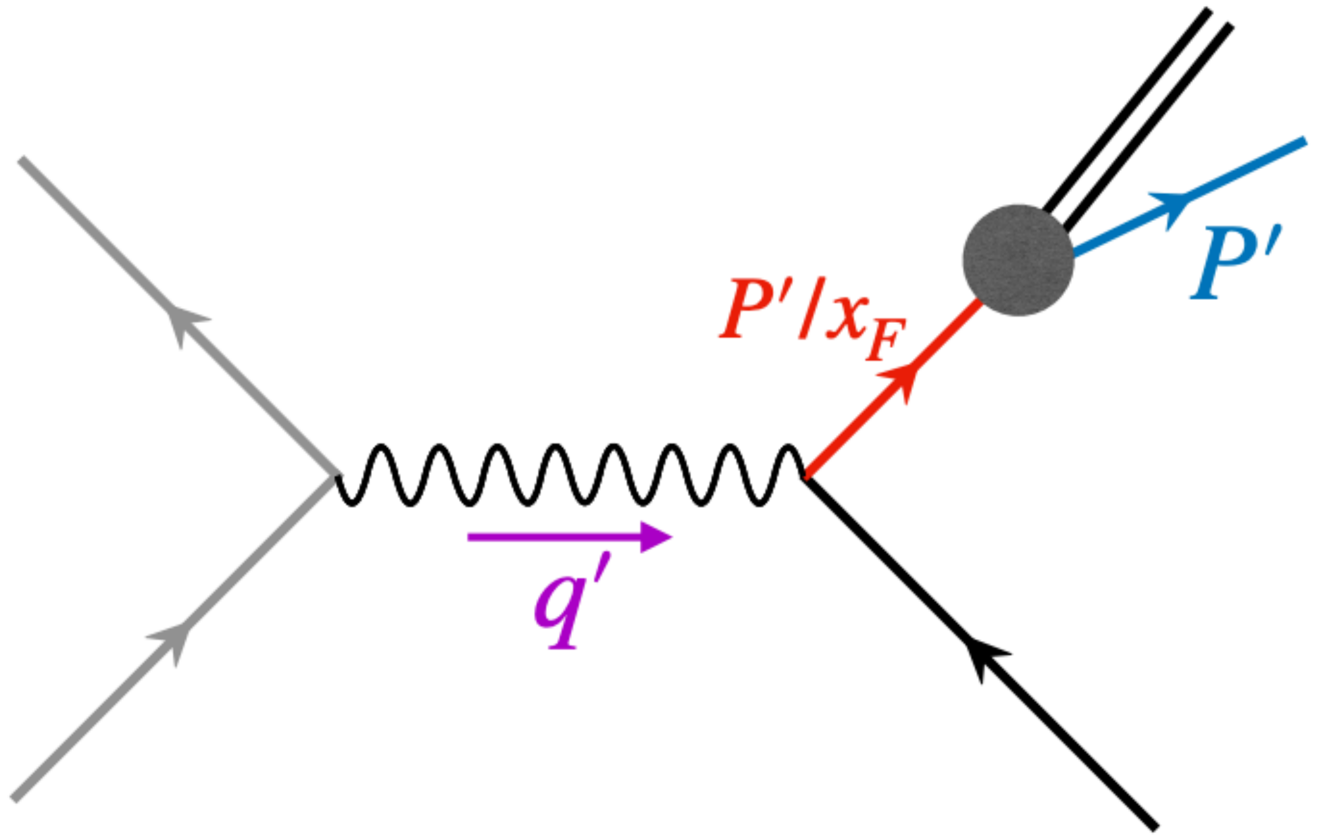
What’s good about analytic results

- Well, there is no other option at the moment. Not clear how to compute numerically with rapidity regulator
- But also because
  - Recycling: master integrals are building blocks. Once known, can be used to calculate other TMDs ([in preparation: M.X. Luo, T.Z. Yang, HXZ, Y.J. Zhu, N3LO Transversity, Helicity])
  - The transcendental weight information could provide a hint for underlying integrability structure. A famous example is DGLAP kernel in N=4 SYM [Kotikov, Lipatov, Onishchenko, Velizhanin, hep-th/0404092]
  - Facilitate study of analytic property of the results, e.g., under crossing
  - Provides exact data for asymptotic limit, such as small-x



$$x_B = \frac{-q^2}{2P \cdot q}$$

(a) DIS



(b) e^+e^-

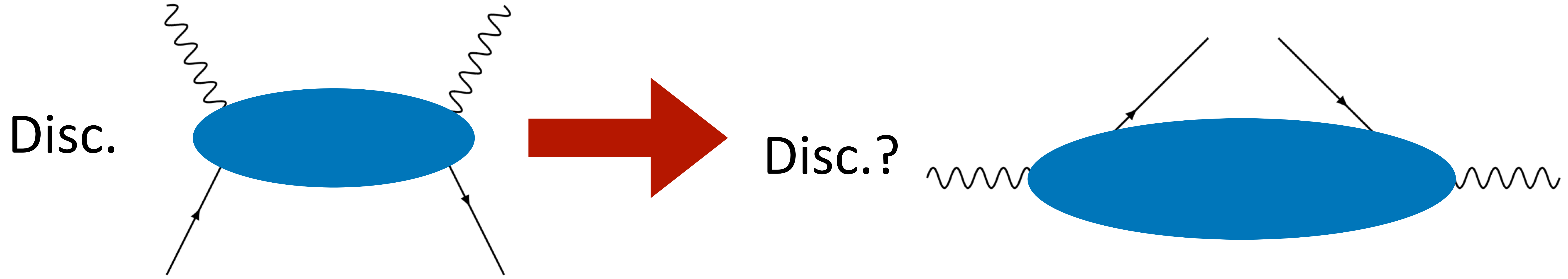
$$x_F = \frac{2P' \cdot q'}{q'^2}$$

$$x_F = \frac{1}{x_B}$$

Can one analytic continue TMD PDFs to TMD FFs by this kinematical relation?

Analytic continuation for splitting functions

- This idea has been applied to relate space-like and time-like DGLAP kernel [Drell, Levi, T.-M. Yan, 1970; Stratmann, Vogelsang, hep-ph/9612250; Blumlein, Ravindran, van Neerven, hep-ph/0004172; Mueller, Pire, Szymanowski, Wagner, 1203.4392]
- First (incomplete) results on NNLO time-like splitting kernel obtained in this way [Mitov, Moch, Vogt, hep-ph/0604053; Moch, Vogt, 0709.3899; Almasy, Moch, Vogt, 1107.2263]

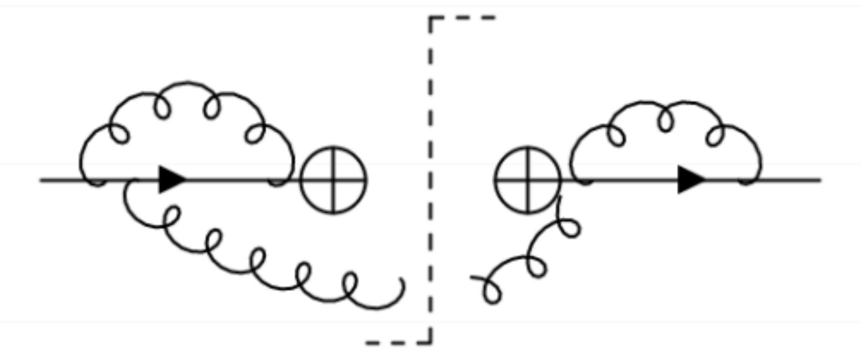


$$\ln(1-x_B) \to \ln\left(1-\frac{1}{x_F}\right) = \ln(1-x_F) - \ln x_F + i\sigma\pi\,, \quad \sigma = 0,1$$

- Several tricks are needed to check/fix the results: reciprocity respect evolution [Dokshitzer, Marchesini, Salam, hep-ph/0511302], momentum conservation sum rule, N=1 Supersymmetry relation
- Still P_{qg} not fully determined, P_{gq} need check

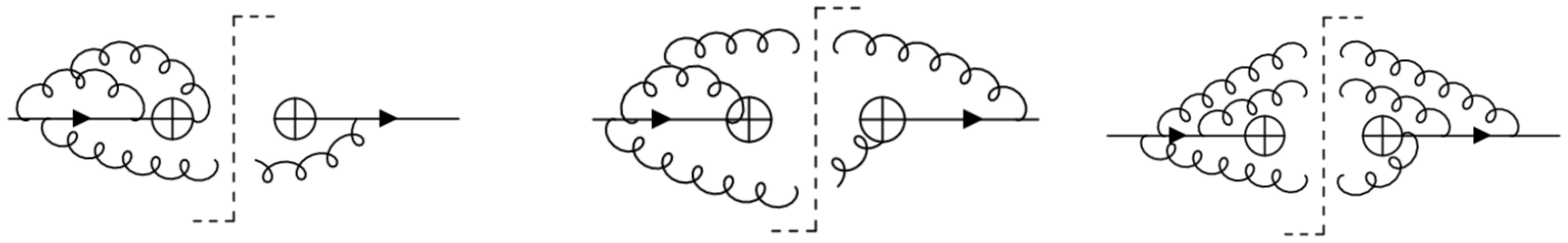
Problem with direct analytic continuation

- It is easy to see the problem with direct analytic continuation from Divide and Conquer point of view



(b) VV*R

Contain both holomorphic and anti-holomorphic



(a) VVR (c) VRR (d) RRR

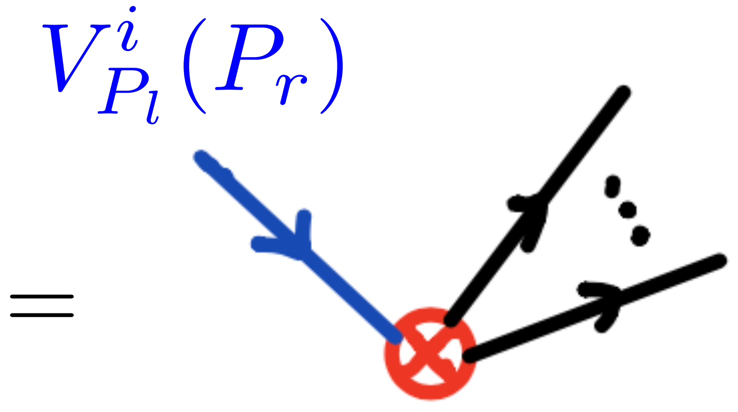
Holomorphic or anti-holomorphic

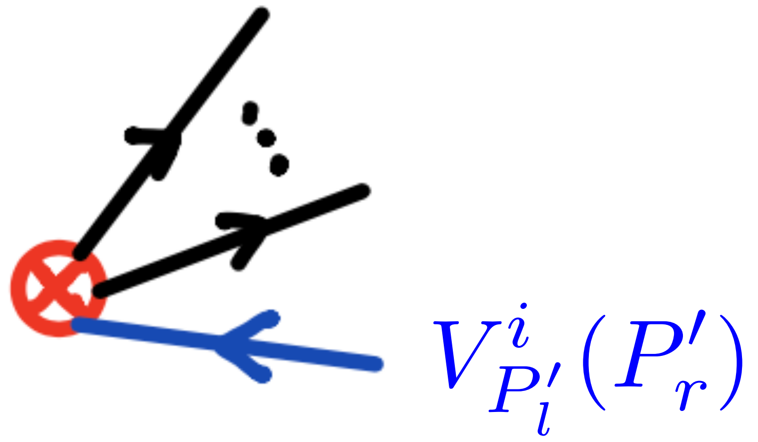
- Analytic continuation for (anti) holomorphic parts possible, as long as
  - Phase space integrals do not induce branch cut as x->1/x
  - Rapidity regulator does not introduce non-analyticity

Analytic continuation of splitting amplitudes

[H. Chen, T.Z. Yang, HXZ, Y.J. Zhu, 2006.10534]

- Splitting amplitudes are local matrix element in SCET

$$\langle X_n|\chi_n(0)|V^i_{P_l}(P_r)\rangle =$$ 

$$\langle X_n V^i_{P'_l}(P'_r)|\chi_n(0)|0\rangle =$$ 

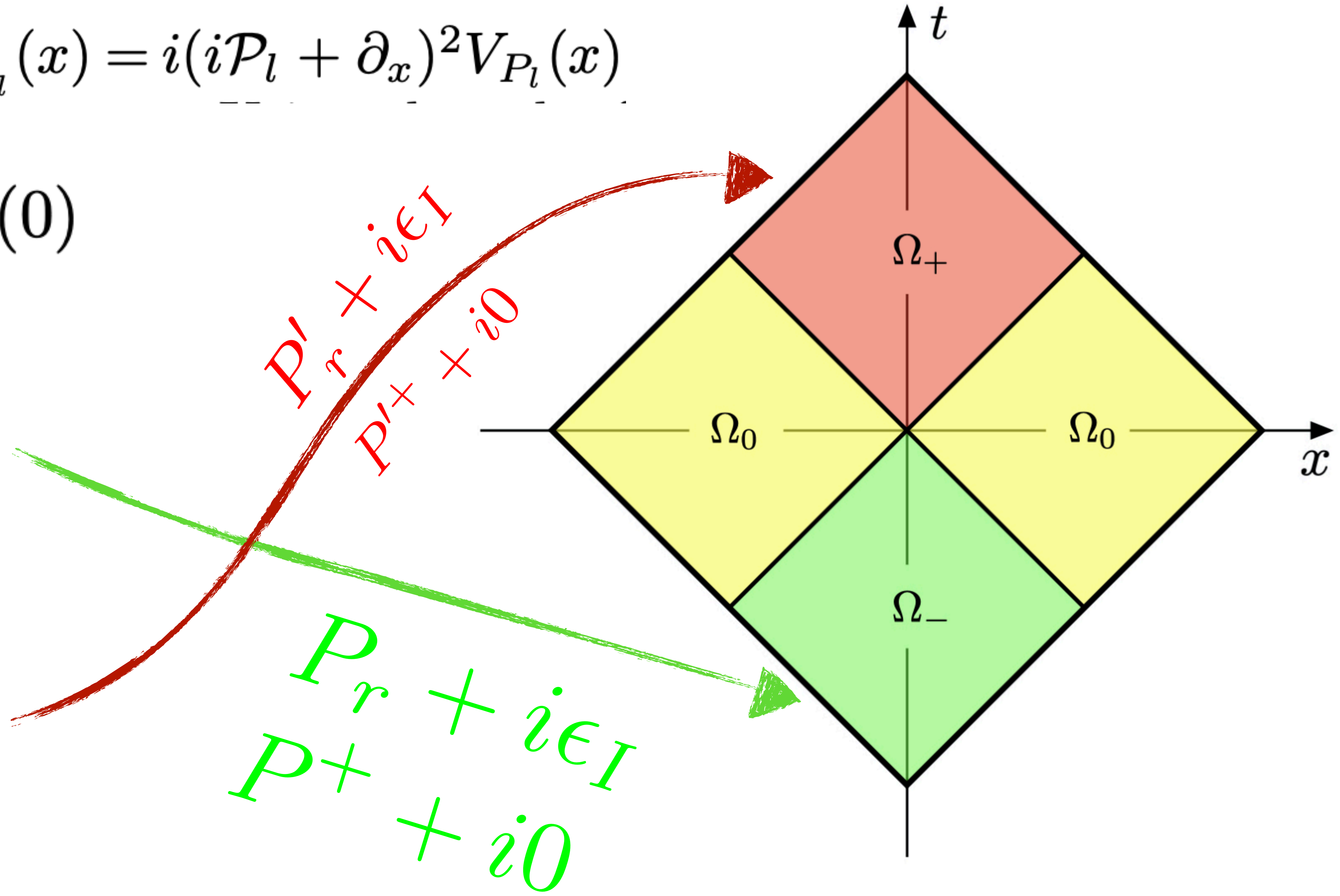
- We are interested in the casual prescription for residual momentum P_r and P'_r

$$\langle X_n|\chi_n(0)|V^i_{P_l}(P_r)\rangle = \int d^dx\, e^{-iP_r\cdot x}\langle X_n|\mathrm{T}\{\chi_n(0)J^i_{P_l}(x)\}|0\rangle \qquad J^i_{P_l}(x) = i(i\mathcal{P}_l+\partial_x)^2 V_{P_l}(x)$$

$$\mathrm{T}\{\chi_n(0)J^i_{P_l}(x)\} = \theta(-x^0)\left[\chi_n(0), J^i_{P_l}(x)\right]_{\mp} \pm J^i_{P_l}(x)\chi_n(0)$$

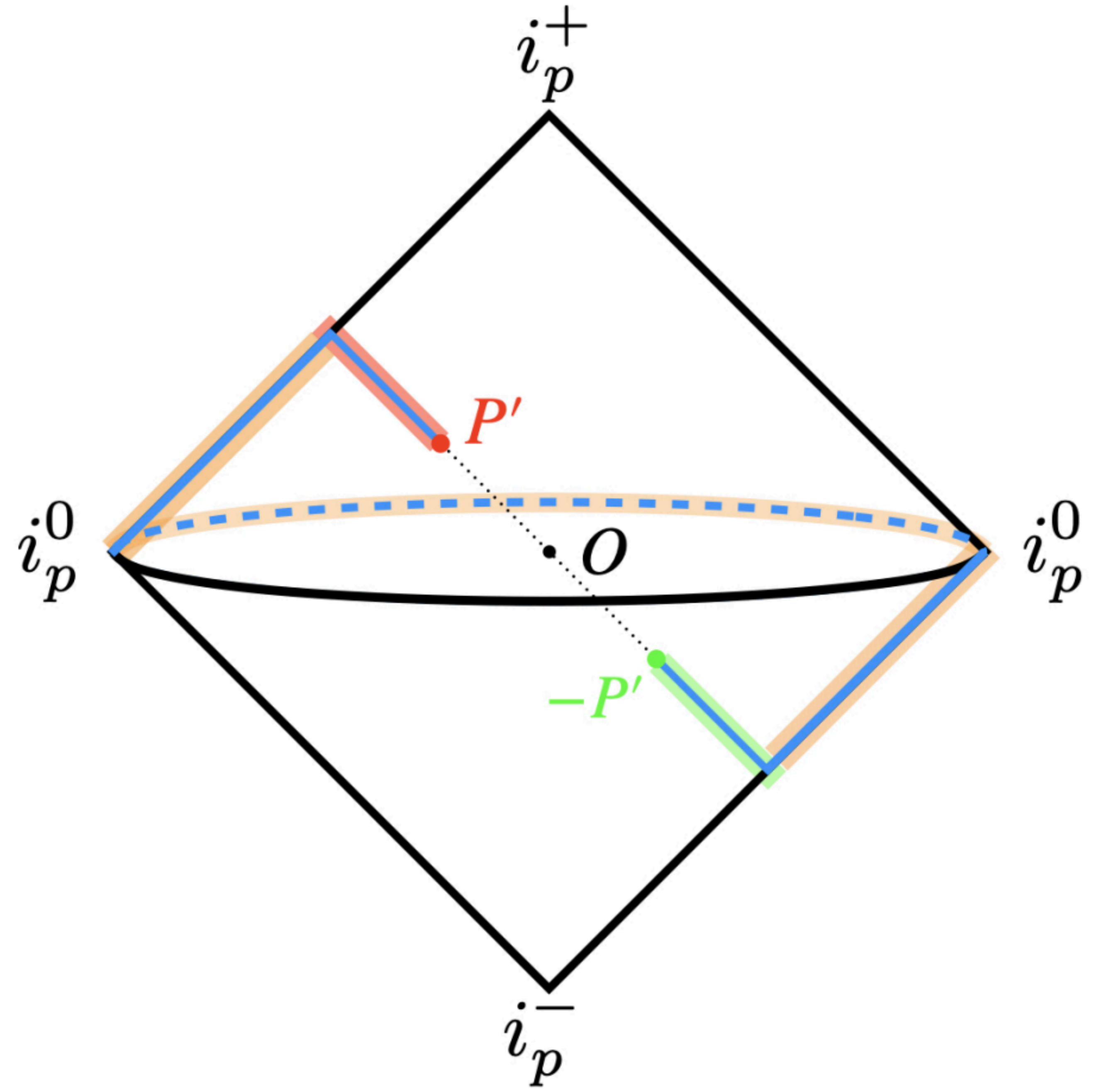
$$\mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i} = \int_{x\in\Omega_-} d^dx\, e^{-iP_r\cdot x}\langle X_n|[\chi(0), J^i_{P_l}(x)]_{\mp}|0\rangle$$

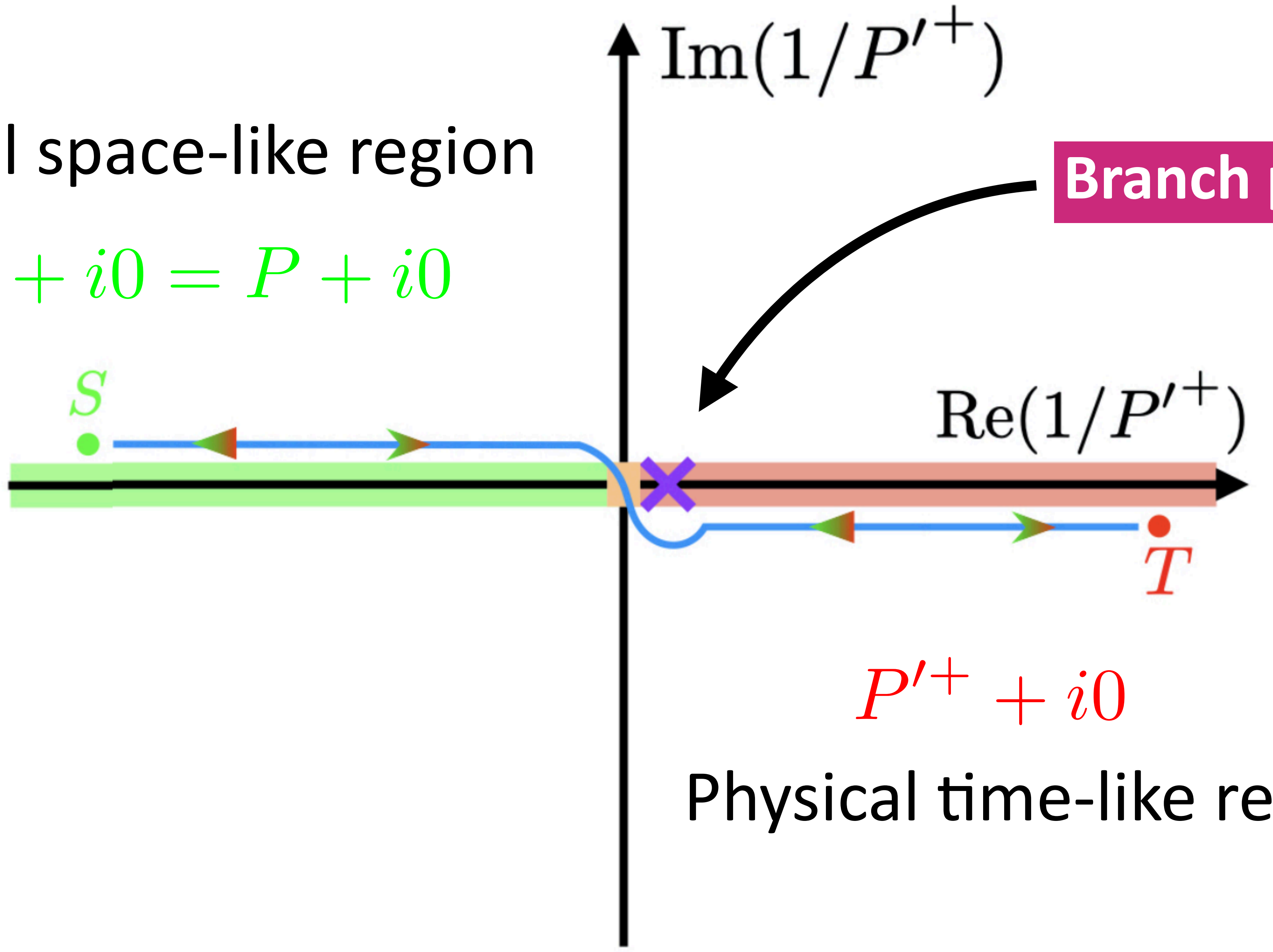
$$\mathbf{Sp}^{\mathrm{T}}_{X_n\bar{i}\leftarrow\bar{q}} = \int_{x\in\Omega_+} d^dx\, e^{iP'_r\cdot x}\langle X_n|[J^{\bar{i}}_{P'_l}(x), \chi(0)]_{\mp}|0\rangle$$



Continuation prescription

[H. Chen, T.Z. Yang, HXZ, Y.J. Zhu, 2006.10534]





$$\underset{\mathrm{T}\to\mathrm{S}}{\mathcal{AC}} \circ \mathbf{Sp}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow\bar{q}}\left(\frac{k_a^+}{P'^+ e^{i0_+}}, \cdots\right) \equiv \mathbf{Sp}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow\bar{q}}\left(\frac{k_a^+}{P'^+ e^{i(\pi+0_+)}}, \cdots\right) = \mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i}\left(\frac{k_a^+}{P^+ e^{i0_+}}, \cdots\right).$$

$$\underset{\mathrm{S}\to\mathrm{T}}{\mathcal{AC}} \circ \mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i}\left(\frac{k_a^+}{P^+ e^{i0_+}}, \cdots\right) \equiv \mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i}\left(\frac{k_a^+}{P^+ e^{-i\pi+i0_+}}, \cdots\right) = \mathbf{Sp}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow\bar{q}}\left(\frac{k_a^+}{P'^+ e^{i0_+}}, \cdots\right).$$

$$x_B = \frac{Q}{2P^+}, \quad x_F = \frac{2P'^+}{Q}$$

$$(1 - x_B) \to \left(\frac{1}{x_F} - 1\right) e^{i\pi}$$

Time-like splitting functions at N3LO

- In this way we determine the full N3LO unpolarized quark and gluon TMD FFs
- Since time-like DGLAP governs the evolution of TMD FFs, we also determine the full time-like DGLAP
- First complete determination of P_{qg}

$$P_{qg}^{T,(2)}\Big|_{CYZZ,2006.10534} - P_{qg}^{T,(2)}\Big|_{AMV,1107.2263}$$
$$= -\frac{\pi^2}{3}(C_A - C_F)\beta_0[-4 + 8z + z^2 + 6(1 - 2z + 2z^2)\ln z]$$

- The difference can't be detected by supersymmetry relation, nor momentum conservation sum rule
- The new time-like Provide new data point for reciprocity

Reciprocity in N=4 SYM and QCD

- Gribov-Lipatov reciprocity $\gamma_{ij}^S(N) = \gamma_{ji}^T(N)$
- Modified from small-x consideration (N->1) [Mueller, 1983; Neill, Ringer, 2003.02275]

$$2\gamma^T(N) = 2\gamma^S(N - 2\gamma_T(N))$$

- Inspired from large-x consideration: reciprocity respect evolution (non-singlet) [Dokshitzer, Marchesini, Salam, hep-ph/0511302]

$$\partial_t D(x, Q^2) = \int_0^1 \frac{dz}{z} \mathcal{P}(z, \alpha_s(z^{-1}Q^2))\, D\left(\frac{x}{z}, z^\sigma Q^2\right)$$

SL: σ=-1 TL:σ=1

- Conformal symmetry: quasipartonic operator classified by SL(2,R) [Basso, Korchemsky, hep-th/0612247]

Conformal spin $2j = N + \Delta = 2N + L + 2\gamma \;\Rightarrow\; 2\gamma^S(N) = f(N + \gamma^S(N))$

- **Assuming that** $2\gamma^T(N) = f(N - \gamma^S(N))$

$$\Rightarrow 2\gamma^T(N) = 2\gamma^S(N - 2\gamma^T(N))$$

Evidence for reciprocity in QCD

- Reciprocity respected anomalous dimension also found in N=4 SYM for twist 3 [Beccaria, Dokshitzer, Marchesini, 0705.2639]
- Reciprocity holds in full QCD for non-singlet [Basso, Korchemsky, hep-th/0612247]
- Using our new P_{qg}, we found new evidence in the singlet sector through NNLO

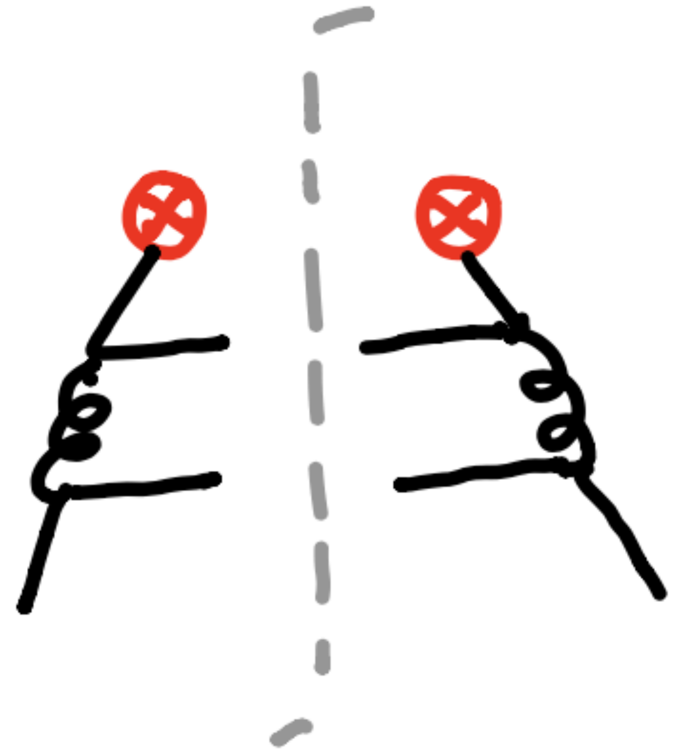
$$2\gamma^{\rm S}_{\pm}(N,\alpha_s) = 2\gamma^{\rm T}_{\pm}(N + 2\gamma^{\rm S}_{\pm}(N,\alpha_s),\alpha_s)\,,$$
$$2\gamma^{\rm T}_{\pm}(N,\alpha_s) = 2\gamma^{\rm S}_{\pm}(N - 2\gamma^{\rm T}_{\pm}(N,\alpha_s),\alpha_s)$$

[H. Chen, T.Z. Yang, HXZ, Y.J. Zhu, 2006.10534]

$$\gamma_\pm = \frac{1}{2}\left(\pm\sqrt{({\rm tr}\widehat{\gamma})^2 - 4{\rm det}\widehat{\gamma}} + {\rm tr}\widehat{\gamma}\right)$$

- Can be proved in weakly coupled CFT combining light-ray OPE [Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov, 1905.01311] and fragmentation factorization for Energy-Energy Correlator [Korchemsky, 1905.01444; Dixon, Moult, HXZ, 1905.01310; H. Chen, Moult, X.Y. Zhang, HXZ, 2004.11381]
- Holds for all N!
- Functional equation relation space-like and time-like quantities
- The additional π^2 term in NNLO P_{qg} essential to make reciprocity work

Small-x TMD PDFs to N3LO



$$xI^{(2)}_{qq}(x) = xI^{(2)}_{qq'}(x) = xI^{(2)}_{q\bar{q}}(x) = xI^{(2)}_{q\bar{q}'}(x) = 2C_F T_F \left(\frac{172}{27} - \frac{8\zeta_2}{3}\right),$$

$$\begin{aligned} xI^{(3)}_{qq}(x) =& xI^{(3)}_{qq'}(x) = xI^{(3)}_{q\bar{q}}(x) = xI^{(3)}_{q\bar{q}'}(x) \\ =& 2T_F \Bigg[\left(\frac{208\zeta_2}{9} + \frac{32\zeta_3}{3} - \frac{17152}{243}\right) C_A C_F \ln x + \left(-16\zeta_2 + \frac{512}{9}\zeta_3 + \frac{32}{3}\zeta_4 - \frac{269}{9}\right) C_F^2 \\ &+ \left(\frac{12008\zeta_2}{81} + 120\zeta_3 + \frac{920\zeta_4}{9} - \frac{456266}{729}\right) C_A C_F + \left(-\frac{32\zeta_2}{9} - \frac{64\zeta_3}{9} + \frac{16928}{729}\right) C_F N_f T_F \Bigg] \end{aligned}$$

- Single logarithmic enhance at small x
- Leading log formula provide in [marzani, 1511.06039]
- Using the LL formula and relevant anomalous dimension, we worked out the LL expansion at N3LO

$$\frac{208\zeta_2}{9} + \frac{32\zeta_3}{9} - \frac{17152}{243} \quad \text{(LL)} \qquad \text{v.s.} \qquad \frac{208\zeta_2}{9} + \frac{32\zeta_3}{3} - \frac{17152}{243} \quad \text{(N3LO, [1912.05778])}$$

- Calls for further efforts to pin down the discrepancy

Also see [Ebert, Mistlberger, Vita, 2006.05329]

Small-z TMD FFs to N3LO

LL NLL NNLL

$$z\widehat{C}^{s(1)}_{gq}(z) = 8C_F \ln z\,,$$

$$z\widehat{C}^{s(2)}_{gq}(z) = C_A C_F\left[-\frac{80}{3}\ln^3 z - \frac{212}{3}\ln^2 z + (32\zeta_2+12)\ln z - 88\zeta_3 - \frac{88}{3}\zeta_2 + \frac{3128}{27}\right] + C_F^2\left[(48-64\zeta_2)\ln z\right],$$

$$z\widehat{C}^{s(3)}_{gq}(z) = C_A^2 C_F\left[32\ln^5 z + \frac{7816}{27}\ln^4 z + \left(\frac{7376}{9} - \frac{2560}{9}\zeta_2\right)\ln^3 z + \left(-\frac{1984}{3}\zeta_2 + \frac{1184}{3}\zeta_3 + \frac{8608}{9}\right)\ln^2 z + \left(\frac{3776}{3}\zeta_2 + \frac{3136}{3}\zeta_3 + 1408\zeta_4 - \frac{123892}{81}\right)\ln z + \frac{7456\zeta_5}{3} + \frac{608}{3}\zeta_2\zeta_3 + \frac{5870}{3}\zeta_4 - \frac{7588}{9}\zeta_3 + \frac{21944}{27}\zeta_2 - \frac{3650707}{729}\right] + C_A C_F^2\left[\left(\frac{1792}{9}\zeta_2 - \frac{1360}{9}\right)\ln^3 z + \left(-\frac{64}{3}\zeta_3 + \frac{1888}{3}\zeta_2 - \frac{1532}{3}\right)\ln^2 z + \left(\frac{488}{3}\zeta_4 - \frac{80}{3}\zeta_3 - \frac{1360}{3}\zeta_2 + \frac{701}{9}\right)\ln z + \frac{6800}{3}\zeta_5 + 992\zeta_3\zeta_2 + \frac{830}{3}\zeta_4 - 1746\zeta_3 - \frac{48568}{27}\zeta_2 + \frac{10141}{9}\right]$$

- Double logarithms at each order, much worse expansion
- Different origin of small-z logarithms compared with PDFs

Resummation of log(z) by consistency conditions

- Time-like small-x resummed for splitting functions [Vogt, 1108.2993; Kom, Vogt, Yeats, 1207.5631]. Same method can be applied here

Un-renormalized collinear factorization

$$\mathcal{F}^s_{i/j}(z,\epsilon) = \frac{1}{Z_j^B}\frac{\mathcal{F}^{s,\mathrm{bare}}_{i/j}(z,\epsilon)}{\mathcal{S}_{0b}} = \sum_k d^s_{ik} \otimes C^s_{kj}(z,\epsilon)$$

Collinear divergent ($\mathcal{F}^s_{i/j}(z,\epsilon)$)

Pure collinear divergent. Structure governed by time-like DGLAP (d^s_{ik})

Infrared finite ($C^s_{kj}(z,\epsilon)$)

n independent coefficient ($c_{gi}^{(1,l,n)}$)

n-th order expansion

$$\mathcal{F}^{s(n)}_{g/i}(z,\epsilon) = \frac{1}{\epsilon^{2n-1}}\sum_{l=0}^{n-1} z^{-1-2(n-l)\epsilon}\Big(\underbrace{c_{gi}^{(1,l,n)}}_{\mathrm{LL}} + \underbrace{\epsilon c_{gi}^{(2,l,n)}}_{\mathrm{NLL}} + \underbrace{\epsilon^2 c_{gi}^{(3,l,n)}}_{\mathrm{NNLL}} + \dots\Big)$$

Actual pole structure:

$$\frac{0}{\epsilon^{2n-1}} + \frac{0}{\epsilon^{2n-2}} + \frac{0}{\epsilon^{2n-3}} + \cdots + \frac{0}{\epsilon^{n+1}} + \frac{\overbrace{P^T \otimes P^T \otimes \cdots \otimes P^T}^{n\ \mathrm{times}}}{\epsilon^n} + \cdots$$

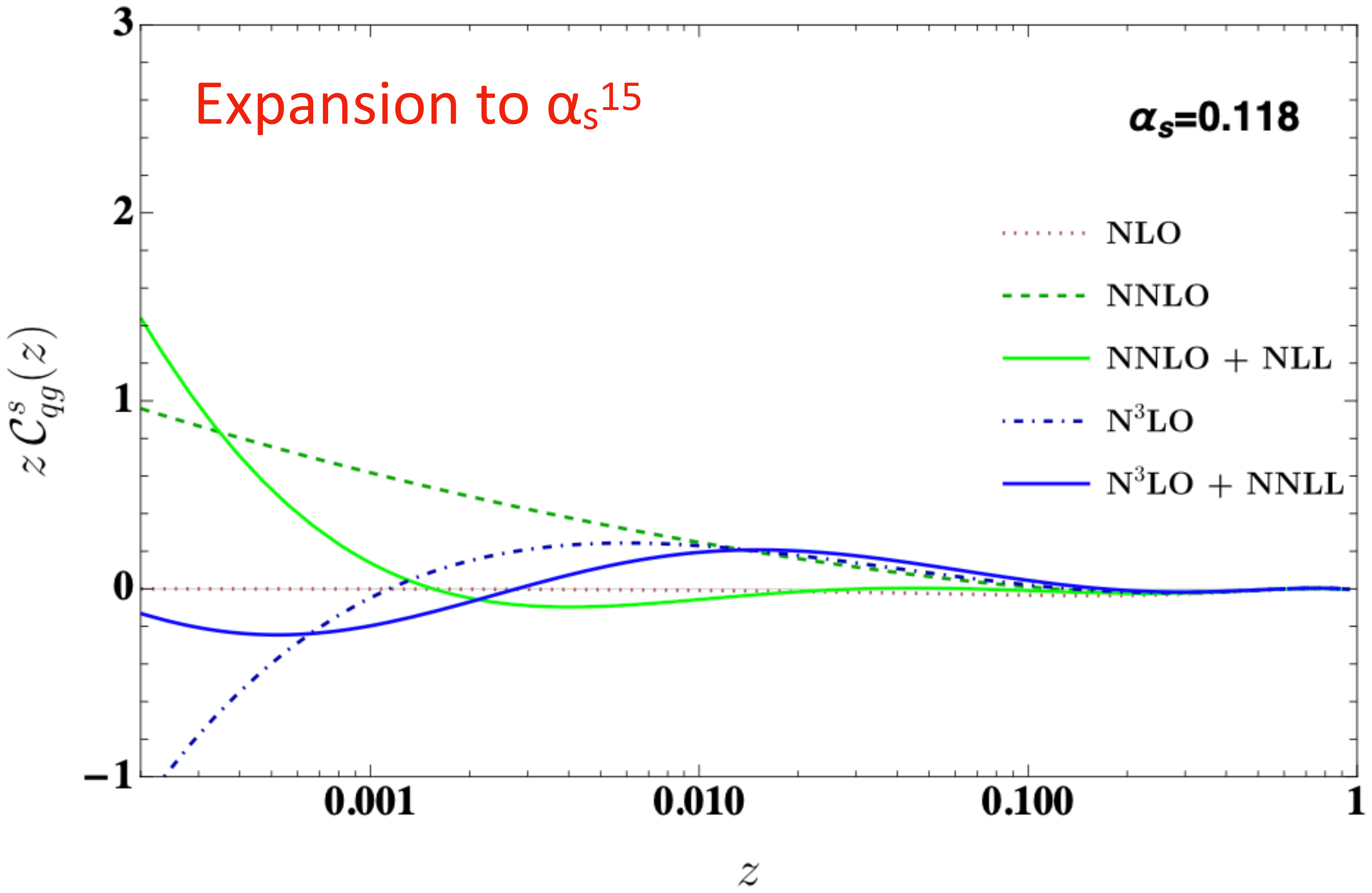
+ running coupling

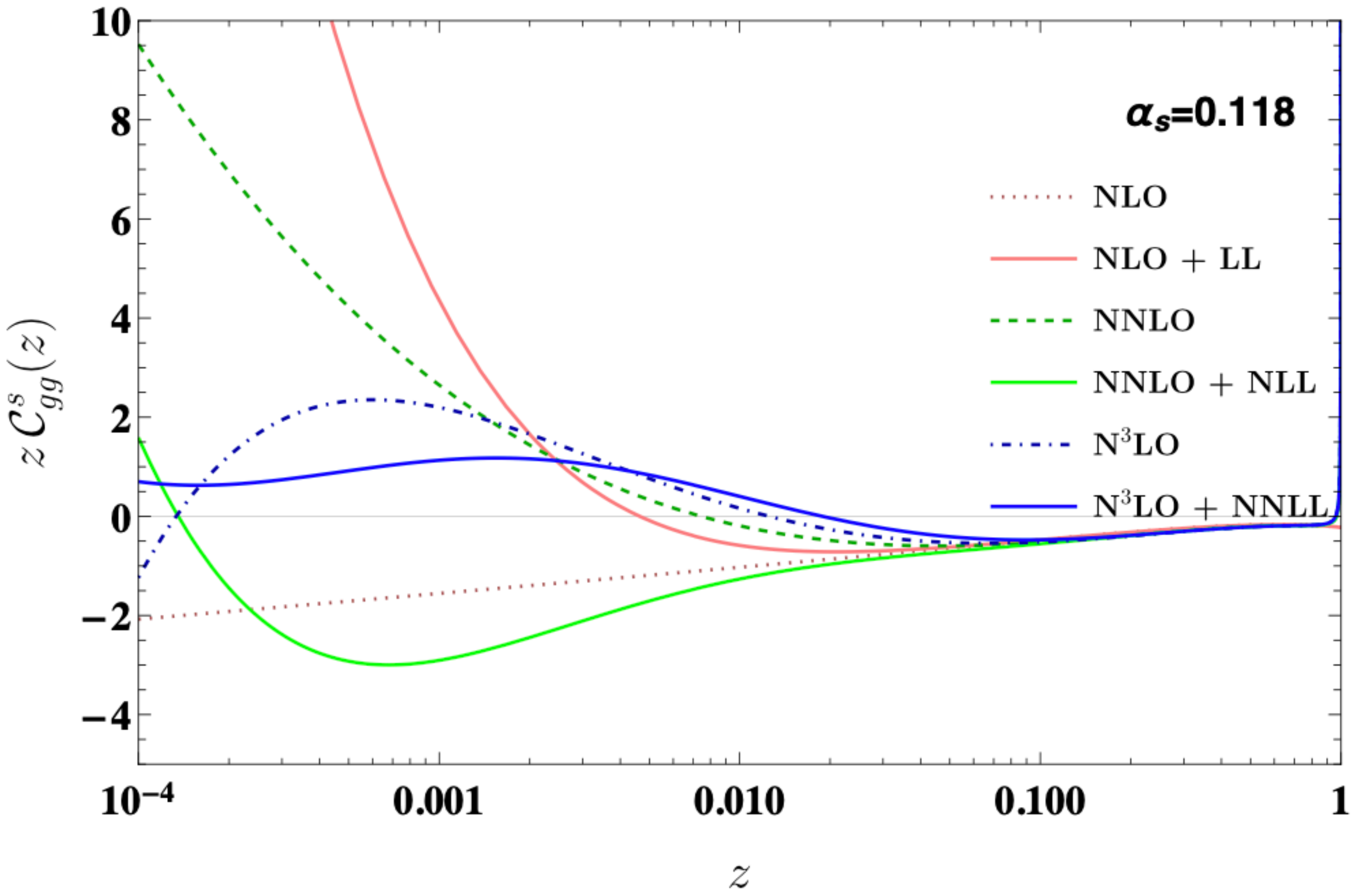
- Uniquely determine all $c_{gi}^{(1,l,n)}$

TMD FFs at N3LO + NNLL

- Time-like log(z) resummation lead to finite results at z=0 ($\bar{N}$=0)

$$\widehat{C}^s_{gg}(\overline{N})|_{\rm LL} = \frac{C_A}{C_F}\widehat{C}^s_{gq}(\overline{N})|_{\rm LL} = \left(1+\frac{32C_A a_s}{\overline{N}^2}\right)^{-1/4} - 1$$





Conclusion

- Analytic results for all unpolarized TMD PDFs + FFs at N3LO (Numerical fits provided)
- Resolve the long missing piece of P_{qg} in NNLO time-like splitting functions
- First evidence for generalized Gribov-Lipatov reciprocity in QCD singlet splitting function up to NNLO
- Small-z resummation for TMD FFs to NNLL
- Outlook:
  - qT subtraction at N3LO
  - All twist 2 TMDs at N3LO
  - Phenomenology for TMD determination

Thank you for your attention!