Lattice QCD thermodynamics

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Outline overall

- lecture 1: introduction to QCD and thermodynamics
- lecture 2: hot Yang-Mills theory on the lattice
- lecture 3: hot QCD on the lattice
- lecture 4: QCD in extreme conditions on the lattice

Outline lecture 2

- "rest" from lecture 1
- lattice discretization of the QCD action
- confinement and deconfinement
- Polyakov loop

Non-interacting systems at finite T

Scalars, free case

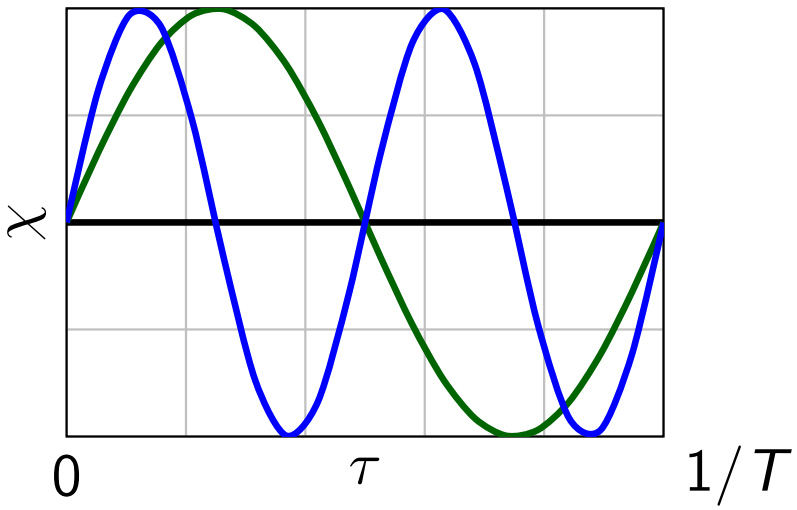
- scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\phi\underbrace{\left[-\partial^2 + m^2\right]}_{M}\phi$$

- eigensystem $M\chi = \lambda^2\chi$

$$\chi = e^{i\omega_n\tau + i\mathbf{p}\mathbf{x}} \qquad \lambda^2 = \omega_n^2 + \mathbf{p}^2 + m^2 \qquad \omega_n = 2n\pi T \quad n \in \mathbb{Z}$$

bosonic Matsubara frequencies



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bosonic Matsubara frequencies

- determinant

$$\log\det M = \operatorname{tr}\log M = T\sum_n \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\log\left[\omega_n^2 + \overbrace{\mathbf{p}^2 + m^2}^{E_p^2}\right]$$

Fermions, free case

- fermion field

$$\mathcal{L} = \bar{\psi}\underbrace{(\partial\!\!\!/ + m)}_{M}\psi \qquad M^\dagger M = (-\partial^2 + m^2)\mathbb{1}_{4\times 4}$$

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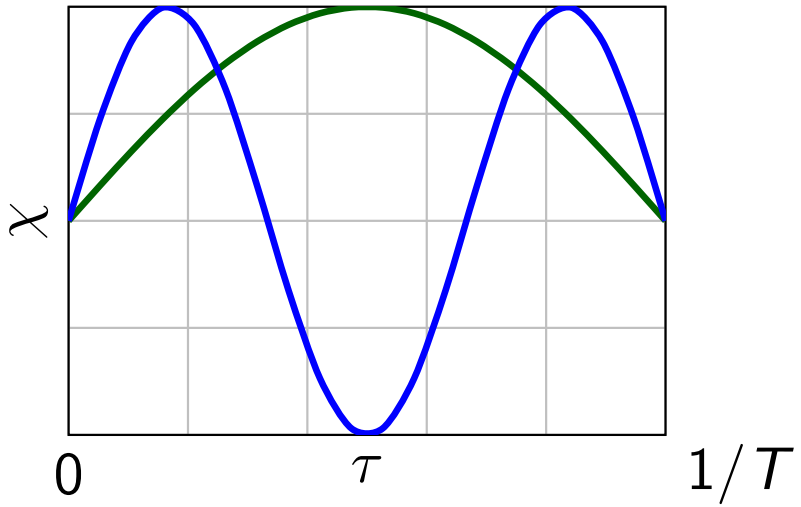
$$\chi = e^{i\bar{\omega}_n\tau + i\mathbf{p}\mathbf{x}} \qquad \lambda^2 = \bar{\omega}_n^2 + \mathbf{p}^2 + m^2 \qquad \bar{\omega}_n = (2n+1)\pi T \quad n \in \mathbb{Z}$$

fermionic Matsubara frequencies



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fermionic Matsubara frequencies

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Vacuum and thermal contributions

- trick

$$\frac{\partial}{\partial E} T \sum_n \log(\omega_n^2 + E^2) = \coth \frac{E}{2T}$$

$$\int \mathrm{d}E \coth \frac{E}{2T} = E + 2T \log(1 - e^{-E/T})$$

- real scalars

$$f_{\mathrm{rs}} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left[\frac{E_p}{2} + T \log(1 - e^{-E_p/T}) \right]$$

- fermions

$$f_{\mathrm{f}} = -4 \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left[\frac{E_p}{2} + T \log(1 + e^{-E_p/T}) \right]$$

- note: Bose/Fermi statistics, sign of vacuum energy, spin and particle/antiparticle degrees of freedom

Gauge field theory

- gauge field A_μ ($\mathrm{U}(1)$) or A^a_μ with $a = 1 \ldots N_c^2 - 1$ ($\mathrm{SU}(N_c)$)
- partition function

$$\mathcal{Z} = \int_{A^a_\mu(x,0)=A^a_\mu(x,1/T)} \mathcal{D}A^a_\mu \, e^{-S[A^a_\mu]}$$

- note 1: this is gauge invariant, but to derive it we needed to fix a gauge
- note 2: to evaluate $\mathcal{Z}$ via perturbation theory, we need to fix a gauge again
- free case

$$f_{\mathrm{U}(1)} = f_{\mathrm{rs}} \cdot (4-2) \qquad f_{\mathrm{SU}(N_c)} = f_{\mathrm{rs}} \cdot (4-2) \cdot (N_c^2 - 1)$$

ghosts cancel half the gauge field contribution

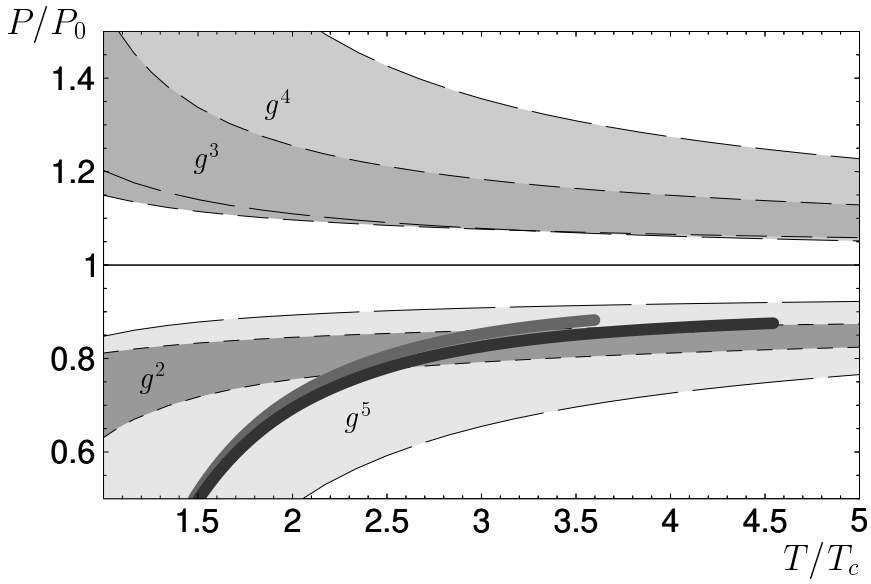
Perturbation theory

- pure glue free pressure

$$p_{\mathrm{SU}(3)} = -f_{\mathrm{SU}(3)} = -16 \cdot \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} T \log(1 - e^{-|\mathbf{p}|/T}) = \frac{8\pi^2}{45} \cdot T^4$$

- perturbation theory in g (vertices A^3 and A^4)

  Blaizot, Iancu, Rebhan PRD '03



bands: renormalization scale depence $g^2(\pi T \leq \bar{\mu} \leq 4\pi T)$

- need for lattice gauge theory

Remark: ultraviolet divergences

- thermal free energy is UV finite ($e^{-E_p/T}$ for large p)
- vacuum free energy is UV divergent

$$f_{\rm vac} = \frac{1}{2}\int_0^\Lambda \frac{4\pi p^2 \mathrm{d}p}{(2\pi)^3}\sqrt{p^2+m^2} = \underline{\mathcal{O}(\Lambda^4) + \mathcal{O}(m^2\Lambda^2) + \mathcal{O}(m^4 \log \Lambda^2) + \mathcal{O}(m^4)}$$

- renormalize free energy at $T = 0$ to zero

$$f^r = f - f(T=0)$$

(normal ordering in operator language)

- for the equation of state

$$p^r = p - p(T=0) \qquad \epsilon^r = \epsilon - \epsilon(T=0)$$

no renormalization necessary for entropy

Finite T QFT: summary

- all information encoded in $\log \mathcal{Z}$
- path integral representation for bosons/fermions with periodic/antiperiodic boundary conditions
- insight from free / weakly interacting cases
  - Matsubara frequencies
  - T-independent divergences
  - slow convergence of perturbation theory

QCD on the lattice

In the continuum

- Euclidean path integral

$$\mathcal{Z} = \int \mathcal{D}A_\mu \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, \exp\left[- \int_0^{1/T} \mathrm{d}x_4 \int_{L^3} \mathrm{d}^3x \, \mathcal{L}(x) \right]$$

- Lagrangian

$$\mathcal{L} = \sum_f \bar{\psi}_f (\not{D} + m_f) \psi_f + \frac{1}{2g^2} \mathrm{Tr}\, G_{\mu\nu} G_{\mu\nu}$$

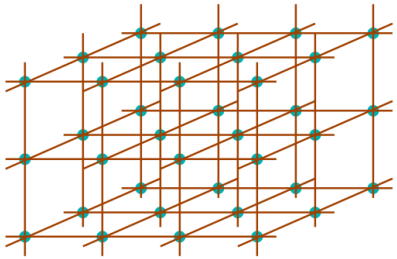
- fields

$$A_\mu = A^a_\mu T^a_{cd}, \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + [A_\mu, A_\nu]$$
$$\psi_f = \psi_{f,\alpha,c}, \quad \not{D}^{\alpha\beta}_{cd} = \gamma^{\alpha\beta}_\mu (\partial_\mu + i A^a_\mu T^a_{cd})$$

- parameters: m_f quark masses and g strong coupling

Discretization

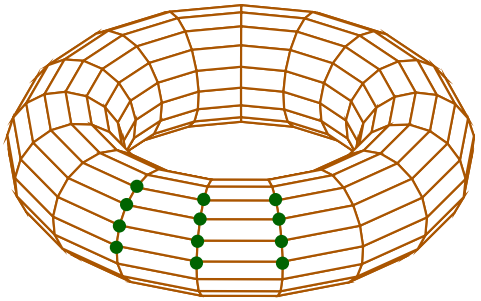
- discretize space $L = a \cdot N_s$ and imaginary time $1/T = a \cdot N_t$
  $x = a \cdot n$



- path integral

$$\mathcal{Z} = \lim_{\substack{a \to 0 \\ N_t \to \infty \\ N_s \to \infty}} \int \prod_{n,\mu} \mathrm{d}A_\mu(n)\, \mathrm{d}\bar{\psi}(n)\, \mathrm{d}\psi(n)\, \exp\left[-S^{\mathrm{lat}}_{F+G}\right]$$

- nonzero T: bosons periodic, fermions antiperiodic in imag. time
- finite volume: periodic boundary conditions in space



Fermion action

Fermion action

- free case: $\not{D}_{cd}^{\alpha\beta} = \gamma_\mu^{\alpha\beta}\partial_\mu\delta_{cd}$
- discretize derivative operator

$$\partial_\mu\psi(n) = \frac{\psi(n+\hat{\mu}) - \psi(n)}{a} + \mathcal{O}(a)$$
$$\partial_\mu\psi(n) = \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + \mathcal{O}(a^2)$$

- symmetric discretization is antihermitean and scales better

$$\mathcal{L}_F^{\mathrm{lat}} = \sum_\mu \bar{\psi}(n)\gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\bar{\psi}(n)\psi(n)$$

- gauge invariance?

Gauge invariance

- variable substitution to parallel transporter

$$U_\mu = \exp(iaA_\mu) \in \mathrm{SU}(3)$$

- gauge transformation (Ω unitary)

$$\psi(n) \to \Omega(n)\psi(n), \qquad \bar{\psi}(n) \to \bar{\psi}(n)\Omega^\dagger(n)$$
$$U(n \to n+\hat{\mu}) = U_\mu(n) \to \Omega(n)U_\mu(n)\Omega^\dagger(n+\hat{\mu})$$

- backward parallel transporter

$$U(n \to n-\hat{\mu}) = U_\mu^\dagger(n-\hat{\mu})$$

- action must be gauge invariant

$$\bar{\psi}(n)U_\mu(n)\psi(n+\hat{\mu}) \quad \to \quad \text{fermion action}$$
$$\mathrm{Tr}\,U(n_0 \to n_1)U(n_1 \to n_2)\ldots U(n_{k-1} \to n_0) \quad \to \quad \text{gluon action}$$

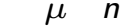
Fermion action

- include interaction with gluons in a gauge invariant way

$$\mathcal{L}_F^{\mathrm{lat}} = \sum_\mu \bar{\psi}(n)\gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})}{2a} + m\bar{\psi}(n)\psi(n)$$

- symbolically

$U_\mu(n) =$

$U_\mu^\dagger(n-\hat{\mu}) =$ 

so the Dirac operator

$\not{D} = \frac{1}{2}\sum_\mu \gamma_\mu \Big[$

$-$

$\Big]$

Gluon action

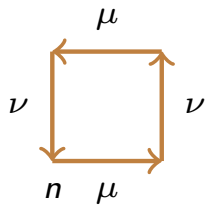
Gluon action

- smallest closed curve: plaquette

$$P_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)$$

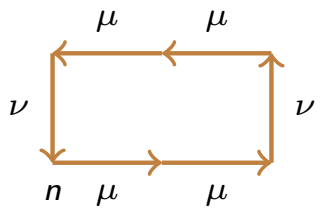
- gauge invariant combination

$$\mathrm{Tr}\, P_{\mu\nu}(n) =$$



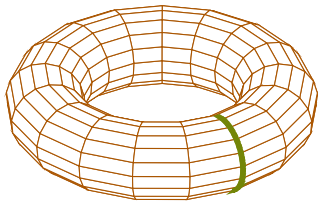
- other possibilities: larger closed loops

$$\mathrm{Tr}\, P_{\mu\nu}^{1\times 2}(n) =$$



or winding loops (Polyakov loop)

$$\mathrm{Tr}\, L_\mu(n) =$$



Gluon action

- plaquette in terms of A_μ

$$P_{\mu\nu}(n) = e^{iaA_\mu(n)} e^{iaA_\nu(n+\hat{\mu})} e^{-iaA_\mu(n+\hat{\nu})} e^{-iaA_\nu(n)}$$

- Baker-Campbell-Hausdorff formula

$$e^{aA} e^{aB} = e^{aA+aB+[aA,aB]/2+\mathcal{O}(a^3)}$$

- expand shifted gauge fields

$$A_\mu(n+\hat{\nu}) \approx A_\mu(n) + a\partial_\nu A_\mu(n) + \mathcal{O}(a^2)$$

- to recover field strength

$$P_{\mu\nu}(n) = \exp\left[ia^2 G_{\mu\nu}(n) + \mathcal{O}(a^3)\right]$$

Gluon action

- take trace to make gauge invariant

$$\mathrm{Re}\,\mathrm{Tr}(\mathbb{1} - P_{\mu\nu}(n)) = \frac{a^4}{2}\mathrm{Tr}\, G_{\mu\nu}(n) G_{\mu\nu}(n) + \mathcal{O}(a^6)$$

- full action

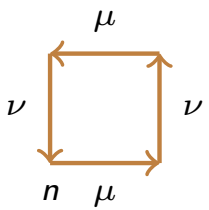
$$S_G^{\mathrm{lat}} = \frac{\beta}{3}\sum_n \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{Tr}(\mathbb{1} - P_{\mu\nu}(n))$$

- strong coupling parameter

$$\beta = \frac{6}{g^2}$$

- path integral over links

$$\mathcal{Z} = \int \mathcal{D}U \exp\left[-\beta \cdot \frac{1}{3}\sum_{n,\mu<\nu} \mathrm{Re}\left(3 - \right.\right.$$



$$\left.\left.\right)\right]$$

Continuum limit

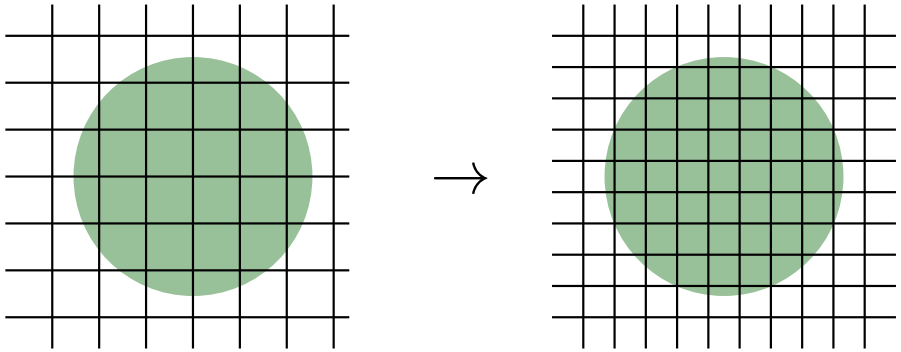
- continuum RG equation perturbatively

$$\bar{\mu}\frac{\partial g_R(\bar{\mu})}{\partial \bar{\mu}} = -b_1 g_R^3(\bar{\mu}) + \dots$$

- lattice RG equation perturbatively

$$a^{-1}\frac{\partial g(a^{-1})}{\partial (a^{-1})} = -b_1 g^3(a^{-1}) + \dots$$

- asymptotic freedom: cont. limit at $g \to 0$, $\beta = 6/g^2 \to \infty$
- continuum limit nonperturbatively



- lattice spacing from scale setting $\boxed{a(\beta)}$

Finite temperature on the lattice

Finite temperature on the lattice

- spatial size: $L = N_s a(\beta)$
- temporal size: $1/T = N_t a(\beta)$

Finite temperature on the lattice

- spatial size: $L = N_s a(\beta)$
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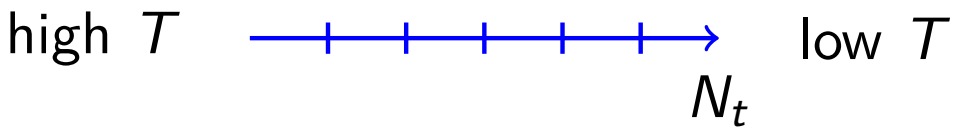
- fixed-β approach: change T by changing N_t
  - only discrete temperatures possible ×
  - all temperatures have same lattice spacing ✓
  - scale setting and renormalization only once ✓



Finite temperature on the lattice

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- temporal size: $1/T = N_t a(\beta)$
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  - all temperatures have same lattice spacing ✓
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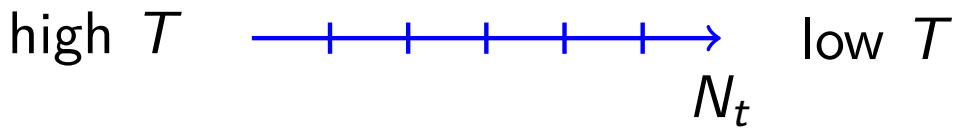
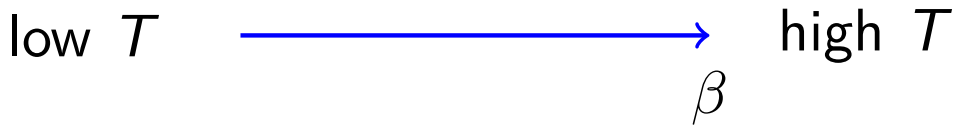


- fixed-N_t approach: change T by changing β
  - continuous temperatures possible ✓
  - different temperatures have different lattice spacing ×
  - scale setting and renormalization for each β ×



Confinement and deconfinement

Confinement

- free energy of thermal medium: F
  free energy in presence of static quark: F_q
- confinement: $F_q - F = \infty$
  deconfinement: $F_q - F < \infty$
- operator language

$$e^{-(F_q - F)/T} = \frac{\operatorname{tr} e^{-\hat{H}/T} \hat{P}_q}{\operatorname{tr} e^{-\hat{H}/T}}$$

  with P_q projector to states including static quark Ukawa '93
- with path integral

$$P(\mathbf{x}) = \operatorname{tr} \mathcal{P} \exp\left(i \int_0^{1/T} \mathrm{d}x_4 A_4(x) \right)$$

Polyakov loop

- on the lattice

$$P(\mathbf{n}) = \mathrm{tr} \prod_{n_4=0}^{N_t-1} U_t(n) =$$

- average Polyakov loop $P = \frac{1}{V} \sum_{\mathbf{n}} P(\mathbf{n})$
- order parameter for confinement

$$\langle P \rangle = e^{-(F_q - F)/T} \quad \begin{cases} = 0 & \text{confinement} \\ \neq 0 & \text{deconfinement} \end{cases}$$

- static quark and antiquark at separation r

$$e^{-(F_{q\bar{q}}(r) - F)/T} = \langle P(0) P^\dagger(r) \rangle \xrightarrow{r \to \infty} \langle P \rangle \langle P^\dagger \rangle + \mathcal{O}(e^{-\sigma r/T})$$

so

$$F_{q\bar{q}}(r \to \infty) - F \propto \begin{cases} \sigma r & \text{confinement} \\ \text{const.} & \text{deconfinement} \end{cases}$$

Center symmetry

- symmetry corresponding to confinement/deconfinement?
- transformation under which S is invariant but P is not

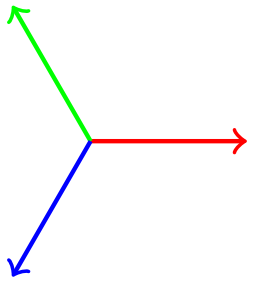
  $\Rightarrow$ center transformation

- elements of <u>center of group</u>

$$V_\zeta \quad \text{so that} \quad [V_\zeta, U] = 0 \quad \forall\, U \in \mathrm{SU}(3)$$

in our case it is

$$\mathbb{Z}(3) = \{\mathbb{1}, e^{i2\pi/3}\mathbb{1}, e^{-i2\pi/3}\mathbb{1}\} = \zeta \cdot \mathbb{1}$$

$\zeta =$ 

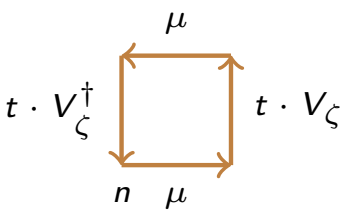
Center symmetry

- center transformation

$$U_t(\mathbf{n}, \bar{n}_4) \to V_\zeta \cdot U_t(\mathbf{n}, \bar{n}_4) \qquad V_\zeta \in \mathbb{Z}(3) \qquad \forall\, \mathbf{n}$$

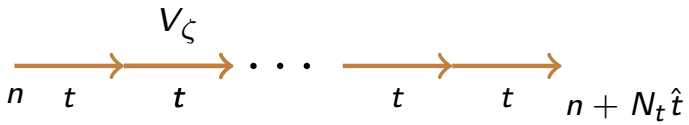
- gauge action is invariant

$$[V_\zeta, U_\mu(n)] = 0 \quad \Rightarrow \quad S_G \to S_G$$



- Polyakov loop not invariant

$$P(\mathbf{n}) \to \zeta \cdot P(\mathbf{n})$$



confinement	$\langle P \rangle = 0$	$\mathrm{Z}(3)$ symmetry intact
deconfinement	$\langle P \rangle \neq 0$	$\mathrm{Z}(3)$ symmetry spontaneously broken

Spontaneous center symmetry breaking

- Polyakov loop in Polyakov gauge

$$P(\mathbf{n}) = \underset{n\quad t}{\longrightarrow} \cdots \underset{t}{\longrightarrow}\underset{t}{\longrightarrow}_{\,n+N_t\hat{t}} \quad \to \quad \overset{\mathbb{1}}{\underset{n\quad t}{\longrightarrow}} \cdots \overset{\mathbb{1}}{\underset{t}{\longrightarrow}}\overset{\prod U}{\underset{t}{\longrightarrow}}_{\,n+N_t\hat{t}}$$

  is a boundary condition

- if $P(\mathbf{n})$ is constant, it can be diagonalized

$$P = \operatorname{tr}\operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{-i(\varphi_1+\varphi_2)})$$

- perturbative treatment at high temperature:
  $\log \mathcal{Z}(\varphi_1, \varphi_2)$ in the background of a constant Polyakov loop
  Roberge, Weiss NPB '86
- three degenerate minima at

$$\varphi_1 = \varphi_2 = 0 \qquad \varphi_1 = \varphi_2 = 2\pi/3 \qquad \varphi_1 = \varphi_2 = -2\pi/3$$

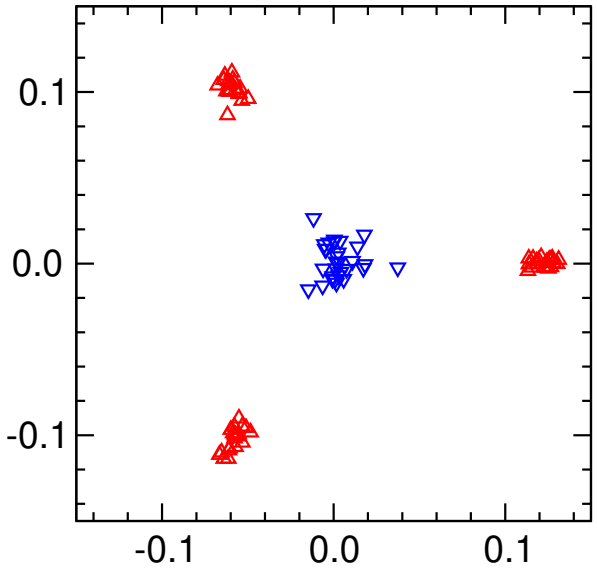
$$P = 3 \qquad P = 3\,e^{2\pi i/3} \qquad P = 3\,e^{-2\pi i/3}$$

Spontaneous center symmetry breaking

- from perturbation theory we except three minima

$$P = 3 \qquad P = 3\, e^{2\pi i/3} \qquad P = 3\, e^{-2\pi i/3}$$

- scatter plot at low T and high T Danzer et al. JHEP '08



- remember Ising model recipe: $\lim_{V\to\infty} \langle |M| \rangle$ at $h = 0$

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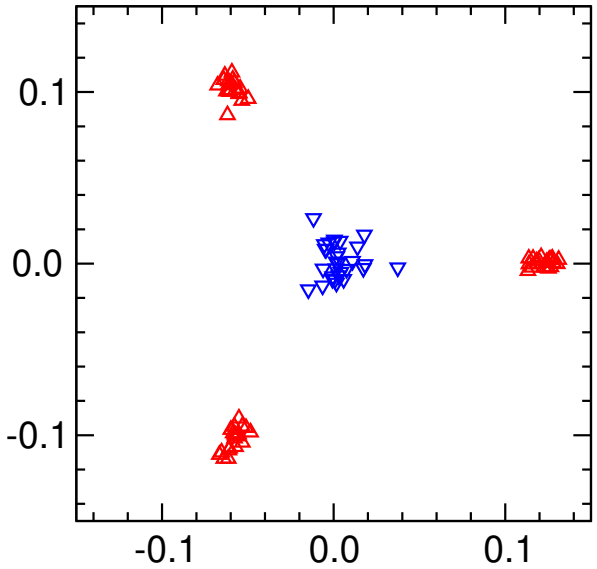


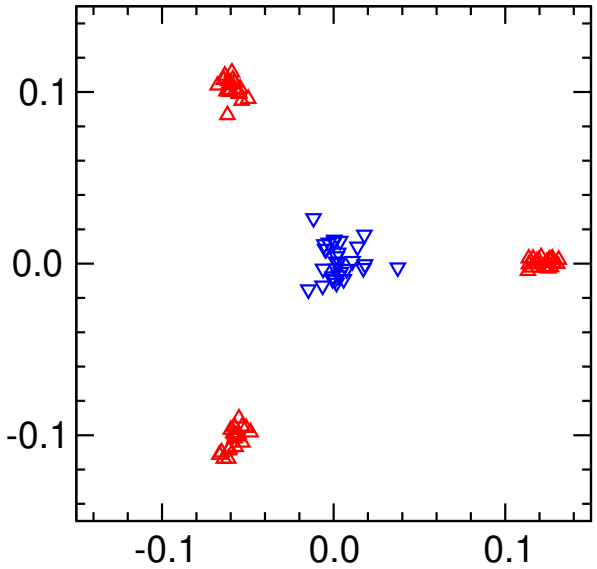
- remember Ising model recipe: $\lim_{V\to\infty} \langle |M| \rangle$ at $h = 0$
- alternative 1: measure $\langle |P| \rangle$

Spontaneous center symmetry breaking

- from perturbation theory we except three minima

$$P = 3 \qquad P = 3\, e^{2\pi i/3} \qquad P = 3\, e^{-2\pi i/3}$$

- scatter plot at low T and high T Danzer et al. JHEP '08



- remember Ising model recipe: $\lim_{V\to\infty} \langle |M| \rangle$ at $h = 0$
- alternative 1: measure $\langle |P| \rangle$
- alternative 2: measure $\langle P_{\rm rot} \rangle$ with rotated Polyakov loop

$$-\pi/3 < \arg P_{\rm rot} < \pi/3$$

Dictionary 1.

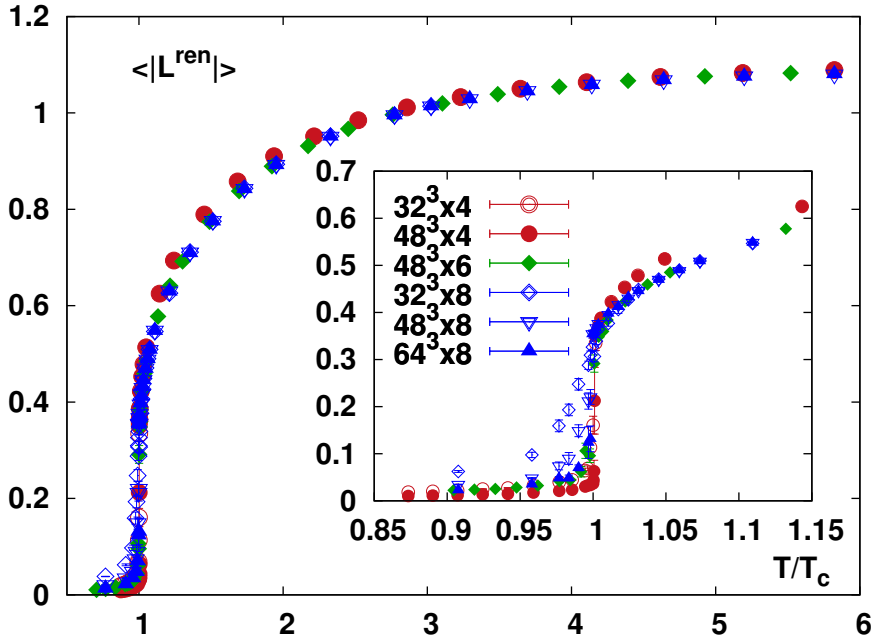
	Ising model	Yang-Mills theory
symmetry group	$\mathrm{Z}(2)$	$\mathrm{Z}(3)$
spontaneous breaking	$\langle M \rangle = \frac{\partial \log \mathcal{Z}}{\partial h}$	$\langle P \rangle$
explicit breaking	h	?
symmetry restoration	at high T	at low T

Deconfinement transition

Results: Polyakov loop

- $\langle |P| \rangle$ as function of T in the fixed N_t-approach

  Lo et al. PRD '13



- note: UV renormalization: F_q additive $\rightsquigarrow$ $\langle P \rangle$ multiplicative

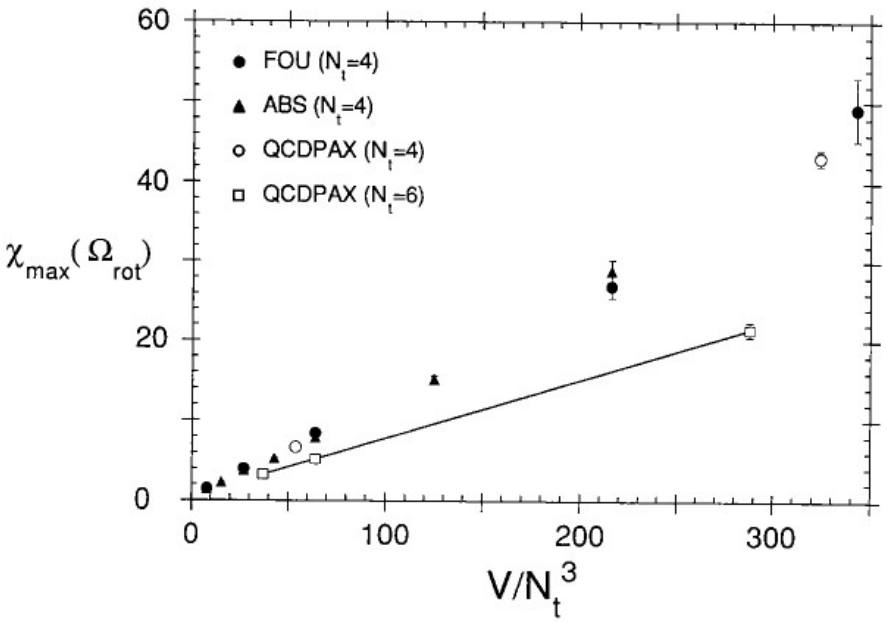
Results: susceptibility

- susceptibility of order parameter

$$\chi_P = \langle P_{\mathrm{rot}}^2 \rangle - \langle P_{\mathrm{rot}} \rangle^2$$

- how does peak height scale with volume?

  Iwasaki et al. PRD '92



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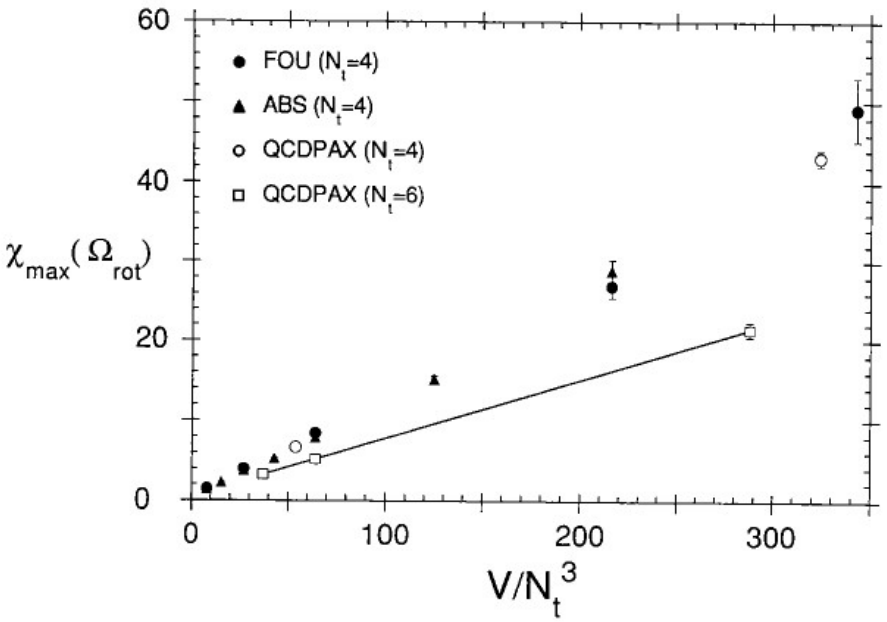

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