

# Lattice QCD thermodynamics

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# Outline overall

- ▶ lecture 1: introduction to QCD and thermodynamics
- ▶ lecture 2: hot Yang-Mills theory on the lattice
- ▶ lecture 3: hot QCD on the lattice
- ▶ lecture 4: QCD in extreme conditions on the lattice

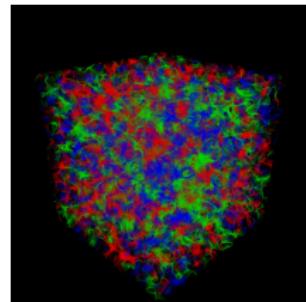
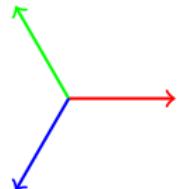
## Outline lecture 3

- ▶ “rest” from lecture 2: center clusters and pure gauge EoS
- ▶ dynamical fermions and chiral symmetry; staggered fermions
- ▶ finite temperature transition in full QCD

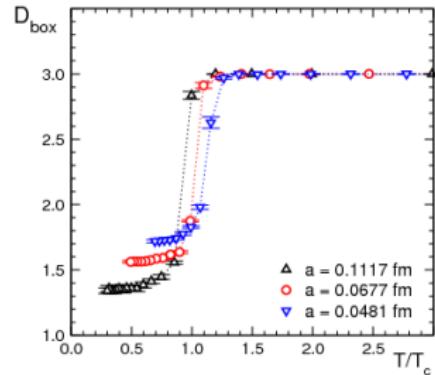
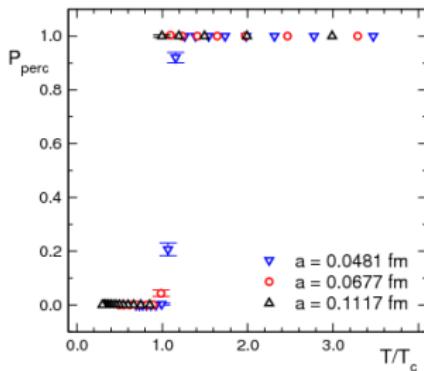
## **Center clusters**

# Center clusters

- ▶ local distribution of  $P(\mathbf{n})$  ↗ Stokes, Kamleh, Leinweber Ann. Phys. '14  
<https://www.youtube.com/watch?v=T4sRON6u0z0>



- ▶ clusters percolate at  $T_c$  and they are fractals
- ↗ Endrődi, Gatringer, Schadler PRD '14

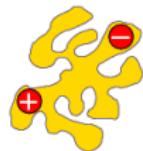


# Center clusters

- insight into confinement and deconfinement mechanism
  - 🔗 Gatringer, Schmidt JHEP '10

$T < T_c$

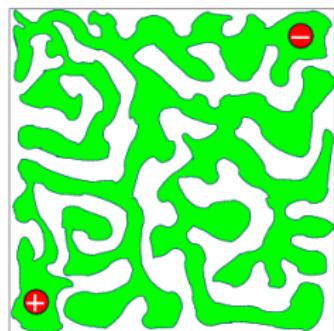
$$\langle L^*(x) L(y) \rangle \neq 0$$



$$\langle L^*(x) L(y) \rangle = 0$$



$T > T_c$



$$\langle P(0)P^\dagger(r) \rangle \propto \exp(-\sigma r/T)$$

## **Equation of state**

## Equation of state, reminder

- ▶ free energy (density)

$$F = -T \log \mathcal{Z} \quad f = \frac{F}{V}$$

- ▶ entropy density

$$s = -\frac{1}{V} \frac{\partial F}{\partial T}$$

- ▶ pressure

$$p = -\frac{\partial F}{\partial V} \xrightarrow{V \rightarrow \infty} -f$$

- ▶ energy density

$$\epsilon = -\frac{1}{V} \frac{\partial \log \mathcal{Z}}{\partial (1/T)} = f + Ts$$

- ▶ interaction measure / trace anomaly

$$I = \text{tr} T_{\mu\nu} = \epsilon - 3p$$

## EoS - methods

- ▶ very high  $T$ : perturbation theory, Hard Thermal Loop resummation
  - 🔗 Braaten, Pisarski PRL '90    ↳ Andersen, Strickland, Su JHEP '10
- ▶ low  $T$ : glueball resonance gas model
- ▶ intermediate  $T$ : lattice gauge theory
- ▶ how to determine  $\log \mathcal{Z}$  via expectation values?
  - ▶ derivative method
  - ▶ integral method
  - ▶ moving frame method

## **Derivative method**

# Derivative method

- ▶ trace anomaly as a derivative

$$\frac{d \log \mathcal{Z}}{d \log a} = a \frac{d \log \mathcal{Z}}{da} = \frac{1}{T} \frac{\partial \log \mathcal{Z}}{\partial(1/T)} + 3L^3 \frac{\partial \log \mathcal{Z}}{\partial(L^3)} = -\frac{V}{T}(\epsilon - 3p)$$

- ▶ how does  $\log \mathcal{Z}$  depend on  $a$ ?

$$\mathcal{Z} = \int \mathcal{D}U \exp \left[ -\beta \cdot \overbrace{\frac{1}{3} \sum_{n,\mu < \nu} \text{Re} \left( 3 - \begin{array}{c} \mu \\ \nu \\ \square \\ n \quad \mu \end{array} \right)}^{S_G} \right]$$

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- ▶ implicitly via  $\beta$

$$\frac{d \log \mathcal{Z}}{d \log a} = \frac{\partial \log \mathcal{Z}}{\partial \beta} \cdot \frac{\partial \beta}{\partial \log a} = \langle -S_G \rangle \cdot \frac{a(\beta)}{a'(\beta)}$$

for which we need to know  $a(\beta)$  from scale setting

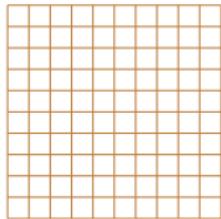
- ▶ so

$$I = \epsilon - 3p = \frac{T}{V} \langle S_G \rangle \frac{a(\beta)}{a'(\beta)}$$

# Derivative method

- ▶ how to get just  $p$  or just  $\epsilon$ ?

$$1/T = N_t a$$

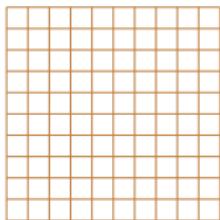


$$L = N_s a$$

# Derivative method

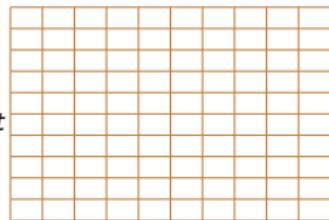
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$$1/T = N_t a$$



$$L = N_s a$$

$$1/T = N_t a_t$$



$$L = N_s a$$

- ▶ anisotropic lattice  $\xi = a/a_t$

$$\epsilon = -\frac{T}{V} \left. \frac{d \log \mathcal{Z}}{d \log a_t} \right|_a \quad 3p = \frac{T}{V} \left. \frac{d \log \mathcal{Z}}{d \log a} \right|_{a_t}$$

these can be evaluated at  $\xi = 1$

∅ Karsch NPB '82    ∅ Engels et al. NPB '82

# Anisotropy coefficients

- ▶ anisotropic lattice action

$$S_G = \xi_0 \cdot \frac{1}{3} \sum_{n,\mu \neq t} \text{Re} \left( 3 - \overbrace{\quad \quad \quad}^{S_G^t} \begin{array}{c} \mu \\ t \\ \square \\ n \quad \mu \end{array} \right) + \frac{1}{\xi_0} \cdot \frac{1}{3} \sum_{\substack{n,\mu < \nu \\ \mu,\nu \neq t}} \text{Re} \left( 3 - \overbrace{\quad \quad \quad}^{S_G^s} \begin{array}{c} \mu \\ \nu \\ \square \\ n \quad \mu \end{array} \right)$$

and we need one more scale setting relation  $\boxed{\xi(\beta, \xi_0)}$

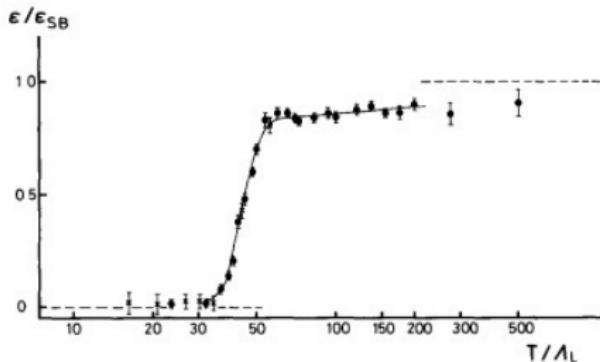
- ▶ energy density

$$\epsilon = -\xi^2 \frac{T}{V} \left[ \langle S_G^s \rangle \frac{\partial(\beta\xi_0)}{\partial \xi} + \langle S_G^t \rangle \frac{\partial(\beta/\xi_0)}{\partial \xi} \right]$$

- ▶ anisotropy coefficients difficult to measure precisely

## Results: derivative method

- ▶ results for energy density ↗ Engels et al. NPB '82



- ▶ remember additive divergences in  $\log \mathcal{Z} \propto a^{-4}$
- ▶ renormalized energy density

$$\epsilon^r = \epsilon - \epsilon(T \approx 0)$$

involving cancellation of  $\mathcal{O}(a^{-4})$  divergences

$$\left\langle S_G^{t,s} \right\rangle_{N_s^3 N_t} - \left\langle S_G^{t,s} \right\rangle_{N_s^4}$$

## **Integral method**

## Integral method

- ▶ integrate back the derivatives ↗ Boyd et al. NPB '96

$$\log \mathcal{Z}(\beta_1) - \log \mathcal{Z}(\beta_0) = \int_{\beta_0}^{\beta_1} d\beta \frac{\partial \log \mathcal{Z}}{\partial \beta}$$

- ▶ works in the fixed  $N_t$ -approach
- ▶ differences of dimensionless pressures

$$p(T_1)a_1^4 - p(T_0)a_0^4 = -\frac{1}{N_s^3 N_t} \int_{\beta_0}^{\beta_1} d\beta \langle S_G \rangle$$

or

$$\frac{p(T_1)}{T_1^4} - \frac{p(T_0)}{T_0^4} = -\frac{N_t^3}{N_s^3} \int_{\beta_0}^{\beta_1} d\beta \langle S_G \rangle$$

- ▶ is this UV finite?

# Renormalization

- ▶  $p(T_1, a) - p(T_0, a)$  UV finite but  $p(T_1, a_1) - p(T_0, a_0)$  divergent
- ▶ need to do  $T \approx 0$  subtraction

$$\frac{p^r(T_1)}{T_1^4} - \frac{p^r(T_0)}{T_0^4} = -\frac{N_t^3}{N_s^3} \int_{\beta_0}^{\beta_1} d\beta \left[ \langle S_G \rangle_{N_s^3 N_t} - \langle S_G \rangle_{N_s^4} \right]$$

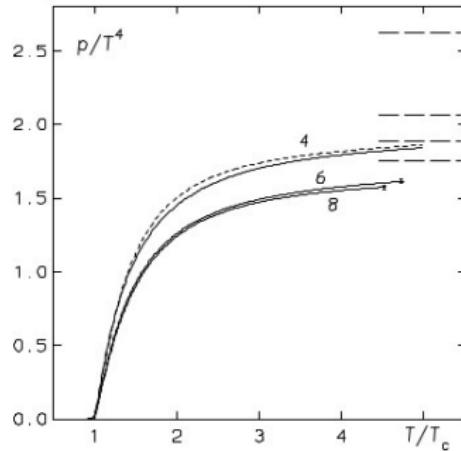
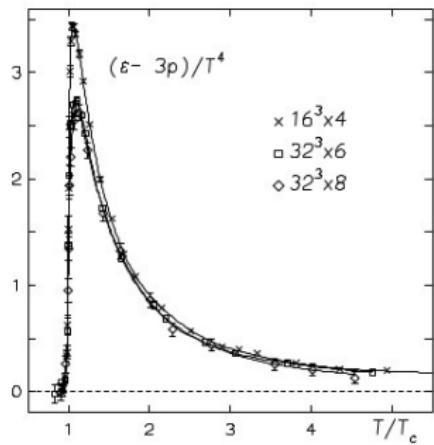
with starting point  $\beta_0$  where  $p^r(T_0)/T_0^4 \approx 0$

- ▶ renormalized interaction measure

$$\frac{I^r}{T^4} = \frac{N_t^3}{N_s^3} \frac{a(\beta)}{a'(\beta)} \left[ \langle S_G \rangle_{N_s^3 N_t} - \langle S_G \rangle_{N_s^4} \right]$$

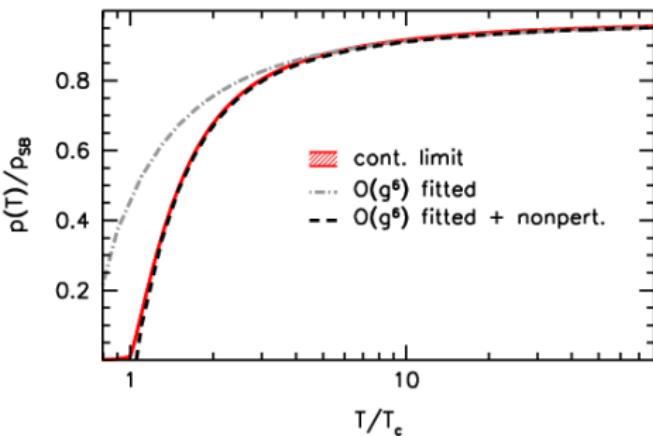
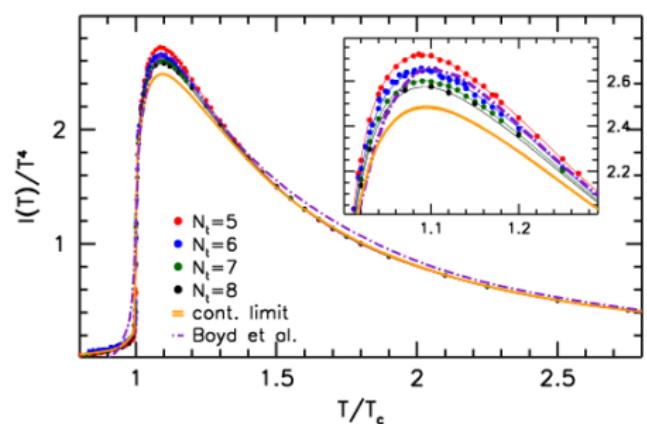
## Results: integral method

- ▶ need interpolation ( $+ a(\beta)$ ) for  $I$ , then numerical integral for  $p$ 
  - 🔗 Boyd et al. NPB '96



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- ▶ need interpolation (+  $a(\beta)$ ) for  $I$ , then numerical integral for  $p$   
🔗 Boyd et al. NPB '96
- ▶ update on finer lattices  
🔗 Borsányi, Endrődi et al. JHEP '12



## Moving frame method

## Thermal medium at relativistic speeds

- so far we have been in the rest frame

$$\langle \hat{\Theta}_{\mu\nu} \rangle = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

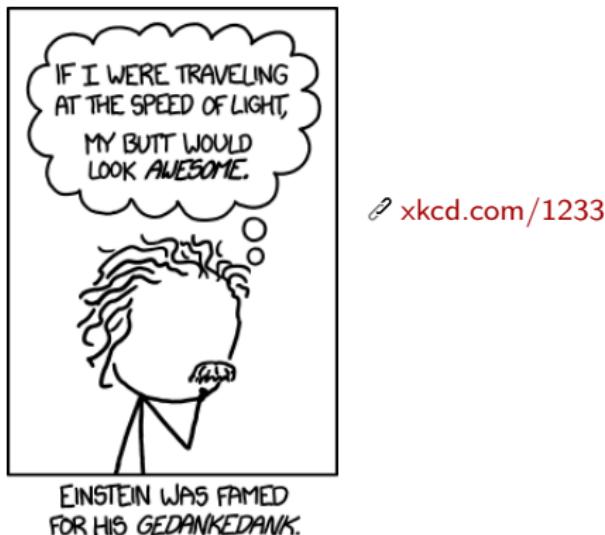
- now consider frame moving with  $\mathbf{v} = (v, 0, 0)$

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$$\langle \hat{\Theta}'_{\mu\nu} \rangle = \Lambda_\mu^\rho \Lambda_\nu^\sigma \langle \hat{\Theta}_{\rho\sigma} \rangle = \begin{pmatrix} \frac{\epsilon + v^2 p}{1-v^2} & v \frac{\epsilon + p}{1-v^2} & 0 & 0 \\ v \frac{\epsilon + p}{1-v^2} & \frac{p + v^2 \epsilon}{1-v^2} & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- recall  $\epsilon = f + Ts = -p + Ts$

therefore

$$\langle \hat{\Theta}_{01} \rangle_v = \frac{v}{1-v^2} (\epsilon + p) = \frac{v}{1-v^2} Ts$$

# Shifted boundary conditions

- partition function becomes

↗ Giusti, Meyer PRL '11   ↗ Giusti, Meyer JHEP '13

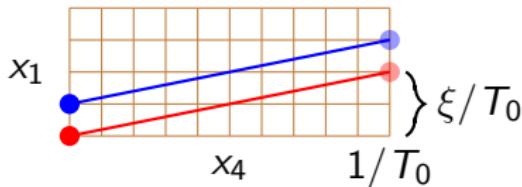
$$\mathcal{Z} = \text{tr exp} \left[ -(\hat{\Theta}_{00} - v\hat{\Theta}_{01})/T_0 \right]$$

in Euclidean space  $v = i\xi$

$$\mathcal{Z} = \text{tr exp} \left[ -(\hat{\Theta}_{00} - i\xi\hat{\Theta}_{01})/T_0 \right]$$

- states with  $x_1$ -momentum  $\Theta_{01}$  weighted by  $e^{i\xi\Theta_{01}/T_0}$   
⇒ shifted boundary conditions

$$U(n_1, 0) = U(n_1 + \xi/T_0, 1/T_0)$$



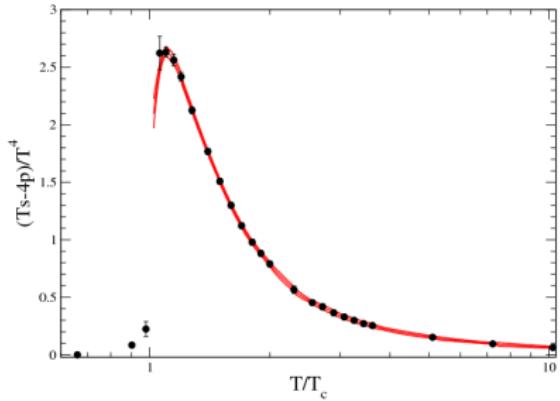
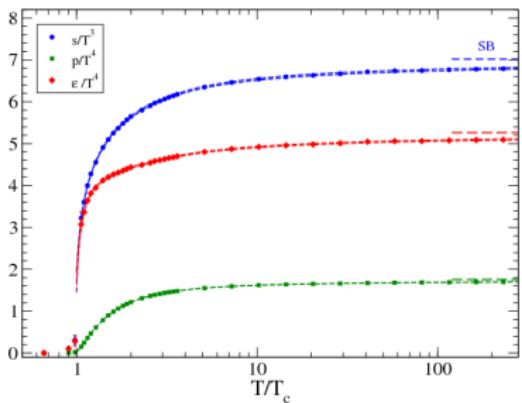
- but watch out:  $1/T = 1/T_0 \cdot \sqrt{1 + \xi^2}$

# Results: moving frame method

- ▶ entropy in moving frame

$$Ts = \frac{1 + \xi^2}{\xi} \left\langle \hat{\Theta}_{01} \right\rangle_\xi \cdot Z_T(a)$$

- ▶ simulations with  $\xi = \{1, \sqrt{2}, \sqrt{3}\} \cdot a$  ↗ Giusti, Pepe JHEP '16
- ▶ multiplicative renormalization  $Z_T$  for operator  $\Theta_{\mu\nu}$
- ▶ recover full EoS from  $s$



# Equation of state: summary

- ▶ derivative method
  - ▶ works with a single ensemble ✓
  - ▶ needs anisotropy coefficients ✗
- ▶ moving frame method
  - ▶ works with a single ensemble ✓
  - ▶ needs renormalization constants ✗
- ▶ integral method: most powerful up to date
  - ▶ only simple expectation values required ✓
  - ▶ needs many ensembles ✗
- ▶ Jarzynski's method ↗ Caselle et al. PRD '18  
and other approaches

## Dynamical fermions

# Full QCD path integral

- ▶ path integral over links and fermion fields

$$\mathcal{Z} = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\beta S_G - \bar{\psi} M \psi}$$

- ▶ Grassmann numbers on a computer?  
⇒ integrate out fermions analytically

$$\mathcal{Z} = \int \mathcal{D}U \det M e^{-\beta S_G}$$

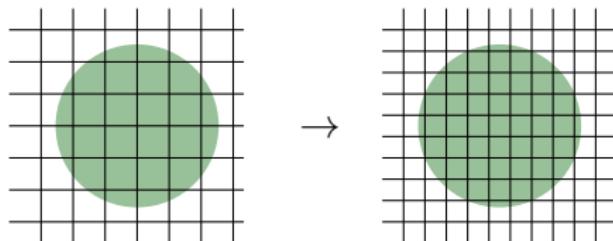
- ▶ fermion matrix

$$M = \text{diag}(\not{D} + m_u, \not{D} + m_d, \not{D} + m_s, \dots)$$

- ▶ note:  $m_f \rightarrow \infty$  of full QCD gives back pure gauge theory

# Continuum limit

- ▶ continuum limit nonperturbatively



- ▶ lattice spacing from scale setting  $a(\beta)$
- ▶ quark masses from line of constant physics  $m_f(\beta)$

tuned to the *physical point*:

$$M_\pi = 139 \text{ MeV}, M_K = 495 \text{ MeV}, M_p = 938 \text{ MeV} \dots$$

## Fermions: dynamical vs. quenched

- ▶ pure Yang-Mills theory

$$\mathcal{Z}_{\text{YM}} = \int \mathcal{D}U e^{-\beta S_G} \quad \langle A[U] \rangle_{\text{YM}} = \frac{1}{\mathcal{Z}_{\text{YM}}} \int \mathcal{D}U e^{-\beta S_G} A[U]$$

- ▶ full QCD

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\beta S_G - S_F} = \int \mathcal{D}U \det M e^{-\beta S_G} \\ \langle A[\bar{\psi}, \psi, U] \rangle &= \frac{1}{\mathcal{Z}} \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\beta S_G - S_F} A[\bar{\psi}, \psi, U] \\ &= \frac{1}{\mathcal{Z}} \int \mathcal{D}U \det M e^{-\beta S_G} B[U] \end{aligned}$$

- ▶ quenched approximation: “drop the det”

$$\langle A[\bar{\psi}, \psi, U] \rangle_{\text{YM}} = \frac{1}{\mathcal{Z}_{\text{YM}}} \int \mathcal{D}U e^{-\beta S_G} B[U]$$

## **Chiral symmetry and its breaking**

## Chiral symmetry: continuum

- ▶ massless fermion action  $\bar{\psi} \not{D} \psi$  (flavor index implicit)
- ▶ invariance under the chiral group

$$\mathrm{SU}_V(N_f) \times \mathrm{SU}_A(N_f) \times \mathrm{U}_V(1) \times \mathrm{U}_A(1)$$

$$\psi \rightarrow e^{i\alpha\tau_a}\psi, \quad \psi \rightarrow e^{i\alpha\tau_a\gamma_5}\psi, \quad \psi \rightarrow e^{i\alpha\mathbb{1}}\psi, \quad \psi \rightarrow e^{i\alpha\mathbb{1}\gamma_5}\psi,$$

axial symmetries:  $\boxed{\{\gamma_5, \not{D}\} = 0}$

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- ▶  $\mathrm{SU}_A(N_f)$  broken spontaneously in the QCD vacuum  $\langle \bar{\psi} \psi \rangle \neq 0$   
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~~ Goldstone bosons:  $N_f^2 - 1$  massless pions
- ▶ massive case:  $m \bar{\psi} \psi$  breaks axial symmetries  
~~ pseudo-Goldstone bosons:  $N_f^2 - 1$  almost massless pions

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- ▶ high temperature: chiral symmetry restoration

## Dictionary 2.

	Ising model	QCD
symmetry group	$Z(2)$	$SU(2)$
spontaneous breaking	$\langle M \rangle = \frac{\partial \log \mathcal{Z}}{\partial h}$	$\langle \bar{\psi}_u \psi_u \rangle = \frac{\partial \log \mathcal{Z}}{\partial m_u}$
Goldstones	—	3
explicit breaking	$h$	$m_u = m_d$
symmetry restoration	at high $T$	at high $T$

# Chiral condensate

- ▶ full QCD expectation value

$$\begin{aligned}\langle \bar{\psi}_u \psi_u \rangle &= \frac{T}{V} \frac{1}{\mathcal{Z}} \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\beta S_G - \bar{\psi} M \psi} \bar{\psi}_u \psi_u \\ &= \frac{T}{V} \frac{1}{\mathcal{Z}} \int \mathcal{D}U \det M e^{-\beta S_G} \operatorname{tr} M_u^{-1} \\ &= \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial m_u}\end{aligned}$$

- ▶ useful:

$$\frac{\partial \log \det M_u}{\partial m_u} = \frac{\partial \operatorname{tr} \log M_u}{\partial m_u} = \operatorname{tr} M_u^{-1} \overbrace{\frac{\partial M_u}{\partial m_u}}^1 = \operatorname{tr} M_u^{-1}$$

- ▶ remember order parameter definition for Ising model

$$\lim_{h \rightarrow 0^+} \lim_{V \rightarrow \infty} \langle M \rangle \quad \text{or} \quad \lim_{V \rightarrow \infty} \langle |M| \rangle_{h=0}$$

- ▶ here only option:

$$\lim_{m_u \rightarrow 0^+} \lim_{V \rightarrow \infty} \langle \bar{\psi}_u \psi_u \rangle$$

# Renormalization

- ▶ remember UV divergences from free case ( $u$  and  $s$  flavors)

$$\log \mathcal{Z}_{\text{vac}}^{\text{free}} = \mathcal{O}(\Lambda^4) + \mathcal{O}((m_u^2 + m_s^2)\Lambda^2) + \mathcal{O}((m_u^2 + m_s^2)^2 \log \Lambda^2) + \text{finite}$$

so the condensate

$$\langle \bar{\psi}_u \psi_u \rangle = \frac{\partial \log \mathcal{Z}_{\text{vac}}^{\text{free}}}{\partial m_u} = \mathcal{O}(m_u \Lambda^2) + \mathcal{O}(m_u(m_u^2 + m_s^2) \log \Lambda^2) + \text{finite}$$

- ▶ multiplicative divergence (interacting case) ↗ Peskin, Schroeder

$$m_f^r = Z_m \cdot m_f \quad \forall f$$

- ▶ fully renormalized combination

$$\left[ \langle \bar{\psi}_u \psi_u \rangle_T - \langle \bar{\psi}_u \psi_u \rangle_{T=0} \right] \cdot m_u$$

- ▶ sometimes also used

$$m_s \langle \bar{\psi}_u \psi_u \rangle_T - m_u \langle \bar{\psi}_s \psi_s \rangle_T$$

cancels quadratic divergence but not the logarithmic one

## Staggered fermions

# Staggered fermions

- ▶ naive Dirac operator: 16-fold doubling

$$\not{D} = \frac{1}{2} \sum_{\mu} \gamma_{\mu} \left[ \begin{array}{c} \xrightarrow{\mu} - \xleftarrow{\mu} \end{array} \right]$$

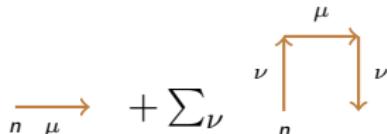
- ▶ staggered Dirac operator (no Dirac indices!): 4-fold doubling

$$\not{D} = \frac{1}{2} \sum_{\mu} \eta_{\mu} \left[ \begin{array}{c} \xrightarrow{\mu} - \xleftarrow{\mu} \end{array} \right] \quad \eta_{\mu}(n) = (-1)^{\sum_{\nu < \mu} n_{\nu}}$$

- ▶ partition function

$$\mathcal{Z} = \int \mathcal{D}U e^{-\beta S_G} \prod_f \sqrt[4]{\det(\not{D} + m_f)}$$

- ▶ rooting: no doubling? but has theoretical problems ↗ Creutz '07
- ▶ note: local averaging of links suppresses discretization errors  
“stout smearing” ↗ Morningstar, Peardon PRD '04



## Staggered chiral symmetry

- ▶ massless staggered fermion action

$$\bar{\psi} \not{D} \psi = \frac{1}{2a} \sum_{n,\mu} \bar{\psi}(n) \eta_\mu(n) \left[ U_\mu(n) \psi(n + \hat{\mu}) - U_\mu^\dagger(n - \hat{\mu}) \psi(n - \hat{\mu}) \right]$$

- ▶ quark number conservation  $U_V(1)$  ✓
- ▶ there is no full chiral symmetry  $SU(N_f) \times SU(N_f)$   
but only remnant:  $U(1)$  even-odd symmetry

$$\psi(n) \rightarrow e^{i\alpha\eta_5(n)} \psi(n) \quad \boxed{\{\not{D}, \eta_5\} = 0} \quad \eta_5(n) = (-1)^{n_1+n_2+n_3+n_4}$$

## Staggered $\eta_5$ -hermiticity

- staggered Dirac operator

$$\not{D}_{nm} = \frac{1}{2a} \sum_{\mu} \eta_{\mu}(n) \left[ U_{\mu}(n) \delta_{n+\hat{\mu},m} - U_{\mu}^{\dagger}(n-\hat{\mu}) \delta_{n-\hat{\mu},m} \right]$$

- $\eta_5$ -hermiticity     $\{\text{forward hopping}\} = \{\text{backward hopping}\}^{\dagger}$

$$\boxed{\eta_5 \not{D} \eta_5 = \not{D}^{\dagger}}$$

- determinant is real

$$\det[\eta_5 \not{D} \eta_5] = \det \not{D}^{\dagger} \quad \rightarrow \quad \det \not{D} = (\det \not{D})^*$$

- in even-odd space

$$\not{D} + m\mathbb{1} = \begin{pmatrix} m & \not{D}_{eo} \\ \not{D}_{oe} & m \end{pmatrix} = -\not{D}^{\dagger} + m\mathbb{1} = \begin{pmatrix} m & -\not{D}_{oe}^{\dagger} \\ -\not{D}_{eo}^{\dagger} & m \end{pmatrix}$$

thus

$$\det M = \det(m^2 - \not{D}_{oe} \not{D}_{eo}) = \underline{\det(m^2 + \not{D}_{oe} \not{D}_{oe}^{\dagger})} > 0$$

## **Finite temperature transition in full QCD**

## **Chiral restoration in full QCD**

## Chiral condensate: staggered

- ▶ full QCD expectation value

$$\langle \bar{\psi}_u \psi_u \rangle = \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial m_u} = \frac{1}{4\mathcal{Z}} \int \mathcal{D}U \prod_f [\det M_f]^{1/4} e^{-\beta S_G} \text{tr } M_u^{-1}$$

- ▶ assume we know how to invert  $\chi = M_u^{-1}\phi$   
how to calculate  $\text{tr } M_u^{-1}$ ?
- ▶ point sources

$$\text{tr } M_u^{-1} = \sum_{x,c} \delta_{xc}^\dagger M_u^{-1} \delta_{xc} \quad \mathcal{O}(3V) \text{ inversions} \quad \times$$

- ▶ random sources (noisy estimators)

$$\sum_{k=1}^{N_{\text{vec}}} \xi_k \xi_k^\dagger = \mathbb{1} + \mathcal{O}(1/N_{\text{vec}}) \quad \text{tr } M_u^{-1} \approx \frac{1}{N_{\text{vec}}} \sum_{k=1}^{N_{\text{vec}}} \xi_k^\dagger M_u^{-1} \xi_k$$

$\mathcal{O}(N_{\text{vec}})$  inversions ✓

# Chiral susceptibility

- ▶ susceptibility of order parameter

$$\begin{aligned}\chi_{\bar{\psi}\psi} &= \frac{\partial \langle \bar{\psi}_u \psi_u \rangle}{\partial m_u} = \frac{\partial}{\partial m_u} \frac{T}{4V} \frac{1}{Z} \int \mathcal{D}U [\det M]^{1/4} e^{-\beta S_G} \text{tr } M_u^{-1} \\ &= \frac{T}{16V} \frac{1}{Z} \int \mathcal{D}U [\det M]^{1/4} e^{-\beta S_G} (\text{tr } M_u^{-1})^2 \quad \blacksquare \\ &\quad - \frac{T}{16V} \frac{1}{Z^2} \left[ \int \mathcal{D}U [\det M]^{1/4} e^{-\beta S_G} \text{tr } M_u^{-1} \right]^2 \quad \blacksquare \\ &\quad + \frac{T}{4V} \frac{1}{Z} \int \mathcal{D}U [\det M]^{1/4} e^{-\beta S_G} \frac{\partial}{\partial m_u} \text{tr } M_u^{-1} \quad \blacksquare \\ &= \frac{T}{16V} \left[ \langle (\text{tr } M_u^{-1})^2 \rangle - \langle \text{tr } M_u^{-1} \rangle^2 \right] - \frac{T}{4V} \langle \text{tr } M_u^{-2} \rangle\end{aligned}$$

- ▶ useful:

$$M_u M_u^{-1} = \mathbb{1} \quad \rightarrow \quad \underbrace{\frac{\partial M_u}{\partial m_u}}_{\mathbb{1}} M_u^{-1} + M_u \frac{\partial M_u^{-1}}{\partial m_u} = 0 \quad \rightarrow \quad \frac{\partial M_u^{-1}}{\partial m_u} = -M_u^{-2}$$

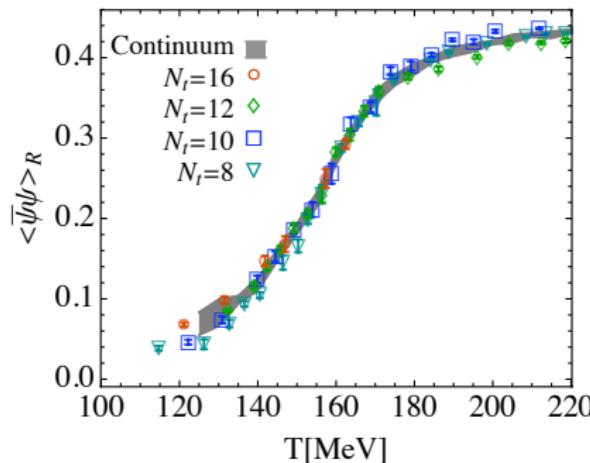
## Results: condensate

- ▶ average light quark condensate after renormalization

$$\langle \bar{\psi} \psi \rangle^r = - \left[ \langle \bar{\psi}_u \psi_u \rangle_T - \langle \bar{\psi}_u \psi_u \rangle_{T=0} \right] \cdot \frac{m_u}{m_\pi^4}$$

watch out: this vanishes at  $T = 0$  and positive at high  $T$

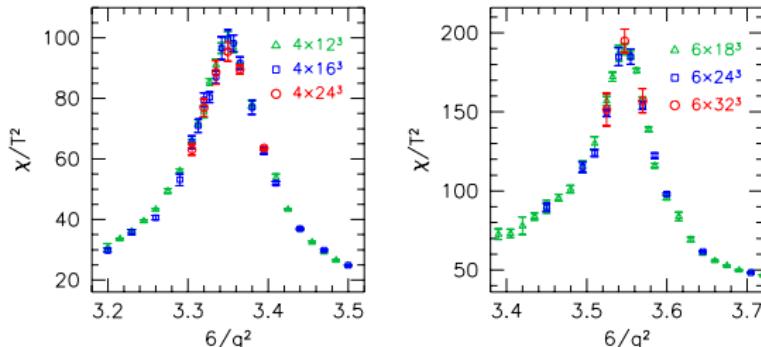
↗ Borsányi et al. JHEP '10



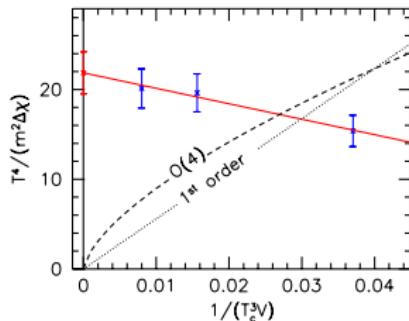
- ▶ does not look like a real phase transition

# Results: order of transition

- ▶ noisy estimators to calculate  $\chi_{\bar{\psi}\psi}$  ↗ Aoki, Endrődi et al. Nature '06



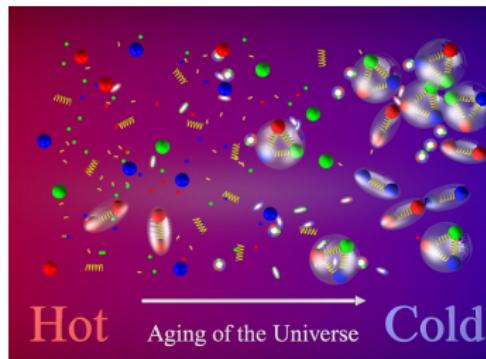
- ▶ volume scaling of peak height  $\chi(V, T_c(V)) \propto L^0$



- ▶ confirmed with other discretizations ↗ Bhattacharya et al. PRL '14 31 / 34

# Crossover transition

- ▶ in full QCD at the physical point, there is no real phase transition but merely an **analytic crossover**

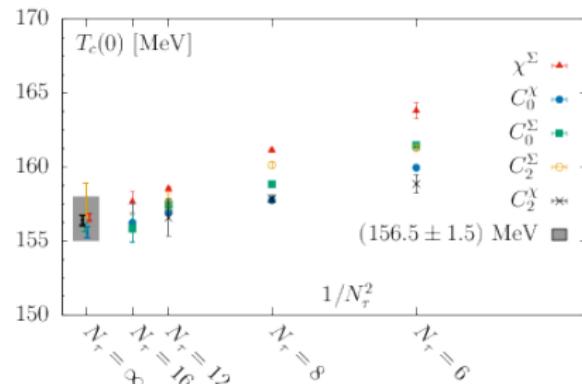
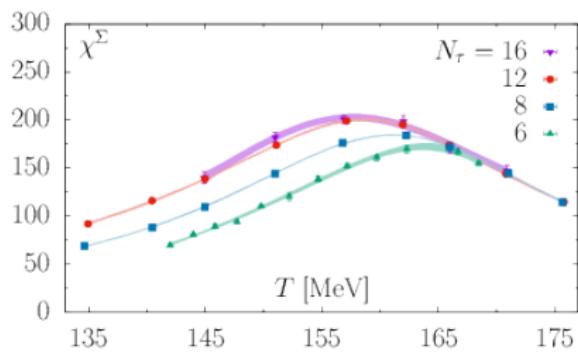


- ▶ there is no bubble formation in the QCD epoch of the early universe  
~~ relevant for cosmology

# Transition temperature

- ▶ at what temperature does the transition take place?
- ▶ there is no unique  $T_c$  but we can define it via
  - ▶ inflection point of  $\langle \bar{\psi} \psi \rangle$
  - ▶ maximum of  $\chi_{\bar{\psi} \psi}$
  - ▶ any characteristic behavior

↗ Bazavov et al. PLB '19



- ▶ transition at  $T_{pc} = 156.5(1.5)$  MeV
- ▶ transition width  $\mathcal{O}(15)$  MeV

## **Deconfinement in full QCD**

## Center symmetry in full QCD

- ▶ remember center transformation

$$U_t(\mathbf{n}, \bar{n}_4) \rightarrow V_\zeta \cdot U_t(\mathbf{n}, \bar{n}_4) \quad V_\zeta \in \mathbb{Z}(3)$$

- ▶ gauge action is invariant
- ▶  $\det M$  in heavy-quark expansion:

$$\det M \propto \det \left( 1 + \frac{\not{D}}{m} \right) = \exp \left[ \text{tr} \log \left( 1 + \frac{\not{D}}{m} \right) \right] = \exp \left[ - \sum_{k>0} \frac{\text{tr}(\not{D}/m)^{2k}}{2k} \right]$$

this includes Polyakov loop

- ▶  $\det M$  not invariant under center transformations  
it serves as explicit breaking for  $Z(3)$  symmetry

which center sector does it prefer?