# Renormalization and Improvement

## Lecture 1

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EuroPLEx Summer School 2021 on
Lattice Field Theory and Applications

University of Edinburgh

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***Origin of ultra-violet divergences in field theories is explained by the theory of renormalization. It also gives a prescription on how to systematically remove them so they don't appear in physical quantities.***

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Schroeder M. Peskin

"***Renormalization is an extensive study, and one can make a career out of it***"

Robert D. Klauber

# OUTLINE OF LECTURE 1

★ The need of Renormalization - Generalities

★ Perturbation theory

★ An example from QED: electron - muon scattering

★ Regularizations

★ Practice problem: photon self-energy

★ QCD - Running coupling (if time permits)

★ Key points of Lecture 1

# Useful Reading Material

- ★ **Lattice Gauge Theories An Introduction**

  **H. J. Rothe**

  https://www.worldscientific.com/worldscibooks/10.1142/1268

- ★ **Quantum Chromodynamics on the Lattice An Introductory Presentation**

  **C. Gattringer and C. Lang**

  https://www.springer.com/us/book/9783642018497

- ★ **Renormalization (An Introduction to Renormalization, the Renormalization Group and the OPE)**

  **J. Collins**

  https://doi.org/10.1017/CBO9780511622656

# Generalities

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- In QFT we encounter infinities that need to be dealt with, if the theory is describing physical processes;
  Such infinities appear in the majority of the matrix elements
  e.g., calculating scattering amplitudes within QED leads to terms

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★ t'Hooft proved that the SM is fully renormalizable

★ Nobel prize 1982: Kenneth Wilson received for Lattice Gauge Theory
LGT regularize infinities, e.g., in QCD

# Generalities

★ Renormalization relevant to many fields
quantum gravity, electroweak, QED, QCD,…

★ A theory is renormalizable if all UV divergence can be canceled with a finite number of counter terms

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**Our main focus will be renormalization in lattice QCD…**
**… but first, let's look at an example from QED**

# QED

# Perturbative treatment of QED

★ A perturbative expansion in terms of the coupling constant is linked to the appearance of certain factors attached to each (relevant) terms of the QED Lagrangian

One can simplify the calculations if we calculate these contributions (vertices) beforehand, and only perform contractions between them

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**(Bare) QED Lagrangian** $$\mathscr{L}_{\text{QED}} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$ No photon self-interaction

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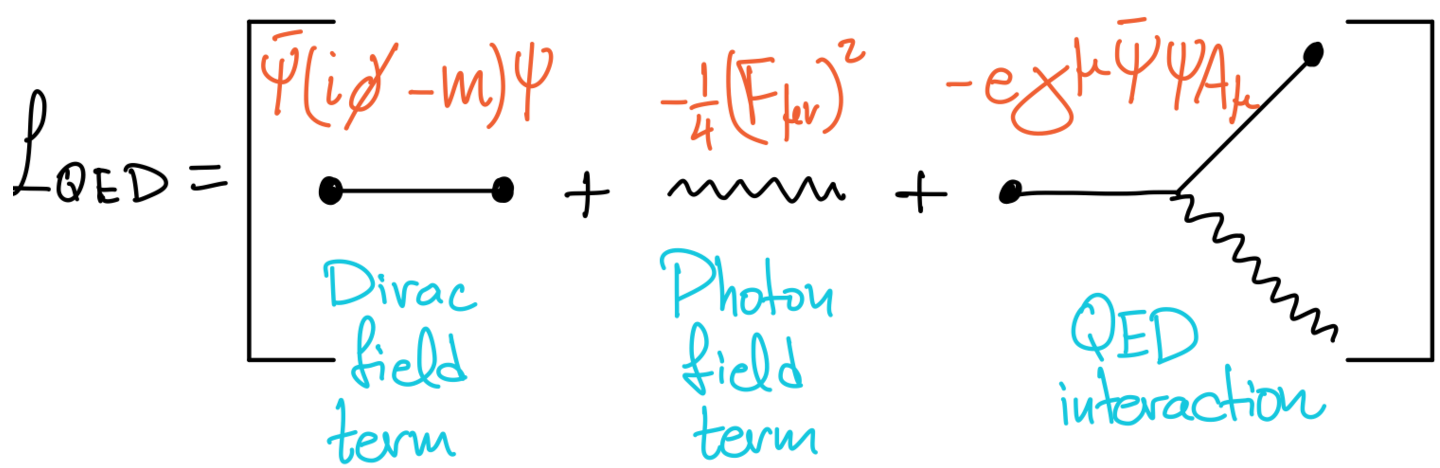
Pictorial representation of QED Lagrangian



# Perturbative treatment of QED

- ★ Scattering processes can be treated perturbatively order-by-order. Each order of the expansion is given by Feynman diagrams, which are pictorial representations of complicated integrals

- ★ Feynman diagrams are formed by contractions of vertices that involve fermion and photon propagators

- ★ Internal loops in diagrams contain an internal (loop) momentum that must be integrated over

- ★ Often, these diagrams have divergences and special tools are needed to deal with them

# Perturbative treatment of QED

**Strategy to eliminate divergences (in a nutshell):**

- A regulator is needed to make the integrals finite
- A renormalization is applied by defining a subtraction procedure for divergence (e.g., by introducing counter-terms)
- There is certain freedom on the renormalization prescription and different schemes differ <u>only</u> by finite contributions.
  The most widely used scheme is $\overline{\mathrm{MS}}$.
- Renormalized results may be compared to experimental data
- Renormalization redefines fields and coupling constants, and introduces a renormalization scale
  (The energy at which the elimination of the UV divergence is performed)

# An example from QED

## Electron - muon scattering

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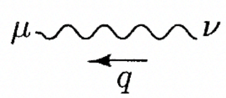
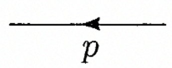
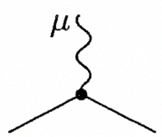
### QED Feynman rules



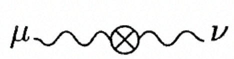
$= \frac{-ig_{\mu\nu}}{q^2+i\epsilon}$ (Feynman gauge)



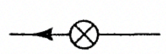
$= \frac{i}{\not{p}-m+i\epsilon}$

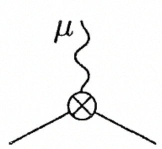


$= -ie\gamma^\mu$



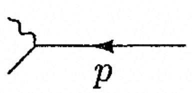
$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$


$= i(\not{p}\delta_2 - \delta_m)$

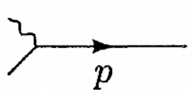


$= -ie\gamma^\mu\delta_1$

External fermions:

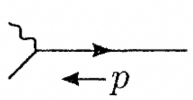


$= u^s(p)$ (initial)

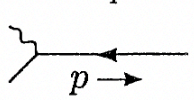


$= \bar{u}^s(p)$ (final) (A.7)

External antifermions:

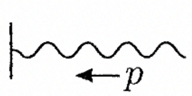


$= \bar{v}^s(p)$ (initial)

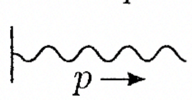


$= v^s(p)$ (final) (A.8)

External photons:



$= \epsilon_\mu(p)$ (initial)

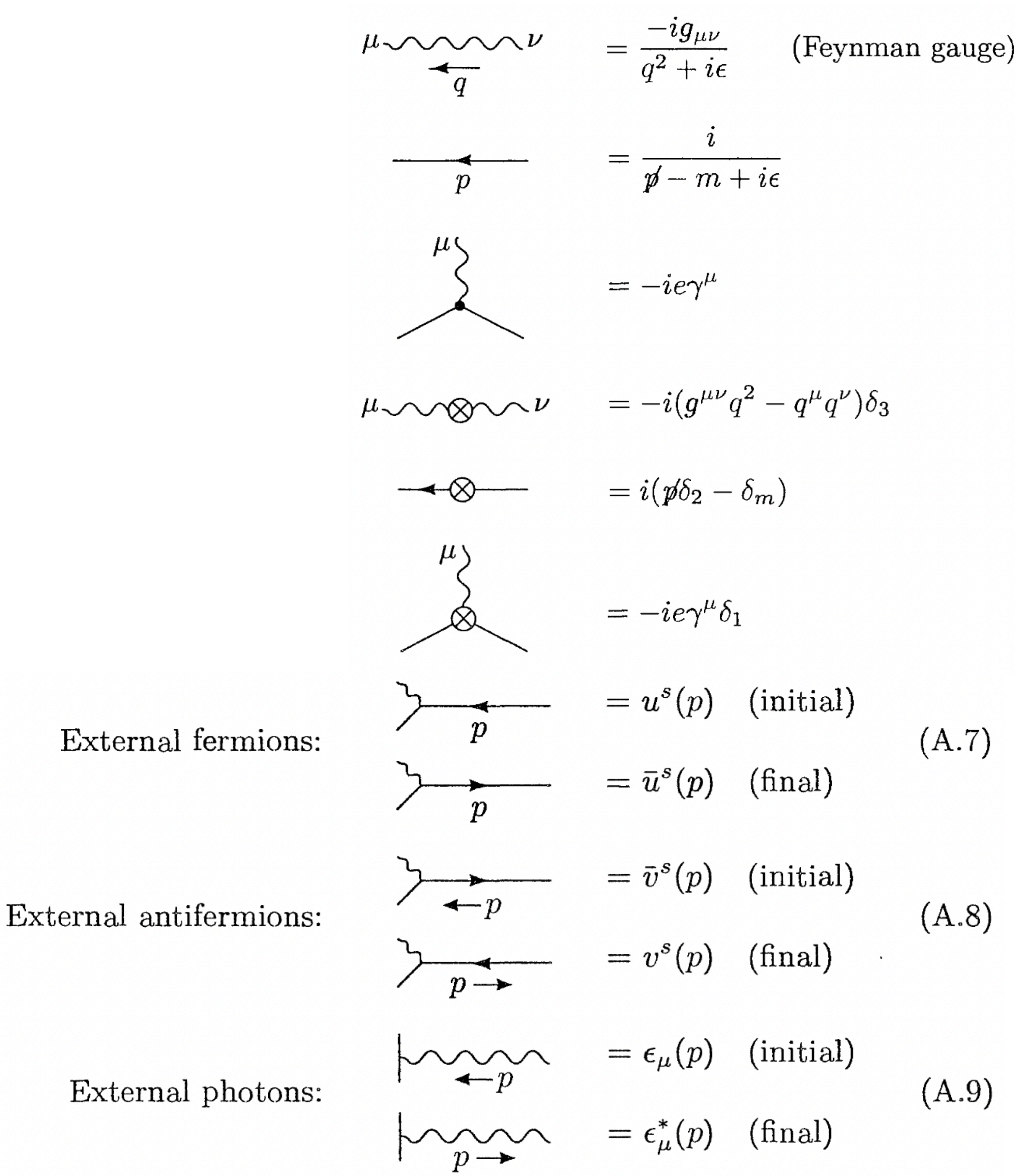


$= \epsilon^*_\mu(p)$ (final) (A.9)

# An example from QED

## Electron - muon scattering

### QED Feynman rules




### Terminology

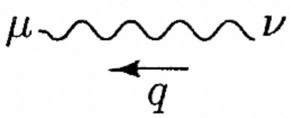
★ Bare quantities (e.g., $g_0$):
Appear in the Lagrangian and are <u>not</u> measurable quantities
Associated with divergences

★ Renormalized quantities (e.g., $g_R$):
Free of divergences and have finite values

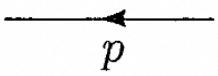
# Electron - muon scattering

*Let's do this exercise together*

# QED Feynman Rules




$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \quad \text{(Feynman gauge)}$$

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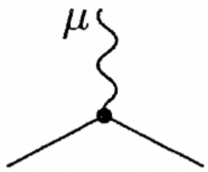


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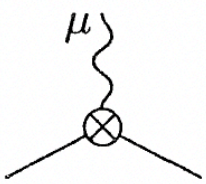


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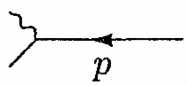


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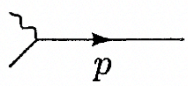


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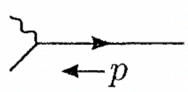


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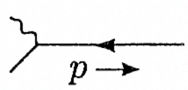


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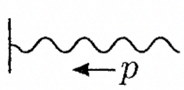


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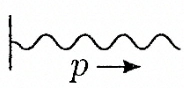


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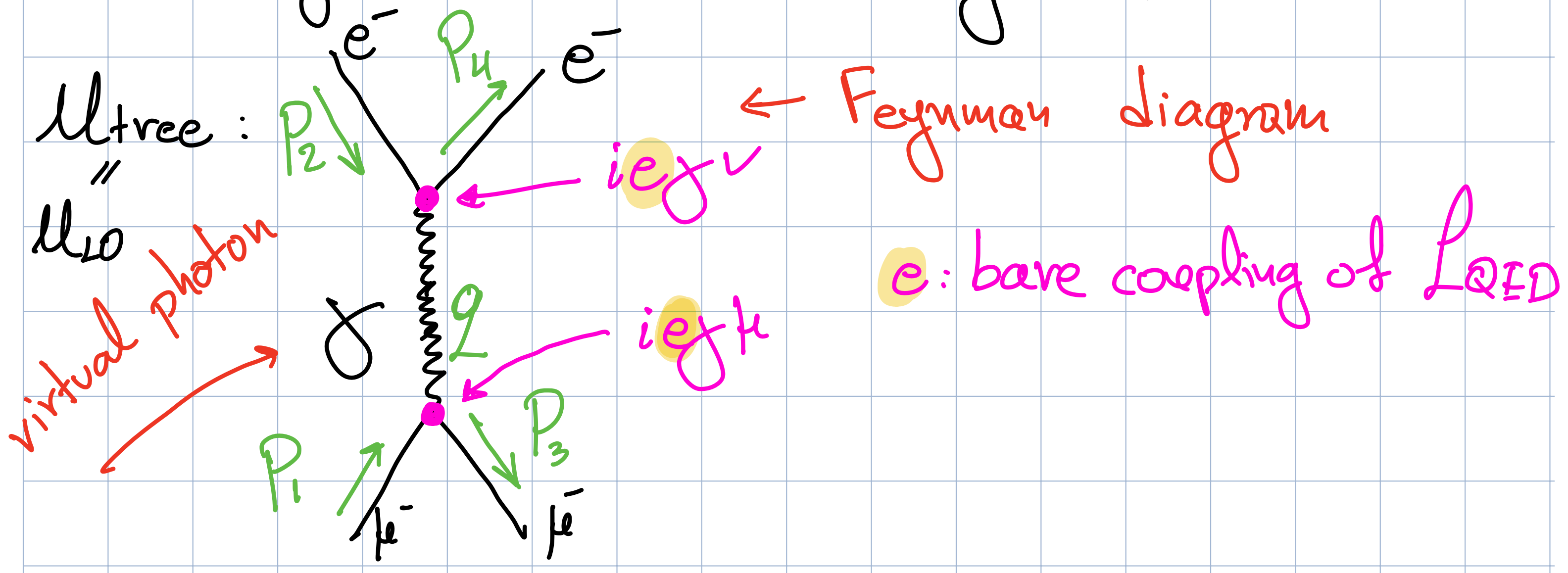
up to 1-loop QED : $e^- \mu^-$ scattering.

- Leading order in scattering amplitude : $\mathcal{M}$

$\mathcal{M}_{tree} = \mathcal{M}_{LO}$ :

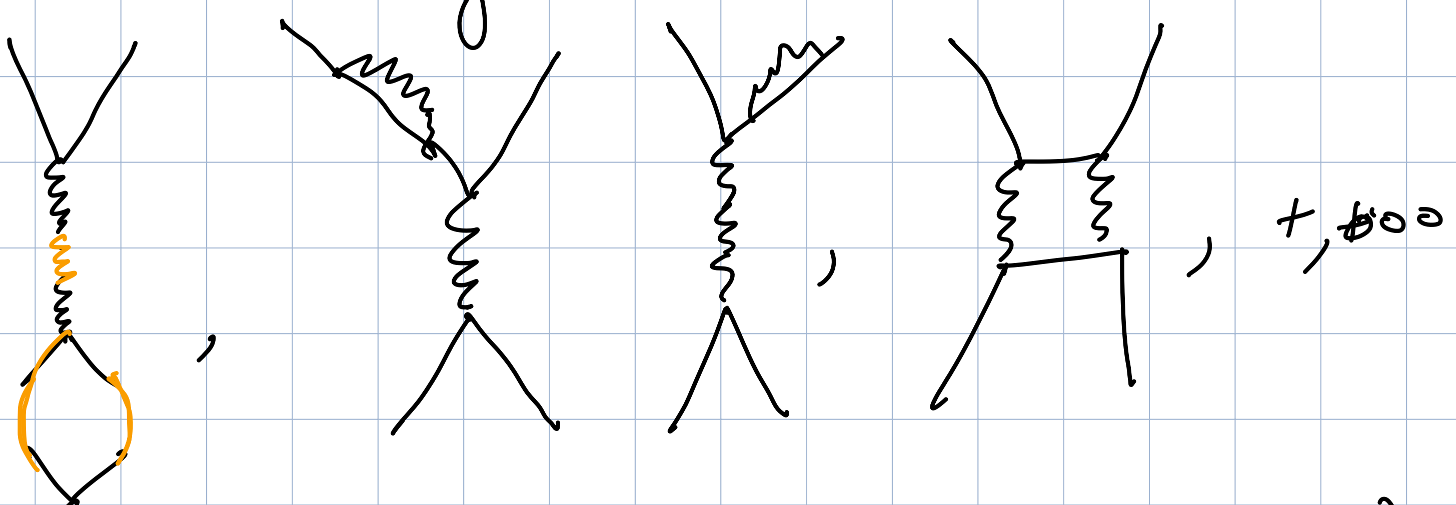


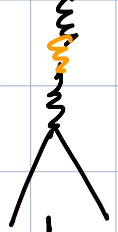

$$\mathcal{M}^{tree}_{LO} = -e^2 \left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right] \frac{g_{\mu\nu}}{q^2} \left[\bar{u}(p_4)\gamma^{\nu}u(p_2)\right] \quad (1)$$

$q = p_2 - p_4 \equiv$ energy scale of the process.

- Subleading Order (NLO) $\mathcal{O}(e^4)$ ($\hookrightarrow$ 4 vertices each with $e$)

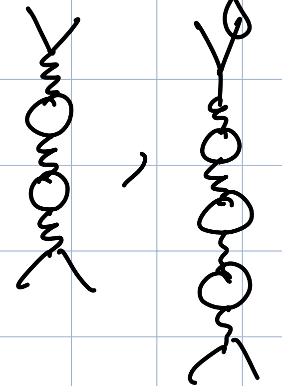
More than one diagrams.





Most interesting as it contains ○ divergences.

(In higher loops:



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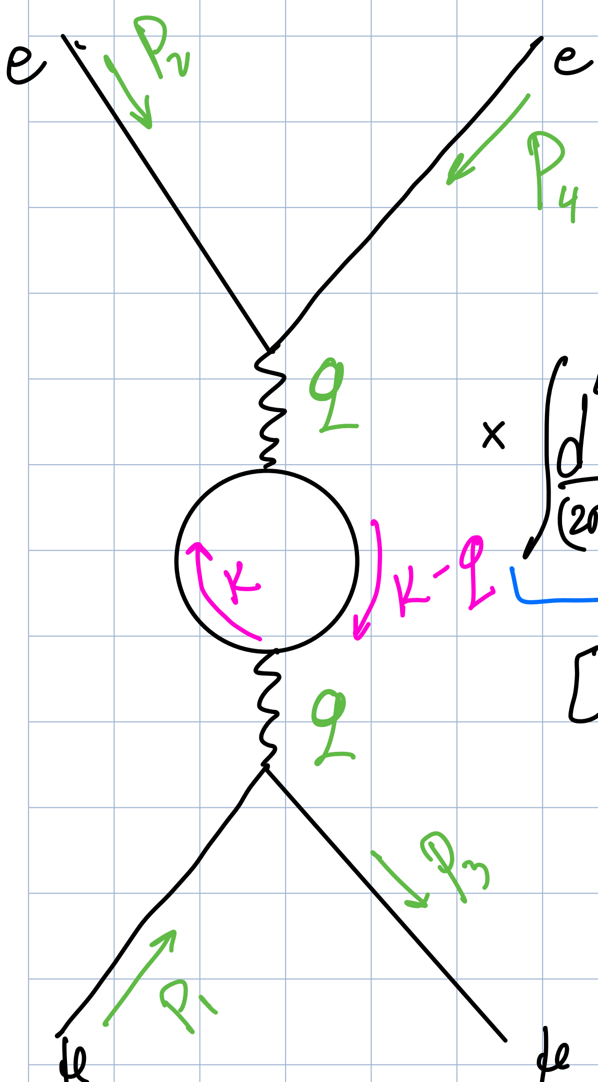



$$\mathcal{M}_{NLO} = -\frac{i e^4}{q^4}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \times$$

$$\times \underbrace{\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu \frac{(\not{k}+m)}{[k^2-m^2]} \gamma_\nu \frac{((\not{k}-\not{q})+m)}{(k-q)^2-m^2}\right]}_{-i g_{\mu\nu} \cdot I(q^2)} \times$$

$$\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right] \quad (2)$$

$I(q^2)$ : has logarithmic divergence

After algebraic manipulations we can rewrite the integral as:

$$I(q^2) = -\frac{1}{12\pi^2}\left[\underbrace{\int_{m^2}^{\infty} \frac{dp}{p}}_{\text{log - divergent}} - 6\underbrace{\int_0^1 dp(1-p)\,p \ln\left(1-\frac{q^2}{m^2}(1-p)p\right)}_{F\left(-\frac{q^2}{m^2}\right)}\right]$$

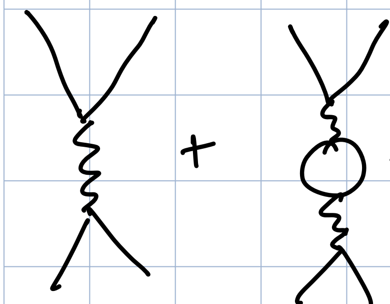
I need to develop a tool to treat the unphysical divergence that appeared in the theoretical calculation.

**Step 1** Regularization of the theory:

One way to do it: momentum cutoff

$$\int_{m^2}^{\infty} \frac{dp}{p} \rightarrow \int_{m^2}^{\Lambda^2} \frac{dp}{p} \rightarrow \ln\left(\frac{\Lambda^2}{m^2}\right) : \text{finite if } \Lambda \not\rightarrow \infty$$

Total Contribution to $\mathcal{M} = \mathcal{M}_{LO} + \mathcal{M}_{NLO} + \ldots$



(1)+(2)

$$= -e^2 \left[\text{spinor for } \mu\right] \frac{g^{\mu\nu}}{q^2} \left( \ldots\ldots \right) \left[\text{spinor for } e^-\right]$$

$$1 + \frac{e^2}{q^2} \frac{1}{12\pi^2} \left[ \ln\left(\frac{\Lambda^2}{m^2}\right) - F\left(\frac{-q^2}{m^2}\right) \right]$$

- Observation: $\mathcal{M}_{NLO}$ is proportional to $\mathcal{M}_{LO}$

$\mathcal{M}_{LO}$: would be the amplitude <u>if</u> there were no divergences and the $e$ is the same as the bare.

- If I redefine what I call an electron charge in theory:

$$e_R^2 \equiv e^2 \to e^2 + \frac{e^4}{q^2}\frac{1}{12\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - F\right]$$

the theory becomes finite.

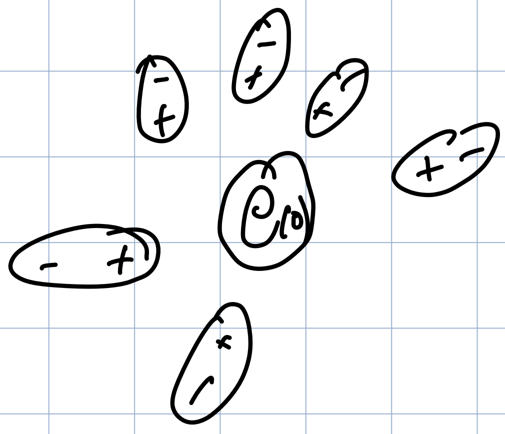
The procedure of rescaling theory parameters to absorbe divergences is the theory of <u>Renormalization</u>

- The renormalization procedure as done here is not <u>exact</u>!
  We only treated DIV up to 1-loop level.
- In QED to all loops we have geometric series (converges)

$$\alpha(q^2) = \frac{\alpha(0)}{1 - \frac{\alpha(0)}{3\pi}\ln\left(\frac{q^2}{m^2}\right)}$$

as $q^2 \uparrow$ $\alpha(0) \downarrow$

Observation: $e_R^2$ is actually $e_R^2(q^2)$
this is the screening effect:
in the vicinity of the charge, the vacuum becomes polarized which screens the charge of $e^-$



The higher the energy scale, the closest I get to e and therefore measure different charge.

# Key Points

★ Physical quantities depend on the energy scale of the experimental process

★ The cutoff $\Lambda$ is present in the bare calculation, but it is removed from renormalized quantities

★ In renormalizable theories, the regulator is removed in systematic way

# Key Points

★ Working in perturbation theory restricts renormalization procedure to a given order (n). In such a case:

- Draw all Feynman diagrams up to n-loop (including tree-level)

- Calculate contributions to amplitudes

- Regularize loop integrals. Results will depend on bare parameters and $\Lambda$

- Combine the bare parameters and regulator to define the renormalized quantities which are finite. Bare parameters are expressed in terms of measurable quantities

# Key Points

★ Once the relation between bare and renormalized quantities is known, one calculates transition amplitudes of interactions by:

- Writing the tree-level expressions with the bare parameters

- Replace the bare parameters with the renormalized ones

- Replace propagators and vertices with modified ones that contain the renormalized quantities