Renormalization and Improvement

Lecture 2

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EuroPLEx Summer School 2021 on
Lattice Field Theory and Applications

University of Edinburgh

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EUROPLEX
Summer School 2021 on lattice field theory and applications

OUTLINE OF LECTURE 1

- ★ Recap
- ★ Regularizations
- ★ Practice problem: photon self-energy
- ★ QCD - Running coupling

Recap of Lecture 1

- ★ Not all theories are renormalizable.
  The Standard Model is fully renormalizable

- ★ Theories are developed in the long-standing effort to describe physical phenomena.
  Divergences in theoretical calculations are accepted as long as there is a systematic way to treat them with a finite number of counter terms.

- ★ Renormalization is necessary to connect theoretical results to physical quantities

- ★ Two-step procedure:
  - implementation of a regulator (PV, DR, etc)
  - Removal of divergences and dependence on regulator with a renormalization scheme (prescription not unique)

Key Points

- ★ Physical quantities depend on the energy scale of the experimental process
- ★ The cutoff Λ is present in the bare calculation, but it is removed from renormalized quantities
- ★ In renormalizable theories, the regulator is removed in systematic way

Key Points

★ Working in perturbation theory restricts renormalization procedure to a given order (n). In such a case:

- Draw all Feynman diagrams up to n-loop (including tree-level)

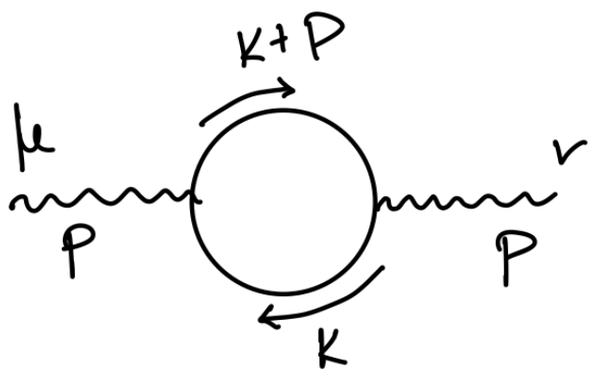
- Calculate contributions to amplitudes

- Regularize loop integrals. Results will depend on bare parameters and Λ

- Combine the bare parameters and regulator to define the renormalized quantities which are finite. Bare parameters are expressed in terms of measurable quantities

Practice Problem

Photon self-energy



$$i\Pi_2^{\mu\nu}(p) = \underbrace{(-1)}_{\text{closed fermion loop}}(-ie^2)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu i\,\frac{(\not k+m)}{k^2+m^2}\,\gamma^\nu i\,\frac{(\not p+\not k+m)}{(p+k)^2+m^2}\right]$$

$$i\Pi_2^{\mu\nu}(p) = i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\underbrace{\Pi_2(p^2)}_{\text{scalar quantity}}$$

★ **Answer for pole contribution:**

$$-\frac{8e^2}{(2\pi)^2}\, i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{6\varepsilon}$$

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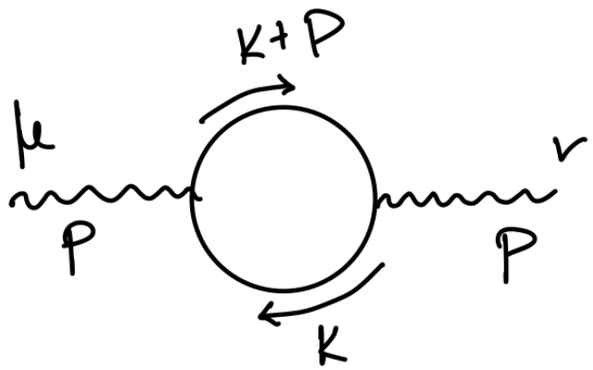

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closed fermion loop

- ★ Calculate the pole in DR
- ★ Based on symmetry properties and Ward identities we expect

$$i\Pi_2^{\mu\nu}(p) = i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\Pi_2(p^2)$$

↳ scalar quantity

Feynman Parameters:

- $$\frac{1}{A\cdot B^n} = \int_0^1 dx\, dy\, \delta(x+y-1)\, \frac{n\, y^{n-1}}{[xA+yB]^{n+1}}$$

D-dimensional Integrals (t'Hooft & Veltman)

- $$\int d^Dp\, \frac{1}{(p^2+2kp+m^2)^a} = \frac{i\pi^{D/2}}{(m^2-k^2)^{a-D/2}}\, \frac{\Gamma(a-D/2)}{\Gamma(a)}$$
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Regularization

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- ★ Pauli-Villar mass regulator:
  Fictitious particle with mass Λ, which makes the integral $\int_{-\infty}^{\infty} dp$ convergent due to addition of a propagator $-\frac{\Lambda^2}{q^2 - \Lambda^2}$ which goes to 1 as $\Lambda \to \infty$

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- ★ Dimensional regularization:
  calculation in $D = 4 - 2\epsilon$ dimensions, $\int \frac{dk^4}{(2\pi)^4} \to \int \frac{dk^D}{(2\pi)^D}$, $\frac{1}{\epsilon^n}, (n > 0)$ divergences.
  ϵ not an integer. The dependence on ϵ is removed by renormalization (e.g., $\overline{\mathrm{MS}}$)
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- ★ Lattice regularization:
  (discretization of space time without the additional parameters in theory
  $\int_{-\infty}^{\infty} dp \to \int_{-\pi/a}^{+\pi/a} dp$ (UV cutoff: a^{-1}, $a \to 0$)
  $\int dp f(p) \to \sum_{n=0}^{N_{\max}} \frac{2\pi}{L} F(p_0 + \frac{2\pi n}{L})$ (IR cutoff: L^{-1}, $L \to \infty$)

QCD

QED vs QCD

QED

- ★ Description of interaction between light and matter
- ★ Types of charge: ±
- ★ Force mediator: photons
- ★ No photon self-interaction
- ★ Abelian U(1) group
- ★ Charge screening effect
- ★ Mass mechanism: dominance of Higgs mechanism
  (e.g., Hydrogen mass ~ mass of e^- plus mass of p^+)

QCD

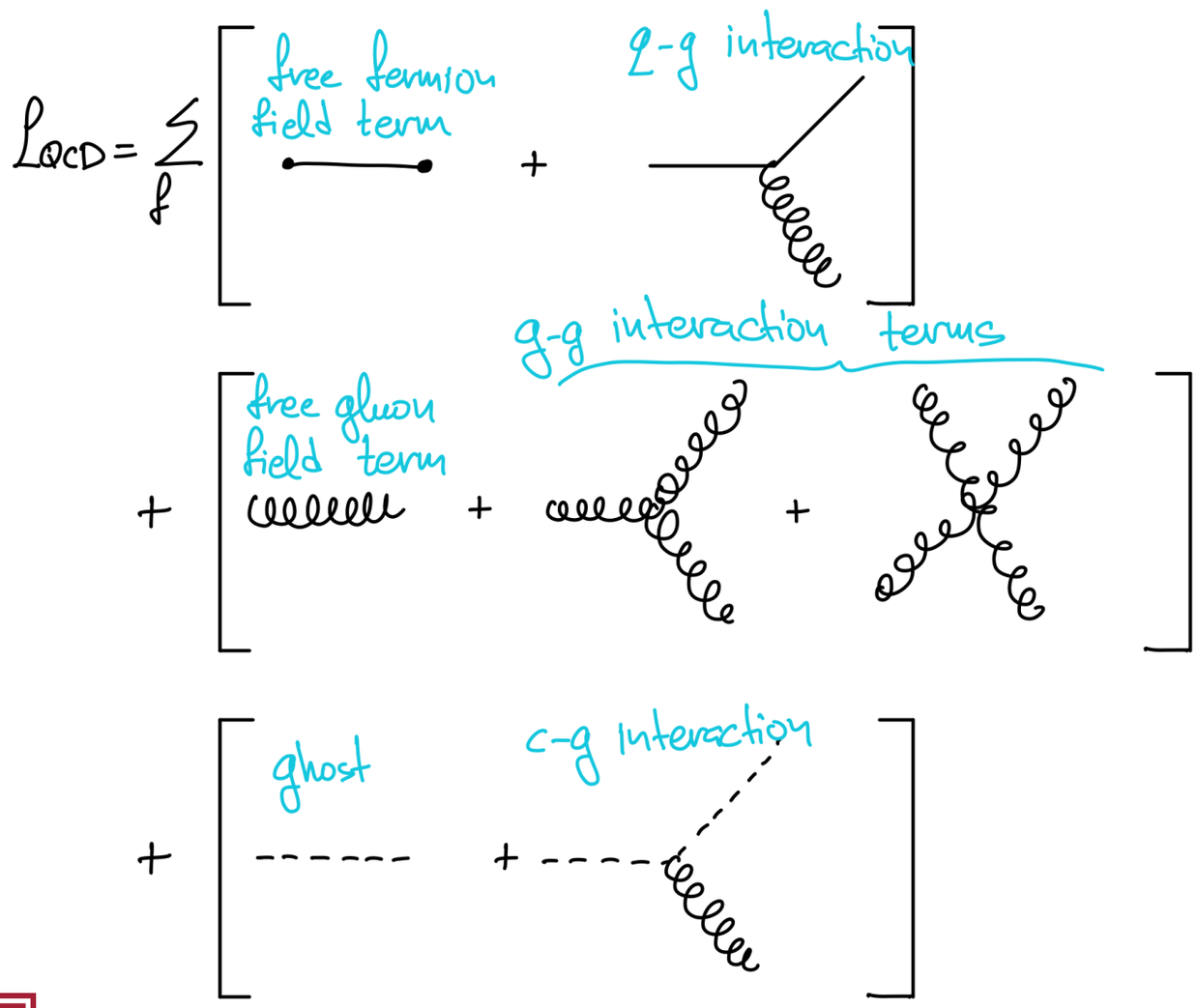
- ★ Description of strong interaction between quarks and gluons
- ★ Types of color charge: RGB
- ★ Force mediator: gluons
- ★ Gluon self-interaction
- ★ Non-Abelian SU(3) group
- ★ Color anti-screening effect
- ★ Mass mechanism: dominance of QCD dynamics
  (e.g., proton mass >> quark mass)

QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c$$

$$F^a_{\mu\nu} \rightarrow \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

★ Pictorial Representation



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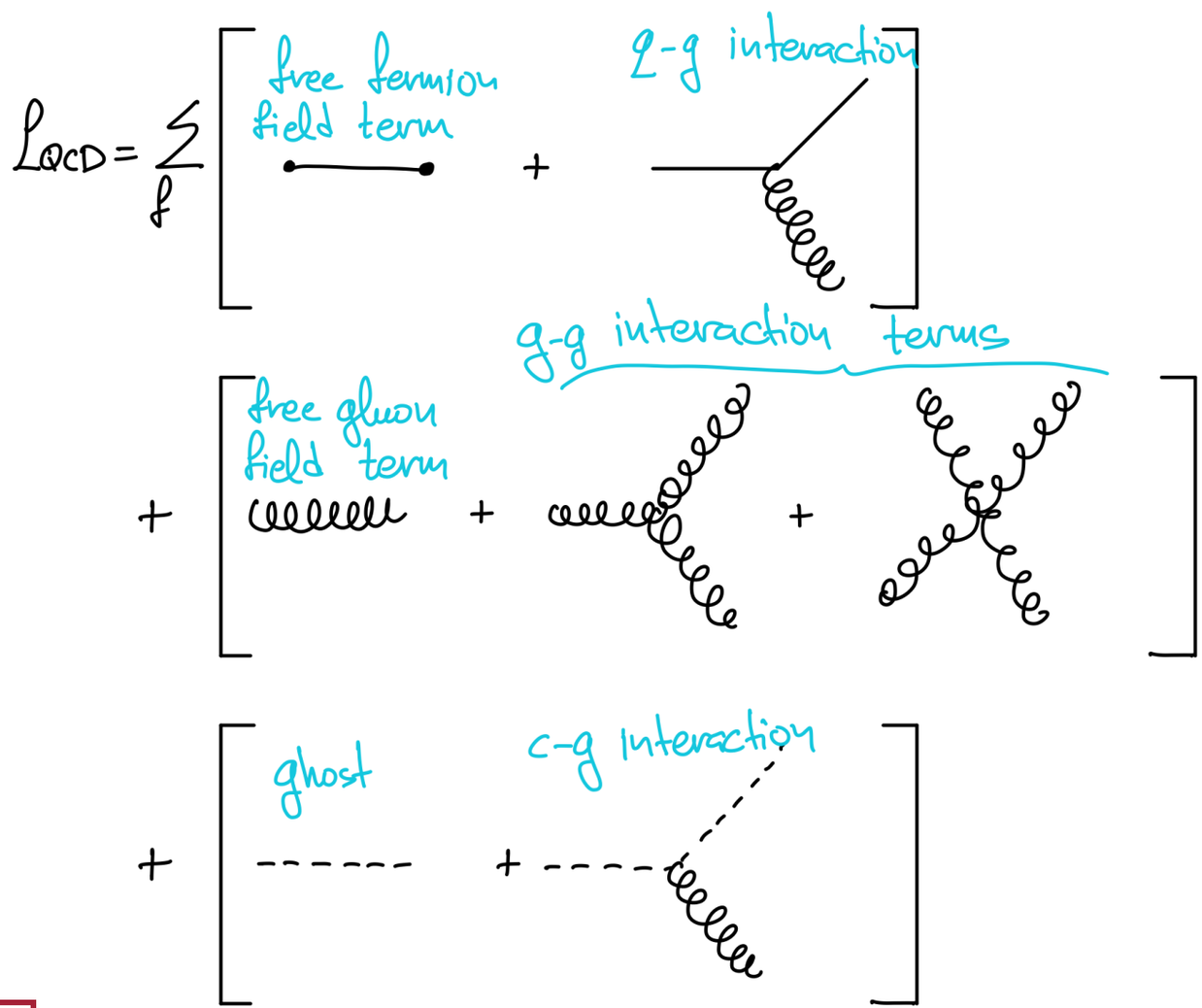
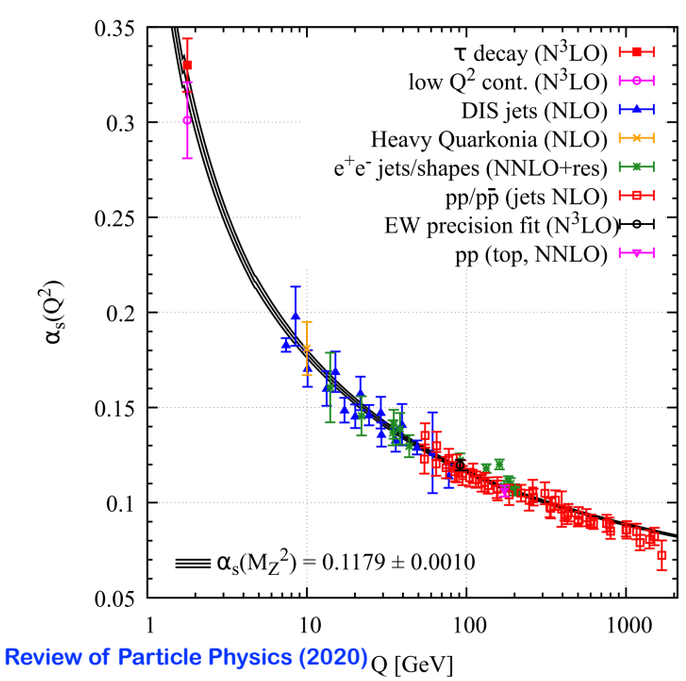


Interaction vertices make QCD calculations more complicated than QED

Confinement enforces non-perturbative solutions



Review of Particle Physics (2020)

Running Coupling - Renormalization Group Equation

- Assume we calculate a dimensionless physical quantity $\mathcal{O}$ in perturbation theory. For simplicity, $\mathcal{O}$ depends on α_s and an energy scale Q ($Q >> 1$), e.g., the energy of scattering process

- $\mathcal{O}$ is renormalized in some scale μ, thus $\mathcal{O} = \mathcal{O}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right)$

- The physical $\mathcal{O}$ must be independent of μ, that is

$$\boxed{\mu^2 \frac{d\mathcal{O}}{d\mu^2} = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial a_s}{\partial \mu^2} \frac{\partial}{\partial a_s}\right)\mathcal{O} = 0}$$

RGE

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RGE

- Any dependence on μ should be canceled by an appropriate dependence of α_s on μ. Frequently, one choses $\mu^2 = Q^2$

Running Coupling - Renormalization Group Equation

- We define the β function as: $$\boxed{\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \equiv \beta(\alpha_s(\mu))}$$
- Practical use of $\beta(\alpha_s)$: if α_s fixed at some scale, this relation can be used to evolve results to another scale through RGE

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★ $\beta(\alpha_s)$ known from perturbation theory with known coefficients:

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 + \mathcal{O}(\alpha_s^5)$$

$$\beta_0 = \frac{33 - 2N_f}{12\pi}$$

$$\beta_1 = \frac{153 - 19N_f}{24\pi^2}$$

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★ One-loop approximation for α_s

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = -b_0 \alpha_s^2 \Rightarrow \int_{\alpha(\mu)}^{\alpha(Q)} \frac{d\alpha_s}{\alpha_s^2} = -b_0 \int_{\mu}^{Q} \frac{d\mu^2}{\mu^2} \Rightarrow$$

$$-\frac{1}{\alpha_s(Q^2)} + \frac{1}{\alpha_s(\mu^2)} = -b_0 \ln\left(\frac{Q^2}{\mu^2}\right) \Rightarrow \boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2) b_0 \ln\left(\frac{Q^2}{\mu^2}\right)}}$$

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Governs the evolution from one energy scale (μ) to another (Q)

Running Coupling - RGE

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\alpha_s(\mu^2)\beta_0 \ln\left(\frac{Q^2}{\mu^2}\right)}}$$

Running Coupling - RGE

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Already at 1-loop we see asymptotic freedom:

★ For $Q^2 >> \mu^2$, $\alpha_s(Q^2)$ decreases if $\beta_0 < 0$.

This happens for $N_f < \frac{11N_C}{2} \leq 16$

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Crucial parameters of QCD: the energy scale at which $\alpha_s(Q^2) \to \infty$: Λ_{QCD}

★ 1-loop level:

$$-\frac{1}{\alpha_s(Q^2)} + \cancel{\frac{1}{\alpha_s(\Lambda_{QCD})}}^{\,0} = -\beta_0 \ln\left(\frac{Q^2}{\Lambda_{QCD}^2}\right) \Rightarrow \alpha_s(Q^2) = \frac{1}{\beta_0 \ln\left(\frac{Q^2}{\Lambda_{QCD}^2}\right)}$$

but $\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\alpha_s(\mu^2)\beta_0 \ln\left(\frac{Q^2}{\mu^2}\right)} \Rightarrow \boxed{\Lambda_{QCD}^2 = \frac{\mu^2}{e^{1/\beta_0 \alpha_s(\mu^2)}}}$

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$\Lambda_{\rm QCD}$ determines the non-perturbative region

For $N_f = 4$,

$\Lambda_{\rm QCD} \sim 300\,\text{MeV}$

Lattice Regularization

- ★ Most regularizations (PV, DR, etc) are perturbative, and remove the divergences order by order in perturbation theory

- ★ Advantage of Lattice regularization: non-perturbative. Particularly useful for studying the low energy regime of QCD, where the hadronic physics is

- ★ Preserves chirality and gauge invariance simultaneously

- ★ The applicability of Lattice QCD can also be perturbative. Perturbative renormalization was traditionally used until about 10 years ago

Renormalization on the Lattice

- ★ Lattice perturbation theory was extensively used to renormalize lattice data of matrix elements of operators, and parameters of the QCD Lagrangian in the past

- ★ In 1995 ideas for non-perturbative renormalization have been implemented
  [Martinelli et al., Nucl. Phys. B445, 81, arXiv:hep-lat/9411010]

- ★ Currently, non-perturbative renormalization prescriptions are mostly used

- ★ Lattice perturbation theory is still a useful tool for several reasons

Renormalization Functions

- ★ Renormalization functions (Z-factors) depend on the lattice formulation and other parameters of the discretized action employed

- ★ Z-factors contain divergent parts as well as finite.
  - The regulator independent divergences are universal
  - Finite parts depend on action parameters

- ★ Unless one studies matrix elements of conserved currents, the operators must be renormalized

- ★ Typically, results are converted to a scheme. Convenient choice: $\overline{\mathrm{MS}}$ scheme, where Wilson coefficients are also calculated

Perturbative Renormalization

- ★ In QCD, there are infinite number of interaction vertices.

  However, in order-by-order in perturbation theory, only a finite number of vertices is needed (vertices are accompanied by factors of the bare coupling constant)

- ★ Lattice QCD perturbation theory is much more complicated than continuum QCD perturbation theory:

  - more vertices and more diagrams

  - expressions for propagators and vertices can become very lengthy

Perturbative Renormalization

RI-MOM scheme

- ★ Regularization Independent momentum subtraction (RI-MOM) schemes naturally defined in perturbation theory

- ★ Calculation of Green functions of operators at given off-shell external states with momentum p

- ★ A condition is applied on the Green functions to match them with their tree-level value

- ★ Examples of RI-type conditions:

For fermion field

$$Z_q^{\mathrm{RI}} = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

For operator $\bar{\psi}\Gamma\psi$

$$(Z_q^{\mathrm{RI}})^{-1}\, Z_\Gamma^{\mathrm{RI}}\, \mathrm{Tr}\left[G_\Gamma^L(p)\, G_\Gamma^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2} = \mathrm{Tr}\left[G_\Gamma^{\mathrm{tree}}(p)\, G_\Gamma^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$